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## 2-Stage RC Ladder: Solution of a Noise Paradox

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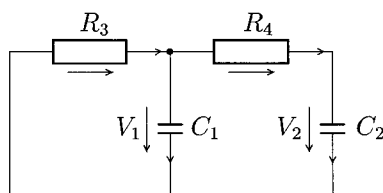
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**Abstract.** This paper addresses the 2-stage RC ladder paradox presented by Abbott et al. at the previous UPoN conference. It is clarified which of the previously obtained contradictory results are correct and which are wrong. A physical interpretation for success and failure of different approaches is given.

### INTRODUCTION

Recently, Abbott et al. [1] have presented a paradox related to the thermal noise analysis of the 2-stage RC ladder in Fig. 1. Using conventional spectral techniques



**FIGURE 1.** 2-stage RC ladder

the correlation matrix of capacitor noise voltages  $v_{1,n}$ ,  $v_{2,n}$  in thermal equilibrium at temperature  $T$  was calculated

$$\overline{(v_{i,n}^2)_{eq}} = kT/C_i, \quad i = 1, 2; \quad \overline{(v_{1,n}v_{2,n})_{eq}} = 0 \quad (1)$$

(In this paper we will use ensemble averages). The results (1) are independent of  $R_3$ ,  $R_4$ , so to calculate the limits  $R_3 \rightarrow \infty$ ,  $R_4 \rightarrow 0$ , two methods were used. In a first approach, the complex noise voltages  $v_{1,n}$ ,  $v_{2,n}$  as functions of  $R_3$ ,  $R_4$  were

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used to calculate spectral densities  $S_{ij}(\omega; R_3, R_4)$ . Then, the  $R_3/R_4$  limits were taken, and finally frequency integration (the limit  $\omega \rightarrow \infty$ ) was performed. With this, negative cross correlation was found for  $R_3 \rightarrow \infty$

$$\begin{aligned} \lim_{R_3 \rightarrow \infty} \overline{(v_{1,n}^2)_{eq}} &= \frac{kTC_2}{C_1(C_1 + C_2)}, & \lim_{R_3 \rightarrow \infty} \overline{(v_{2,n}^2)_{eq}} &= \frac{kTC_1}{C_2(C_1 + C_2)}, \\ \lim_{R_3 \rightarrow \infty} \overline{(v_{1,n}v_{2,n})_{eq}} &= -\frac{kT}{C_1 + C_2}, \end{aligned} \quad (2)$$

and positive cross correlation for  $R_4 \rightarrow 0$

$$\lim_{R_4 \rightarrow 0} \overline{(v_{1,n}^2)_{eq}} = \lim_{R_4 \rightarrow 0} \overline{(v_{2,n}^2)_{eq}} = \lim_{R_4 \rightarrow 0} \overline{(v_{1,n}v_{2,n})_{eq}} = kT/(C_1 + C_2). \quad (3)$$

As it might be a problem to perform the  $R_3/R_4$  limits before the frequency integration, the order of resistance limits and frequency limits was interchanged: Since the correlation functions (1) are independent of the  $R_i$ , it was wrongly inferred that no  $R_i$  limits can be taken on this basis. (The  $R_i$  independence of (1) simply means that the limits do not change the result (1), which agrees with (4).) Hence, the complex noise voltages  $v_{1,n}$ ,  $v_{2,n}$  were split into contributions due to  $R_3$  and  $R_4$ , respectively. Then, spectral densities and correlation functions were calculated as functions of  $R_3$ ,  $R_4$ . Taking the  $R_3/R_4$  limits, the cross correlations became

$$\lim_{R_3 \rightarrow \infty} \overline{(v_{1,n}v_{2,n})_{eq}} = 0 = \lim_{R_4 \rightarrow 0} \overline{(v_{1,n}v_{2,n})_{eq}}. \quad (4)$$

The result (4) seems in agreement with the vanishing cross correlation (1), but (4) is an apparent contradiction to (2), (3). Open questions are to find out whether the results (2), (3) or the results (4) are correct, to explain the relation to (1), to give a sensible physical interpretation of the results and to draw conclusions for handling noise sources in circuit theory. In addition, in [1] speculations about a relation between the RC ladder paradox and Penfield's (first) motor paradox [2] were made.

