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Distributed finite-time containment control for double-integrator multi-agent systems

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Abstract

In this paper, the distributed finite-time containment control problem for double-integrator multi-agent systems with multiple leaders and external disturbances is discussed. In the presence of multiple dynamic leaders, by utilizing the homogeneous control technique, a distributed finite-time observer is developed for the followers to estimate the weighted average of the leaders' velocities at first. Then based on the estimates and the generalized adding a power integrator approach, distributed finite-time containment control algorithms are designed to guarantee that the states of the followers converge to the dynamic convex hull spanned by those of the leaders in finite time. Moreover, as a special case of multiple dynamic leaders with zero velocities, the proposed containment control algorithms also work for the case of multiple stationary leaders without using the distributed observer. Simulations demonstrate the effectiveness of the proposed control algorithms.

Index Terms

Distributed control, containment control, finite-time control, second-order systems, multi-agent systems

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I. Introduction

In recent years, as a specific problem of complex networks, distributed cooperative control for multi-agent systems has attracted more and more research attention. This is due to its broad applications (e.g., formation control [1]–[4], flocking [5], [6], rendezvous [7], [8], etc.) and its advantages (e.g., better efficiency, higher robustness, less communication requirement, etc.) compared with the traditional centralized coordination control approaches.

In the distributed cooperative control field, most of the existing results reported in the literature concentrate on two fundamental problems. One is the consensus problem for leaderless multi-agent systems, which is also called as the synchronization problem in complex networks (for more details about synchronization in complex networks, see [9], [10]). The consensus of multi-agent systems means that all the agents reach the agreement on a common state by implementing appropriate consensus control laws. Recently, consensus algorithms have been extensively studied for first-order [11]–[13], second-order [13]–[18] and high-order [19], [20] multi-agent systems. The other fundamental problem is the consensus tracking problem for leader-follower multi-agent systems. In this case, the control objective is to drive the states of the followers to track the state of the single leader. Control algorithms for this problem have been reported in [13], [14], [18]–[26].

Different from the leaderless and the leader-follower consensus problems, a more challenging problem in distributed cooperative control is the containment control problem for multi-agent systems with multiple leaders, which is also an extension of the leader-follower consensus problem to the multi-leader case. In this case, the control objective is to drive the states of the followers into the convex hull spanned by those of the leaders. The containment control problem is also very important since the multiple leaders are useful to achieve effectively the containment or guidance of an agent group in a target region [27]. Moreover, the study of containment control stems from numerous natural phenomena and potential applications. For examples, the male silkworm months will end up in the convex hull spanned by all the female silkworm moths by detecting pheromone released by females; for a vehicle group moving to a target place, the followers will stay in the safe area formed by the leaders when close to the hazardous obstacles, where the vehicles which are equipped necessary sensors to detect the obstacles paly the role of the leaders and the others are the followers. As a result, the containment control is meaningful for many practical problems, such as UAV [28], autonomous underwater vehicle (AUV) [29] formation control, and robot swarms [30].

For the first-order multi-agent systems, several containment control algorithms have been proposed recently in [31]–[33]. Since many practical individual systems, especially mechanical systems, are of second-order dynamics, it is significant and necessary to study containment control algorithms for the second-order multi-agent systems [33]–[37]. In [33], both continuous-time and sampled-data based containment control algorithms were proposed for double-integrator multi-agent systems with multiple dynamic leaders. [34] investigated the containment control problem for double-integrator multi-agent systems under random switching topologies. In [35], attitude containment control algorithms were proposed for multiple rigid bodies. [36] focused on the problem of distributed second-order multi-agent tracking of a convex set specified by multiple dynamic leaders under jointly connected switching topologies. [37] studied the containment control problem for networked Lagrangian systems with multiple dynamic leaders in the presence of parametric uncertainties under a directed graph. Note that all the control algorithms proposed in the aforementioned literature provide asymptotic convergence, which means that convergence rates of the closed-loop systems are at best exponential with infinite settling time. In other words, the states of the followers can not converge to the convex hull spanned by those of the leaders in finite time. To this end, considering the convergence rates, finite-time containment control algorithms are more desirable.

Besides faster convergence rates, the closed-loop systems with finite-time convergence usually demonstrate some other superiorities, such as better disturbance rejection properties and better robustness against uncertainties [18], [38]. Because of the above superiorities, some kinds of finite-time containment control algorithms have been developed for second-order multi-agent systems [39], [40]. In [39], for the multiple rigid bodies with multiple stationary and dynamic leaders, homogeneous and nonsingular terminal sliding mode control techniques were used to design finite-time attitude containment algorithms, respectively. However, two main problems exist there. One is that in the stationary leader case, the proposed distributed control law for each follower needs the information from its neighbors' neighbors, which is difficult to obtain in practice. The other is that the switching control scheme proposed for the dynamic leader case is also somehow impractical, since to obtain the switching time instant (or the finite settling time of the distributed sliding mode observer), some global information of the agent communication topology and the bound on the accelerations of the leaders are required to be known. In [40], for the double-integrator multi-vehicle systems with multiple dynamic leaders under a fixed directed network topology, based on the homogeneous control method presented in [41], finite-time containment control laws were proposed. Nevertheless, the dynamic leaders are required to have an identical velocity there, which is somehow harsh for practical implementations.

Considering various potential applications of the containment control, the superiorities of the finite-time control, and also to improve the aforementioned problems in the existing literature on the distributed finite-time containment control for second-order multi-agent systems, in this paper, distributed finite-time containment control algorithms are proposed for double-integrator multi-agent systems with multiple dynamic or stationary leaders in the presence of external disturbances. At first, in the dynamic leader case, the finite-time containment control is achieved by integrating the adding a power integrator technique [42], [43] (to design the distributed control laws), the homogeneous control method [41] (to design the distributed observer), and the graph theory. Then the proposed control algorithms are shown to be also able to cope with the stationary leader case without using the distributed observer.

The main contributions of this paper are fourfold. Firstly, this paper extends the result in our previous work [18] from the single-leader case to the multi-leader case. Specifically, compared with the single-leader case, the communication subgraph of the followers in the multi-leader case is not required to be connected while the whole agent communication topology becomes more complex due to the presence of multiple leaders. In other words, in contrast with their single-leader counterparts, the main difficulty in design and analysis of algorithms with multiple leaders is that the followers need to use more limited information to achieve more complicated collective behaviors. A case in point is that, in the dynamic leader case, the finite-time convergence proof on the distributed observer is more difficult than its single-leader counterpart (see more in Remark 2). Secondly, in the case of multiple stationary leaders, for each follower, the proposed distributed control law in this paper requires less information than that designed in [39], namely, only information from its neighbors (see more in Remark 6). Thirdly, in this paper, in the case of multiple dynamic leaders, based on a distributed finite-time observer without using any global information of the agent communication topology and the bound information on the accelerations of the leaders, non-switching containment control laws are proposed, which are more convenient in practical implementations than those designed by using the switching control scheme presented in [39] (see more in Remark 5). The last but not least, also in the dynamic leader case, the proposed distributed control laws in this paper achieve finite-time convergence without requiring that the velocities of the dynamic leaders be identical, while which is required in [40].

The remainder of this paper is organized as follows. In Section II, some useful preliminaries and problem formulation are exhibited. In Section III, the main result, i.e., the distributed finite-time containment control scheme, is presented. Some simulations are performed in Section IV. Finally, conclusions are drawn in Section V.

II. Preliminaries and problem formulation

A. Notations

Denote $\mathrm{sig}^{\alpha}(x) = \mathrm{sgn}(x)|x|^{\alpha}$, where $x, \alpha \in R$ and $\mathrm{sgn}(\cdot)$ is the standard sign function. Given a vector $x = [x_1, \cdots, x_n]^T \in R^n$ and $\alpha \in R$, let $x^{\alpha} = [x_1^{\alpha}, \cdots, x_n^{\alpha}]^T, \mathrm{sig}^{\alpha}(x) = [\mathrm{sig}^{\alpha}(x_1), \cdots, \mathrm{sig}^{\alpha}(x_n)]^T$, especially, $\mathrm{sgn}(x) = [\mathrm{sgn}(x_1), \cdots, \mathrm{sgn}(x_n)]^T$. Let $\|x\| = \sqrt{x^{\mathrm{T}}x}$ denote the Euclidean norm of vector x. Let $P > 0$ denote a symmetric positive definite matrix P. Let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively. For brevity, let $\mathbf{1}_n = [1, \cdots, 1]^T \in R^n$. Let I_p denote $p \times p$ identity matrix, where p is a positive integer.

B. Useful lemmas and definitions

Lemma 1: [38] Consider the system $\dot{x} = f(x), f(0) = 0, x \in R^n$, there exist a positive definite continuous function $V(x) : U \to R$, real numbers $c > 0$ and $\alpha \in (0, 1)$, and an open neighbor $U_0 \subset U$ of the origin such that $\dot{V}(x) + c(V(x))^{\alpha} \leq 0, x \in U_0 \setminus \{0\}$. Then $V(x)$ approaches 0 in finite time. In addition, the finite settling time T satisfies that $T \leq \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$.

Lemma 2: [43] For any real numbers $x_i, i = 1, \cdots, n$ and $0 < q \leq 1$, the following inequality holds $(\sum_{i=1}^{n} |x_i|)^q \leq \sum_{i=1}^{n} |x_i|^q$. When $0 < q = q_1/q_2 \leq 1$, where q_1, q_2 are odd integers, then $|x^q - y^q| \leq 2^{1-q}|x - y|^q$.

