Lattice vs. Continuum: Landau Gauge Fixing and 't Hooft-Polyakov Monopoles

A Dissertation Submitted for the Degree of Doctor of Philosophy

by

Dhagash B. Mehta

Supervisors: Dr. Lorenz von Smekal
and
Prof. Anthony G. Williams

Centre for the Subatomic Structure of Matter,
School of Chemistry and Physics,
University of Adelaide.
March, 2009.

To
my family and friends ...

Contents

Abstract **viii**

Statement of Originality **x**

Acknowledgments **xi**

1 Introduction **1**
1.1 Motivation 1
1.2 Landau Gauge-fixing 2
1.3 't Hooft-Polyakov Monopoles 6
1.4 Structure Of The Thesis 6

2 Landau Gauge Fixing **8**
2.1 Gauge Field Theories 8
2.1.1 Faddeev-Popov Gauge-fixing Procedure 10
2.1.2 Gribov Copies 12
2.2 Gauge Field Theory On The Lattice 13
2.2.1 Landau Gauge Fixing On The Lattice 14
2.2.2 The Neuberger 0/0 Problem 16
2.3 Topological Interpretation 17
2.3.1 Morse Theory and The Neuberger 0/0 Problem 17
2.3.2 Modification Via Stereographic Projection 19
2.3.3 Betti Numbers and Gribov Copies 20
2.4 Remarks 22
2.5 Summary 23

3 Lattice Landau Gauge For Lower Dimensional Models **25**
3.1 Lattice Landau Gauge Fixing for Compact U(1) 25
3.2 Anti-periodic Boundary Conditions In One Dimension 27
3.2.1 Standard Lattice Landau Gauge (SLLG) 27
3.2.2 Modified Lattice Landau Gauge (MLLG) 30
3.3 Periodic Boundary Conditions In One Dimension 31
3.3.1 SLLG 31

3.3.2 MLLG 33
3.4 Topological Interpretation Revisited 34
3.4.1 One-dimensional SLLG 35
3.4.2 One-dimensional MLLG 36
3.5 Faddeev-Popov Procedure for the MLLG 36
3.6 $1 + 1$ Complete Coulomb Gauge 39
3.6.1 Anti-periodic Boundary Conditions 39
3.6.2 Periodic Boundary Conditions 40
3.7 Remarks 42
3.8 Summary 43

4 Algebraic Geometry and Lattice Landau Gauge Fixing 45
4.1 Landau Gauge Fixing Equations as Polynomial Equations 45
4.2 Groebner Basis 50
4.2.1 SLLG 51
4.2.2 MLLG 52
4.2.3 Random Orbit Case 53
4.2.4 Higher Dimensional Lattices 54
4.3 More About The Solution Space 54
4.3.1 Gribov copies and Algebraic Geometry 56
4.4 Remarks 57
4.5 Summary 59

5 Polynomial Homotopy Continuation Method 61
5.1 Polynomial Homotopy Continuation 61
5.1.1 Homotopy Continuation 62
5.1.2 Multivariable Polynomial Homotopy Continuation 63
5.1.3 Polyhedral Homotopy 65
5.2 Results 67
5.2.1 SLLG With Anti-periodic Boundary Conditions 67
5.2.2 SLLG With Periodic Boundary Conditions 71
5.2.3 MLLG 71
5.3 Some Remarks 73
5.4 Summary 76

6 't Hooft-Polyakov Monopoles In Lattice SU(N)+adjoint Higgs Theory 78
6.1 Magnetic Charges In The Continuum 79
6.2 Magnetic Charge On The Lattice 82
6.3 Monopole Mass 84
6.4 Twisted Boundary Conditions 85
6.4.1 Fully C-periodic Boundary Conditions 86
6.4.2 Magnetic Flux 88

6.4.3 Allowed Magnetic Charges 89
6.4.4 Relation To The Continuum and Zeroes of Higgs 92
6.5 Summary 93

7 Conclusions **94**

Appendices **98**

A No Neuberger $0/0$ Problem in Higher Dimensional Lattices **98**

B One-dimensional Periodic Boundary conditions **100**
B.1 SLLG 100
B.1.1 Classification Of Gribov Copies of The SLLG 103
B.2 MLLG 104

C Miscellaneous: Algebraic Geometry **106**
C.1 Transforming Problems To Algebraic Geometry 106
C.2 Special Appearance: Numerical Algebraic Geometry 115
C.2.1 SLLG and NAG 117
C.2.2 Discussion 117

D Mixed Boundary Conditions and Magnetic Monopole Charge **119**
D.1 Mixed boundary conditions 119
D.1.1 z Direction C-periodic and x,y Directions Periodic 119
D.1.2 y,z Directions C-periodic and x Direction Periodic 121