## TIME DOMAIN APPROACH

In [1], frequency domain approaches were used to treat the 2-stage RC ladder circuit. Two limits had to be taken, either  $R_3 \rightarrow \infty$  or  $R_4 \rightarrow 0$  and  $\omega_{max} \rightarrow \infty$  in the frequency integral. Comparing (2), (3) with (4), we infer that these limits cannot be interchanged in general. The most appropriate way to cope with the problem is to use stochastic differential equations (SDEs), which form the correct mathematical description for thermal noise problems and do not distinguish components at different frequencies. While the time dependent solutions of SDEs include the transient behavior of the noisy voltages, the results (1)-(4) obtained by frequency domain methods only apply to the stationary state at thermodynamic

equilibrium. Switching from the frequency domain to the time domain means that frequency limit  $\omega_{max} \rightarrow \infty$  is replaced by time limit  $t \rightarrow \infty$ .

Consider the 2-stage RC ladder in Fig.1. Let there be initial voltages  $V_1(t = 0) = V_1^0$ ,  $V_2(t = 0) = V_2^0$  across the capacitors  $C_1$ ,  $C_2$ , respectively. Following the circuit equations

$$C_1 dV_1/dt = -V_1/R_3 - (V_1 - V_2)/R_4, \quad V_1(0) = V_1^0, \quad (5)$$

$$C_2 dV_2/dt = (V_1 - V_2)/R_4, \quad V_2(0) = V_2^0, \quad (6)$$

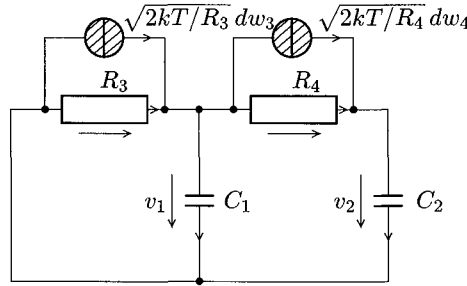
for  $t > 0$  these initial voltages will relax to equilibrium (in the sense of dynamical systems), which is  $V_1^{eq} = 0$ ,  $V_2^{eq} = 0$ . Assume that the circuit is in contact with a heat bath of absolute temperature  $T$ . Then, equilibrium will be a thermodynamic equilibrium. Due to the thermal motion of electrons the resistors exhibit thermal noise.

In a neighborhood of thermodynamic equilibrium, the time dependence of the noisy capacitor voltages  $v_1 = V_1 + v_{1,n}$ ,  $v_2 = V_2 + v_{2,n}$  can be calculated from the stochastic differential equations

$$C_1 dv_1 = \left( -\frac{v_1}{R_3} - \frac{v_1 - v_2}{R_4} \right) dt + \sqrt{\frac{2kT}{R_3}} dw_3 - \sqrt{\frac{2kT}{R_4}} dw_4, \quad (7)$$

$$C_2 dv_2 = \frac{v_1 - v_2}{R_4} dt + \sqrt{\frac{2kT}{R_4}} dw_4. \quad (8)$$

$dw_3$  and  $dw_4$  are differentials of independent Wiener processes, i.e.  $dw_3/dt$  and  $dw_4/dt$  are independent Gaussian white noise processes. As the SDEs are linear there is no difference between the Itô and the Stratonovich interpretation of (7), (8). The equations (7), (8), represent the conventional noise equivalent network in Fig. 2. Since our interest is limited to thermodynamic equilibrium, it is not



**FIGURE 2.** Circuit representation of SDEs (7), (8)

necessary to calculate the transient noise behavior. Instead, it is sufficient to use the equipartition theorem of equilibrium thermodynamics, which says that the *independent* capacitors  $C_1$ ,  $C_2$  exhibit thermal voltage fluctuations

$$\overline{(v_{i,n}^2)_{eq}} = kT/C_i, \quad i = 1, 2; \quad \overline{(v_{1,n}v_{2,n})_{eq}} = 0. \quad (9)$$

(9) can also be obtained by solving the SDE system (7), (8) for  $v_1(t)$ ,  $v_2(t)$ , by calculating the time dependent correlations  $\overline{(v_i(t)v_j(t))}$  and finally by taking the limits  $t \rightarrow \infty$ . Clearly, (9) is in agreement with (1).