Lemma 3: [43] If $c > 0, d > 0$ and $\gamma(x, y) > 0$ is a real-valued function for $x \in R, y \in R$, then $|x|^c|y|^d \leq \frac{c\gamma(x,y)|x|^{c+d}}{c+d} + \frac{d\gamma^{-c/d}(x,y)|y|^{c+d}}{c+d}$.

Lemma 4: [44] Given matrices A and B with compatible sizes, then $(A \otimes B)^T = A^T \otimes B^T, (A \otimes I_p)(B \otimes I_p) = AB \otimes I_p$, where $\otimes$ denotes the Kronecker product.

Definition 1: [39] Let X be a set in a real vector space $V \subseteq R^p$, where p is a positive integer. The convex hull $\text{Co}(X)$ of the set X is defined as $\text{Co}(X) = \{\sum_{i=1}^{k} a_i x_i | x_i \in X, a_i \in R, a_i \geq 0, \sum_{i=1}^{k} a_i = 1, k = 1, 2, \cdots\}$.

C. Graph theory notions

Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph, where $\mathcal{V} = \{v_1, \cdots, v_n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and $\mathcal{A} = [a_{ij}] \in R^{n\times n}$ is the weighted adjacency matrix of the graph G. The node indexes belong to a nonempty finite index set $\Gamma = \{1, \cdots, n\}$. An edge (v_i, v_j) denotes that node v_j can access information from node v_i and v_i is said to be a neighbor of v_j, but not necessarily vice versa. The set of neighbors of node v_i is denoted as $N_i = \{v_j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\}$. In addition, an undirected graph G is defined such that $(v_j, v_i) \in \mathcal{E}$ implies $(v_i, v_j) \in \mathcal{E}$. In a directed graph, a directed path is a sequence of edges of the form $(v_{k_1}, v_{k_2}), (v_{k_2}, v_{k_3}), \cdots, k_i \in \Gamma$. An undirected path in an undirected graph is defined analogously. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{n\times n}$ associated with the directed graph G is defined such that $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ while $a_{ij} = 0$ otherwise. For an undirected graph, we assume that $a_{ij} = a_{ji}$. Moreover, we assume that $a_{ii} = 0, \forall i \in \Gamma$. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in R^{n\times n}$ associated with $\mathcal{A}$ is defined as $l_{ii} = \sum_{j\in N_i} a_{ij}$ and $l_{ij} = -a_{ij}$, where $i \neq j$. Obviously, zero is an eigenvalue of $\mathcal{L}$ with an associated eigenvector $\mathbf{1}_n$. Note that matrix $\mathcal{L}$ is symmetric for an undirected graph while not necessarily symmetric for a directed graph.

D. Problem formulation

First, in this paper, the multi-agent systems to be studied are of the form

$$\dot{x}_i(t) = v_i(t),\ \dot{v}_i(t) = u_i(t) + d_i(t),\ i \in F \bigcup L, \tag{1}$$

where $x_i(t) = [x_{i1}(t), \cdots, x_{ip}(t)]^T, v_i(t) = [v_{i1}(t), \cdots, v_{ip}(t)]^T, u_i(t) = [u_{i1}(t), \cdots, u_{ip}(t)]^T \in R^p$ are the position, velocity and control input, respectively, p is a positive integer, $d_i(t) = [d_{i1}(t), \cdots, d_{ip}(t)]^T \in R^p$ represents a bounded external disturbance satisfying $\|d_i(t)\| \leq h$ with h being a positive constant, associated with the i-th agent, $i \in F \bigcup L$, and $F = \{1, \cdots, n\}$ and $L = \{n+1, \cdots, n+m\}$ represent the follower set and the leader set, respectively. For the leaders, the following natural assumption is made.

Assumption 1: For the leaders of multi-agent system (1), $x_j(t), v_j(t), \dot{v}_j(t), j \in L$ are all bounded $\forall\ t \in [0, \infty)$.

Second, the communication topology of multi-agent system (1) can be described by a directed graph $G_{n+m} = (\mathcal{V}_{n+m}, \mathcal{E}_{n+m}, \mathcal{A}_{n+m})$ with m leader nodes and n follower nodes. A node is called a follower if the node has at least one neighbor. Otherwise, the node is called a leader. $\mathcal{A}_{n+m} = [a_{ij}] \in R^{(n+m)\times(n+m)}$ and $\mathcal{L}_{n+m} = [l_{ij}] \in R^{(n+m)\times(n+m)}$ denote the adjacency and the Laplacian matrices of the graph G_{n+m}, respectively. For brevity, we use $\mathcal{A}$ and $\mathcal{L}$ to replace $\mathcal{A}_{n+m}$ and $\mathcal{L}_{n+m}$ later in this paper, respectively. Let $G_n^F = (F, \mathcal{E}^F, A_F)$ and $G_m^L = (L, \mathcal{E}^L, A_L)$ denote the follower and the leader communication topologies, respectively. Assume that the leaders do not communicate with each other, which implies that $\mathcal{E}^L = \emptyset$. The communication between different followers are bidirectional, namely, G_n^F is an undirected graph. In addition, the communication between a leader and a follower is unidirectional with the leader issuing the communication. Thus, each entry of the last m rows of the Laplacian matrix $\mathcal{L}_{n+m}$ is zero and $\mathcal{L}_{n+m} = \begin{bmatrix} \mathcal{T} & \mathcal{T}_d \\ 0_{m\times n} & 0_{m\times m} \end{bmatrix}$, where $\mathcal{T} = [T_{ij}] \in R^{n\times n}, \mathcal{T}_d \in R^{n\times m}$. On the communication topology G_{n+m} of system (1), the following natural assumption is made.

Assumption 2: For each follower of multi-agent system (1), there exists at least one leader that has a path to the follower.

Lemma 5: [39] Under Assumption 2, matrix $\mathcal{T}$ is positive definite. In addition, each entry of $-\mathcal{T}^{-1}\mathcal{T}_d$ is nonnegative and each row sum of $-\mathcal{T}^{-1}\mathcal{T}_d$ is equal to one.

For brevity, we denote $x_F = [x_1^T, \cdots, x_n^T]^T, v_F = [v_1^T, \cdots, v_n^T]^T, x_L = [x_{n+1}^T, \cdots, x_{n+m}^T]^T, v_L = [v_{n+1}^T, \cdots, v_{n+m}^T]^T$, $x_d = [x_{d1}^T, \cdots, x_{dn}^T]^T = -(\mathcal{T}^{-1}\mathcal{T}_d \otimes I_p)x_L$ and $v_d = [v_{d1}^T, \cdots, v_{dn}^T]^T = \dot{x}_d$, where $x_{di} = [x_{di1}, \cdots, x_{dip}]^T, v_{di} = [v_{di1}, \cdots, v_{dip}]^T \in R^p, i \in F$. From Definition 1 and Lemma 5, $x_F \to x_d$ (i.e., $x_i \to x_{di}, i \in F$) means that $x_i, i \in F$ converge to the convex hull $\mathrm{Co}\{x_j, j \in L\}$.

Based on the above descriptions, the objective of this paper is to achieve distributed finite-time containment control for multi-agent system (1), i.e., to design the distributed control laws for system (1) such that $x_i \to \mathrm{Co}\{x_j, j \in L\}, i \in F$ (specifically, $x_i \to x_{di}, v_i \to v_{di}, i \in F$) in finite time under Assumptions 1-2.

III. Main result

Without loss of generality, the leaders of multi-agent system (1) are presumed to be dynamic. Actually, stationary leaders can also be regarded as dynamic ones but with zero velocities. The distributed finite-time control design mainly consists of two parts. First, a distributed finite-time observer is proposed for the followers to obtain the accurate estimates of the weighted average of the leaders' velocities in finite time. Second, based on the estimated weighted average of the leaders' velocities, distributed finite-time containment control laws are proposed for system (1).

A. Distributed finite-time observer design

Let $\hat{v}_{di} = [\hat{v}_{di1}, \cdots, \hat{v}_{dip}]^T, i \in F$ denote the estimate of v_{di} with respect to the i-th follower and $\hat{v}_d = [\hat{v}_{d1}, \cdots, \hat{v}_{dn}]^T$. For multi-agent system (1), the distributed observer is designed as

$$\dot{\hat{v}}_{di} = \frac{1}{T_{ii}} \sum_{j \in F \bigcup L} a_{ij} \dot{\hat{v}}_{dj} - \frac{k}{T_{ii}} \text{sig}^{\alpha} \left(\sum_{j \in F \bigcup L} a_{ij} (\hat{v}_{di} - \hat{v}_{dj}) \right), \ i \in F, \tag{2}$$

where $\hat{v}_{dj} = v_j, j \in L, k > 0, 0 < \alpha < 1$, T_{ii} is the (i,i) entry of matrix $\mathcal{T}$ defined in Lemma 5. For brevity, denote $e_i = [e_{i1}, \cdots, e_{ip}]^T = \sum_{j \in F \bigcup L} a_{ij}(\hat{v}_{di} - \hat{v}_{dj}), i \in F$ and $e = [e_1^T, \cdots, e_n^T]^T$.