E Another Approach To Modify Lattice Landau Gauge **123**

F Published Articles of Author **125**

Bibliography **137**

List of Acronyms

1. BKK - Bernstein-Khovanskii-Kushnirenko
2. BRST - Becchi-Rouet-Stora-Tyutin
3. CAD - Cylindrical Algebraic Decomposition
4. CBB - Classical Bezout Bound
5. CGB - Comprehensive Groebner Basis
6. DSE - Dyson-Schwinger Equation
7. FMR - Fundamental modular region
8. LSZ - Lehmann-Symanzik-Zimmermann
9. MLLG - Modified lattice Landau gauge
10. NAG - Numerical Algebraic Geometry
11. NPHC - Numerical Polynomial Homotopy Continuation
12. QE - Qunatifier Elimination
13. QED - Quantum Electrodynamics
14. QCD - Quantum Chromodynamics
15. RPXYM - Random phase XY model
16. SLLG - Standard lattice Landau gauge
17. SMV - Stable Mixed Volume

List of Figures

5.1 The horizontal axis denotes the solution number (arbitrarily given) and the vertical axis has the polynomial version (the determinants for the trigonometric version differers by a constant factor only in any case) of the Faddeev-Popov determinant, for the 3×3 lattice for the SLLG, trivial orbit, anti-periodic boundary conditions case. As with all other plots hereafter, this plot does not include the two global maxima and two global minima, since these four points have considerably higher magnitude, an obstacle to viewing the rest of the symmetry clearly. However, the Faddeev-Popov determinants at those solutions as well cancel each other exactly. 69

5.2 *Special random* orbit, 3×3 lattice, SLLG, anti-periodic boundary conditions, with the polynomial version of the Faddeev-Popov determinant. The *randomness* is now apparent here compared to the trivial orbit case. 70

5.3 A random orbit, 3×3 lattice, SLLG, anti-periodic boundary conditions, with the polynomial version of the Faddeev-Popov determinant. 72

5.4 Trivial orbit, 3×3 lattice, SLLG, periodic boundary conditions, with the polynomial version of the Faddeev-Popov determinant. . 72

5.5 Random orbit, 3×3 lattice, SLLG, periodic boundary conditions, with the polynomial version of the Faddeev-Popov determinant. . 74

6.1 Integration curve used to calculate the flux through half of the box. 88

D.1 Integration curves for one and two C-periodic directions. 120

List of Tables

5.1 Summary of the 3×3 lattice, SLLG with anti-periodic boundary conditions, trivial orbit. 67

5.2 Summary of the number of solutions with i negative eigenvalues, for the SLLG, 3×3 lattice, trivial orbit, anti-periodic boundary conditions. 69

5.3 Summary of the solutions for the SLLG, 3×3 lattice, special random orbit, anti-periodic boundary conditions. 69

5.4 Summary of the number of solutions with i negative eigenvalues for the SLLG, 3×3 lattice, anti-periodic boundary conditions, special random orbit. 70

5.5 Summary of the solutions for the SLLG, 3×3 lattice, random orbit, anti-periodic boundary conditions. 71

5.6 Summary of the number of solutions with i negative eigenvalues for the SLLG, 3×3 lattice, anti-periodic boundary conditions, random orbit. 71

5.7 Summary of the number of solutions with i positive eigenvalues, denoted as P_i, for the SLLG, 3×3 lattice, trivial orbit, periodic boundary conditions. 71

5.8 Summary of the number of solutions with i negative eigenvalues for the SLLG, 3×3 lattice, trivial orbit, periodic boundary conditions. 73

5.9 Summary of the number of solutions with i positive eigenvalues, P_i, for the SLLG, 3×3 lattice, random orbit, periodic boundary conditions. 73

5.10 Summary of the number of solutions with i negative eigenvalues for the 3×3 lattice, random orbit, periodic boundary conditions. 73

C.1 Summary of the solutions for SO(3). 116

C.2 Summary of the solutions for the SLLG with periodic boundary conditions for the one-dimensional $n = 3$ lattice, from NAG. Note that this summary remains valid for both the trivial orbit and random orbit cases. 117

Abstract

In this thesis we study the connection between continuum quantum field theory and corresponding lattice field theory, specifically for two cases: Landau gauge fixing and 't Hooft-Polyakov monopoles.

