Regularized Equivariant Euler Classes and Gamma Functions

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For my parents,
MR LOOI CHEE SIN *and* MDM LEE LAI CHAN
on the occasion of their 30th anniversary.

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Contents

Abstract

We consider the regularization of some equivariant Euler classes of certain infinite-dimensional vector bundles over a finite-dimensional manifold M using the framework of zeta-regularized products [35, 53, 59]. An example of such a regularization is the Atiyah–Witten regularization of the $\mathbb{T}$-equivariant Euler class of the normal bundle $\nu(TM)$ of M in the free loop space LM [2].

In this thesis, we propose a new regularization procedure — W-regularization — which can be shown to reduce to the Atiyah–Witten regularization when applied to the case of $\nu(TM)$. This new regularization yields a new multiplicative genus (in the sense of Hirzebruch [26]) — the $\hat{\Gamma}$-genus — when applied to the more general case of a complex spin vector bundle of complex rank ≥ 2 over M, as opposed to the case of the complexification of TM for the Atiyah–Witten regularization. Some of its properties are investigated and some tantalizing connections to other areas of mathematics are also discussed.

We also consider the application of W-regularization to the regularization of $\mathbb{T}^2$-equivariant Euler classes associated to the case of the double free loop space LLM. We find that the theory of zeta-regularized products, as set out by Jorgenson–Lang [35], Quine et al [53] and Voros [59], amongst others, provides a good framework for comparing the regularizations that have been considered so far. In particular, it reveals relations between some of the genera that appeared in elliptic cohomology, allowing us to clarify and prove an assertion of Liu [44] on the $\hat{\Theta}$-genus, as well as to recover the Witten genus. The $\hat{\Gamma}_2$-genus, a new genus generated by a function based on Barnes' double gamma function [5, 6], is also derived in a similar way to the $\hat{\Gamma}$-genus.

Signed Statement

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Chapter 1

Introduction

This thesis is concerned with applying regularization methods to infinite products of equivariant characteristic classes of vector bundles over a finite-dimensional manifold M. Amongst our results, we find that, by proposing a new regularization method, called *W-regularization*, we managed to derive two new multiplicative genera (see §4.2 for a definition): the $\hat{\Gamma}$-genus, described in Chapter 6 and the paper [46], and the $\hat{\Gamma}_2$-genus, which appears in Chapter 7.

This work grew out of an investigation into the circle-equivariant de Rham cohomology of the free smooth loop space LM of a compact finite-dimensional manifold M. However, as one of the objects of our investigation was the Atiyah–Witten regularization of the $\mathbb{T}$-equivariant Euler class of the normal bundle of M in LM, our study soon led us to the theory of zeta-regularization, far removed from the index theory that motivated Atiyah in his paper [2].

One of the tools that yielded some insight into the Atiyah–Witten regularization was the abstract theory of zeta-regularized products developed by authors like Jorgenson–Lang [35], Quine et al. [53] and Voros [59]. We find that this theory helps to place our proposed W-regularization in the context of the established zeta-regularization methods, which is fortuitous given what W-regularization has yielded.

Indeed, W-regularization led us to discover a new multiplicative genus. We call this the $\hat{\Gamma}$-genus, as it bears a resemblance to both the $\hat{A}$-genus and the Γ-genus of Libgober [42]. The generating function of the genus, naturally called the $\hat{\Gamma}$-function, turns out to have surprising connections to algebraic geometry and number theory.

An extension of our construction to the double free loop space LLM also allowed us to recover some genera in elliptic cohomology. For instance, we were able to clarify and prove a statement of Liu [44] on the $\hat{\Theta}$-genus, as well as recover the Witten genus (see Chapter 7). The language of zeta-regularized products also allowed us to express the relations between these genera more clearly. We found that we could also derive a new multiplicative genus, which is generated by a function based on Barnes' double gamma function, by applying W-regularization to the case of the double loop space.

In what follows, we shall give a more detailed account of the locus of ideas that come together in this thesis. First, we review in §1.1 some of the developments in the study of $\mathbb{T}$-equivariant cohomology and its relation to the free loop space. Next, in §1.2, we consider the development of the theory of zeta-regularization of infinite products and how this theory has influenced our work. We outline our results in §1.3 and conclude this chapter by giving a guide to the contents of this thesis in §1.4.

1.1 Equivariant cohomology and loop spaces

The study of the equivariant cohomology of the free loop space LM was carried out by many authors in the 1980's. A confluence of intriguing developments may serve, perhaps, to explain this. One is the proof of the Duistermaat–Heckman theorem [18, 19], which showed that stationary phase approximation is exact under certain conditions, and the subsequent realization of this result as a key feature of equivariant cohomology. Another is the keen interest in loop spaces coming from theoretical physics. Also, the discovery of cyclic homology by Connes [17] and the elucidation of the relation of this new theory to the $\mathbb{T}$-equivariant cohomology of LM [23, 33, 45] occupied the attentions of many authors as well.

The key feature of equivariant cohomology is that it satisfies the *localization theorem*. This allows the computation of the equivariant cohomology of a manifold from the ordinary cohomology of simpler spaces. To give an example, the theorem asserts, for the group $\mathbb{T}$, that a suitably defined form of the $\mathbb{T}$-equivariant cohomology of a manifold M carrying a $\mathbb{T}$-action can be computed from the ordinary cohomology of its fixed point set F. More precisely, it is known (see, for example, Atiyah–Bott [3]) that the $\mathbb{T}$-equivariant de Rham cohomology of a compact finite-dimensional manifold M can be defined as

$$H^*_{\mathbb{T}}(M) = H^*(\Omega_{\mathbb{T}}(M)[u], d_{\mathbb{T}}).$$

Here, the complex is the ring of polynomials in an indeterminate u of degree 2 with coefficients in the $\mathbb{T}$-invariant differential forms on M, and $d_{\mathbb{T}}$ is an equivariant version of the exterior derivative. The *localized* $\mathbb{T}$-equivariant cohomology is then defined by *localizing* the complex at u (i.e. inverting u):

$$H^*_{\mathbb{T},loc}(M) = H^*(\Omega_{\mathbb{T}}(M)[u, u^{-1}], d_{\mathbb{T}}).$$

The localization theorem then states that if $i\colon F \hookrightarrow M$ is the inclusion of the fixed point set in M, then its pullback is an isomorphism on cohomology:

$$i^*\colon H^*_{\mathbb{T},loc}(M) \xrightarrow{\cong} H^*_{\mathbb{T},loc}(F) \cong H^*(F)[u, u^{-1}].$$

It turns out that the integration formula of Duistermaat and Heckman can then be viewed as a consequence of the localization theorem. In fact, the formula embodies the localization theorem in a more useful form. As Berline and Vergne [8] realized, after having independently derived the same formula, the $\mathbb{T}$-equivariant Euler class can be recovered from it. This prompted the realization that the Duistermaat–Heckman formula could be considered to be a result in equivariant cohomology.

Subsequently, Witten was inspired by these results and index theory to attempt to apply the localization theorem for equivariant cohomology to spaces of infinite dimensions, in particular, to loop spaces [61]. Note that there is a natural action of the circle $\mathbb{T}$ on LM, given by the rotation of loops, and that its fixed point set is just M, viewed as the submanifold of constant loops. Thus, a suitably defined form of equivariant cohomology for loop spaces might be computable using only knowledge of the cohomology of the underlying manifold. Witten's considerations turned out to be very fruitful indeed, initiating a stream of mathematical activity that included the birth and development of elliptic cohomology [39].

One of Witten's ideas was elaborated in a paper of Atiyah [2], which outlined the Atiyah–Witten regularization of the $\mathbb{T}$-equivariant Euler class of the normal bundle of M in LM. In this paper, Atiyah described a procedure, due to Witten, by which the Atiyah–Singer equivariant index theorem can be derived formally. This theorem may be formulated in terms of an equation relating the equivariant index of the Dirac operator D on a spin manifold M and a multiplicative genus, the $\hat{A}$-genus. The (equivariant) index of an operator can be viewed as the difference in dimension between its kernel and cokernel, while a genus (in the sense of Hirzebruch [26]) is a homomorphism from the oriented cobordism ring to the real numbers, defined by a (formal) power series. The $\hat{A}$-genus $\hat{A}(M)$ of a compact oriented finite-dimensional manifold M is then defined by the $\hat{A}$-function

$$\hat{A}(z) := \frac{z/2}{\sinh(z/2)},$$

and the index theorem is the identity

$$\operatorname{index}(D) = \hat{A}(M).$$

By applying the Duistermaat–Heckman theorem formally to calculate the equivariant index of the (undefined) Dirac operator on the free loop space, and applying zeta-function regularization to deal with a divergent infinite product that results, Atiyah was able to recover the $\hat{A}$-genus, and thus formally derive the equivariant index theorem.

All of this motivated Jones–Petrack [34] to construct a version of $\mathbb{T}$-equivariant cohomology for infinite-dimensional manifolds that satisfies the localization theorem. In their search for such a version of equivariant cohomology for LM, they were motivated also by Goodwillie's negative result that the usual localized $\mathbb{T}$-equivariant cohomology does not satisfy the localization theorem for LM [23]. The construction that they eventually found may be considered to be a *completed* version of $\mathbb{T}$-equivariant cohomology, which is how we shall describe this variant in the sequel.

1.2 Zeta-regularization and infinite products

The zeta-regularization that Atiyah applied heuristically in [2] was not initially designed to be used in such situations. The first hints of zeta-regularization may be found in the paper of Ray and Singer [54], where it was used to define the analytic torsion of a manifold. Here and in the work of many subsequent authors, zeta-regularization appeared in terms of an operator.

Subsequently, however, some authors began studying zeta-regularization in a more abstract setting. This seemed to have been triggered by the appearance of a long-forgotten special function — Barnes' double gamma function [5, 6] — in calculations relating to the determinant of the Laplacian on Riemann surfaces. In studying this, Voros compiled a summary of the theory of zeta-regularization in his work on the factorization of the Selberg zeta function [59]. As part of that exposition, he demonstrated a theorem which gave a clear picture of the structure of *zeta-regularized products*. These are divergent infinite products, constructed out of sequences of complex numbers, that were assigned values using zeta-regularization. This was later refined by Quine et al. [53], who computed many examples, and Jorgenson and Lang [35], who highlighted the importance of the hybrid Laplace–Mellin transform in the theory.

In light of these developments, Atiyah's calculations in the paper [2] can now be viewed as an application of zeta-regularization to an infinite product of factors involving characteristic classes. As we eventually found out from our investigations, the Atiyah–Witten regularization can be described within the framework of zeta-regularized products. It is not, however, a zeta-regularized product in the strict sense, and a good part of our investigation is devoted to describing this difference.

Our initial motivation for this investigation was, however, not from the consideration of zeta-regularized products. Rather, it was an attempt at generalizing the work of Jones–Petrack, which had been developed with an eye towards a possible application to index theory (see their paper written jointly with Getzler [22]). Thus, it sufficed for them that the $\hat{A}$-genus could be recovered. In view of what we have discussed so far, this appeared to be a rather narrow scope for an application of their work.

Our point of departure, then, is the observation that the construction makes use of the *complexification* of a real vector bundle. The natural question is then to ask what follows from applying this construction to complex vector bundles which are not complexifications of real vector bundles. We find rather quickly that we run into problems with the convergence of infinite products. This is the point at which the theory of zeta-regularized products makes its appearance.

1.3 Results

Our first result in this thesis is that our proposed regularization — W-regularization — yields a new multiplicative genus when it is applied to the case of complex spin vector bundles of complex rank ≥ 2 over a base manifold M. We note that W-regularization is different from the zeta-regularization of infinite products in [53, 59], which actually yields another genus: the Γ-genus that was introduced and studied by Libgober in the context of mirror symmetry [42]. However, W-regularization is closer in spirit to the Atiyah–Witten regularization, and actually reduces to the latter in the case of the complexification of a real bundle. The resulting new genus — the $\hat{\Gamma}$-genus — also comes with some curious properties.

We also find a surprisingly elegant interpretation of these two regularizations in the setting of Hoffman's work [28, 29] on multiple zeta values (MZVs), which came to our attention after we had constructed the W-regularization. In the course of studying MZVs, Hoffman found that by making use of symmetric functions (see Appendix A for a summary of the relevant theory or the book [47]) to define a homomorphism $Z\colon \mathrm{Sym} \to \mathbb{R}$, he could express MZVs as elements in the image of the Z-homomorphism. In particular, the image of the generating function of the elementary symmetric function

$$C(t) = \prod_{i=1}^{\infty} (1 + x_i t)$$

under Z is

$$Z(C(t)) = \frac{1}{\Gamma(1+t)},$$

which is the generating function of the Γ-genus. We find that the Z-homomorphism can be modified to obtain a map, which we call the $\hat{Z}$-map, that produces the regularization leading to the $\hat{\Gamma}$-genus.

We also studied the case of the double free loop space LLM and found that we were able to recover some of the genera that appeared in elliptic cohomology. Specifically, we were able to clarify and prove an assertion of Liu [44] on the $\hat{\Theta}$-genus, as well as recover the Witten genus, by applying the various regularizations, which we have considered so far, to the $\mathbb{T}^2$-equivariant Euler class. By mimicking the construction that produced the $\hat{\Gamma}$-genus in the case of LM, we were able to derive another new multiplicative genus — the $\hat{\Gamma}_2$-genus — with a generating function based on Barnes' double gamma function [5, 6].

1.4 Outline

As the work in this thesis draws together many streams of ideas in mathematics, a substantial amount of background material has to be considered before the main results are presented. The background material is presented in Chapters 2 to 5, the main results are described in Chapters 6 and 7, and Chapter 8 concludes the thesis. The reader is referred to Appendix A for notation in the thesis that deviates from the traditional one in the theory of symmetric functions.

The theory of zeta-regularization forms a key component of the work in this thesis. Much of this relies on the theory of the Laplace and Mellin transforms, which we review in Chapter 2. These transforms form the basis for the theory of zeta-regularized products [53, 59], which we review in Chapter 3, following a summary of the theory of Weierstrass products, which may be seen as the precursor of zeta-regularization theory. In fact, this link becomes manifest in the statement of Theorem 3.4.3, the structure theorem of zeta-regularized products. We conclude the chapter by computing several useful examples taken from Quine et al. [53].

Hirzebruch's theory of multiplicative sequences and genera also plays an important role in this thesis. We review this theory in Chapter 4, phrasing our account in the language of generating functions and symmetric functions. We also give a recursive algorithm, due to Hirzebruch and refined by Libgober–Wood [43], that allows us to calculate the polynomials in a multiplicative sequence.

Chapter 5 is a review of equivariant de Rham cohomology, which we need as part of the background on equivariant characteristic classes. The main feature of this cohomology theory is that a suitably localized variant of equivariant cohomology would satisfy a localization theorem, which allows the calculation of the equivariant cohomology of a manifold in terms of the ordinary cohomology of its fixed point set under a group action. We conclude this chapter by showing how one may obtain the equivariant Euler class from the localization theorem.