Proposition 1: Under Assumption 2, the distributed observer (2) is globally finite-time convergent, namely, $\hat{v}_{di} \to v_{di}, i \in F$ in a finite time T_0 satisfying $T_0 = \max_{\forall i \in F, 1 \le l \le p} \left\{ \frac{|e_{il}(0)|^{1-\alpha}}{k(1-\alpha)} \right\}$.

Proof: Based on the definition of e_i, the equality (2) can be rewritten as

$$\dot{e}_i = -k \text{sig}^{\alpha}(e_i), \ i \in F. \tag{3}$$

By direct integration on (3), the convergence time T_i of e_i is $T_i = \max_{1 \le l \le p} \left\{ \frac{|e_{il}(0)|^{1-\alpha}}{k(1-\alpha)} \right\}, i \in F$. Hence, $e \to 0$ in a finite time $T_0 = \max_{\forall i \in F} T_i = \max_{\forall i \in F, 1 \le l \le p} \left\{ \frac{|e_{il}(0)|^{1-\alpha}}{k(1-\alpha)} \right\}$.

Denote $\bar{v}_{dj} = [\bar{v}_{dj1}, \cdots, \bar{v}_{djp}]^T = \hat{v}_{dj} - v_{dj}, j \in F$. Then $e_i = \sum_{j \in F \bigcup L} l_{ij} \hat{v}_{dj} = \sum_{j \in F} T_{ij} \bar{v}_{dj} + \sum_{j \in F} T_{ij} v_{dj} + \sum_{j \in L} l_{ij} v_j, i \in F$, where l_{ij}, T_{ij} are the (i,j) elements of matrices $\mathcal{L}$ and $\mathcal{T}$, respectively. Denote $\mathcal{B} = [b_{ij}] = -\mathcal{T}^{-1}\mathcal{T}_d \in R^{n \times m}$. Then $v_d = (\mathcal{B} \otimes I_p) v_L$ and thus $\sum_{j \in F} T_{ij} v_{dj} = \sum_{j=1}^n T_{ij} \sum_{k=1}^m b_{jk} v_{k+n} = \sum_{k=1}^m \left(\sum_{j=1}^n T_{ij} b_{jk} \right) v_{k+n}$. Also note that $\sum_{j=1}^n T_{ij} b_{jk} = -l_{ik+n}, i \in F, k = 1, \cdots, m$, where l_{ik+n} is equal to the (i,k) entry of matrix $\mathcal{T}_d$ because of $\mathcal{T}(-\mathcal{T}^{-1}\mathcal{T}_d) = -\mathcal{T}_d$. Then $\sum_{j \in F} T_{ij} v_{dj} = -\sum_{j \in L} l_{ij} v_j, i \in F$, which implies that $e_i = \sum_{j \in F} T_{ij} \bar{v}_{dj}, i \in F$.

From the above proof, we have $e = (\mathcal{T} \otimes I_p)(\hat{v}_d - v_d)$. Since $\mathcal{T}$ is invertible under Assumption 2, then $\hat{v}_d \to v_d$ (i.e., $\hat{v}_{di} \to v_{di}, i \in F$) in the finite time T_0. This completes the proof. ■

Remark 1: It is notable that the proof on $e = (\mathcal{T} \otimes I_p)(\hat{v}_d - v_d)$ is more difficult than its single-leader counterpart in [18], due to the more complex agent communication topology in the case of multiple leaders. Next, the structure of observer (2) will be analyzed. On one hand, observer (2) is designed based on the homogeneous control method [41]. Specifically, for the i-th ($i \in F$) follower, the terms $-\frac{k}{T_{ii}}\text{sig}^{\alpha}(\sum_{j\in F\bigcup L} a_{ij}(\hat{v}_{di} - \hat{v}_{dj}))$ in observer (2) aim to guarantee that the follower can obtain the accurate estimate of v_{di} in finite time, and the computation of $\dot{\hat{v}}_{di}$ depends on both its neighbors' states and their derivatives as in [16], [18]. Actually, the derivatives can be calculated by numerical differentiation. On the other hand, the distributed observer (2) is not suitable for the case of followers with cycles in their communication subgraph, because in this case, the interconnections among the followers are highly coupled and the computation of $\dot{\hat{v}}_{di}$ depends on the computation of $\dot{\hat{v}}_{dk}$ (for some k), which in turn depends on the computation of $\dot{\hat{v}}_{di}$, which is technically impractical.

Remark 2: From the proof of Proposition 1, the finite-time settling time T_0 of observer (2) depends on the communication topology structure, the initial states $\hat{v}_{di}(0), i \in F, v_j(0), j \in L$, and the control parameters k, α. By defining $f(\alpha) = \frac{x_0^{1-\alpha}}{k(1-\alpha)}$ for $\alpha \in (0,1)$ with $x_0 > 0$ being a constant, it follows that $\frac{\partial f(\alpha)}{\partial \alpha} = \frac{x_0^{1-\alpha}(1-\ln x_0^{1-\alpha})}{k(1-\alpha)^2}$. Therefore, under any admissible communication topology satisfying Assumption 2 and initial states, if fixing α, T_0 decreases as k increases. However, if fixing k, the monotonicity of T_0 on α is complex, which is also related to the communication topology structure and the aforementioned initial states.

B. *Distributed finite-time containment control design*

Based on the developed distributed finite-time observer (2), for multi-agent system (1) with multiple dynamic leaders, the control law u_i for the i-th follower is designed as

$$\begin{aligned} u_i = \dot{\hat{v}}_{di} - k_2 \left[(v_i - \hat{v}_{di})^{1/q} + k_1^{1/q} \left(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j) \right) \right]^{2q-1} \\ - k_4 \text{sgn} \left[(v_i - \hat{v}_{di})^{1/q} + k_1^{1/q} \left(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j) \right) \right], \ i \in F, \end{aligned} \tag{4}$$

where $\hat{v}_{di}$ is the estimate of $v_{di}, i \in F$ generated from observer (2), and the control parameters satisfy $k_1 \geq \frac{2^{1-q}}{1+q} + \frac{(\beta+n\eta)q}{1+q} + k_3, k_2 \geq (2-q)2^{1-q}k_1^{1+1/q}\left[\frac{(k_1+n\eta)2^{1-q}q}{k_1(1+q)} + \frac{\sigma}{k_1} + k_3\right], k_3 > 0, k_4 \geq h, \beta = \max_{\forall i \in F}\left\{\sum_{j\in F\bigcup L} a_{ij}\right\}, \eta = \max_{\forall i,j\in F}\{a_{ij}\}, \sigma = \frac{(\beta+n\eta)(k_1+2^{1-q})+\beta q 2^{1-q}}{1+q}, 1/2 < q = q_1/q_2 < 1$ with positive odd integers q_1, q_2.

Proposition 2: For multi-agent system (1), if Assumptions 1-2 hold and the control law $u_i, i \in F$ is designed as (4), then $x_i(t), v_i(t), i \in F$ are bounded $\forall\ t \in [0, \infty)$.

Proof: See Appendix. ■

With the help of Proposition 2, the main result of the paper can be stated as the following theorem.

Theorem 1: For multi-agent system (1) with multiple dynamic leaders, if Assumptions 1-2 hold and the control law $u_i, i \in F$ is designed as (4), then $x_i \to \text{Co}\{x_j, j \in L\}$ in finite time, more specifically, $x_i \to x_{di}, v_i \to v_{di}, i \in F$ in finite time.

Proof: For the case of $0 < t < T_0$, it follows from Proposition 2 that $x_i(t), v_i(t), i \in F$ are bounded. Next, we focus on the global finite-time convergence proof for the case of $t \geq T_0$.

When $t \geq T_0$, it follows from Proposition 1 that $\hat{v}_{di} = v_{di}, i \in F$. Denote $\bar{x}_i = [\bar{x}_{i1}, \cdots, \bar{x}_{ip}]^T = x_i - x_{di}, \bar{v}_i = [\bar{v}_{i1}, \cdots, \bar{v}_{ip}]^T = \dot{\bar{x}}_i, \bar{u}_i = [\bar{u}_{i1}, \cdots, \bar{u}_{ip}]^T = u_i - \dot{v}_{di}, i \in F, \bar{x}_j = 0, j \in L$, $\bar{x}_F = [\bar{x}_1^T, \cdots, \bar{x}_n^T]^T, \bar{v}_F = [\bar{v}_1^T, \cdots, \bar{v}_n^T]^T$, and $\bar{x}_L = [\bar{x}_{n+1}^T, \cdots, \bar{x}_{n+m}^T]^T$. By applying the new notations to system (1), the tracking error dynamics of the followers and the leaders can be respectively written as

$$\dot{\bar{x}}_i(t) = \bar{v}_i(t),\ \dot{\bar{v}}_i(t) = \bar{u}_i(t) + d_i(t),\ i \in F, \tag{5}$$

$$\bar{x}_j(t) = 0,\ \dot{\bar{x}}_j(t) = 0,\ \ddot{\bar{x}}_j(t) = 0,\ j \in L. \tag{6}$$

The following proof is based on the generalized adding a power integrator technique ([42], [43]), which is composed of two steps. First, a virtual velocity $\bar{v}_i^*$ is designed for each follower. Second, the distributed law is designed for each follower such that $\bar{v}_i \to \bar{v}_i^*$ in finite time and then global finite-time convergence of the closed-loop system (4)-(5) is guaranteed.