To study non-perturbative phenomena such as the confinement mechanism of quarks and gluons and dynamical chiral symmetry breaking in Quantum Chromodynamics (QCD), there are two major approaches: the Dyson-Schwinger equations (DSEs) approach, which is based on the covariant continuum formulation, and lattice gauge theory. The strength and beauty of lattice gauge theory is due to the fact that gauge invariance is manifest and fixing a gauge is not required. In the covariant continuum formulation of gauge theories, on the other hand, one has to deal with the redundant degrees of freedom due to gauge invariance and has to fix gauge (most popularly, Landau gauge). There, the gauge-fixing machinery is based on the so-called Faddeev-Popov procedure or more generally, the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. Beyond perturbation theory this is aggravated by the existence of so-called Gribov copies, however, that satisfy the same gauge-fixing condition, but are related by gauge transformations, and are thus physically equivalent. When attempting to fix Landau gauge on the lattice to make a connection with its continuum counterpart, this ambiguity manifests itself in the Neuberger 0/0 problem that asserts that the expectation value of any physical observable will always be of the indefinite form 0/0. We explain the topological nature of this problem and how the complete cancellation of Gribov copies can be avoided in a modified lattice Landau gauge based on a new definition of gauge fields on the lattice as stereographically projected link variables. For compact U(1), where the Gribov copy problem is related to the classification the local minima of XY spin glass models, we explicitly show that there still remain Gribov copies but their number is exponentially reduced in lower dimensional models. We then formulate the corresponding Faddeev-Popov procedure on the lattice, for these models. Moreover, we explicitly demonstrate that the proposed modification circumvents the Neuberger 0/0 problem for lattices of arbitrary dimensions for compact U(1). Applied to the maximal Abelian subgroup this will avoid the perfect cancellation amongst the remaining Gribov copies for SU(N), and so the corresponding BRST formulation is also then possible for generic SU(N), in particular, for the Standard Model groups.

For higher dimensional lattices, the gauge fixing conditions for both the standard and the modified lattice Landau gauges are systems of multivariate nonlinear equations, solving which in general is a highly non-trivial task. However, we show that these systems can be interpreted as systems of polynomial equations. They can then be solved exactly by computational Algebraic Geometry, the Groebner basis technique in particular, and numerically by the Polynomial Homotopy Continuation method.

't Hooft-Polyakov monopoles play an important role in high energy physics due to their presence in grand unified theories and their usefulness in studying non-perturbative properties of quantum field theories through electric-magnetic dualities. In the second part of the thesis, we study adjoint Higgs models, which exhibit 't Hooft-Polyakov monopoles, and have been extensively analyzed using semi-classical analysis in the continuum. However, to study them in a fully non-perturbative fashion, it is essential to put the theory on the lattice. Here, we investigate twisted C-periodic boundary conditions in SU(N) gauge field theory with an adjoint Higgs field and show that for even N with a suitable twist one can impose a non-zero magnetic charge relative to each of $N-1$ residual U(1)'s in the broken phase, thereby creating 't Hooft-Polyakov magnetic monopoles. This makes it possible then to use lattice Monte-Carlo simulations to study the properties of these monopoles in the full quantum theory and compare them with the existing results in the continuum.

Statement of Originality

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying; subject to the provisions of the Copyright Act 1968.

I also give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library catalogue, the Australasian Digital Theses Program (ADTP) and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

Dhagash B. Mehta

Acknowledgements

Firstly, I am very thankful to my primary supervisor, Lorenz von Smekal, who first convinced me and then helping and supporting me to work on these extremely interesting problems. Mainly due to countless discussions with him and his encouragements this thesis has made thus far.

I thank my co-supervisor, Anthony G. Williams, who supported me in many ways including providing me assistance and support in many administrative problems. I am also very thankful to my both supervisors for giving me the creative freedom. I thank Arttu Rajantie who invited me to visit Imperial College London to work in collaboration with him and being my co-supervisor for more than 8 months during my ph.d.; Martin Schaden for very fruitful discussions at early and very important stage of this thesis during his visit to Adelaide; and Jon-Ivar Skullerud for his critical remarks on the thesis.

I thank Andre Sternbeck and Samuel Edwards for their wonderful collaborations, discussions, inputs and great help in proof-reading this thesis. Thanks to my friends at the CSSM, Ahmed Bakry, Marco Bartolozzi, Jonathan Carroll, Ben Crouch, Tat Chew, Marco Ghiotti, John Hedditch, Mariusz Hoppe, Antonio Laguna, Ben Lasscock, Selim Mehbub, Peter Moran, Alan O'Cais, Maria Parapilly, Skye Platten et al. who made my stay at CSSM a wonderful experience and also proof-read my thesis. Thanks to my office-mates at Theoretical Division at Imperial College London who made my visit to London a memorable experience.

I thank my postgraduate coordinator Rodney Crewther and Max Lohe for their kind help and guidance at many stages; to Varghese Mathai and all the postgraduate students in the Mathematics department of University of Adelaide; to Billy Hanan for his kind help in proof-reading a draft of this thesis; to Ramona Adorjan, Patrick Fitzhenry and the whole SAPAC team for their numerous technical help related to computers; and to Sara Boffa, Brownyn Gibson, Sharon Johnson, and Silvana Santucci for their tremendous administrative help.

I acknowledge the E-IPRS scholarships administrators who financially supported me throughout my ph.d. candidature and British Council and the University of Adelaide who funded me for my visit to London.

At personal level, I acknowledge the love and encouragement from my parents, wife, brother and my extended family. Thanks to my friends spread across the globe for their love, support and friendship.