All the material considered in Chapters 2 to 5 is then brought together in Chapter 6. We begin this chapter by reviewing the idea behind the Atiyah–Witten regularization of the $\mathbb{T}$-equivariant Euler class of the normal bundle of M in LM. We then propose a new regularization, which can be applied to the more general case of complex spin vector bundles of complex rank ≥ 2 over M, as opposed to just the complexification of TM. The $\hat{\Gamma}$-genus is thus derived and we discuss some of its interesting properties, as well as those of the Γ-genus of Libgober [42].

In Chapter 7, we consider the regularization of the $\mathbb{T}^2$-equivariant Euler class of the normal bundle of M in the double free loop space LLM. Using the framework of the theory of zeta-regularized products, we give a unified treatment of the various genera coming from different regularization procedures. In particular, we clarify and prove a statement of Liu [44] regarding the $\hat{\Theta}$-genus. Next, we apply the same idea used to generalize the Atiyah–Witten regularization to do the same in the case of LLM. We find that we obtain a genus generated by a function based on Barnes' double gamma function $\Gamma_2(z; u, v)$. We call this genus the $\hat{\Gamma}_2$-genus.

Finally, we conclude this work in Chapter 8 with a summary and discussion of our results. We highlight the connections with other areas of mathematics that our work has revealed, which naturally leads to some interesting speculations. In Appendix A, we review some aspects of the theory of symmetric functions used throughout this work so as to indicate the deviations from standard notation that we have employed to avoid conflict of notations. We also list, in this appendix, some of the multiplicative sequences, and the first few polynomials in these sequences, that appear in this thesis.

Chapter 2

The Integral Transforms of Laplace and Mellin

The theory of zeta-regularized products draws heavily on the theory of the Laplace and Mellin transforms, which we review in this chapter. We begin this chapter by reviewing some properties of the Laplace transform in §2.1, which concludes with the proof of Watson's Lemma (Theorem 2.1.6).

In §2.2, we discuss the Laplace–Mellin transform, following Jorgenson–Lang [35], who identified this hybrid of the Laplace and Mellin transforms as the fundamental building block of zeta-regularization. We prove some of its properties, including an asymptotic formula that incorporates Watson's Lemma. This formula will be very useful in the theory of zeta-regularized products that we shall review in Chapter 3.

2.1 The Laplace transform

In this section, we review some rudiments in the theory of the Laplace transform, concluding with a proof of Watson's lemma, which gives an asymptotic expansion for the Laplace integral. This will be needed in the theory of zeta-regularized products.

We recall the definition of the Laplace transform (cf. [40, p. 395] and [55, p. 391]).

Definition 2.1.1. Suppose f is a piecewise continuous function on $(0, \infty)$ and satisfies the condition that as $t \to \infty$, there are constants C and k such that

$$|\, f(t)\, | \leq Ce^{kt}.$$

Then the *Laplace transform* of f is given by the integral

$$\mathcal{L}f(z) := \int_0^\infty e^{-zt} f(t)dt.$$

It is known that the Laplace transform converges for some half-plane (cf. [40, p. 395] and [55, p. 391]), so we introduce some notation:

Definition 2.1.2. The *abscissa of simple convergence* σ of the Laplace transform of $f(t)$

$$\mathcal{L}f(z) = \int_0^\infty e^{-zt} f(t)dt$$

is a real number such that the integral is convergent for all z with $\mathrm{Re}(z) > \sigma$ and divergent for all z with $\mathrm{Re}(z) < \sigma$.

The next few lemmas (cf. [55, §8.11]) are preparation for the proof of Watson's lemma.

Lemma 2.1.3. *Suppose $f(t)$ is a piecewise continuous function on $(0,\infty)$ having a Laplace transform $\mathcal{L}f(z)$ with abscissa of simple convergence σ. Let $C > 0$ be a positive number such that $C > \sigma$. Then, as $v \to \infty$,*

$$\int_0^v f(t)dt = O(e^{Cv}).$$

Proof. If σ is negative, then the integral $\int_0^\infty f(t)dt$ must exist by the definition of the abscissa of simple convergence. The Laplace integral of f is therefore bounded, so the lemma holds.

We are left with the case when $\sigma \geq 0$. Note that [55, (7.15-1)]

$$\sigma = \lim_{v\to\infty} \sup \frac{\log \left| \int_0^v f(t)dt \right|}{v}$$

or equivalently, that there is some $\epsilon > 0$ such that as $v \to \infty$,

$$\left| \int_0^v f(t)dt \right| < e^{(\sigma+\epsilon)v}$$

By hypothesis, $C > \sigma$, so we can choose ϵ such that $\sigma + \epsilon < C$. Therefore, there exists a $\delta > 0$ such that as $v \to \infty$,

$$\left| \int_0^v f(t)dt \right| < e^{(\sigma+\epsilon-C)v} e^{Cv} < \delta e^{Cv},$$

which completes the proof. □

Lemma 2.1.4. *Let $z \in \mathbb{C}$ be such that* $\mathrm{Re}(z) > \max(0,\sigma)$. *Then we have*

$$\int_0^\infty e^{-zt} f(t)dt = z\int_0^\infty e^{-zt} \int_0^t f(s)dsdt. \tag{2.1.1}$$

Proof. Note that $\mathcal{L}f(z)$ is holomorphic in the half-plane $\mathrm{Re}(z) > \sigma$. Thus, if we can show that both sides of (2.1.1) are holomorphic functions and coincide on the real numbers in the half-plane $\mathrm{Re}(z) > \max(0,\sigma)$, then they must be equal everywhere in that half-plane. Hence, it suffices to prove the lemma for real values of z.

Suppose then that $z \in \mathbb{R}$ with $z > \max(0, \sigma)$. Integrating by parts gives

$$\int_0^v e^{-zt} f(t)dt = e^{-zv} \int_0^v f(t)dt + z \int_0^v e^{-zt} \int_0^t f(s)dsdt. \tag{2.1.2}$$

Note that there exists a number $C > 0$ such that $\sigma < C < z$. By Lemma 2.1.3, we see that

$$\lim_{v\to\infty} e^{-zv} \int_0^v f(t)dt \leq \lim_{v\to\infty} e^{(C-z)v} = 0.$$

As $v \to \infty$, then, the first term in (2.1.2) vanishes and we obtain (2.1.1). This completes the proof of the lemma. □

We now show that

Lemma 2.1.5. *As $z \to \infty$ in the sector $|\arg(z)| < \pi/2$,*

$$\varphi(z) = \int_1^\infty e^{-zt} f(t)dt = O(e^{-\operatorname{Re}(z)}).$$

Proof. Consider the function $g(t)$ defined by

$$g(t) = \begin{cases} 0 & \text{when } 0 < t \leq 1, \\ f(t) & \text{when } t > 1. \end{cases}$$

This gives

$$\int_0^\infty e^{-zt} g(t)dt = \varphi(z),$$

Applying Lemma 2.1.4, we see that, for $\operatorname{Re}(z) > \max(0, \sigma)$,

$$\varphi(z) = z \int_1^\infty e^{-zt} \int_1^t f(s)dsdt.$$

By Lemma 2.1.3, if $C > \max(0, \sigma)$, then as $t \to \infty$,

$$\int_1^t f(s)ds = O(e^{Ct}).$$

Hence, we have that for $t \geq 1$,

$$\left| \int_1^t f(s)ds \right| < Be^{Ct},$$

for some constant B. Therefore, for $|z|$ sufficiently large, we see that

$$|\varphi(z)| \leq |z| B \int_1^\infty e^{-(\operatorname{Re}(z)-C)t} dt = \frac{B|z|}{\operatorname{Re}(z) - C} e^{-(\operatorname{Re}(z)-C)}.$$

If we assume that $\mathrm{Re}(z) \geq 2C$, then in the sector $|\arg(z)| < \pi/2$, we see that

$$\frac{|z|}{\mathrm{Re}(z) - C} \leq \frac{2|z|}{\mathrm{Re}(z)} \leq \frac{2}{\cos|\arg(z)|},$$

so that

$$|\varphi(z)| \leq (2Be^C \sec|\arg(z)|)e^{-\mathrm{Re}(z)}.$$

Therefore, we have shown that $\varphi(z) = O(e^{-\mathrm{Re}(z)})$. □

We can now present a proof of Watson's lemma, following the approach in [9], which shows that the asymptotic expansion of the Laplace transform of a function $f(t)$ is given by taking the Laplace transform of each term in the asymptotic expansion of $f(t)$.

Theorem 2.1.6. (Watson's Lemma, cf. [9]) *Suppose $f(t)$ is a function of a real variable t such that, as $t \to 0$, $f(t)$ has the asymptotic expansion*

$$f(t) \sim \sum_{n=0}^{\infty} c_{k_n} t^{k_n}, \tag{2.1.3}$$

where $-1 < k_0 < k_1 < \ldots \to \infty$. Furthermore, suppose that the Laplace transform of $f(t)$

$$\mathcal{L}f(z) = \int_0^\infty e^{-zt} f(t)dt$$

converges on some half-plane. Then, as $z \to \infty$ in the sector $|\arg(z)| < \pi/2$, $\mathcal{L}f(z)$ has the asymptotic expansion

$$\mathcal{L}f(z) \sim \sum_{n=0}^{\infty} c_{k_n} \Gamma(k_n + 1) z^{-(k_n+1)}.$$

Proof. We split the Laplace integral into two terms:

$$\mathcal{L}f(z) = I_1(z) + I_2(z),$$

where

$$I_1(z) = \int_0^1 e^{-zt} f(t)dt, \qquad I_2(z) = \int_1^\infty e^{-zt} f(t)dt.$$

Recall from Lemma 2.1.5 that as $z \to \infty$ in the sector $|\arg(z)| < \pi/2$,

$$\int_1^\infty e^{-zt} f(t)dt = O(e^{-\mathrm{Re}(z)}),$$

so it remains to estimate $I_1(z)$.

First, we note that as $t \to 0$, $f(t)$ has an asymptotic expansion given by (2.1.3). This allows us to write, for any positive integer N,

$$f(t) = \sum_{n=0}^{N} c_{k_n} t^{k_n} + R_N(t),$$

where $R_N(t) = O(t^{\operatorname{Re} k_{N+1}})$. Putting this into $I_1(z)$, we see that

$$I_1(z) = \sum_{n=0}^{N} c_{k_n} \int_0^1 t^{k_n} e^{-zt} dt + \int_0^1 R_N(t) e^{-zt} dt. \tag{2.1.4}$$

Next, note that

$$\begin{aligned} \int_0^1 t^{k_n} e^{-zt} dt &= \int_0^\infty t^{k_n} e^{-zt} dt - \int_1^\infty t^{k_n} e^{-zt} dt \\ &= \Gamma(k_n+1) z^{-(k_n+1)} + O(e^{-\operatorname{Re}(z)}), \end{aligned} \tag{2.1.5}$$

where the estimate for the second term follows by substituting t^{k_n} for $f(t)$ in Lemma 2.1.5. Since $R_N(t) = O(t^{\operatorname{Re} k_{N+1}})$,

$$|\, R_N(t) \,| \le c_N t^{\operatorname{Re} k_{N+1}}$$

for some constant c_N. Therefore,

$$\begin{aligned} \left| \int_0^1 R_N(t) e^{-zt} dt \right| &\le c_N \int_0^\infty t^{\operatorname{Re} k_{N+1}} e^{-zt} dt \\ &= c_N \Gamma(\operatorname{Re}(k_{N+1}) + 1) z^{-(\operatorname{Re} k_{N+1}+1)} \\ &= O(z^{-(\operatorname{Re} k_{N+1}+1)}), \end{aligned} \tag{2.1.6}$$

and we see that, as $z \to \infty$,

$$\mathcal{L}f(z) = \sum_{n=0}^{N} c_{k_n} \Gamma(k_n+1) z^{-(k_n+1)} + O(z^{-(\operatorname{Re} k_{N+1}+1)}).$$

This is equivalent to saying that $\mathcal{L}f(z)$ has the asymptotic expansion

$$\mathcal{L}f(z) \sim \sum_{n=0}^{\infty} c_{k_n} \Gamma(k_n+1) z^{-(k_n+1)}$$

as $z \to \infty$ in the sector $|\arg(z)\,| < \pi/2$, so the proof is complete. $\square$

2.2 The Laplace–Mellin transform

In this section, we define the Laplace–Mellin transform, following the approach of Jorgenson and Lang [35]. It turns out that this transform is crucial to an understanding of the theory of zeta-regularized products, so we shall prove some of its properties as a preliminary to discussing zeta-regularized products in the sequel.

We first give the definition of the following transforms, deferring the discussion of convergence conditions to later in this section.

Definition 2.2.1. The *Mellin transform* of $f(t)$ is given by

$$\mathcal{M}f(s) := \int_0^\infty f(t)t^s \frac{dt}{t}.$$

Definition 2.2.2. The *Laplace–Mellin transform* of $f(t)$ is given by

$$\mathcal{LM}f(s,z) := \int_0^\infty f(t)e^{-zt}t^s \frac{dt}{t}.$$

We prove a useful lemma on the simplest Laplace–Mellin transform.

Lemma 2.2.3. *Let $k, p \in \mathbb{C}$ and $\mathrm{Re}(z), \mathrm{Re}(s+p) > 0$. Then we have*

$$\int_0^\infty ke^{-zt}t^{s+p}\frac{dt}{t} = k\frac{\Gamma(s+p)}{z^{s+p}}. \tag{2.2.1}$$

Proof. We recall that the gamma function can be represented as the integral

$$\Gamma(s+p) = \int_0^\infty e^{-t}t^{s+p}\frac{dt}{t}.$$

Under the substitution $t \mapsto zt$, where $\mathrm{Re}(z), \mathrm{Re}(s+p) > 0$,

$$\Gamma(s+p) = \int_0^\infty e^{-zt}z^{s+p}t^{s+p}\frac{dt}{t} = z^{s+p}\int_0^\infty e^{-zt}t^{s+p}\frac{dt}{t},$$

so we find that

$$\frac{\Gamma(s+p)}{z^{s+p}} = \int_0^\infty e^{-zt}t^{s+p}\frac{dt}{t}, \tag{2.2.2}$$

which completes the proof. □

Remark 2.2.4. We prove several more identities that we shall need later on. Let $\mathrm{Re}(s), \mathrm{Re}(z) > 0$. Then we note that

$$\int_1^\infty e^{-zt}t^s\frac{dt}{t} = \frac{\Gamma(s)}{z^s} - \int_0^1 e^{-zt}t^s\frac{dt}{t}. \tag{2.2.3}$$

We observe that this integral has a meromorphic continuation to all $s \in \mathbb{C}$ and all $z \in \mathbb{C}$. To see this, we can expand the term e^{-zt}, in the integral on the right-hand side, as a Taylor series and integrate term by term to get

$$\begin{aligned}\int_0^1 e^{-zt} t^s \frac{dt}{t} &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} t^{s+k-1} dt \\ &= \sum_{k=0}^{\infty} \left[\frac{(-z)^k}{k!} \frac{t^{s+k}}{s+k} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \frac{1}{s+k}.\end{aligned} \tag{2.2.4}$$

Using integration by parts also gives us a meromorphic continuation:

$$\begin{aligned} s \int_0^1 e^{-zt} t^s \frac{dt}{t} &= t^s e^{-zt} \Big]_0^1 - \int_0^1 (-z) e^{-zt} t^{s+1} \frac{dt}{t} \\ &= e^{-z} + z \int_0^1 e^{-zt} t^{s+1} \frac{dt}{t}.\end{aligned} \tag{2.2.5}$$

Before we continue to exhibit more properties of the Laplace–Mellin transform, we need the following technical lemma.