Step 1. (Virtual velocity design) Choose the following Lyapunov function

$$V_0 = \frac{1}{2}\bar{x}_F^T(\mathcal{T} \otimes I_p)\bar{x}_F = \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\|\bar{x}_i - \bar{x}_j\|^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=n+1}^{n+m} a_{ij}\|\bar{x}_i\|^2. \tag{7}$$

By Assumption 2, V_0 is positive definite and differentiable. In addition, $V_0 \leq \frac{1}{2}\lambda_{\max}(\mathcal{T})\bar{x}_F^T\bar{x}_F$, where $\lambda_{\max}(\mathcal{T}) > 0$ since $\mathcal{T} > 0$ (by Lemma 5). The derivative of V_0 along system (5) is

$$\dot{V}_0 = \bar{x}_F^T(\mathcal{T} \otimes I_p)\dot{\bar{x}}_F = \sum_{i=1}^{n}\left[\sum_{j=1}^{n+m} a_{ij}(\bar{x}_i - \bar{x}_j)^T\right]\bar{v}_i. \tag{8}$$

By denoting $w_i = [w_{i1}, \cdots, w_{ip}]^T = \sum_{j=1}^{n+m} a_{ij}(\bar{x}_i - \bar{x}_j), i \in F$, it follows that $[w_1^T, \cdots, w_n^T]^T = (\mathcal{T} \otimes I_p)\bar{x}_F$ and then $\sum_{i=1}^{n} w_i^T w_i = \bar{x}_F^T(\mathcal{T} \otimes I_p)^T(\mathcal{T} \otimes I_p)\bar{x}_F = \bar{x}_F^T(\mathcal{T}^2 \otimes I_p)\bar{x}_F$. Thus,

$$\sum_{i=1}^{n} w_i^T w_i \geq \lambda_{\min}(\mathcal{T}^2)\bar{x}_F^T\bar{x}_F \geq \frac{2\lambda_{\min}(\mathcal{T}^2)V_0}{\lambda_{\max}(\mathcal{T})}, \tag{9}$$

where $\lambda_{\min}(\mathcal{T}^2) > 0$ since $\mathcal{T}^2 > 0$. Take the virtual velocity as

$$\bar{v}_i^* = [\bar{v}_{i1}^*, \cdots, \bar{v}_{ip}^*]^T = -k_1 w_i^q, \ i \in F, \tag{10}$$

where $k_1 > 0$ to be determined, and $1/2 < q = q_1/q_2 < 1$ with positive odd integers q_1, q_2. With the help of (10), (8) becomes

$$\dot{V}_0 = -k_1 \sum_{i=1}^{n} w_i^T w_i^q + \sum_{i=1}^{n} w_i^T(\bar{v}_i - \bar{v}_i^*). \tag{11}$$

Step 2. (Control law design) Denote $\xi_i = [\xi_{i1}, \cdots, \xi_{ip}]^T = \bar{v}_i^{1/q} - \bar{v}_i^{*1/q}, i \in F$ and $r = 1 + q$. Choose the following Lyapunov function

$$V = V_0 + \sum_{i=1}^{n}\sum_{l=1}^{p} V_{il}, \tag{12}$$

where $V_{il} = \frac{1}{(2-q)2^{1-q}k_1^{1+1/q}} \int_{\bar{v}_{il}^*}^{\bar{v}_{il}} (s^{1/q} - \bar{v}_{il}^{*1/q})^{2-q}\mathrm{d}s, \bar{v}_{il}^* = -k_1 w_{il}^q, i \in F, l = 1, \cdots, p$. From Propositions B1 and B2 in [42], V_{il} (also V) is differentiable, positive definite and proper $\forall i \in F, l = 1, \cdots, p$. Moreover, based on the fact $0 < q < 1$ and Lemma 2, it can be obtained that

$$V_{il} \leq \frac{1}{(2-q)2^{1-q}k_1^{1+1/q}}|\bar{v}_{il} - \bar{v}_{il}^*||\xi_{il}|^{2-q} \leq \frac{1}{(2-q)k_1^{1+1/q}}\xi_{il}^2, \ i \in F, \ l = 1, \cdots, p. \tag{13}$$

Then by integrating (9) and (13), there is $c = \max\left\{\frac{\lambda_{\max}(\mathcal{T})}{2\lambda_{\min}(\mathcal{T}^2)}, \frac{1}{(2-q)k_1^{1+1/q}}\right\}$ such that

$$V = V_0 + \sum_{i=1}^{n}\sum_{l=1}^{p} V_{il} \leq c\sum_{i=1}^{n}\sum_{l=1}^{p}(w_{il}^2 + \xi_{il}^2), \tag{14}$$

Next, we estimate the terms in $\dot{V} = \dot{V}_0 + \sum_{i=1}^{n}\sum_{l=1}^{p}\dot{V}_{il}$ from left to right. First, by Lemmas 2-3, it follows from (11) that

$$\begin{aligned}\dot{V}_0 &\leq -k_1\sum_{i=1}^{n}\sum_{l=1}^{p} w_{il}^r + 2^{1-q}\sum_{i=1}^{n}\sum_{l=1}^{p}|w_{il}||\xi_{il}|^q \\ &\leq -k_1\sum_{i=1}^{n}\sum_{l=1}^{p} w_{il}^r + 2^{1-q}\sum_{i=1}^{n}\sum_{l=1}^{p}\left(\frac{w_{il}^r}{r} + \frac{q\xi_{il}^r}{r}\right).\end{aligned} \tag{15}$$

Second, taking the derivative of V_{il} along system (5) yields

$$\dot{V}_{il} = -\frac{1}{2^{1-q}k_1^{1+1/q}}\frac{\mathrm{d}\bar{v}_{il}^{*1/q}}{\mathrm{dt}}\int_{\bar{v}_{il}^*}^{\bar{v}_{il}}(s^q - \bar{v}_{il}^{*1/q})^{1-q}\mathrm{d}s + \frac{\xi_{il}^{2-q}(\bar{u}_{il} + d_{il})}{(2-q)2^{1-q}k_1^{1+1/q}}, \ i \in F, \ l = 1, \cdots, p. \tag{16}$$

From (10), it can be obtained that $\mathrm{d}\bar{v}_{il}^{*1/q}/\mathrm{dt} = -k_1^{1/q}\sum_{j=1}^{n+m} a_{ij}(\bar{v}_{il} - \bar{v}_{jl}) \leq k_1^{1/q}(\beta|\bar{v}_{il}| + \eta\sum_{j=1}^{n}|\bar{v}_{jl}|), i \in F$, where $\beta = \max_{\forall i\in F}\left\{\sum_{j\in F\bigcup L} a_{ij}\right\}$ and $\eta = \max_{\forall i,j\in F}\{a_{ij}\}$. In addition, by Lemma 2, it holds that $\int_{\bar{v}_{il}^*}^{\bar{v}_{il}}(s^{1/q} - \bar{v}_{il}^{*1/q})^{1-q}\mathrm{d}s \leq |\bar{v}_{il} - \bar{v}_{il}^*||\xi_{il}|^{1-q} \leq 2^{1-q}|\xi_{il}|$. Based on the above analysis, it follows from (16) that

$$\dot{V}_{il} \leq \frac{1}{k_1}\left(\beta|\bar{v}_{il}| + \eta\sum_{j=1}^{n}|\bar{v}_{jl}|\right)|\xi_{il}| + \frac{\xi_{il}^{2-q}(\bar{u}_{il} + d_{il})}{(2-q)2^{1-q}k_1^{1+1/q}},\ i \in F,\ l = 1, \cdots, p. \tag{17}$$

From (10) and Lemma 2, it holds that $|\bar{v}_{jl}| \leq |\bar{v}_{jl}^*| + |\bar{v}_{jl} - \bar{v}_{jl}^*| \leq k_1|w_{jl}|^q + 2^{1-q}|\xi_{jl}|^q, j \in F, l = 1, \cdots, p$. By Lemma 3, it follows that $|\bar{v}_{jl}||\xi_{il}| \leq (k_1|w_{jl}|^q + 2^{1-q}|\xi_{jl}|^q)|\xi_{il}| \leq \frac{k_1 q}{r}w_{jl}^r + \frac{2^{1-q}q}{r}\xi_{jl}^r + \frac{k_1+2^{1-q}}{r}\xi_{il}^r$. Then, by applying the above inequalities to (17) yields $(i \in F, l = 1, \cdots, p)$

$$\dot{V}_{il} \leq \frac{\beta q}{r}w_{il}^r + \frac{\sigma}{k_1}\xi_{il}^r + \frac{\eta q}{r}\sum_{j=1}^{n}w_{jl}^r + \frac{\eta 2^{1-q}q}{k_1 r}\sum_{j=1}^{n}\xi_{jl}^r + \frac{\xi_{il}^{2-q}(\bar{u}_{il} + d_{il})}{(2-q)2^{1-q}k_1^{1+1/q}}, \tag{18}$$

where $\sigma = \frac{(\beta+n\eta)(k_1+2^{1-q})+\beta q 2^{1-q}}{r}$.