### Noise analysis in the limit $R_3 \rightarrow \infty$

Before we do the noise analysis in the limit  $R_3 \rightarrow \infty$ , let us think about the physics of the problem: Open circuiting  $R_3$  as depicted in Fig. 3 destroys the fluctuational independence of the capacitors. Any change in charge on  $C_1$  means

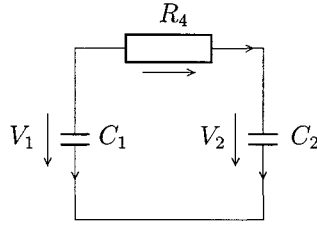


FIGURE 3. RC ladder for  $R_3 \rightarrow \infty$

an oppositely directed change in charge on  $C_2$ . Hence we can expect anti-correlation of the noise voltages  $v_{n,1}$ ,  $v_{n,2}$ . This cannot only be inferred from Fig. 3 but also from the corresponding SDEs

$$C_1 dv_1 = -(v_1 - v_2)/R_4 dt - \sqrt{2kT/R_4} dw_4, \quad v_1(0) = V_1^0, \quad (10)$$

$$C_2 dv_2 = (v_1 - v_2)/R_4 dt + \sqrt{2kT/R_4} dw_4, \quad v_2(0) = V_2^0, \quad (11)$$

which result from (7), (8) by  $R_3 \rightarrow \infty$ .

To calculate the noise correlations we solve the SDE system (10), (11)

$$v_1(t) = \frac{V_1^0}{C_1 + C_2} (C_1 + C_2 e^{-t/R_4 C_s}) + \frac{V_2^0}{C_1 + C_2} (C_2 - C_2 e^{-t/R_4 C_s}) - C_1^{-1} \sqrt{2kT/R_4} e^{-t/R_4 C_s} \int_0^t e^{\tilde{t}/R_4 C_s} dw_4(\tilde{t}), \quad (12)$$

$$v_2(t) = \frac{V_1^0}{C_1 + C_2} (C_1 - C_1 e^{-t/R_4 C_s}) + \frac{V_2^0}{C_1 + C_2} (C_2 + C_1 e^{-t/R_4 C_s}) + C_2^{-1} \sqrt{2kT/R_4} e^{-t/R_4 C_s} \int_0^t e^{\tilde{t}/R_4 C_s} dw_4(\tilde{t}), \quad (13)$$

and obtain in the limit  $t \rightarrow \infty$

$$\overline{(v_{1,n}^2)_{eq}} = \frac{kTC_2}{C_1(C_1 + C_2)}, \quad \overline{(v_{2,n}^2)_{eq}} = \frac{kTC_1}{C_2(C_1 + C_2)}, \quad \overline{(v_{1,n}v_{2,n})_{eq}} = -\frac{kT}{C_1 + C_2}, \quad (14)$$

where the series capacitance

$$C_s := C_1C_2/(C_1 + C_2) \quad (15)$$

has been introduced. The result (14) proves the correctness of (2) and demonstrates the failure of the first equation in (4).

However if the circuit is assumed to begin at  $t = 0$  by removing a finite resistor  $R_3$ , the the voltages  $V_1^0, V_2^0$  will be random variables such that  $\overline{(V_1^0)^2} = kT/C_1$  and  $\overline{(V_2^0)^2} = kT/C_2$ . It follows that  $\overline{(V_{s,eq})^2} = kT/(C_1 + C_2)$  and we obtain

$$\overline{v_1^2(t)} = kT/C_1, \quad \overline{v_2^2(t)} = kT/C_2, \quad \overline{v_1(t)v_2(t)} = 0, \quad (16)$$

and for this special case, the result in (16) is independent of  $t$ . On the other hand, if it is assumed  $V_1^0 = V_2^0 = 0$  then  $v_1(t) = v_{1,n}(t)$  and  $v_2(t) = v_{2,n}(t)$  and as  $t \rightarrow \infty$  we obtain the same result as in (14).