Lemma 2.2.5. ([40, XV Lemma 1.1]) *Let T be a possibly infinite interval on the real numbers and $U \subseteq \mathbb{C}$ be an open set of complex numbers. Let $f : T \times U \to \mathbb{C}$ be a continuous function and*

$$F(z) = \int_T f(t, z) dt$$

be its integral over T. Suppose, in addition, that

1. *$F(z)$ converges uniformly on every compact subset K of U.*

2. *For each $t \in T$, the function $f(t, z)$ is an analytic function of z.*

Then $F(z)$ is analytic on U.

Proof. Let $\{T_n\}$ be a sequence of finite closed intervals with $\lim_{n\to\infty} T_n = T$, D be a disc in the z-plane with $\overline{D} \subseteq U$, and γ be the boundary of D. By Cauchy's formula, we have that for each $z \in D$,

$$f(t, z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t, \zeta)}{\zeta - z} d\zeta.$$

Hence,

$$F(z) = \frac{1}{2\pi i} \int_T \int_\gamma \frac{f(t, \zeta)}{\zeta - z} d\zeta dt.$$

Denote the radius of γ by R, and the centre of γ by z_0. If $z \in D$, $|\,z - z_0\,| \leq R/2$, so that

$$\left|\frac{1}{\zeta - z}\right| \leq \frac{2}{R}.$$

Define, for each n, the function

$$F_n(z) := \frac{1}{2\pi i}\int_{T_n}\int_{\gamma}\frac{f(t,\zeta)}{\zeta - z}d\zeta dt.$$

Under the given restriction on z, we can use Fubini's theorem to interchange the integrals:

$$F_n(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - z}\left[\int_{I_n} f(t,\zeta)dt\right]d\zeta.$$

By hypothesis 1, the integrals over T_n converge uniformly to the integral over T, so, for $|\,z - z_0\,| \leq R/2$, F_n converge uniformly to F. Hence, F is analytic, thus completing the proof of this lemma. □

We now show that if $f(t)$ has an asymptotic expression as $t \to 0$, then its Laplace–Mellin transform $\mathcal{LM}f(s,z)$ has a meromorphic continuation and an asymptotic expansion as $z \to \infty$.

Lemma 2.2.6. [35, Lemma 1.3] *Let f be a piecewise continuous function on $(0,\infty)$. Suppose that the following conditions hold:*

1. *The function $f(t)$ is bounded as $t \to \infty$.*
2. *For some numbers $b_k \in \mathbb{C}$, $k,\ell \in \mathbb{C}$, with $\mathrm{Re}(\ell) > \mathrm{Re}(k)$, the function $f(t)$ satisfies the asymptotic condition*

$$f(t) = b_k t^k + O(t^{\mathrm{Re}(\ell)})$$

as $t \to 0$.

Then the following statements are true:

1. *For $\mathrm{Re}(s) > -\mathrm{Re}(k)$ and $\mathrm{Re}(z) > 0$, $\mathcal{LM}f(s,z)$ is absolutely convergent.*
2. *For $\mathrm{Re}(s) > -\mathrm{Re}(\ell)$, $\mathcal{LM}f(s,z)$ has a meromorphic continuation given by*

$$\mathcal{LM}f(s,z) = b_k\frac{\Gamma(s+k)}{z^{s+k}} + g(s,z).$$

Here, for any fixed z, $g(s,z)$ is holomorphic in s for $\mathrm{Re}(s) > -\mathrm{Re}(\ell)$.

3. *When $\mathrm{Re}(s) > -\mathrm{Re}(\ell)$ and $\mathrm{Re}(z) > 0$, the only possible singularities of $\mathcal{LM}f(s,z)$ are poles at $s = -(n+k)$ of order 1, where $n \geq 0$ is an integer.*

Proof. The Laplace–Mellin transform can be split into a sum of integrals:

$$\begin{aligned}
\mathcal{LM}f(s,z) &= \int_0^\infty (f(t)-b_k t^k)e^{-zt}t^s\frac{dt}{t} + \int_0^\infty e^{-zt}b_k t^{s+k}\frac{dt}{t} \\
&= \int_0^1 (f(t)-b_k t^k)e^{-zt}t^s\frac{dt}{t} \\
&\quad + \int_1^\infty (f(t)-b_k t^k)e^{-zt}t^s\frac{dt}{t} + b_k\frac{\Gamma(s+k)}{z^{s+k}} \\
&= b_k\frac{\Gamma(s+k)}{z^{s+k}} + I_1(s,z) + I_2(s,z),
\end{aligned}$$

where

$$\begin{aligned}
I_1(s,z) &= \int_0^1 (f(t)-b_k t^k)e^{-zt}t^s\frac{dt}{t}, \\
I_2(s,z) &= \int_1^\infty (f(t)-b_k t^k)e^{-zt}t^s\frac{dt}{t}.
\end{aligned}$$

We note that by Lemma 2.1.5, $I_2(s,z)$ is holomorphic in s, and converges uniformly for all s with $\mathrm{Re}(s) > -\mathrm{Re}(k)$ and all z with $\mathrm{Re}(z) > 0$. It remains to consider the integral $I_1(s,z)$, but Lemma 2.2.5 shows that $I_1(s,z)$ is holomorphic in s on the half-plane $\mathrm{Re}(s) > -\mathrm{Re}(\ell)$. Setting $g(s,z) = I_1(s,z) + I_2(s,z)$ proves assertions 1 and 2.

For assertion 3, note that $b_k\Gamma(s+k)z^{-(s+k)}$ is the only term that can contribute to $\mathcal{LM}f(s,z)$ having poles. Thus, the only possible poles of $\mathcal{LM}f(s,z)$ are simple and must be at $s = -(k+n)$ for all integers $n \geq 0$. □

Combining this with Watson's Lemma, we have the following statement on the asymptotic expansion of the Laplace–Mellin transform of a function:

Corollary 2.2.7. *Let $f(t)$ be a piecewise continuous function on $(0,\infty)$ such that as $t \to 0$,*

$$f(t) \sim \sum_{n=0}^\infty c_{k_n} t^{k_n},$$

where $k_0 < k_1 < \ldots \to \infty$. Then, as $z \to \infty$ in the sector $|\arg(z)| < \pi/2$,

$$\mathcal{LM}f(s,z) \sim \sum_{n=0}^\infty c_{k_n}\Gamma(s+k_n)z^{-(s+k_n)}.$$

This result will be a useful tool in helping us to calculate zeta-regularized products in Chapter 3.

Chapter 3

Zeta Regularization of Infinite Products

In this chapter, we review the theory of the zeta-function regularization of infinite products. We begin, firstly, with a review of the classical theory of infinite products in §3.1, and give several useful examples. This is not only useful for understanding zeta-regularized products, but also re-surfaces when we discuss Hirzebruch's theory of multiplicative genera in Chapter 4. Thus, we shall also give the infinite product representation of several entire functions that we shall use later in defining multiplicative genera.

In §3.2, we consider infinite products defined over a lattice in $\mathbb{C}$. These products are taken over a pair of indices ranging over the integers, and the theory is sufficiently involved that its discussion merits an entire section. We shall need this in our consideration of $\mathbb{T}^2$-equivariant classes.

We then arrive at the notion of a zeta-regularizable sequence of complex numbers in §3.3, and discuss some properties of the zeta-regularized product associated to such a sequence. In §3.4, we prove a structure theorem for zeta-regularized products, following the approaches of Quine et al. [53] and Voros [59]. This also shows the relation of zeta-regularized products with Weierstrass products.

In §3.5, we conclude the chapter with some examples of zeta-regularized products, which are taken from Quine et al. [53]. Several important functions are derived as zeta-regularized products, including the Γ-function, Barnes' double gamma function $\Gamma_2(z; u, v)$ and the Dedekind eta function $\eta(\tau)$.

3.1 Infinite products and entire functions

In this section, we shall review the factorization of an entire function into an infinite product. Our references here include [10, 37, 40, 48].

Recall that an *entire function* of a complex variable $f(z)$ can be represented by a power series

$$f(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots$$

which converges for all $z \in \mathbb{C}$. A special class of entire functions having a finite number of zeroes is the class of polynomial functions. By the fundamental theorem of algebra, any polynomial function

$$p(z) = c_0 + c_1 z + \cdots + c_m z^m$$

of degree m on the complex plane has a factorization in terms of its zeroes z_i, counting multiplicities, i.e.

$$p(z) = c_m \prod_{i=1}^{m} (z - z_i).$$

The Weierstrass factorization theorem can be viewed as an extension of this theorem to any entire function. However, the general situation is far more complicated. For instance, entire functions need not have zeroes, and such a function $f(z)$ can always be written as the exponential of another entire function $h(z)$:

$$f(z) = e^{h(z)}.$$

Thus, it follows that an entire function $f(z)$ with a finite number of distinct zeroes $z_1, \ldots, z_n$, with multiplicities $a_1, \ldots, a_n$, has the general form (see, e.g. [48, Theorem II.9.9])

$$f(z) = e^{h(z)} (z - z_1)^{a_1} \cdots (z - z_n)^{a_n}.$$

When the entire function $f(z)$ has an infinite number of zeroes, however, and $f(z)$ is not identically zero, then the zeroes of such an entire function cannot have an accumulation point in $\mathbb{C}$, so that they must tend to infinity (see, e.g. [48, Theorem I.17.1]). In particular, since there are no accumulation points in $\mathbb{C}$, every compact subset of $\mathbb{C}$ must contain only a finite number of zeroes. Thus, we can order the sequence of zeroes $\{z_k\}$ in terms of increasing absolute value, so that for all k, $|\, z_k \,| \leq |\, z_{k+1} \,|$.

We now quote *Weierstrass' factorization theorem* for entire functions.

Theorem 3.1.1. *Let $f(z)$ be an entire function with zeroes given by the sequence*

$$\underbrace{0, \ldots, 0}_{a \text{ times}}, z_1, \ldots, z_n, \ldots,$$

where $a \geq 0$ is an integer and for all k, $|\, z_k \,| \leq |\, z_{k+1} \,|$. Then $f(z)$ has an infinite product representation of the form

$$f(z) = e^{h(z)} z^a \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \cdots + \frac{z^n}{n z_n^n}\right) \right\},$$

where $h(z)$ is an entire function.

We shall not prove this theorem, but only explain the key ideas behind the proof, referring the reader to the references cited earlier for the details. In particular, we shall explain how the exponential factors arise, following the arguments given in [48, Section II.46].

Remark 3.1.2. The idea of the proof is to show that the sequence of entire functions $\{f_\nu(z)\}$, given by

$$f_\nu(z) = z^a \prod_{n=1}^{\nu} \left\{ \left(1 - \frac{z}{z_n}\right) e^{\phi_n(z)} \right\}$$

for polynomials $\phi_n(z)$ and $\nu = 1, 2, \ldots$, converges uniformly to an entire function

$$f_\infty(z) = z^a \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{z_n}\right) e^{\phi_n(z)} \right\}.$$

Note that by the discussion in the introductory paragraphs to this section, any entire function $f(z)$ is unique up to an exponential factor, so we can restrict our analysis to proving the above.

We note that we have not yet chosen the polynomials $\phi_n(z)$. The trick is to do this such that $\{f_\nu(z)\}$ converges uniformly on every compact subset of $\mathbb{C}$. Then, Weierstrass' theorem on uniformly convergent sequences of analytic functions (see, e.g. [48, Theorem I.15.8]) implies that $f_\infty(z)$ is entire.

To do this, fix a disk $D_R := \{z \in \mathbb{C} : |\, z\,| < R\}$. Choose $\mu(R)$ to be the smallest integer such that for any $n > \mu(R)$, $|\, z_n\,| > 2R$. Then, for $\nu > \mu(R)$ and $z \in D_R$, we can rewrite $f_\nu(z)$ as

$$f_\nu(z) = f_{\mu(R)}(z) \exp\left\{ \sum_{n=\mu(R)+1}^{\nu} \left[\ln\left(1 - \frac{z}{z_n}\right) + \phi_n(z) \right] \right\}.$$

Observing that

$$\ln\left(1 - \frac{z}{z_n}\right) = -\sum_{m=1}^{\infty} \frac{z^m}{m z_n^m},$$

we note that a suitable choice for $\phi_n(z)$ would be

$$\phi_n(z) = \frac{z}{z_n} + \cdots + \frac{z^n}{n z_n^n},$$

since this gives the estimate

$$\left| \ln\left(1 - \frac{z}{z_n}\right) + \phi_n(z) \right| \leq \sum_{m=n+1}^{\infty} \frac{|\, z\,|^m}{m\,|\, z_n\,|^m} < \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n},$$

so that

$$\sum_{m=\mu(R)+1}^{\infty} \left| \ln\left(1 - \frac{z}{z_n}\right) + \phi_n(z) \right| < \sum_{m=1}^{\infty} \frac{1}{2^m} < \infty,$$

ensuring that the series

$$\sum_{m=\mu(R)+1}^{\infty}\left[\ln\left(1-\frac{z}{z_n}\right)+\phi_n(z)\right] \tag{3.1.1}$$

converges uniformly on D_R. The proof follows by exponentiating the above series and analyzing $f_\infty(z)$, or by noting that the uniform convergence of the above series in D_R implies the uniform convergence of the infinite product

$$\prod_{n=1}^{\infty}\left\{\left(1-\frac{z}{z_n}\right)\exp\left(\frac{z}{z_n}+\cdots+\frac{z^n}{nz_n^n}\right)\right\}$$

to an entire function [48, Theorem I.17.4], which proves that $f_\infty(z)$ is entire.

Note 3.1.3. For entire functions that we are interested in, the above theorem is too general to help us in computing their infinite product. In particular, the exponential factors are too unwieldy to be calculated, given that the polynomials $\phi_n(z)$ have degrees that grow arbitrarily large. A sharper version of this theorem exists for a class of entire functions, which shall be sufficient for our purposes here, but this requires further analysis, which we shall summarize, following [48, Sections 47–8].