Note that $|d_i| \leq h, i \in F$. Then, putting (12), (15), and (18) together yields

$$\begin{aligned}\dot{V} \leq &- \left[k_1 - \frac{2^{1-q}}{r} - \frac{(\beta + n\eta)q}{r}\right]\sum_{i=1}^{n}\sum_{l=1}^{p}w_{il}^r + \left[\frac{(k_1 + n\eta)2^{1-q}q}{k_1 r} + \frac{\sigma}{k_1}\right]\sum_{i=1}^{n}\sum_{l=1}^{p}\xi_{il}^r \\ &+ \frac{1}{(2-q)2^{1-q}k_1^{1+1/q}}\sum_{i=1}^{n}\sum_{l=1}^{p}\xi_{il}^{2-q}\bar{u}_{il} + \frac{h}{(2-q)2^{1-q}k_1^{1+1/q}}\sum_{i=1}^{n}\sum_{l=1}^{p}|\xi_{il}|^{2-q}.\end{aligned} \tag{19}$$

Similar to the proof in Proposition 1, we have $\sum_{j\in F} T_{ij}x_{dj} = -\sum_{j\in L} l_{ij}x_j, i \in F$. Then it follows that $\sum_{j\in F\bigcup L} a_{ij}(\bar{x}_i - \bar{x}_j) = \sum_{j\in F} T_{ij}\bar{x}_j = \sum_{j\in F\bigcup L} a_{ij}(x_i - x_j)$. If u_i is taken as (4), by noting that $\bar{u}_i = u_i - \dot{v}_{di}, i \in F$, control input $\bar{u}_i$ of system (5) can be described as

$$\bar{u}_{il} = -k_2\xi_{il}^{2q-1} - k_4\mathrm{sgn}(\xi_{il}),\ i \in F, l = 1, \cdots, p, \tag{20}$$

where $k_1 \geq \frac{2^{1-q}}{r} + \frac{(\beta+n\eta)q}{r} + k_3, k_2 \geq (2-q)2^{1-q}k_1^{1+1/q}\left[\frac{(k_1+n\eta)2^{1-q}q}{k_1 r} + \frac{\sigma}{k_1} + k_3\right], k_3 > 0, k_4 \geq h$.

Substituting (20) into (19) yields

$$\dot{V} \leq - k_3\sum_{i=1}^{n}\sum_{l=1}^{p}(w_{il}^r + \xi_{il}^r). \tag{21}$$

Based on the fact $0 < r/2 < 1$ and Lemma 2, it follows from (14) and (21) that

$$\dot{V} + \frac{k_3}{c^{r/2}}V^{r/2} \leq 0, \tag{22}$$

which means that V reaches zero in finite time (by Lemma 1). Then $\bar{x}_F \to 0, \bar{v}_F \to 0$, i.e., $x_i \to x_{di}$ and $v_i \to v_{di}, i \in F$ in finite time with control law (4). This completes the proof. ■

Remark 3: On one hand, the terms $\dot{\hat{v}}_{di} - k_2[(v_i - \hat{v}_{di})^{1/q} + k_1^{1/q}(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j))]^{2q-1}$ in the distributed control law (4) are used to force the i-th ($i \in F$) follower to converge to the leaders' convex hull $\text{Co}\{x_j, j \in L\}$ in finite time, and these terms are obtained through a recursive design process based on the generalized adding a power integrator technique [42], [43]. On the other hand, the discontinuous term $k_4\text{sgn}(\cdot)$ in control law (4) is used to dominate the external disturbance d_i contained in the i-th follower dynamics of system (1) such that the global finite-time convergence of the whole closed-loop system can be achieved. In the absence of external disturbances, this discontinuous term is not needed.

Remark 4: Let T denote the settling time of the closed-loop system (1)-(2)-(4). From (22) and Lemma 1, it follows that $T \le T_0 + \frac{2c^{(1+q)/2}V(T_0)^{(1-q)/2}}{k_3(1-q)}$ with T_0 being the finite settling time of observer (2) and $c = \max\left\{\frac{\lambda_{\max}(\mathcal{T})}{2\lambda_{\min}(\mathcal{T}^2)}, \frac{1}{(2-q)k_1^{1+1/q}}\right\}$. For $t \ge T_0$, we define $K(t) = \sum_{i=1}^{n}\left[\frac{\lambda_{\max}(\mathcal{T})}{2}\|x_i(t) - x_{di}(t)\|^2 + \frac{1}{(2-q)k_1^{1+1/q}}\|(v_i(t) - v_{di}(t))^{1/q} + k_1^{1/q}\sum_{j=1}^{n+m} a_{ij}(x_i(t) - x_j(t))\|^2\right]$. Note that $\sum_{j=1}^{n+m} a_{ij}(\bar{x}_i - \bar{x}_j) = \sum_{j=1}^{n+m} a_{ij}(x_i - x_j)$. Then, from (7), (12), and (13), it can be obtained that $V(t) \le K(t), \forall t \ge T_0$. Therefore, an upper bound of T can be given by $T \le T_0 + \frac{2c^{(1+q)/2}K(T_0)^{(1-q)/2}}{k_3(1-q)}$. From the above analysis and noting that k_1, k_2 depend on k_3, then T mainly depends on the communication topology structure, the agent initial states, T_0 and the control parameters k_3, q.

Remark 5: In the case of multiple dynamic leaders, if Assumptions 1-2 hold, according to the control design in [39], for the i-th ($i \in F$) follower in system (1) without external disturbances, the switching finite-time containment control law can be written as

$$u_i = \begin{cases} -k_{p_i}x_i - k_{d_i}v_i, & t \le T^*, \\ -\alpha^{-1}b^{-1}\text{sig}^{2-\alpha}(v_i - \hat{v}_{di}) - \mu\text{sgn}\left(\sum_{j=1}^{n} T_{ij}s_j\right) - \varrho\text{sig}^{\gamma}\left(\sum_{j=1}^{n} T_{ij}s_j\right), & t > T^*, \end{cases} \tag{23}$$

where $k_{p_i}, k_{d_i}, b, \varrho > 0, 1 < \alpha < 2, \mu > \sup_{i\in L, l=1,2,3}|\dot{v}_{il}|, 0 < \gamma < 1, s_i = \sum_{j\in F\bigcup L} a_{ij}(x_i - x_j) + b\sum_{j=1}^{n} T_{ij}\text{sig}^{\alpha}(v_j - \hat{v}_{dj})$, T_{ij} is the (i,j) entry of the matrix $\mathcal{T}$ defined in Lemma 5, and $\hat{v}_{di}, i \in F$ denotes estimate of v_{di} with respect to the i-th follower, which is generated from the following distributed sliding mode observer with a finite settling time T^*

$$\dot{\hat{v}}_{di} = -k\text{sgn}\left[\sum_{j\in F\bigcup L} a_{ij}(\hat{v}_{di} - \hat{v}_{dj})\right], \ i \in F, \tag{24}$$

where $\hat{v}_{dj} = v_j, j \in L, \sup_{i\in L, l=1,2,3} |\dot{v}_{il}| < k < \mu$ and the initial states satisfy $\hat{v}_{di}(0) = 0, i \in F$. According to analysis in [39], $T^* = \frac{\sqrt{3n}\|v_d\|_\infty \lambda_{\max}(\mathcal{T})}{(p-\|\dot{v}_d\|_\infty)\lambda_{\min}(\mathcal{T})}$, where $\|\cdot\|_\infty$ represents the infinity norm. As the authors said in [39], the switching time T^* depends on $\|\dot{v}_d\|_\infty$ and global information $\lambda_{\max}(\mathcal{T}), \lambda_{\min}(\mathcal{T})$. However, it is usually difficult to obtain this global information and hence the switching time T^* in a distributed way. In addition, the control law $u_i = -k_{p_i} x_i - k_{d_i} v_i, i \in F$ is used to guarantee the state boundedness of the closed-loop system (1)-(23)-(24) when $t \le T^*$, but it may negatively affect the followers' tracking performances. Therefore, compared with the switching control law (23), the non-switching control law (4) is more convenient to be taken into practice.

Actually, the result of Theorem 1 also covers the case of multiple stationary leaders. The only difference is that for multi-agent system (1) with multiple stationary leaders (i.e., the leaders' velocities are all zeros), the distributed observer is not needed anymore. More specifically, without further proof, the following corollary can be given.

Corollary 1: For multi-agent system (1) with multiple stationary leaders, if Assumption 2 holds and the control law $u_i, i \in F$ is designed as

$$\begin{aligned} u_i = &- k_2 \left[v_i^{1/q} + k_1^{1/q} \left(\sum_{j \in F \bigcup L} a_{ij}(x_i - x_j) \right) \right]^{2q-1} \\ &- k_4 \mathrm{sgn} \left[v_i^{1/q} + k_1^{1/q} \left(\sum_{j \in F \bigcup L} a_{ij}(x_i - x_j) \right) \right], \ i \in F, \end{aligned} \tag{25}$$

where the control parameters are the same as those defined in (4), then $x_i \to \mathrm{Co}\{x_j, j \in L\}$ in finite time, more specifically, $x_i \to x_{di}, v_i \to 0, i \in F$ in finite time.