It is instructive to see how the same result can be obtained using the equipartition law: In a first step we combine the series capacitors to single series capacitor  $C_s$ . Then, the mean square series voltage fluctuation is  $\overline{(v_{s,n}^2)_{eq}} = \overline{((v_{n,1} + v_{n,2})^2)_{eq}} = kT/C_s$ . To calculate the contributions of the single capacitors we assume a noise source  $\overline{(v_{s,n}^2)_{eq}}$  across the two capacitors in series. The potential divider rule then yields  $\overline{(v_{n,1}^2)_{eq}}, \overline{(v_{n,2}^2)_{eq}}$  as in (14), whereas the cross correlation (14) is obtained from  $2\overline{(v_{1,n}v_{2,n})_{eq}} = \overline{(v_{s,n}^2)_{eq}} - \overline{(v_{n,1}^2)_{eq}} - \overline{(v_{n,2}^2)_{eq}}$ .

### Noise analysis in the limit $R_4 \rightarrow 0$

In the limit  $R_4 \rightarrow 0$ , the capacitors become parallel with equal noise voltages, which are fully correlated. Looking at the SDE system (7), (8) we observe divergence in the limit  $R_4 \rightarrow 0$ . A mathematically correct noise description of the circuit's transient behavior in the sense of an independent initial value problem does not exist! (The calculations leading to (3) were restricted to thermodynamic equilibrium, i.e. to the steady state.) This is true for time and frequency domain approaches. As the capacitors are parallel, the initial voltages  $V_1^0, V_2^0$  cannot be chosen independently. Again, (1), which only holds for different  $v_{1,n}, v_{2,n}$ , is no longer applicable. To do the noise analysis we introduce a parallel capacitor  $C_p := C_1 + C_2$  and treat the resulting two element network, see Fig. 4. Corresponding SDE is

$$C_p dv_p = -\frac{v_p}{R_3} dt + \sqrt{\frac{2kT}{R_3}} dw_3, \quad v_p(0) = V_1^0 = V_2^0. \quad (17)$$

As the circuit contains only one capacitor, the equipartition theorem is applicable

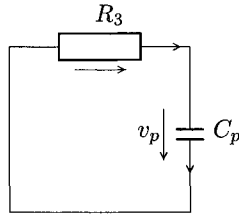


FIGURE 4.  $C_1$  and  $C_2$  combined to the parallel capacitor  $C_p$

$$\overline{(v_{p,n}^2)_{eq}} = \frac{kT}{C_p} = \frac{kT}{C_1 + C_2} = \overline{(v_{1,n}^2)_{eq}} = \overline{(v_{2,n}^2)_{eq}} = \overline{(v_{1,n}v_{2,n})_{eq}}. \quad (18)$$

Comparing (18) with (3) we see that (3) is correct, while the 2nd equation in (4) is wrong. At first glance, the non-vanishing cross correlation (18) looks like a contradiction to the vanishing cross correlations in (1) and (9). There is no contradiction any more if we keep in mind that (1) and (9) were obtained for a (different) circuit with independent capacitors. The pre-requisites yielding (1) and (9) are not valid in the limit  $R_4 \rightarrow 0$ .

## CONCLUSIONS AND OPEN QUESTIONS

We have demonstrated that limits of circuit parameter values, which change the topology of a circuit, should always be taken before a noise analysis is performed. Doing the noise analysis first and taking the limits in circuit parameter values afterwards means to interchange two limits, which is wrong when the change in circuit topology “transforms” fluctuationaly independent dynamical elements into fluctuationaly dependent ones. Such a change is always accompanied by a qualitative change of the poles and zeros of the spectral densities related to the mean square fluctuations. In such cases, smooth changes of the resistance values  $R_3 \rightarrow \infty$  and  $R_4 \rightarrow 0$  yield discontinuous changes of the spectral densities, hence a discontinuous change of the total mean square fluctuations. The “2-stage RC ladder paradox”, which is caused by the interchange of forbidden limits, is related to “Penfield’s (second) motor paradox” [3], and illustrates that the resolution of the paradox is not to be found in circuit analysis. It is not related to Penfield’s (first) motor paradox [2], which demonstrates how a physically incomplete noise modeling results in a “contradiction” to the second law of thermodynamics.

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