The first step in sharpening the theorem is to consider the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{|\,z_n\,|^b},$$

where $\{z_n\}$ is an increasing nonzero sequence of complex numbers, and $b \geq 0$ is an integer. It is clear that if the above series converges for some $b_0 > 0$, then it must converge for all $b > b_0$. We make the following definition.

Definition 3.1.4. The *divergence exponent* $\beta \geq 0$ of the series

$$\sum_{n=1}^{\infty}\frac{1}{|\,z_n\,|^b},$$

where $\{z_n\}$ is an increasing nonzero sequence of complex numbers, is the largest integer b for which this series diverges. If the series diverges for all $b > 0$, then we define its divergence exponent to be $\beta = \infty$.

The second step is to consider the order of the entire function $f(z)$. We begin with the following definition.

Definition 3.1.5. The maximum modulus $M(r)$ of an entire function $f(z)$ is given by

$$M(r) = \max_{|\,z\,|=r}|\,f(z)\,|\,.$$

A standard result on the growth of entire functions asserts that if $f(z)$ is an entire function that has an essential singularity at infinity, i.e. that $f(z)$ does not reduce to a polynomial, then $f(z)$ grows faster than any fixed power of r. More precisely,

$$\lim_{r\to\infty} \frac{\ln M(r)}{\ln r} = \infty.$$

Definition 3.1.6. Suppose $f(z)$ is an entire function with maximum modulus $M(r)$, and there exists a real number $a > 0$ for which

$$M(r) < e^{ra}$$

for sufficiently large r. Then $f(z)$ has finite *order* ν, where ν is defined to be

$$\nu := \inf a \geq 0.$$

We write $\lfloor \nu \rfloor$ for the largest integer not exceeding ν.

We now have all the ingredients needed to quote *Hadamard's factorization theorem* for entire functions of finite order:

Theorem 3.1.7. *Let $f(z)$ be an entire function of finite order ν, with zeroes given by an increasing sequence of complex numbers*

$$\underbrace{0, \ldots, 0}_{a\,\text{times}}, z_1, \ldots, z_n, \ldots,$$

and let β be the divergence exponent of the series

$$\sum_{n=1}^{\infty} \frac{1}{\mid z_n \mid^{\beta}}.$$

Then $f(z)$ can be written in the form

$$f(z) = e^{h(z)} z^a \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \cdots + \frac{z}{\beta z_n^{\beta}}\right) \right\},$$

for a polynomial $h(z)$ of degree not exceeding $\lfloor \nu \rfloor$, and the exponential factors vanish if $\beta = 0$.

We shall also omit the proof of this result, referring the reader to the references cited earlier. We shall remark, however, that the theorem is now much sharper, since the polynomials in the exponential factors have the same fixed degree. In fact, the uniform convergence of the series (3.1.1) now depends on the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{\mid z_n \mid^{\beta+1}},$$

instead of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}.$$

We give some examples to illustrate the theorem. Example 3.1.8 is taken from [48, Volume II, Section 48], but is also worked out in [37, Section 3, 1st Example] in the guise of $f(z) = \sin \pi z$.

Example 3.1.8. Let $f(z) = \sin z$, or

$$f(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

The following analysis closely follows that in [48]. We note, firstly, that this is an entire function of order 1, since for $z = x + iy$, the inequality

$$|\sinh y| \leq |\sin z| = \sqrt{\sinh^2 y + \sin^2 x} \leq \sqrt{\sinh^2 y + 1} = \cosh y$$

implies that

$$\frac{e^r - 1}{2} < \frac{e^r - e^{-r}}{2} \leq M(r) = \max_{|z|=r} |\sin z| \leq \frac{e^r + e^{-r}}{2} < \frac{e^r + 1}{2},$$

so that

$$M(r) < e^{r^1}.$$

Next, $f(z)$ has zeroes at $z = 0, \pi, -\pi, 2\pi, -2\pi, \ldots, n\pi, -n\pi, \ldots$. This increasing sequence of zeroes has divergence exponent 1, since the series

$$\sum_{n=1}^{\infty} \frac{1}{2(n\pi)^2}$$

converges but the series

$$\sum_{n=1}^{\infty} \frac{1}{2(n\pi)}$$

diverges. By the factorization theorem, $f(z)$ can then be represented in the form

$$\begin{aligned} f(z) &= e^{h(z)} z \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{n\pi}\right) \exp\left(\frac{z}{n\pi}\right) \left(1 + \frac{z}{n\pi}\right) \exp\left(-\frac{z}{n\pi}\right) \right\} \\ &= e^{h(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right) \end{aligned}$$

We remark here that there is a cancellation of the convergence factors owing to the symmetrical occurrence of the zeroes.

To find $h(z)$, note that since $f(z)$ has order 1, $h(z)$ has at most degree 1, i.e.

$$h(z) = h_0 + h_1 z.$$

Observe that the quotient of power series

$$\exp(h(z)) = \frac{\sin z}{z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right)}$$

is even, so that

$$\exp(h_0 + h_1 z) = \exp(h_0 - h_1 z),$$

and thus $\exp(2h_1 z) = 1$, i.e. $h_1 = 0$. Taking the limit as $z \to 0$ yields $\exp(h_0) = 1$, hence $h_0 = 0$. Therefore

$$f(z) = \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right).$$

The following example follows easily from the above.

Example 3.1.9. Consider the function $\sinh z = -i \sin(iz)$. It follows from Example 3.1.8 that $\sinh z$ has the infinite product

$$\sinh z = -i(iz) \prod_{n=1}^{\infty} \left(1 - \frac{(iz)^2}{(n\pi)^2}\right) = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right).$$

In particular, note that the $\hat{A}$-function can be written as the infinite product

$$\hat{A}(z) := \frac{z/2}{\sinh(z/2)} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(2n\pi)^2}\right)$$

The next example is taken from [37, Section 3, 3rd Example]. For a different approach, see [48, Section II.53].

Example 3.1.10. Consider the gamma function, defined as the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, e^{z \ln n}}{z(z+1)(z+2)\cdots(z+n)}.$$

This is a meromorphic function having poles of order 1 at $z = 0, -1, -2, -3, \ldots$. Hence, its inverse $1/\Gamma(z)$ is an entire function with zeroes at $z = 0, -1, -2, -3, \ldots$. Thus, $1/\Gamma(z)$ can be represented as an infinite product

$$\frac{1}{\Gamma(z)} = e^{h(z)} z \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right\}.$$

Following [37], observe that

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to\infty} \frac{z(z+1)(z+2)\cdots(z+n)}{n!\, e^{z\ln n}} \\
&= \lim_{n\to\infty} \frac{z\cdot n!\cdot(1+z)(1+z/2)\cdots(1+z/n)}{n!\, e^{z\ln n}} \\
&= \lim_{n\to\infty} e^{-z\ln n} z(1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right) \\
&= \lim_{n\to\infty} \exp\left[z\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)\right]\cdot z\cdot\prod_{m=1}^{n}\left[\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right)\right].
\end{aligned}$$

Note that the limit

$$\gamma := \lim_{n\to\infty} 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n$$

is just Euler's constant, so

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right\},$$

i.e. $h(z) = \gamma z$.

The following example is an extension of the above.

Example 3.1.11. The functional equation of the gamma function

$$z\Gamma(z) = \Gamma(1+z)$$

implies that $\Gamma(1+z)$ is a meromorphic function with simple poles at all the negative integers. Thus, $1/\Gamma(1+z)$ is an entire function with zeroes of multiplicity one at the negative integers, and can be represented as the infinite product

$$\frac{1}{\Gamma(1+z)} = e^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right\}.$$

Since $e^{-\gamma z}$ is an entire function, and the product of entire functions is also entire, we conclude that

$$\frac{1}{\hat{\Gamma}(z)} = \frac{1}{e^{\gamma z}\Gamma(1+z)} = \prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right\}.$$

3.2 Infinite products over a lattice

In this section, we consider some infinite products taken over a lattice $\Lambda \subset \mathbb{C}$, which we shall need when considering zeta-regularized products over a lattice and $\mathbb{T}^2$-equivariant classes. The approach taken here is adopted from Weil's exposition

[60] of Eisenstein's approach to elliptic functions (see also [20]). The material here is sufficiently different from the preceding section that it merits a treatment in a separate section.

We set out some notation for this section. We shall write u and v for the generators of the lattice Λ, with $\operatorname{Im} v/u > 0$, and denote v/u by τ. When we wish to take the sum or product over all values except for $n = 0$ or $(m, n) = (0, 0)$, we shall indicate this by a prime sign over the summation or product sign. For example,

$$\sum_{m,n\in\mathbb{Z}}{}' \text{ denotes summing over all pairs of integers except for (0,0),}$$

$$\text{while } \prod_{m,n\in\mathbb{Z}}{}' \text{ denotes taking the product over all pairs of integers except for (0,0).}$$

We need a few definitions of some frequently used functions.

Definition 3.2.1. The *Dedekind eta function* is the function

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q = e^{2\pi i\tau}. \tag{3.2.1}$$

Definition 3.2.2. The *Jacobi theta function* $\theta(z,\tau)$ is defined by the series [14, Chapter V, (1.1)]

$$\theta(z,\tau) := \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}, \tag{3.2.2}$$

which can be expressed by the infinite product [14, Chapter V, (6.4)]

$$\theta(z,\tau) = q^{1/8}\cdot 2\sin\pi z\cdot\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2\pi i z})(1-q^n e^{-2\pi i z}), \tag{3.2.3}$$

where z is a complex variable.

Definition 3.2.3. The *Weierstrass sigma function* is the function

$$\sigma(z) := z\prod_{m,n\in\mathbb{Z}}{}'\left\{\left(1-\frac{z}{m+n\tau}\right)\exp\left[\frac{z}{m+n\tau}+\frac{1}{2}\left(\frac{z}{m+n\tau}\right)^2\right]\right\} \tag{3.2.4}$$

Our aim here is to show how the infinite product

$$\begin{aligned}\varphi(z) &:= z\prod_{m,n\in\mathbb{Z}}{}'\left(1+\frac{z}{m+n\tau}\right)\\ &= \sigma(z)\exp\left\{\sum_{m,n\in\mathbb{Z}}{}' -\left[\frac{z}{m+n\tau}+\frac{1}{2}\left(\frac{z}{m+n\tau}\right)^2\right]\right\}\end{aligned}$$

can be written in terms of $\theta(z,\tau)$ and $\eta(\tau)$.

We begin by considering the related function in two complex variables w and z

$$f(w,z) := \prod_{m,n\in\mathbb{Z}} \left(1 + \frac{w}{z+m+n\tau}\right).$$

The following lemma summarizes the relation between $f(w,z)$ and $\varphi(z)$.

Lemma 3.2.4. *The functions $f(w,z)$ and $\varphi(z)$ satisfy the following relations:*

$$f(w,z) = \frac{\varphi(z+w)}{\varphi(z)}, \quad \varphi(w) = z f(w,z)\big|_{z=0}. \tag{3.2.5}$$

Proof. Note that

$$\begin{aligned}
\frac{\varphi(z+w)}{\varphi(z)} &= \frac{z+w}{z} \prod_{m,n\in\mathbb{Z}}{}' \left(\frac{1+\frac{z+w}{m+n\tau}}{1+\frac{z}{m+n\tau}} \right) \\
&= \left(1+\frac{w}{z}\right) \prod_{m,n\in\mathbb{Z}}{}' \left(1 + \frac{w}{z+m+n\tau}\right) \\
&= \prod_{m,n\in\mathbb{Z}}{}' \left(1 + \frac{w}{z+m+n\tau}\right) \\
&= f(w,z).
\end{aligned}$$

For the second relation, note that

$$\begin{aligned}
z f(w,z)\big|_{z=0} &= (z+w) \prod_{m,n\in\mathbb{Z}}{}' \left(1 + \frac{w}{z+m+n\tau}\right)\Bigg|_{z=0} \\
&= w \prod_{m,n\in\mathbb{Z}}{}' \left(1 + \frac{w}{m+n\tau}\right) \\
&= \varphi(w).
\end{aligned}$$

This completes the proof of the lemma. □

To derive the next set of identities, we note the following useful expression for $f(w,z)$:

Lemma 3.2.5. *We can write $f(w,z)$ as a double infinite product:*

$$f(w,z) = \prod_{n\in\mathbb{Z}} \prod_{m\in\mathbb{Z}} \left(1 + \frac{w}{z+m+n\tau}\right).$$

Proof. Note that the sine function can be written as the infinite product

$$\sin \pi z = \pi z \prod_{m=-\infty}^{\infty}{}' \left(1 + \frac{z}{m}\right).$$

Then we observe that

$$\begin{aligned}
\frac{\sin \pi(z + n\tau + w)}{\sin \pi(z + n\tau)} &= \frac{z + n\tau + w}{z + n\tau} \prod_{m=-\infty}^{\infty}{}' \left(\frac{1 + \frac{z+n\tau+w}{m}}{1 + \frac{z+n\tau}{m}}\right) \\
&= \left(1 + \frac{w}{z + n\tau}\right) \prod_{m=-\infty}^{\infty}{}' \left(1 + \frac{w}{z + m + n\tau}\right) \\
&= \prod_{m=-\infty}^{\infty} \left(1 + \frac{w}{z + m + n\tau}\right).
\end{aligned}$$

If we put

$$P_n := \prod_{m=-\infty}^{\infty} \left(1 + \frac{w}{z + m + n\tau}\right),$$

then $f(w, z)$ can be written as

$$f(w, z) = \prod_{n=-\infty}^{\infty} P_n.$$

□

We can now give a more explicit formula for $f(w, z)$, using its expression as an iterated product.