Remark 6: Control law (25) is obtained by letting $\hat{v}_{di} = 0, \dot{\hat{v}}_{di} = 0, i \in F$ in control law (4) and thus (25) has almost the same structure as (4): the terms $-k_2[v_i^{1/q} + k_1^{1/q}(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j))]^{2q-1}$ are used to guarantee the finite-time convergence and the discontinuous term $k_4\mathrm{sgn}(\cdot)$ is used to dominate the external disturbances. Since Assumption 1 naturally holds for stationary leaders, it is omitted in Corollary 1. By Remark 4, an upper bound for the finite settling time T of the closed-loop system (1)-(25) can be directly obtained: $T \le \frac{2c^{(1+q)/2}K(0)^{(1-q)/2}}{k_3(1-q)}$, where $K(t)$ (note that $t \ge T_0 = 0$ and $v_{di} = 0, i \in F$ here) is the same as that defined in Remark 4. In the case of multiple stationary leaders, if Assumption 2 holds, according to the control design in [39], for the i-th ($i \in F$) follower in system (1) without external disturbances, the finite-time containment control law can be written as

$$u_i = -b_1 \sum_{j \in L \bigcup F} a_{ij} \left[\mathrm{sig}^{\alpha_1} \left(\sum_{k \in L \bigcup F} a_{ik}(x_i - x_k) \right) - \mathrm{sig}^{\alpha_1} \left(\sum_{k \in L \bigcup F} a_{jk}(x_j - x_k) \right) \right] - b_2 \sum_{j \in L \bigcup F} a_{ij} \left[\mathrm{sig}^{\alpha_2} \left(\sum_{k \in L \bigcup F} a_{ik}(v_i - v_k) \right) - \mathrm{sig}^{\alpha_2} \left(\sum_{k \in L \bigcup F} a_{jk}(v_j - v_k) \right) \right], \ i \in F, \tag{26}$$

where $b_1, b_2 > 0, 0 < \alpha_2 < 1$ and $\alpha_1 = \frac{\alpha_2}{2 - \alpha_2}$. Note that for each follower, control law (26) needs the information from its neighbors' neighbors, which is usually difficult to obtain in practice. In contrast, for each follower, control law (25) only requires information from its neighbors, which makes (25) easier for practical implementations.

IV. Numerical simulations

In this section, some simulations are conducted to illustrate the effectiveness of the control scheme proposed in Theorem 1. We consider a group of 3-D agents with 4 leaders and 6 followers, i.e., $m = 4, n = 6$. The communication topology among the agents is shown in Fig. 1 with $F = \{1, 2, 3, 4, 5, 6\}$ and $L = \{7, 8, 9, 10\}$. The external disturbances are assumed to be $d_i(t) = [0.1\sin(t), 0.1\cos(0.5t), 0.1\sin(t) + 0.1\cos(0.5t)]^T, i \in F$. It is easy to obtain that $\|d_i(t)\| \leq 0.2, i \in F$.

A. *Simulations in the case of multiple dynamic leaders*

The leaders are assumed to have constant velocities $v_7(t) = [0.51, 0.61, -0.09]^T, v_8(t) = [0.49, 0.6, -0.08]^T, v_9(t) = [0.5, 0.59, -0.1]^T, v_{10}(t) = [0.5, 0.6, -0.09]^T, \forall t \geq 0$ (note that their velocities are different from each other's) and their initial coordinates are the four vertices of a tetrahedron: $x_7(0) = [0, 0, 0]^T, x_8(0) = [0, 3, 0]^T, x_9(0) = [3\sqrt{3}/2, 3/2, 0]^T$, and $x_{10}(0) = [\sqrt{3}/2, 3/2, \sqrt{6}]^T$. The followers are assumed to be static at $t = 0$ and their initial coordinates are $x_1(0) = [-1.2, 3.6, 1.8]^T, x_2(0) = [0.1, 1.3, 3]^T, x_3(0) = [1.6, 2.1, 2.6]^T, x_4(0) = [1.5, 0.3, -1.5]^T, x_5(0) = [2.2, 3.1, 2.2]^T, x_6(0) = [-1, -0.5, 0.8]^T$. The initial states of the distributed observer (2) are set to zeros, i.e., $\hat{v}_{di}(0) = 0_{3\times 1}, i \in F$.

For the distributed control law (4), take $q = 9/11, k_3 = 1.5$. By calculation, it can be obtained that $\beta = \max_{\forall i \in F} \left\{ \sum_{j \in F \bigcup L} a_{ij} \right\} = 0.24$ and $\eta = \max_{\forall i, j \in F} \{a_{ij}\} = 0.1$. According to the

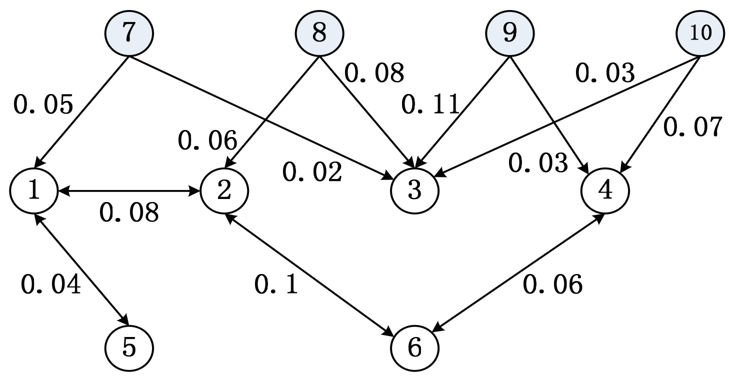


Fig. 1. The communication topology among the agents with $F = \{1, 2, 3, 4, 5, 6\}, L = \{7, 8, 9, 10\}$.

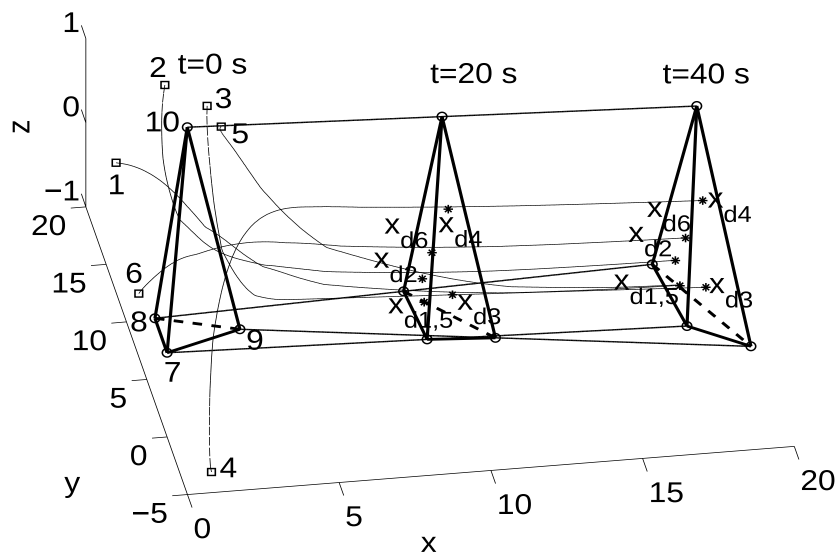
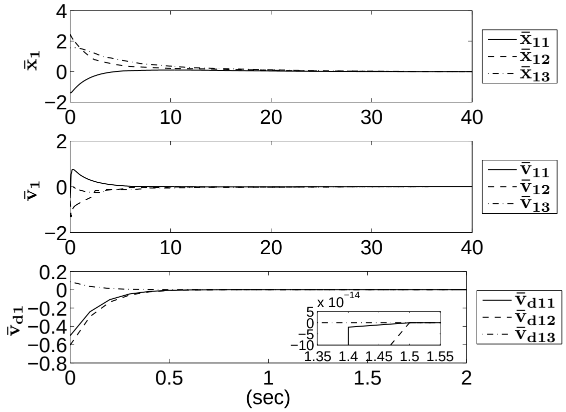
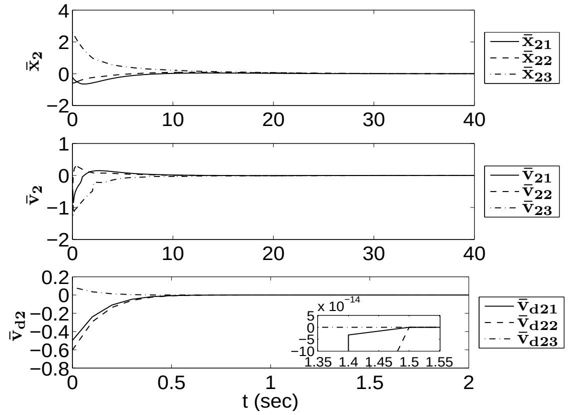


Fig. 2. 3-D phase-plot for the agents with control law (4). The small blocks represent the followers, the small circles represent the leaders, and the small asterisks represent the weighted average of the leaders' coordinates (note that $x_{d1} = x_{d5}$ and thus being denoted by $x_{d1,5}$).

