PROBABILITY OF EXTINCTION OF ADVANTAGEOUS GENES

The recurrence formula

$$u_{n+1} = e^{m(u_n - 1)} \qquad (1)$$

gives the probability of extinction by generation n of a rare mutation enjoying selective advantage in the ratio m : 1. For a hypothetical self-sterility allele produced by recombination between two out of only three alleles, the value of m is 1.5.

The results of applying equation (1) ten times, starting with $u_o = 0$, are shown in the table, with some derived quantities to be explained below.

n	u	v $= \alpha - u$	r^n $r = m\alpha$	x
0	0	.41718,83561	1	
1	.22213,16025	.19405,67536	.62578,25341	
2	.31182,82897	.10536,00664	.39160,37800	
3	.35620,19031	.06098,64530	.24505,88058	
4	.38072,64581	.03646,18080	.15335,35205	
5	.39498,45460	.02220,38101	.09596,54671,6	.02125,46991
6	.40352,20393	.01366,63168	.06005,35132,4	.01330,12316
7	.40872,32751	.00846,50810	.03758,04397,0	.00832,36813
8	.41192,45460	.00526,38101	.02351,71827,9	.00520,88146
9	.41390,73258	.00328,10303	.01471,66422,4	.00325,95852
10	.41514,01889	.00204,81677	.00920,94176,76	.00203,97914

The solution of the equation

$$u = e^{m(u-1)} \qquad (2)$$

is

$$.41718,83561 = \alpha ,$$

the probability of ultimate extinction, as appears in the third column, showing the frequencies of extinction not yet realized, namely

$$v = \alpha - u.$$

The recurrence equation for v is

$$v_{n+1} = \alpha(1 - e^{-mv_n}). \qquad (3)$$

Expanding the exponential it is clear that v ultimately decreases annually in the ratio

$$r = \alpha m = .62578,25341 \qquad (4)$$

of which the powers are shown in the fourth column.

Solution of equation (3) in the form

$$v = A\gamma^{n} + B\gamma^{2n} + C\gamma^{3n} + D\gamma^{4n} + \dots$$

yields

$$B = \frac{m}{2(1-\gamma)}\,A^2 \,,$$

$$C = \frac{m^2(2+\gamma)}{6(1-\gamma)(1-\gamma^2)}\,A^3 \,,$$

$$D = \frac{m^3(6 + 6\gamma + 5\gamma^2 + \gamma^3)}{24(1-\gamma)(1-r^2)(1-r^3)}\,A^4 \,,$$

so that if $\underline{x}$ stand for $A\gamma^{n}$, the solution is

$$v = x + \frac{m}{2(1-\gamma)}\,x^2 + \frac{m^2(2+\gamma)}{6(1-r)(1-r^2)}\,x^3 + \frac{m^3(6 + 6\gamma + 5\gamma^2 + \gamma^3)}{24(1-r)(1-r^2)(1-r^3)}\,x^4\dots \tag{5}$$

or, inverting the equation

$$x = v - \frac{m}{2(1-\gamma)}\,v^2 + \frac{m^2(1+2\gamma)}{6(1-\gamma)(1-\gamma^2)}\,v^3 - \frac{m^3(1+6\gamma+5\gamma^2+6\gamma^3)}{24(1-\gamma)(1-\gamma^2)(1-\gamma^3)}\,v^4 + \dots \tag{6}$$

or $x = v - \overset{b_1}{\cancel{b_2}}\,v^2 + \overset{b_2}{\cancel{b_3}}\,v^3 - \overset{b_3}{\cancel{b_4}}\,v^4.$

Numerically,

$$b_1 = \frac{m}{2(1-\gamma)} = 2.00418,2243$$

$$b_2/b_1 = \frac{m(1+2\gamma)}{3(1-\gamma^2)} = 1.85041,0139$$

$$b_3/b_2 = \frac{m(1+6\gamma+5\gamma^2+6\gamma^3)}{4(1-\gamma^3)(1+2\gamma)} = 1.80530,1511$$

For any given value of $\underline{v}$, therefore, the corresponding value of $\underline{x}$ may be calculated provided that $\underline{v}$ is sufficiently small for the terms omitted from the series to be neglected. Values of $\underline{x}$ for $n = 5\dots10$, calculated from (6), have been inserted in the table. The quotient found on dividing x by γ^{n}, are ~~nearly~~ sufficiently constant from 7 onwards, as nearly as the limited precision of $\underline{v}$ will allow, thus we have

m		n	
5'	2.21480,390	7	2.21489,726
6	2.21489,650	8	2.21489,734
		9	2.21489,734
		10	2.21489,729

Seeing that $\underline{v}$ has only 8 digits, and that ~~these~~ there must be some cumulative errors in the successive calculations from the recurrence formula (1), the value of A, which can seldom be needed with more than four figure accuracy, may be taken to be

$$A = 2.21489,73 \quad ,$$

and for any larger value of n, $\underline{x}$ can be calculated as Ar^n, whence $\underline{v}$ may be obtained from formula (5).

Summary of constants :

		Symbol
Limiting value of u	.41718,83561	α
" " " v_{n+1}/v_n	.62578,25341	r
" " " v_n/r^n	2.21489,73	A

together with the coefficients of (5) and (6) known explicitly in terms of these.