Lemma 3.2.6. *Let $q := e^{2\pi i\tau}$. The function $f(w, z)$ can be expressed as the product*

$$f(w, z) = \frac{e^{\pi i(z+w)} - e^{-\pi i(z+w)}}{e^{\pi iz} - e^{-\pi iz}} \prod_{n=1}^{\infty} \frac{\left(1 - q^n e^{2\pi i(z+w)}\right)\left(1 - q^n e^{-2\pi i(z+w)}\right)}{\left(1 - q^n e^{2\pi iz}\right)\left(1 - q^n e^{-2\pi iz}\right)}. \tag{3.2.6}$$

Furthermore, $\varphi(z)$ can be written as

$$\varphi(z) = \frac{1}{2\pi i} \left(e^{\pi iz} - e^{-\pi iz}\right) \prod_{n=1}^{\infty} \frac{\left(1 - q^n e^{2\pi iz}\right)\left(1 - q^n e^{-2\pi iz}\right)}{(1 - q^n)^2}. \tag{3.2.7}$$

Proof. To prove (3.2.6), we begin by evaluating the product P_0:

$$P_0 = \frac{\sin \pi(z + w)}{\sin \pi z} = \frac{e^{\pi i(z+w)} - e^{-\pi i(z+w)}}{e^{\pi iz} - e^{-\pi iz}}.$$

Next, for $n \neq 0$, the product $P_n P_{-n}$ evaluates to

$$\begin{aligned}
P_n P_{-n} &= \frac{\sin \pi(z+n\tau+w)}{\sin \pi(z+n\tau)} \cdot \frac{\sin \pi(z-n\tau+w)}{\sin \pi(z-n\tau)} \\
&= \frac{q^{n/2}e^{\pi i(z+w)} - q^{-n/2}e^{-\pi i(z+w)}}{q^{n/2}e^{\pi iz} - q^{-n/2}e^{-\pi iz}} \cdot \frac{q^{-n/2}e^{\pi i(z+w)} - q^{n/2}e^{-\pi i(z+w)}}{q^{-n/2}e^{\pi iz} - q^{n/2}e^{-\pi iz}} \\
&= \frac{\left(q^{-n/2}e^{-\pi i(z+w)}\right)\left(1-q^n e^{2\pi i(z+w)}\right)}{\left(q^{-n/2}e^{-\pi iz}\right)\left(1-q^n e^{2\pi iz}\right)} \times \\
&\quad \frac{\left(q^{-n/2}e^{\pi i(z+w)}\right)\left(1-q^n e^{-2\pi i(z+w)}\right)}{\left(q^{-n/2}e^{\pi iz}\right)\left(1-q^n e^{-2\pi iz}\right)} \\
&= \frac{\left(1-q^n e^{2\pi i(z+w)}\right)\left(1-q^n e^{-2\pi i(z+w)}\right)}{\left(1-q^n e^{2\pi iz}\right)\left(1-q^n e^{-2\pi iz}\right)}.
\end{aligned}$$

Thus, since $f(w,z) = P_0 \prod_{n=1}^{\infty} P_n P_{-n}$, we see that

$$f(w,z) = \frac{e^{\pi i(z+w)} - e^{-\pi i(z+w)}}{e^{\pi iz} - e^{-\pi iz}} \prod_{n=1}^{\infty} \frac{\left(1-q^n e^{2\pi i(z+w)}\right)\left(1-q^n e^{-2\pi i(z+w)}\right)}{\left(1-q^n e^{2\pi iz}\right)\left(1-q^n e^{-2\pi iz}\right)}.$$

To prove (3.2.7), we first observe that

$$zf(w,z) = \frac{z\left(e^{\pi i(z+w)} - e^{-\pi i(z+w)}\right)}{e^{\pi iz} - e^{-\pi iz}} \prod_{n=1}^{\infty} \frac{\left(1-q^n e^{2\pi i(z+w)}\right)\left(1-q^n e^{-2\pi i(z+w)}\right)}{\left(1-q^n e^{2\pi iz}\right)\left(1-q^n e^{-2\pi iz}\right)}.$$

The infinite product at $z=0$ becomes

$$\prod_{n=1}^{\infty} \frac{\left(1-q^n e^{2\pi iw}\right)\left(1-q^n e^{-2\pi iw}\right)}{\left(1-q^n\right)^2}.$$

At $z=0$, the factor in front of the infinite product is

$$\left(e^{\pi iw} - e^{-\pi iw}\right) \cdot \lim_{z\to 0} \frac{z}{e^{\pi iz} - e^{-\pi iz}}.$$

To evaluate the limit, we note that the denominator, which we write as $g(z)$, can be expanded as

$$g(z) = e^{\pi iz} - e^{-\pi iz} = 2\pi iz + O(z^2),$$

so $g'(0) = 2\pi i$. Applying l'Hôpital's rule, we see that the limit evaluates to a constant

$$\lim_{z\to 0} \frac{z}{e^{\pi iz} - e^{-\pi iz}} = \frac{1}{2\pi i}.$$

The identity (3.2.7) follows after relabelling w as z:

$$\varphi(z) = \frac{1}{2\pi i}\left(e^{\pi iz} - e^{-\pi iz}\right) \prod_{n=1}^{\infty} \frac{\left(1-q^n e^{2\pi iz}\right)\left(1-q^n e^{-2\pi iz}\right)}{(1-q^n)^2}.$$

□

We can now express $\varphi(z)$ in terms of $\theta(z,\tau)$ and $\eta(\tau)$.

Proposition 3.2.7.

$$\varphi(z) = (2\pi)^{-1}\frac{\theta(z,\tau)}{q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3} = (2\pi)^{-1}\frac{\theta(z,\tau)}{\eta(\tau)^3}.$$

Equivalently,

$$\prod_{m,n\in\mathbb{Z}}{}' \left(1+\frac{z}{m+n\tau}\right) = \frac{\varphi(z)}{z} = (2\pi z)^{-1}\frac{\theta(z,\tau)}{\eta(\tau)^3}.$$

Proof. This follows from (3.2.1), (3.2.3) and (3.2.7). □

3.3 Zeta-regularizable sequences

In this section, we define the notion of a zeta-regularizable sequence of numbers, i.e. a sequence to which a zeta-regularized product can be associated. We then give some properties of zeta-regularized products and consider a related sequence, which we show is zeta-regularizable whenever the original sequence is.

Let $L = \{\lambda_k\}$ be a sequence of non-zero complex numbers with indices in a countable set K.

Definition 3.3.1. The *zeta function* $Z_L(s)$ of L is defined to be

$$Z_L(s) = \sum_{k\in K}\lambda_k^{-s}.$$

Often, the sequence L is the set of eigenvalues of an operator, so that the product of the elements of L is then the determinant of the operator. This infinite product usually diverges, so some form of regularization is needed. To do this rigorously, we need to introduce some conditions on $Z_L(s)$.

Definition 3.3.2. The sequence L is said to be *zeta-regularizable* if its zeta function $Z_L(s)$ has an analytic continuation to a meromorphic function (or a *meromorphic continuation*) on a half plane containing the origin such that this meromorphic function satisfies the following properties:

1. It has poles of order at most one, i.e. all poles are simple.

2. It is analytic at the origin.

The following classical definition of a zeta-regularized product is then well-defined.

Definition 3.3.3. The *zeta-regularized product* associated to the sequence L is defined to be

$$\prod_{k\in K}{}_z \lambda_k := \exp(-Z_L'(0)).$$

The following lemma gives some properties of the zeta-regularized product.

Lemma 3.3.4. *Let* $\{\arg\lambda_k\}$ *be bounded. Then the zeta-regularized product has the following properties:*

1. *For any zeta-regularizable sequence* $\{\lambda_k\}$ *and any nonzero complex number* $a \in \mathbb{C}$ *such that* $\arg a\lambda_k = \arg a + \arg \lambda_k$,

$$\prod_{k\in K}{}_z\, a\lambda_k = a^{Z_L(0)} \prod_{k\in K}{}_z\, \lambda_k. \tag{3.3.1}$$

2. *Suppose a zeta-regularizable sequence* $\{\lambda_k\} = \{\lambda_{1,i}\} \cup \{\lambda_{2,j}\}$ *is the union of two zeta-regularizable sequences with indices* $i \in I$ *and* $j \in J$ *such that* $K = I \cup J$. *Note that the elements in* I *and* J *may be ordered differently from the order in* K. *Then*

$$\prod_{k\in K}{}_z\, \lambda_k = \prod_{i\in I}{}_z\, \lambda_{1,i} \prod_{j\in J}{}_z\, \lambda_{2,j}. \tag{3.3.2}$$

Proof. To show the first property, we let

$$Z_a(s) = \sum_{k\in K} (a\lambda_k)^{-s}.$$

Differentiating this at $s = 0$ gives

$$Z_a'(0) = -\sum_{k\in K} (\log a + \log \lambda_k) = -(\log a \cdot Z_L(0) + Z_L'(0)),$$

so we find that

$$\prod_{k\in K}{}_z\, a\lambda_k = \exp(-Z_a'(0)) = a^{Z_L(0)} \prod_{k\in K}{}_z\, \lambda_k.$$

The second property follows immediately from the assumptions on the sequences. □

Remark 3.3.5. These properties will be very useful in computing zeta-regularized products, but care has to be taken in the choice of $\arg \lambda_k$ and the computation of $Z_L(0)$. For most purposes here, we shall take $|\arg \lambda_k| < \pi/2$ for k large.

We can also associate another useful function to the sequence L.

Definition 3.3.6. [35] The *theta function* associated to L is the series

$$\theta_L(t) = \sum_{k\in K} e^{-\lambda_k t}.$$

Remark 3.3.7. Following [35, 53], we would like to assume that $\theta_L(t)$ is absolutely convergent for $t > 0$; that $\theta_L(t)$ has a full asymptotic expansion

$$\theta_L(t) \sim \sum_{n=0}^{\infty} c_{k_n} t^{k_n}$$

for $k_0 < k_1 < \ldots \to \infty$ as $t \to 0$; and that there is some $\alpha > 0$ for which

$$\lim_{t \to 0} \sum_{k=0}^{\infty} \left| e^{-\lambda_k t} \right| t^{\alpha} = 0.$$

It turns out that these conditions are all satisfied by the sequences that we shall consider, which are themselves taken from [53], where these conditions are also assumed.

In particular, these conditions guarantee that there is a relation between the zeta and theta functions of L given by the identity

$$Z_L(s) = \frac{1}{\Gamma(s)} \mathcal{M}\theta_L(s),$$

where $\mathcal{M}\theta_L(s)$ is the Mellin transform of the theta function. It also gives us a formula for finding $Z(0)$, which turns out to be related to the asymptotic expansion of $\theta_L(s)$:

$$Z(0) = c_0. \tag{3.3.3}$$

We refer the reader to [35, 53] for the details.

Very often, we would also like to consider sequences like $\{\lambda_k - z\}$. We make the following definition.

Definition 3.3.8. The *z-shifted sequence* of L is the sequence $L_z = \{\lambda_k - z\}$. The *z-shifted zeta function* is then the Hurwitz-type zeta function

$$Z(s, -z) = \sum_{k \in K} (\lambda_k - z)^{-s}.$$

The following theorem shows that $Z(s, -z)$ satisfies the conditions of Definition 3.3.2 whenever $Z(s)$ does.

Theorem 3.3.9. [53, Theorem 1] *Let $L = \{\lambda_k\}$ be a zeta-regularizable sequence, such that $\{\arg \lambda_k\}$ is bounded, and $L_z = \{\lambda_k - z\}$ be the associated z-shifted sequence. If, for $|\lambda_k|$ large, $[\arg(\lambda_k - z) - \arg \lambda_k] \to 0$, then $\{\lambda_k - z\}$ is also a zeta-regularizable sequence.*

Proof. Let us fix z and expand $(\lambda_k - z)^{-s}$ as a Taylor series:

$$
\begin{aligned}
(\lambda_k - z)^{-s} = \lambda_k^{-s}\left(1 - \frac{z}{\lambda_k}\right)^{-s} &= \lambda_k^{-s}\sum_{n=0}^{\infty}\frac{(-s)(-s-1)\cdots(-s-n+1)}{n!}\left(-\frac{z}{\lambda_k}\right)^n \\
&= \lambda_k^{-s}\sum_{n=0}^{\infty}\frac{(-1)^{2n}(s+n-1)\cdots(s+1)s}{n!}\left(\frac{z}{\lambda_k}\right)^n \\
&= \sum_{n=0}^{\infty}\binom{s+n-1}{n}\lambda_k^{-(s+n)}z^n.
\end{aligned}
$$

Let $\sigma = \inf\left\{s \in \mathbb{R} \mid \sum_{k=0}^{\infty} |\,\lambda_k\,|^{-s} < \infty\right\}$, $\beta = \max(0,\sigma)$ and $h = [\beta]$ be the greatest integer such that $h \leq \beta$. We let

$$
f_k(s) = (\lambda_k - z)^{-s} - \sum_{n=0}^{h}\binom{s+n-1}{n}\lambda_k^{-(s+n)}z^n. \tag{3.3.4}
$$

Using the estimate $\left|\binom{s+n+h}{n+h+1}\right| \leq (|\,s\,|+1)^{h+1}\binom{|\,s\,|+n-1}{n}$, we see that

$$
\begin{aligned}
|\,f_k(s)\,| &= \left|\sum_{n=h+1}^{\infty}\binom{s+n-1}{n}\lambda_k^{-(s+n)}z^n\right| \\
&\leq \sum_{n=0}^{\infty}(|\,s\,|+1)^{h+1}\binom{|\,s\,|+n-1}{n}\left|\lambda_k^{-(s+n+h+1)}z^{n+h+1}\right| \\
&= (|\,s\,|+1)^{h+1}\sum_{n=0}^{\infty}\binom{|\,s\,|+n-1}{n}\left|\lambda_k^{-(s+h+1)}\right||\,z\,|^{h+1}\left|\frac{z}{\lambda_k}\right|^n \\
&= (|\,s\,|+1)^{h+1}\left|\lambda_k^{-(s+h+1)}\right||\,z\,|^{h+1}\sum_{n=0}^{\infty}\binom{|\,s\,|+n-1}{n}\left|\frac{z}{\lambda_k}\right|^n \\
&= (|\,s\,|+1)^{h+1}\left|\lambda_k^{-(s+h+1)}\right||\,z\,|^{h+1}\left(1-\left|\frac{z}{\lambda_k}\right|\right)^{-|\,s\,|}.
\end{aligned}
$$

For $s \in K$ in a compact set $K \subset \mathbb{C}$, we have

$$
\left|\,\lambda_k^{-s}\,\right| = |\,\lambda_k\,|^{-\operatorname{Re}(s)}\,e^{\arg\lambda_k\cdot\operatorname{Im}(s)}.
$$

As $\{\arg\lambda_k\}$ is bounded, we find that

$$
\left|\,\lambda_k^{-s}\,\right| \leq C_1\,|\,\lambda_k\,|^{-\operatorname{Re}(s)},
$$

so that, for $|\,\lambda_k\,|$ sufficiently large, we have the estimate

$$
|\,f_k(s)\,| \leq C_2\,|\,z\,|^{h+1}\,|\,\lambda_k\,|^{-(\operatorname{Re}(s)+h+1)},
$$

which shows that the function

$$F(s) = \sum_{k\in K} f_k(s) = \sum_{k\in K}\left\{(\lambda_k - z)^{-s} - \sum_{n=0}^{h}\binom{s+n-1}{n}\lambda_k^{-(s+n)}z^n\right\} \tag{3.3.5}$$

is absolutely and uniformly convergent to an analytic function of s on compact subsets of the half plane $\mathrm{Re}(s) > \beta-(h+1)$. Note that, since $-2 < \beta-(h+1) \le -1$, it follows that $\mathrm{Re}(s) > -1$, so that the half plane of convergence of $F(s)$ contains the origin. Since the function

$$\begin{aligned} F(s) &= \sum_{k\in K}(\lambda_k - z)^{-s} - \sum_{n=0}^{h}\binom{s+n-1}{n}\left[\sum_{k\in K}\lambda_k^{-(s+n)}\right]z^n \\ &= \sum_{k\in K}(\lambda_k - z)^{-s} - \sum_{n=0}^{h}\binom{s+n-1}{n} Z(s+n)z^n \end{aligned} \tag{3.3.6}$$

gives a meromorphic continuation of $Z(s,-z) = \sum_{k\in K}(\lambda_k - z)^{-s}$, this completes the proof of the theorem. □

Remark 3.3.10. We can find an asymptotic expansion for the zeta function associated to the z-shifted sequence L_z, which will be useful in helping us to compute the examples. Consider the Laplace–Mellin transform $\mathcal{LM}\theta_L(s,z)$ of the theta function of L. Then we find that (see [35, Corollary 1.7])

$$Z(s,z) = \frac{1}{\Gamma(s)}\mathcal{LM}\theta_L(s,z).$$

With the assumptions on $\theta_L(t)$ made in Remark 3.3.7, we can now apply Corollary 2.2.7 to see that as $z\to\infty$,

$$Z(s,z) \sim \sum_{n=0}^{\infty} c_{k_n}\frac{\Gamma(s+k_n)}{\Gamma(s)} z^{-(s+k_n)} \tag{3.3.7}$$

for $|\arg z| < \pi/2$.