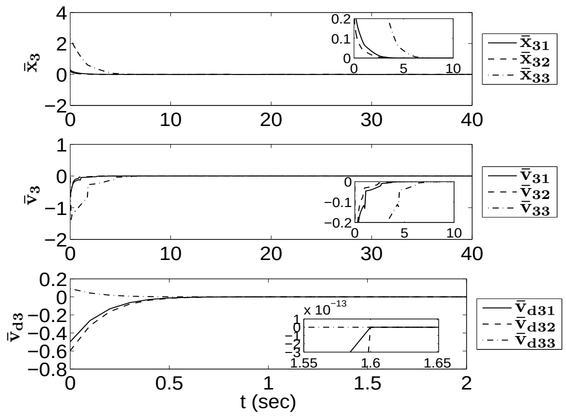
sufficient conditions given in Theorem 1, the other parameters of control law (4) can be chosen as $k_1 = 2.5019, \sigma = 1.8024, k_2 = 29.3536, k_4 = 0.2$. For observer (2), take $k = 5, \alpha = 0.9$. By Proposition 1, the settling time T_0 of observer (2) is $T_0 = \max_{\forall i \in F, 1 \le l \le 3} \left\{ \frac{|e_{il}(0)|^{1-\alpha}}{k(1-\alpha)} \right\}$, where $e_{il}(0) = \sum_{j \in F \bigcup L} a_{ij}(\hat{v}_{dil}(0) - \hat{v}_{djl}(0)), i \in F, \hat{v}_{djl}(0) = v_{jl}(0), j \in L, l = 1, 2, 3$. With the chosen



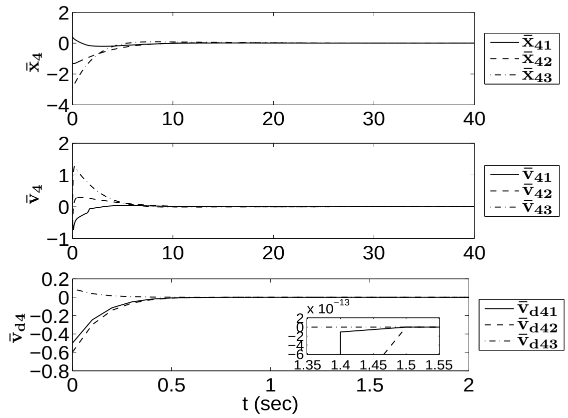
(a) $\bar{x}_1, \bar{v}_1, \bar{v}_{d1}$



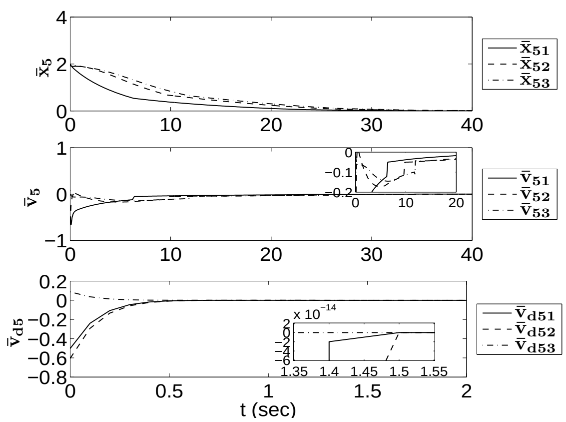
(b) $\bar{x}_2, \bar{v}_2, \bar{v}_{d2}$



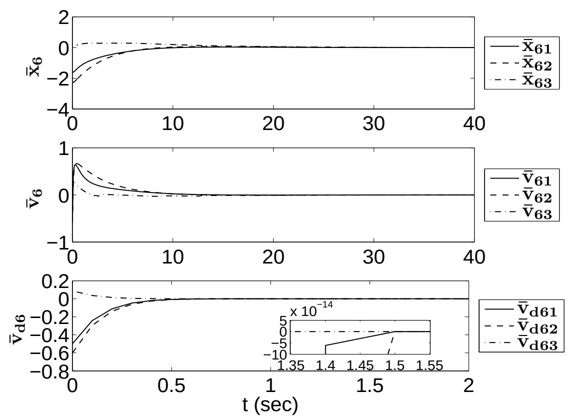
(c) $\bar{x}_3, \bar{v}_3, \bar{v}_{d3}$



(d) $\bar{x}_4, \bar{v}_4, \bar{v}_{d4}$



(e) $\bar{x}_5, \bar{v}_5, \bar{v}_{d5}$



(f) $\bar{x}_6, \bar{v}_6, \bar{v}_{d6}$

Fig. 3. Response curves of the followers with control law (4).

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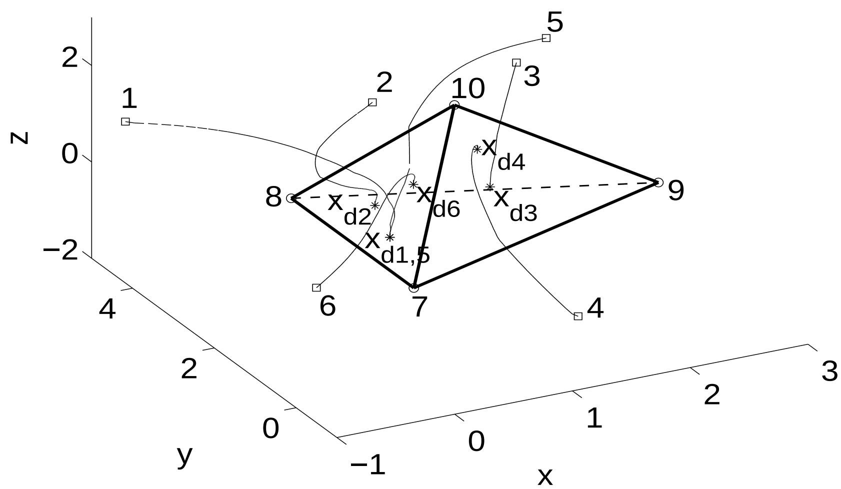


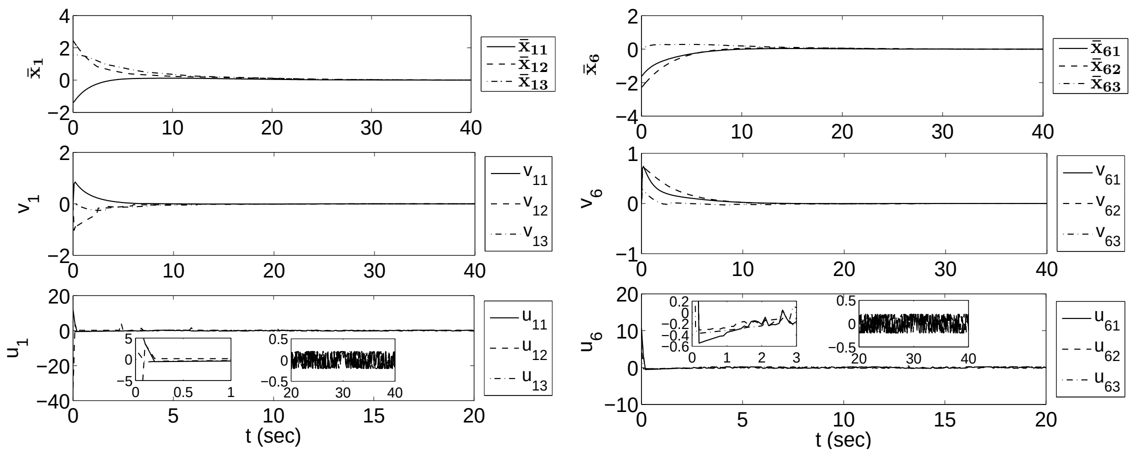
Fig. 4. 3-D phase-plot for the agents with control law (25). The small blocks represent the followers, the small circles represent the leaders, and the small asterisks represent the weighted average of the leaders' coordinates.

parameters and initial states, it can be obtained that $T_0 = 1.6466$.

The simulation results are shown in Figs. 2-3. It can be seen that with control law (4), $r_i \to r_{di}, v_i \to v_{di}, i \in F$, i.e., the states of the followers converge to the dynamic convex hull spanned by those of the dynamic leaders, in finite time.

B. Simulations in the case of multiple stationary leaders

In this subsection, the leaders are assumed to be static. All the initial states are the same as those set in the above subsection and the parameters for control law (25) are also the same as those taken for control law (4) in the above subsection. The simulation results are shown in Figs. 4-5. It can be seen that with control law (25), $r_i \to r_{di}, v_i \to 0, i \in F$, i.e., the states of the followers converge to the static convex hull spanned by those of the stationary leaders, in finite time. The chattering in the curves of the control inputs is caused by the discontinuous term $\mathrm{sgn}(\cdot)$ contained in control law (25).



(a) $\bar{x}_1, v_1, u_1$ (b) $\bar{x}_6, v_6, u_6$

Fig. 5. Response curves of followers 1, 6 with control law (25).

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V. Conclusions

In this paper, distributed finite-time containment control algorithms have been proposed for double-integrator multi-agent systems with multiple dynamic or stationary leaders. It has been shown that with the proposed control algorithms, the states of the followers can converge to the convex hull spanned by the those of the leaders in finite time for both cases in the presence of external disturbances .