3.4 The structure of zeta-regularized products

In this section, we state and prove a theorem that describes the structure of the zeta-regularized product associated to a shifted sequence $L_z = \{\lambda_k - z\}$ in terms of the product associated to $L = \{\lambda_k\}$. We follow the approach of Quine et al. [53], but make use of some notation introduced by Jorgenson and Lang [35].

We begin by recalling that in the previous section, we constructed the function

$$F(s) = \sum_{k\in K}\left\{(\lambda_k - z)^{-s} - \sum_{n=0}^{h}\binom{s+n-1}{n}\lambda_k^{-(s+n)}z^n\right\}$$

It turns out that $F(s)$ is closely related to the Weierstrass canonical product

$$W_L(z) = \prod_{k\in K}\left\{\left(1-\frac{z}{\lambda_k}\right)\exp\left[\sum_{n=1}^{h}\frac{1}{n}\left(\frac{z}{\lambda_k}\right)^n\right]\right\} \tag{3.4.1}$$

associated to the sequence L. In fact, we have

Lemma 3.4.1.

$$W_L(z) = \exp(-F'(0)). \tag{3.4.2}$$

Proof. Differentiating the expression (3.3.5) with respect to s gives

$$\begin{aligned} F'(s) = \sum_{k\in K}\Bigg\{&-\log(\lambda_k - z)(\lambda_k - z)^{-s} + (\log\lambda_k)\lambda_k^{-s}\sum_{n=0}^{h}\binom{s+n-1}{n}\left(\frac{z}{\lambda_k}\right)^n \\ &- \lambda_k^{-s}\sum_{n=1}^{h}\left[\frac{d}{ds}\binom{s+n-1}{n}\right]\left(\frac{z}{\lambda_k}\right)^n\Bigg\} \end{aligned}$$

Recall that for positive integers $n > 0$,

$$\begin{aligned} \binom{s+n-1}{n} &= \frac{1}{n!}s(s+1)\cdots(s+n-1) \\ &= \frac{1}{n!}\left\{s\cdot(n-1)! + s^2\left(\sum_{m=1}^{n-1}\frac{(n-1)!}{m}\right) + O(s^3)\right\}, \end{aligned} \tag{3.4.3}$$

so that $\binom{s+n-1}{n}\big|_{s=0} = 0$ and

$$\frac{d}{ds}\binom{s+n-1}{n} = \frac{1}{n}\left\{1 + 2s\left(\sum_{m=1}^{n-1}\frac{1}{m}\right) + O(s^2)\right\}. \tag{3.4.4}$$

By convention, $\binom{s-1}{0} = 1$, so we have

$$F'(0) = \sum_{k\in K}\left\{-\log(\lambda_k - z) + \log\lambda_k - \sum_{n=1}^{h}\frac{1}{n}\left(\frac{z}{\lambda_k}\right)^n\right\},$$

and we see from (3.4.1) that $W_L(z) = \exp(-F'(0))$, as required. □

Before we state the next theorem, we need some notation for the coefficients of the Laurent series of a function at a point $s = s_0$.

Definition 3.4.2. Let $f(s)$ be a one-parameter function and $g(s, z)$ be a two-parameter function. We write $R_j(s_0)$ (respectively, $R_j(s_0; z)$) for the coefficient of $(s-s_0)^j$ in the Laurent series of $f(s)$ (respectively, $g(s,z)$), so that their Laurent series are given by

$$f(s)=\sum_{j=-\infty}^{\infty} R_j(s_0)(s-s_0)^j, \quad g(s,z)=\sum_{j=-\infty}^{\infty} R_j(s_0;z)(s-s_0)^j.$$

We shall also write

$$R_{j,f}(s_0)=R_j(s_0), \quad R_{j,g}(s_0;z)=R_j(s_0;z)$$

if the functions involved are not clear from the context.

The following theorem shows the relation between the zeta-regularized products of a sequence L and its z-shifted sequence L_z.

Theorem 3.4.3. [53, Theorem 2] *With notation as in Theorem 3.3.9, suppose that $F(s)$ has at most simple poles at integer points. Then*

$$\prod_{k\in K}{}_z(\lambda_k-z)=\left[\prod_{k\in K}{}_z\lambda_k\right]\left[e^{-Q_L(z)}W_L(z)\right],$$

where $Q_L(z)$ is the expression

$$Q_L(z)=\sum_{n=1}^{h} R_{0,Z}(n)\frac{z^n}{n}+\sum_{n=1}^{h} R_{-1,Z}(n)\left(\sum_{m=1}^{n-1}\frac{1}{m}\right)\frac{z^n}{n},$$

and $Z(s)$ is the zeta function of $L=\{\lambda_k\}$.

Proof. Differentiating the expression (3.3.6) at $s=0$ gives

$$F'(0)=Z'(0,-z)-Z'(0)-\sum_{n=1}^{h} z^n\left[\frac{d}{ds}\binom{s+n-1}{n}Z(s+n)\right]\Bigg|_{s=0}.$$

We calculate the derivative in the above expression by making use of the fact that $Z(s)$ has poles of at most order 1, which we assumed in defining the zeta-regularizability of a sequence. For $s\to 0$, then, we can write

$$Z(s+n)=\frac{R_{-1,Z}(n)}{s}+R_{0,Z}(n)+O(s).$$

Hence, for $n\neq 1$, we have that

$$\binom{s+n-1}{n}Z(s+n)=\frac{R_{-1,Z}(n)}{n}\left(1+s\sum_{m=1}^{n-1}\frac{1}{m}\right)+s\frac{R_{0,Z}(n)}{n}+O(s^2),$$

so that

$$\frac{d}{ds}\binom{s+n-1}{n} Z(s+n)\bigg|_{s=0} = \frac{1}{n}\left[R_{-1,Z}(n)\left(\sum_{m=1}^{n-1}\frac{1}{m}\right) + R_{0,Z}(n)\right].$$

When $n = 1$, the derivative is just

$$\frac{d}{ds}\binom{s+n-1}{n} Z(s+1) = R_{0,Z}(1).$$

Therefore, we have

$$-Z'(0,-z) = -Z'(0) - \left[\sum_{n=1}^{h}\frac{R_{0,Z}(n)}{n}z^n + \sum_{n=2}^{h}\frac{R_{-1,Z}(n)}{n}\left(\sum_{m=1}^{n-1}\frac{1}{m}\right)z^n\right] - F'(0).$$

Exponentiating both sides completes the proof of the theorem. □

3.5 Examples

We calculate some examples of zeta-regularized products, taken from [53], to illustrate the theory and provide results to be used in the sequel.

Example 3.5.1. Our first example is

$$\prod_{n=1}^{\infty}{}_z (nu+z) = \left(\frac{2\pi}{u}\right)^{1/2}\frac{1}{\Gamma(1+\frac{z}{u})}. \tag{3.5.1}$$

Proof. By Theorem 3.4.3, we see that

$$\prod_{n=1}^{\infty}{}_z (nu+z) = \left[\prod_{n=1}^{\infty}{}_z\, nu\right] e^{R_{0,Z}(1)z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{nu}\right)\exp\left(-\frac{z}{nu}\right)\right].$$

Here, $Z(s)$ is the zeta function

$$Z(s) = \sum_{n=1}^{\infty}(nu)^{-s} = u^{-s}\sum_{n=1}^{\infty}n^{-s} = u^{-s}\zeta(s)$$

associated to the sequence $\{nu\}_{n=1}^{\infty}$ and $\zeta(s)$ is the Riemann zeta function.

To calculate this, note that

$$\prod_{n=1}^{\infty}{}_z\, nu = u^{\zeta(0)}\prod_{n=1}^{\infty}{}_z\, n = \left(\frac{2\pi}{u}\right)^{1/2},$$

where we recall that $\zeta(0) = -1/2$ and that as $\zeta'(0) = -\log\sqrt{2\pi}$,

$$\prod_{n=1}^{\infty}{}_z\, n = \sqrt{2\pi}.$$

Next, recall that the Riemann zeta function has a pole at $s = 1$ and an expansion as $s \to 1$ given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1). \tag{3.5.2}$$

Note that $R_{0,\zeta}(1) = \gamma$, so that we have

$$R_{0,Z}(1) = \frac{\gamma}{u},$$

which completes the proof of (3.5.1). $\square$

The next series of examples are products taken over a pair of indices which range over the natural numbers.

Example 3.5.2. Let $\tau \in \mathbb{C}$ such that $\tau \notin (-\infty, 0]$ and $|\arg \tau| < \pi$. Then

$$\prod_{m,n=1}^{\infty}{}_{z} (m + n\tau) = (2\pi)^{-\frac{1}{4}} \tau^{\frac{1}{4}+\frac{1}{12}\left(\tau+\frac{1}{\tau}\right)} e^{P_1(\tau)} \prod_{n=1}^{\infty} \left[\Gamma(1+n\tau)^{-1} e^{P_2(n\tau)}\right], \tag{3.5.3}$$

where

$$\begin{aligned} P_1(\tau) &= \left(\zeta'(-1) - \frac{1}{12}\right)\tau - \frac{\gamma}{12\tau} \\ P_2(n\tau) &= \left(n\tau + \frac{1}{2}\right)\log n\tau + n\tau - \frac{1}{2}\log 2\pi + \frac{1}{12n\tau} \end{aligned} \tag{3.5.4}$$

Proof. Consider the zeta functions

$$Z(s) = \sum_{m,n=1}^{\infty} (m + n\tau)^{-s}, \qquad Z_1(s,z) = \sum_{m=1}^{\infty} (m+z)^{-s}.$$

Recall from (3.3.7) that $Z_1(s,z)$ has the asymptotic expansion

$$Z_1(s,z) \sim \sum_{n=0}^{\infty} \frac{\Gamma(j_n + s)}{\Gamma(s)} c_{j_n} z^{-(j_n+s)}$$

where c_{j_n} is the coefficient of t^{j_n} in the expansion of

$$\sum_{m=1}^{\infty} e^{-mt} = t^{-1} - \frac{1}{2} + \frac{1}{12}t + O(t^3).$$

We now define the function

$$B(s,z) = Z_1(s,z) - \frac{1}{s-1} z^{1-s} + \frac{1}{2} z^{-s} - \frac{s}{12} z^{-(1+s)} = O(|\,z\,|^{-3-s}),$$

by subtracting the terms corresponding to $j_n = -1, 0, 1$ from $Z_1(s, z)$. Similarly, define the function

$$A(s) = Z(s) - \frac{\tau^{1-s}}{s-1}\zeta(s-1) + \frac{1}{2}\tau^{-s}\zeta(s) - \frac{s}{12}\tau^{-(1+s)}\zeta(s+1). \tag{3.5.5}$$

Observe that, for $\mathrm{Re}(s) > -2$,

$$\begin{aligned} A(s) &= \sum_{n=1}^{\infty} B(s, n\tau) \\ &= \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} (m + n\tau)^{-s} \right) - \frac{\tau^{1-s}}{s-1} n^{1-s} + \frac{1}{2}\tau^{-s} n^{-s} \right. \\ &\qquad \left. - \frac{s}{12}\tau^{-(1+s)} n^{-(1+s)} \right], \end{aligned} \tag{3.5.6}$$

where the function $\zeta(s)$ is the Riemann zeta function.

Now, we evaluate $A'(0)$ in two different ways. We can derive it, firstly, from (3.5.5), but we have to make use of (3.5.2), the expansion of the Riemann zeta function as $s \to 1$. Thus, we have to rewrite (3.5.5) as

$$\begin{aligned} A(s) &= Z(s) - \frac{\tau^{1-s}}{s-1}\zeta(s-1) + \frac{1}{2}\tau^{-s}\zeta(s) - \frac{s}{12}\tau^{-(1+s)} \left(\frac{1}{s} + \gamma + O(s) \right) \\ &= Z(s) - \frac{\tau^{1-s}}{s-1}\zeta(s-1) + \frac{1}{2}\tau^{-s}\zeta(s) - \frac{1}{12}\tau^{-(1+s)} - \frac{\gamma s}{12}\tau^{-(1+s)} \\ &\quad + O(s^2\tau^{-(1+s)}). \end{aligned}$$

Differentiating this expression gives

$$\begin{aligned} A'(s) = Z'(s) &- \frac{\tau^{1-s}}{s-1}\zeta'(s-1) + \zeta(s-1)\tau^{1-s} \left[\frac{\log\tau}{s-1} + \frac{1}{(s-1)^2} \right] \\ &+ \frac{1}{2}\left[\tau^{-s}\zeta'(s) - \zeta(s)(\log\tau)\tau^{-s} \right] + \frac{1}{12\tau^{1+s}}(\log\tau) \\ &- \frac{\gamma}{12\tau^{1+s}} + \frac{\gamma s}{12}(\log\tau)\tau^{-(1+s)} + O(s^2(\log\tau)\tau^{-(1+s)}). \end{aligned}$$

Setting $s = 0$, and using the identities

$$\zeta(-1) = -\frac{1}{12}, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log 2\pi,$$

we find that

$$\begin{aligned} A'(0) &= Z'(0) + \tau\zeta'(-1) + \zeta(-1)\tau\left(-\log\tau + 1\right) + \frac{1}{2}\zeta'(0) - \frac{1}{2}\zeta(0)\log\tau \\ &\quad + \frac{1}{12\tau}(\log\tau) - \frac{\gamma}{12\tau} \\ &= Z'(0) + \tau\left(\zeta'(-1) - \frac{1}{12} \right) + \log\tau \left[\frac{1}{4} + \frac{1}{12}\left(\tau + \frac{1}{\tau} \right) \right] \\ &\quad - \frac{1}{4}\log 2\pi - \frac{\gamma}{12\tau}. \end{aligned} \tag{3.5.7}$$

On the other hand, from (3.5.6), we have

$$
\begin{aligned}
A'(s) = \sum_{n=1}^{\infty} \Bigg\{ & Z_1'(s, n\tau) + \frac{1}{s-1}(\log n\tau)(n\tau)^{1-s} - \frac{(n\tau)^{1-s}}{(s-1)^2} \\
& - \frac{1}{2}(\tau^{-s}(\log n)n^{-s} + n^{-s}(\log \tau)\tau^{-s}) - \frac{1}{12}\tau^{-(1+s)}n^{-(1+s)} \\
& + \frac{s}{12}\left[\tau^{-(1+s)}(\log n)n^{-(1+s)} + n^{-(1+s)}(\log \tau)\tau^{-(1+s)}\right] \Bigg\}.
\end{aligned}
$$

Recalling that $-Z_1'(0,z) = \log(\sqrt{2\pi}/\Gamma(1+z))$, we find that

$$
\begin{aligned}
A'(0) &= \sum_{n=1}^{\infty} \left[Z_1'(0, n\tau) - n\tau \log(n\tau) - n\tau - \frac{1}{2}(\log n\tau) - \frac{1}{12n\tau} \right] \\
&= \sum_{n=1}^{\infty} \Bigg[-\frac{1}{2}\log 2\pi + \log \Gamma(1+n\tau) - \left(n\tau + \frac{1}{2}\right)\log n\tau \\
&\qquad -n\tau - \frac{1}{12n\tau} \Bigg].
\end{aligned}
\tag{3.5.8}
$$

Equating the two formulas (3.5.7) and (3.5.8) for $A'(0)$ gives us the equation

$$
\begin{aligned}
& Z'(0) + \tau\left(\zeta'(-1) - \frac{1}{12}\right) + \log \tau \left[\frac{1}{4} + \frac{1}{12}\left(\tau + \frac{1}{\tau}\right)\right] - \frac{1}{4}\log 2\pi - \frac{\gamma}{12\tau} \\
&\quad = \sum_{n=1}^{\infty} \left[\log \Gamma(1+n\tau) - \frac{1}{2}\log 2\pi - \left(n\tau + \frac{1}{2}\right)\log n\tau - n\tau - \frac{1}{12n\tau} \right].
\end{aligned}
$$

Exponentiating both sides gives us equation (3.5.3). $\square$

Example 3.5.3. Let $u, v \in \mathbb{C}$ with $|\arg v - \arg u| < \pi$ and $v/u = \tau \notin (-\infty, 0]$. Then

$$
\begin{aligned}
& \prod_{\substack{m,n=0 \\ (m,n)\neq(0,0)}}^{\infty}{}_{z} (mu + nv) \\
&= 2\pi u^{-\frac{3}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})}\tau^{-\frac{1}{2}} \prod_{m,n=1}^{\infty}{}_{z}(m + n\tau) \\
&= (2\pi)^{\frac{3}{4}} u^{-\frac{3}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})}\tau^{-\frac{1}{4}+\frac{1}{12}\left(\tau+\frac{1}{\tau}\right)} e^{P_1(\tau)} \prod_{n=1}^{\infty}\left[\Gamma(1+n\tau)^{-1}e^{P_2(n\tau)}\right],
\end{aligned}
\tag{3.5.9}
$$

where P_1 and P_2 are the polynomials given in (3.5.4). We note that this product is $\rho_2(u, v)$, the constant term in the asymptotic expansion of Barnes' double gamma function.

Proof. By (3.3.2), we can split the product into

$$\begin{aligned}\prod_{\substack{m,n=0\\(m,n)\neq(0,0)}}^{\infty}{}_{z}\,(mu+nv) &= \prod_{m=1}^{\infty}{}_{z}\, mu \prod_{n=1}^{\infty}{}_{z}\, nv \prod_{m,n=1}^{\infty}{}_{z}(mu+nv)\\ &= 2\pi u^{-\frac{1}{2}} v^{-\frac{1}{2}} \prod_{m,n=1}^{\infty}{}_{z}(mu+nv).\end{aligned}$$

We claim that

$$\prod_{m,n=1}^{\infty}{}_{z}(mu+nv) = u^{\frac{1}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})} \prod_{m,n=1}^{\infty}{}_{z}(m+n\tau).$$

To see this, consider the zeta function associated to this zeta-regularized product

$$Z(s) = \sum_{m,n=1}^{\infty} (mu+nv)^{-s}.$$

We may assume that $\mathrm{Re}(u), \mathrm{Re}(v) > 0$ when computing $Z(0)$, since the latter does not change when $u \mapsto ku, v \mapsto kv$ for some $k \neq 0$. As $t \to 0$, the associated theta function has the expansion

$$\begin{aligned}\sum_{m,n=1}^{\infty} e^{-(mu+nv)t} &= \left(\sum_{m=1}^{\infty} e^{-mut}\right)\left(\sum_{n=1}^{\infty} e^{-nvt}\right)\\ &= \frac{1}{e^{ut}-1}\frac{1}{e^{vt}-1}\\ &= \frac{1}{uv}\left(\frac{1}{t^2}\right) - \left(\frac{1}{u}+\frac{1}{v}\right)\frac{1}{2t} + \frac{1}{4} + \frac{1}{12}\left(\tau+\frac{1}{\tau}\right) + O(t).\end{aligned}$$

From formula (3.3.3), we see that $Z(0) = \frac{1}{4} + \frac{1}{12}\left(\tau + \frac{1}{\tau}\right)$, thus proving our claim. It follows that

$$\begin{aligned}\prod_{\substack{m,n=0\\(m,n)\neq(0,0)}}^{\infty}{}_{z}\,(mu+nv) &= 2\pi u^{-\frac{1}{2}} v^{-\frac{1}{2}} \left[u^{\frac{1}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})} \prod_{m,n=1}^{\infty}{}_{z}(m+n\tau)\right]\\ &= 2\pi u^{-\frac{3}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})} \tau^{-\frac{1}{2}} \prod_{m,n=1}^{\infty}{}_{z}(m+n\tau),\end{aligned}$$

which gives us the first equality in (3.5.9). The second equality then follows by using (3.5.3), thus completing the proof. □

Example 3.5.4. Barnes [5, 6] has defined a generalization of the gamma function. The definition of *Barnes' double gamma function* $\Gamma_2(z;u,v)$ is given by the formula

$$\begin{aligned}\frac{1}{\Gamma_2(z;u,v)} = ze^{\gamma_{22}z+\gamma_{21}\frac{z^2}{2}} \prod_{m,n=0}^{\infty}{}' \Bigg\{ \left(1+\frac{z}{mu+nv}\right) \\ \exp\left[-\frac{z}{mu+nv}+\frac{1}{2}\left(\frac{z}{mu+nv}\right)^2\right]\Bigg\}\end{aligned} \tag{3.5.10}$$

Here, γ_{21} and γ_{22} are the double modular constants introduced by Barnes [5, 6]. From Theorem 3.4.3, we see that

$$\frac{1}{\Gamma_2(z;u,v)} = \frac{\prod_{m,n=0}^{\infty}{}_z\,(z+mu+nv)}{\prod_{m,n=0}^{\infty}{}_z\,(mu+nv)}.$$

The next example is a product taken over a pair of indices ranging over the integers.

Example 3.5.5. Let $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ and $-\pi \leq \arg(m+n\tau) < \pi$. Then

$$\prod_{m,n\in\mathbb{Z}}{}'_z(m+n\tau) = 2\pi i \prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})^2 = 2\pi i\eta(\tau)^2\exp\left(-\frac{\pi i\tau}{6}\right). \tag{3.5.11}$$

where $\eta(\tau)$ is the Dedekind eta function as defined in (3.2.1).

Proof. Let $Z(s,n\tau)$ denote the analytic continuation of the zeta function

$$Z(s,n\tau) = \sum_{m=-\infty}^{\infty}(m+n\tau)^{-s}.$$

We can split this up into

$$Z(s,n\tau) = Z_1(s,n\tau) + (n\tau)^{-s} + Z_2(s,n\tau),$$

where

$$Z_1(s,n\tau) = \sum_{m=1}^{\infty}(m+n\tau)^{-s}, \quad Z_2(s,n\tau) = \sum_{m=1}^{\infty}(-m+n\tau)^{-s}.$$

Note that for $\mathrm{Re}(s) > -3$ and $n\tau \to \infty$,

$$Z_1(s,n\tau) = \frac{(n\tau)^{1-s}}{s-1} - \frac{1}{2}(n\tau)^{-s} + \frac{s}{12}(n\tau)^{-1-s} + O(|\,n\tau\,|^{-3-s}),$$

while

$$Z_2(s, n\tau) = -\frac{(n\tau)^{1-s}}{s-1} - \frac{1}{2}(n\tau)^{-s} - \frac{s}{12}(n\tau)^{-1-s} + O(|\,n\tau\,|^{-3-s}),$$

so that

$$Z(s, n\tau) = O(|\,n\tau\,|^{-3-s})$$

is an entire function of s and the sum

$$\sum_{n=-\infty}^{\infty}{}' \sum_{m=-\infty}^{\infty} (m + n\tau)^{-s} = \sum_{n=-\infty}^{\infty}{}' Z(s, n\tau)$$

is absolutely convergent when $\mathrm{Re}(s) > -2$. We can therefore take the iterated product

$$\prod_{n=-\infty}^{\infty}{}' \prod_{m=-\infty}^{\infty}{}_z (m + n\tau) = \left[\prod_{n=1}^{\infty} \prod_{m=-\infty}^{\infty}{}_z (m + n\tau)\right] \left[\prod_{n=1}^{\infty} \prod_{m=-\infty}^{\infty}{}_z (m - n\tau)\right].$$

To compute the zeta-regularized product, we need to consider

$$\prod_{m=-\infty}^{\infty}{}_z (m + z),$$

for $\mathrm{Im}(z) > 0$ and $-\pi < \arg(m + z) < \pi$. Note that $(-m + z) = e^{\pi i}(m - z)$, so

$$\begin{aligned}
\prod_{m=-\infty}^{\infty}{}_z (m + z) &= z \prod_{m=1}^{\infty}{}_z (m + z) \prod_{m=1}^{\infty}{}_z e^{\pi i}(m - z) \\
&= \frac{1}{z}\, e^{\pi i(z-\frac{1}{2})} \prod_{m=0}^{\infty}{}_z (m + z) \prod_{m=0}^{\infty}{}_z (m - z) \\
&= e^{\pi i(z-\frac{1}{2})} \frac{2\pi}{\Gamma(z)\Gamma(1-z)} \\
&= e^{\pi i(z-\frac{1}{2})} \frac{(e^{\pi i z} - e^{-\pi i z})}{i} \\
&= 1 - e^{2\pi i z}.
\end{aligned}$$

Here, we have made use of the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and Example 3.5.1. Finally, since

$$\begin{aligned}
\prod_{m=-\infty}^{\infty}{}'_z\, m &= \left[\prod_{m=1}^{\infty}{}_z\, m\right] \left[\prod_{m=1}^{\infty}{}_z\, e^{-\pi i} m\right] \\
&= (e^{-\pi i})^{-1/2} \sqrt{2\pi} \cdot \sqrt{2\pi} \\
&= 2\pi i,
\end{aligned}$$

we see that

$$
\begin{aligned}
\prod_{m,n\in\mathbb{Z}}{}_z^{\prime}(m+n\tau) &= \prod_{m=-\infty}^{\infty}{}_z^{\prime} m \left[\prod_{n=1}^{\infty}\prod_{m=-\infty}^{\infty}{}_z (m+n\tau)\right]\left[\prod_{n=1}^{\infty}\prod_{m=-\infty}^{\infty}{}_z (m-n\tau)\right] \\
&= 2\pi i\left[\prod_{n=1}^{\infty} 1-e^{2\pi i n\tau}\right]\left[\prod_{n=1}^{\infty} 1-e^{2\pi i n\tau}\right] \\
&= 2\pi i\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})^2 \\
&= 2\pi i\eta(\tau)^2\exp\left(-\frac{\pi i\tau}{6}\right)
\end{aligned}
$$

This completes the proof of (3.5.11). $\square$

Chapter 4

Multiplicative Sequences and Characteristic Classes

In this chapter, we review Hirzebruch's theory of multiplicative sequences and multiplicative genera. We also compute several examples that we shall subsequently need.

We begin with a review of Hirzebruch's theory on multiplicative sequences and the associated genera [26] in §4.1. The key point is that, to every power series with constant term equal to unity, we can associate a multiplicative sequence of polynomials. Thus, we recall that the function

$$\hat{A}(z) := \frac{z/2}{\sinh(z/2)}$$

has an associated multiplicative sequence, and show that a multiplicative sequence can also be associated to

$$\frac{1}{\hat{\Gamma}(z)} = \frac{1}{e^{\gamma z}\Gamma(1+z)}.$$

In §4.2, we review some properties of a multiplicative genus. The crucial observation in Hirzebruch's theory is that every multiplicative sequence gives rise to a multiplicative genus, which is the evaluation of a polynomial of characteristic classes against the fundamental class of a manifold. In particular, it follows that $\hat{A}(z)$ and $\hat{\Gamma}(z)$ both define multiplicative genera. This also justifies the traditional abuse of notation noted in Remark 4.1.7.

In §4.3, we give a recursive formula, due to Hirzebruch and Libgober–Wood [43], that allows us to calculate the polynomials in a multiplicative sequence. We compute the multiplicative sequence corresponding to $1/\Gamma(1+z)$ and $1/\hat{\Gamma}(z)$ as examples.

4.1 Multiplicative sequences

In this section, we shall review Hirzebruch's theory of multiplicative sequences of polynomials, following [26, Section 1]. This relates formal power series with multiplicative sequences, which can be used to generate multiplicative genera. Next, we review Hirzebruch's idea of a multiplicative genus, defined using a multiplicative sequence of polynomials, and indicate why the genus so defined is multiplicative.

Following Hirzebruch [26, Section 1], we begin by considering a commutative ring R with identity. We set $c_0 = 1$ to be the identity and adjoin indeterminates $c_1, c_2, \ldots$ to R to form $\mathcal{R} := R[c_1, c_2, \ldots]$, the polynomial ring generated by R and the c_i's. We observe that we can grade $\mathcal{R}$ by the weight of a polynomial, defined in the following way.

Definition 4.1.1. The *weight* of a product of indeterminates $c_{j_1} c_{j_2} \cdots c_{j_k}$ is given by

$$\operatorname{wt}(c_{j_1} c_{j_2} \cdots c_{j_k}) := \sum_{i=1}^{k} j_i.$$

If we now write $\mathcal{R}_j$ for the additive group of homogeneous polynomials with weight j, then

$$\mathcal{R} = \bigoplus_{j=0}^{\infty} \mathcal{R}_j,$$

where we set $\mathcal{R}_0 := R$. Note that $\mathcal{R}_i \mathcal{R}_j \subset \mathcal{R}_{i+j}$.