Appendix

Proof of Proposition 2. For brevity, we denote $x_F = [x_1^T, \cdots, x_n^T]^T, v_F = [v_1^T, \cdots, v_n^T]^T, x_L = [x_{n+1}^T, \cdots, x_{n+m}^T]^T, v_L = [v_{n+1}^T, \cdots, v_{n+m}^T]^T, u_F = [u_1^T, \cdots, u_n^T]^T, d_F = [d_1^T, \cdots, d_n^T]^T$ for the following proof. First, we define $\rho(x_F, v_F) = \frac{1}{2}x_F^T x_F + \frac{1}{2}v_F^T v_F$. Note that $\|d_i\| \leq h, i \in F$. Taking derivative of ρ along system (1) yields

$$\dot{\rho} = x_F^T v_F + v_F^T(u_F + d_F) \leq \rho + \sum_{i=1}^{n} \|v_i\|(\|u_i\| + h). \quad \text{(A.1)}$$

Next, we begin to estimate $\|u_i\|, i \in F$. From (4), it can be obtained that

$$\|u_i\| \leq \|\dot{\hat{v}}_{di}\| + k_4 + k_2 \left\| \left[(v_i - \hat{v}_{di})^{1/q} + k_1^{1/q} \left(\sum_{j \in F \bigcup L} a_{ij}(x_i - x_j) \right) \right]^{2q-1} \right\|, \ i \in F. \quad \text{(A.2)}$$

We now estimate the last term of (A.2). By Lemma 2, $\forall y = [y_1, \cdots, y_p]^T \in R^p, a \geq 0$, we have

$$\|y^a\| = \left[\sum_{l=1}^{p}(y_l^a)^2\right]^{1/2} \leq \sum_{l=1}^{p}|y_l|^a \leq p\left(\sum_{l=1}^{p}y_l^2\right)^{a/2} = p\|y\|^a, . \tag{A.3}$$

Clearly, (A.3) holds by letting $y = (v_i - \hat{v}_{di})^{1/q} + k_1^{1/q}(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j)) \in R^p, a = 2q-1$ or $y = v_i - \hat{v}_{di} \in R^p, a = 1/q$. Based on the fact $0 < 2q-1 < 1$ and Lemma 2, it follows that $p\|(v_i - \hat{v}_{di})^{1/q} + k_1^{1/q}(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j))\|^{2q-1} \leq p\|(v_i - \hat{v}_{di})^{1/q}\|^{2q-1} + pk_1^{2-1/q}\|\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j)\|^{2q-1}$. Note that $0 < 2 - 1/q < 1$. With the help of (A.3) and Lemma 2, it can be verified that $\|(v_i - \hat{v}_{di})^{1/q}\|^{2q-1} \leq p^{2q-1}(\|v_i\| + \|\hat{v}_{di}\|)^{2-1/q} \leq p^{2q-1}(\|v_i\|^{2-1/q} + \|\hat{v}_{di}\|^{2-1/q})$. In addition, we have $\|\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j)\| \leq \beta\sum_{j=1}^{n+m}(\|x_i\| + \|x_j\|) = \beta(n+m)\|x_i\| + \beta\sum_{j=1}^{n}\|x_j\| + \beta\sum_{j=n+1}^{n+m}\|x_j\|$, where $\beta = \max_{\forall i\in F}\left\{\sum_{j\in F\bigcup L} a_{ij}\right\}$. Based on the above analysis, we have

$$\begin{aligned} p\left\|(v_i - \hat{v}_{di})^{1/q} + k_1^{1/q}\left(\sum_{j\in F\bigcup L} a_{ij}(x_i - x_j)\right)\right\|^{2q-1} &\leq p^{2q}\left(\|v_i\|^{2-1/q} + \|\hat{v}_{di}\|^{2-1/q}\right) \\ &+ pk_1^{2-1/q}\beta^{2q-1}\left[(n+m)^{2q-1}\|x_i\|^{2q-1} + \sum_{j=1}^{n}\|x_j\|^{2q-1} + \sum_{j=n+1}^{n+m}\|x_j\|^{2q-1}\right],\ i \in F. \end{aligned} \tag{A.4}$$

Then, putting (A.2)-(A.4) together yields

$$\begin{aligned} \|u_i\| \leq\ & \|\dot{\hat{v}}_{di}\| + k_4 + k_2p^{2q}\left(\|v_i\|^{2-1/q} + \|\hat{v}_{di}\|^{2-1/q}\right) + k_2pk_1^{2-1/q}\beta^{2q-1}\Bigg[(n+m)^{2q-1}\|x_i\|^{2q-1} \\ & + \sum_{j=1}^{n}\|x_j\|^{2q-1} + \sum_{j=n+1}^{n+m}\|x_j\|^{2q-1}\Bigg] \\ \leq\ & \delta_1 + k_2p^{2q}\|v_i\|^{2-1/q} + k_2pk_1^{2-1/q}\beta^{2q-1}\left[(n+m)^{2q-1}\|x_i\|^{2q-1} + \sum_{j=1}^{n}\|x_j\|^{2q-1}\right],\ i \in F, \end{aligned} \tag{A.5}$$

where $\delta_1 > 0$ satisfying $\delta_1 \geq \|\dot{\hat{v}}_{di}\| + k_4 + k_2p^{2q}\|\hat{v}_{di}\|^{2-1/q} + k_2pk_1^{2-1/q}\beta^{2q-1}\sum_{j=n+1}^{n+m}\|x_j\|^{2q-1}$.

From the proof of Proposition 1, we have $\hat{v}_d = (\mathcal{T}\otimes I_p)^{-1}e - (\mathcal{T}^{-1}\mathcal{T}_d\otimes I_p)v_L$ and $\dot{\hat{v}}_d = -k(\mathcal{T}\otimes I_p)^{-1}\mathrm{sig}^{\alpha}(e) - (\mathcal{T}^{-1}\mathcal{T}_d\otimes I_p)\dot{v}_L$. Due to global convergence of observer (2) and Assumption

1, the existence of δ_1 is guaranteed. Then it follows from (A.1), (A.5) and Lemma 3 that

$$\begin{aligned}\dot{\rho} \leq\ & \rho + (\delta_1 + h)\sum_{i=1}^{n}\|v_i\| + k_2p^{2q}\sum_{i=1}^{n}\|v_i\|^{3-1/q} + \frac{k_2pk_1^{2-1/q}\beta^{2q-1}}{2q}\sum_{i=1}^{n}[(n+m)^{2q-1}+n]\|v_i\|^{2q} \\ & + \frac{k_2pk_1^{2-1/q}\beta^{2q-1}(2q-1)}{2q}\sum_{i=1}^{n}\left[(n+m)^{2q-1}\|x_i\|^{2q} + \sum_{j=1}^{n}\|x_j\|^{2q}\right] \\ \leq\ & \rho + (\delta_1 + h)\sum_{i=1}^{n}\|v_i\| + k_2p^{2q}\sum_{i=1}^{n}\|v_i\|^{3-1/q} + \frac{k_2pk_1^{2-1/q}\beta^{2q-1}[(n+m)^{2q-1}+n]}{2q} \\ & \times \sum_{i=1}^{n}(\|x_i\|^{2q} + \|v_i\|^{2q}). \qquad \text{(A.6)}\end{aligned}$$

In addition, it holds that $\max\{\|x_i\|^a, \|v_i\|^a\} \leq (\|x_i\|^2 + \|v_i\|^2)^{a/2}, \forall\ a \geq 0$. Then based on the fact $0 < q, (3-1/q)/2 < 1$ and (A.6), it follows that

$$\begin{aligned}\dot{\rho} \leq\ & \rho + (\delta_1 + h)\sum_{i=1}^{n}(\|x_i\|^2 + \|v_i\|^2)^{1/2} + k_2p^{2q}\sum_{i=1}^{n}(\|x_i\|^2 + \|v_i\|^2)^{(3-1/q)/2} \\ & + \frac{k_2pk_1^{2-1/q}\beta^{2q-1}[(n+m)^{2q-1}+n]}{q}\sum_{i=1}^{n}(\|x_i\|^2 + \|v_i\|^2)^{q}. \qquad \text{(A.7)}\end{aligned}$$

Note that $\|x\|_p = \left(\sum_{l=1}^{m}|x_l|^p\right)^{1/p}, \forall x = [x_1, \cdots, x_m]^T$ with $p \geq 1, m \in N^+$ denotes p-norm in R^m. Based on the equivalence between any two different norms in R^p and Lemma 2, we can find $\delta_2 > 0$ such that $\sum_{i=1}^{n}(\|x_i\|^2 + \|v_i\|^2)^{1/2} \leq \sum_{i=1}^{n}(\|x_i\| + \|v_i\|) \leq \delta_2\rho^{1/2}$. Similarly, $\sum_{i=1}^{n}(\|x_i\|^2 + \|v_i\|^2)^{(3-1/q)/2} \leq \delta_3\rho^{(3-1/q)/2}, \sum_{i=1}^{n}(\|x_i\|^2 + \|v_i\|^2)^{q} \leq \delta_4\rho^q$ hold with appropriate $\delta_3 > 0, \delta_4 > 0$. Then, it follows from (A.7) that

$$\dot{\rho} \leq \rho + (\delta_1 + h)\delta_2\rho^{1/2} + k_2p^{2p}\delta_3\rho^{(3-1/q)/2} + \frac{k_2pk_1^{2-1/q}\beta^{2q-1}[(n+m)^{2q-1}+n]\delta_4}{q}\rho^q. \qquad \text{(A.8)}$$

From Lemma 3, $\rho^b = \rho^b \cdot 1^{1-b} \leq b\rho + 1 - b, \forall\ 0 < b \leq 1$. Then it follows from (A.8) that

$$\dot{\rho} \leq \delta_5\rho + \delta_6, \qquad \text{(A.9)}$$

where $\delta_5 = 1 + \frac{(\delta_1+h)\delta_2}{2} + \frac{3q-1}{2q}k_2p^{2q}\delta_3 + k_2pk_1^{2-1/q}\beta^{2q-1}[(n+m)^{2q-1}+n]\delta_4$ and $\delta_6 = \frac{(\delta_1+h)\delta_2}{2} + \frac{1-q}{2q}k_2p^{2q}\delta_3 + \frac{k_2pk_1^{2-1/q}\beta^{2q-1}[(n+m)^{2q-1}+n]\delta_4(1-q)}{q}$. By noting that $\delta_5, \delta_6 \in (0, \infty)$, it follows from (A.9) immediately that ρ is bounded, which implies that $x_i(t), v_i(t), i \in F$ are bounded $\forall\ t \in [0, +\infty)$. This completes the proof.

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