Definition 4.1.2. For $j = 0, 1, 2, \ldots$, a *sequence of polynomials* $\{\Phi_j\}$ in $\mathcal{R}$ is such that

$$\Phi_j = \begin{cases} 1, & j = 0 \\ \Phi_j(c_1, \ldots, c_j) \in \mathcal{R}_j, & j \geq 1 \end{cases}$$

We shall write $\Phi_j(c_1, \ldots, c_j)$ when we would like to emphasize the number of indeterminates that Φ_j takes, and Φ_j otherwise.

We can use a sequence of such polynomials to define a morphism of formal power series.

Definition 4.1.3. For any sequence of polynomials $\{\Phi_j\}$ in $\mathcal{R}$, the Φ-*morphism* is defined to be

$$\Phi\left(\sum_{j=0}^{\infty} c_j z^j\right) := 1 + \sum_{j=1}^{\infty} \Phi_j(c_1, c_2, \ldots, c_j) z^j.$$

We shall now define multiplicative sequences of polynomials in $\mathcal{R}$.

Definition 4.1.4. Let z, a_i, b_j be indeterminates. Suppose we have formal power series

$$A = \sum_{i=0}^{\infty} a_i z^i, \quad B = \sum_{j=0}^{\infty} b_j z^j \quad C = \sum_{k=0}^{\infty} c_k z^k$$

satisfying the identity C = AB, i.e.

$$\sum_{k=0}^{\infty} c_k z^k = \left(\sum_{i=0}^{\infty} a_i z^i\right)\left(\sum_{j=0}^{\infty} b_j z^j\right).$$

Then the sequence of polynomials $\{\Phi_j\}$ in $\mathcal{R}$ is said to be *multiplicative* if the corresponding Φ-morphism is a multiplicative homomorphism of formal power series, i.e.

$$\Phi(C) = \Phi(A)\Phi(B).$$

More explicitly,

$$\sum_{k=0}^{\infty} \Phi_k(c_1, c_2, \ldots, c_k)z^k = \left[\sum_{i=0}^{\infty} \Phi_i(a_1, a_2, \ldots, a_i)z^i\right]\left[\sum_{j=0}^{\infty} \Phi_j(b_1, b_2, \ldots, b_k)z^j\right].$$

We shall refer to these sequences simply as *multiplicative sequences* in the sequel.

We shall show that there is a close relation between formal power series and multiplicative sequences. First, we make the following definition.

Definition 4.1.5. The *generating function* of the multiplicative sequence $\{\Phi_j\}$ is defined to be

$$Q(z) := \Phi(1+z) = 1 + \sum_{j=1}^{\infty} \Phi_j(1, 0, \ldots, 0)z^j.$$

The following proposition summarizes the relation between formal power series beginning with 1 and multiplicative sequences. This also justifies the above definition, as $Q(z)$ is indeed the unique function generating the multiplicative sequence $\{\Phi_j\}$.

Proposition 4.1.6. *[26, Lemmata 1.2.1 and 1.2.2] Let $Q(z)$ be a formal power series of the form*

$$Q(z) = 1 + \sum_{j=1}^{\infty} a_j z^j,$$

Then, for every multiplicative sequence $\{\Phi_j\}$, there exists a formal power series $Q(z)$ of this form, which completely determines $\{\Phi_j\}$. Conversely, for every such formal power series $Q(z)$, there is a multiplicative sequence $\{\Phi_j\}$ such that $\Phi(1+z) = Q(z)$.

Proof. [26, p. 10] Let $\{\Phi_j\}$ be a multiplicative sequence and consider, for any integer $n \geq 1$, the formal factorization of the polynomial

$$1 + c_1 z + \ldots + c_n z^n = \prod_{j=1}^{n} (1 + \alpha_j z), \tag{4.1.1}$$

where we now view each c_i as an elementary symmetric polynomial in the α_j's. Since $\{\Phi_j\}$ is multiplicative, it follows from Definition 4.1.4 that

$$\begin{aligned} \sum_{j=0}^{n} \Phi_j(c_1, \ldots, c_i)z^j + \sum_{j=n+1}^{\infty} \Phi_j(c_1, \ldots, c_n, 0, \ldots, 0)z^j &= \prod_{j=1}^{n} \Phi(1 + \alpha_j z) \\ &= \prod_{j=1}^{n} Q(\alpha_j z). \end{aligned} \tag{4.1.2}$$

Hence, for $j \leq n$, the polynomial Φ_j is completely determined as a symmetric polynomial in the α_j's, and therefore as a polynomial in the c_i's. However, since we did not require n to be fixed, this shows that the multiplicative sequence is completely determined by its characteristic power series.

Conversely, consider a power series beginning with constant term 1,

$$Q(z) = 1 + \sum_{j=1}^{\infty} a_j z^j.$$

Note that the coefficient of z^j in the product

$$\prod_{j=1}^{n} Q\left(\alpha_j z\right)$$

is a homogeneous symmetric polynomial in the α_j's. The above product can be formally factorized as in (4.1.1), so that the coefficient of z^j can be expressed as a polynomial $\Phi_{j,n}(c_1, \ldots, c_j)$ uniquely. For $n \geq j$, we observe that $\Phi_{j,n}$ does not depend on n, so we can set $\Phi_j = \Phi_{j,n}$ for $n \geq j$. The multiplicative sequence required is then the sequence $\{\Phi_j\}$. To see this, we note that (4.1.2) is true by construction. Then the conditions of Definition 4.1.4 hold if the a_j's and b_j's there are replaced by zero for large values of j. Thus, $\{\Phi_j\}$ is indeed a multiplicative sequence. □

Remark 4.1.7. Since $\{\Phi_j\}$ and $Q(z)$ are uniquely determined by each other, we shall not hesitate in abusing notation, where no confusion shall arise, by writing $\Phi(z)$ instead of $Q(z)$.

We give some examples of such generating functions.

Example 4.1.8. The $\hat{A}$-function

$$\hat{A}(z) = \frac{z/2}{\sinh(z/2)} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(2n\pi)^2}\right)$$

has a power series expansion and, by considering the infinite product, we observe that $\hat{A}(0) = 1$. Thus $\hat{A}(z)$ defines a multiplicative sequence, usually denoted by $\{\hat{A}_j\}$.

Example 4.1.9. The functions

$$\frac{1}{\Gamma(1+z)} \qquad \text{and} \qquad \frac{1}{\hat{\Gamma}(z)} = \frac{1}{e^{\gamma z}\Gamma(1+z)}$$

are entire functions, and so have a power series expansion. Furthermore, when $z = 0$, each of these functions take the value 1. Hence they both define multiplicative sequences.

4.2 Multiplicative genera

In this section, we consider $\Phi(z)$ as a generating function of polynomials of characteristic classes. Let M be a compact connected oriented manifold of finite even dimension, and $E \to M$ be a complex vector bundle of (complex) rank n over M, i.e. each fiber is isomorphic to $\mathbb{C}^n$.

Remark 4.2.1. We record this remark for subsequent use. Recall that, by the splitting principle, E can be formally decomposed as a direct sum of n line bundles over M

$$E = \bigoplus_{j=1}^{n} L_j$$

for the purposes of calculations of characteristic classes (see, for example, [12, Chapter IV]). In particular, if $c(E)$ is the total Chern class of E and $c(L_j) = c_1(L_j)$ is the first Chern class of L_j, then

$$c(E) = \prod_{j=1}^{n} \left(1 + c_1(L_j)\right).$$

It is then useful to introduce the following notion.

Definition 4.2.2. The Φ-*class* of $E \to M$ is defined to be

$$\Phi(E) = \prod_{j=1}^{n} \Phi\left(1 + c_1(L_j)\right) = \sum_{j=1}^{n} \Phi_j(c_1(E), \dots, c_j(E)).$$

If $E = \eta \otimes \mathbb{C}$ is the complexification of a real vector bundle $\eta \to M$, then we define its Φ-class to be

$$\Phi(E) = \sum_{j=1}^{n} \Phi_j(p_1(E), \dots, p_j(E)).$$

In particular, we shall write

$$\Phi(TM) = \Phi(M).$$

Next, we shall define the notion of a multiplicative genus. Suppose, firstly, that $\Phi(z)$ is an even function.

Definition 4.2.3. Let $\Phi(z)$ be even, and M be a $4n$-dimensional manifold with Pontrjagin classes $p_1(M), p_2(M), \dots, p_n(M)$. Then the Φ-*genus* is defined to be

$$\Phi[M] = \int_M \Phi(M),$$

where the integral denotes the evaluation of the Φ-class of TM against the fundamental class $[M]$ of the manifold M.

Remark 4.2.4. Note that this coincides with the definition given in [26], since only the term $\Phi_n(p_1(M), \ldots, p_n(M))$ is not annihilated under evaluation against $[M]$.

It is a standard result in topology that the value obtained from such an evaluation depends only on $\Phi_n(p_1(M), \ldots, p_n(M))$ when M is connected and oriented (see, e.g. [26, Section 5]).

Example 4.2.5. From the infinite product representation of $\hat{A}(z)$,

$$\hat{A}(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(2n\pi)^2}\right),$$

we see that the $\hat{A}$-function is even, so the $\hat{A}$-genus is defined for Pontrjagin classes.

Now we consider the case when $\Phi(z)$ is not an even function.

Definition 4.2.6. Let $\Phi(z)$ be a function that is not even, and M be an almost complex $2n$-manifold with Chern classes $c_1(M), c_2(M), \ldots, c_n(M)$. Then the Φ-*genus* is defined to be

$$\Phi[M] = \int_M \Phi(M).$$

Example 4.2.7. The functions

$$\frac{1}{\Gamma(1+z)} \quad \text{and} \quad \frac{1}{\hat{\Gamma}(z)}$$

are not even. In particular, the power series representations of these functions do not vanish at odd degrees. Hence, they define the multiplicative Γ-genus and $\hat{\Gamma}$-genus, respectively, which are polynomials of Chern classes evaluated on the fundamental class.

Remark 4.2.8. Hirzebruch showed that every multiplicative sequence $\{\Phi_j\}$ defines a multiplicative Φ-genus in the following sense: if M and N are two almost complex manifolds, and $M \times N$ has the product almost complex structure, then

$$\Phi[M \times N] = \Phi[M]\Phi[N].$$

In particular, all the previous examples of genera are multiplicative. We omit the proof here, referring the reader to Hirzebruch's demonstration of this property [26, Section 5].

4.3 Computing multiplicative sequences

In this section, we present a formula, due to Hirzebruch [26] and Libgober–Wood [43], for generating the polynomials in a multiplicative sequence. We then work out some examples for later use.

We note here that some basic theory of symmetric functions will be used in this section. We refer the reader to Appendix A for the details and the notation that we shall use. For more details, the reader is referred to the book by Macdonald [47].

Remark 4.3.1. Within this section, we shall revert to distinguishing between the formal power series $Q(z)$ and the associated Φ-morphism generating the multiplicative sequence $\{\Phi_j\}$ of polynomials.

We need to introduce some notation. We expand the generating function as

$$Q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$$

and its logarithmic derivative as

$$\frac{d}{dz} \ln Q(z) = \sum_{n=0}^{\infty} \ell_{n+1} z^n.$$

We now observe that

$$\frac{d}{dz} \ln \prod_{j=1}^{m} Q(\alpha_j z) = \sum_{j=1}^{m} \frac{d}{dz} \ln Q(\alpha_j z) = \sum_{n=0}^{\infty} \ell_{n+1} s_{n+1} z^n, \tag{4.3.1}$$

where

$$s_n = \sum_{j=1}^{m} \alpha_j^n$$

is the nth power sum symmetric polynomial of the α_j's.

Remark 4.3.2. We note here that the usual notation in combinatorics for the nth power sum symmetric polynomial is p_n (see, for example, Macdonald [47]). However, this conflicts with the usual convention in topology for the nth Pontrjagin class, so we shall follow the notation used by Libgober–Wood [43] (see also Appendix A).

We now observe that the identity

$$\prod_{j=1}^{m} (1 + \alpha_j z) = 1 + \sum_{k=1}^{m} c_k z^k$$

holds whenever we are given such a product. In fact, the c_k's are just the elementary symmetric polynomials in the α_j's. Applying the Φ-morphism associated to $Q(z)$ gives

$$\begin{aligned}\prod_{j=1}^{m} \Phi(1 + \alpha_j z) &= \prod_{j=1}^{m} Q(\alpha_j z) \\ &= \sum_{k=0}^{m} \Phi_k(c_1, \ldots, c_k) z^k \\ &\quad + \sum_{k=m+1}^{\infty} \Phi_k(c_1, \ldots, c_k, 0, \ldots, 0) z^k.\end{aligned}$$

We note, following Hirzebruch [26, p. 10], that Φ_k is well-defined for $k \leq m$. In particular, we would like to compute Φ_m, and so we can neglect the terms of degree $\geq m+1$. With this in mind, we observe that the logarithmic derivative in (4.3.1) is the quotient

$$\begin{aligned}\frac{d}{dz}\ln\prod_{j=1}^{m}\Phi(1+\alpha_j z) &= \frac{d}{dz}\ln\prod_{j=1}^{m}Q(\alpha_j z)\\ &= \frac{\sum_{j=1}^{m} j\Phi_j z^{j-1}+\cdots}{\sum_{j=0}^{m}\Phi_j z^j+\cdots}.\end{aligned}$$

Comparing coefficients for z^{m-1} gives the recursive formula

$$m\Phi_m = \sum_{j=1}^{m}\ell_j s_j \Phi_{m-j}. \tag{4.3.2}$$

By using Newton's identities (A.1.2), which express the power sum symmetric polynomials s_i's in terms of the c_i's, we can then give an expression for Φ_m in terms of the c_i's.

Example 4.3.3. We shall compute some multiplicative sequences generated from the gamma function. Consider the generating functions

$$\frac{1}{\Gamma(1+z)} \quad \text{and} \quad \frac{1}{e^{\gamma z}\Gamma(1+z)}.$$

Note that each of these functions take the value 1 at $z=0$, so that these functions are power series that begin with 1, and can therefore be generating functions for multiplicative sequences.

We now consider the logarithmic derivatives of the above functions. First, recall (cf. Erdélyi et al. [21]) that for $|\,z\,| < 1$,

$$\ln\Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta(n)z^n.$$

In fact, we do not need the analytic caveat, since Hirzebruch's theory requires only formal power series. Thus, using the notation in the description of the algorithm, we have

$$\ell_j = \begin{cases}\gamma, & j=1\\ (-1)^{j-1}\zeta(j), & j\geq 2\end{cases}$$

We then apply formula (4.3.2) to get the first few polynomials of the $\tilde{\Gamma}$-sequence, which is Libgober's Γ-sequence [42]:

$$\begin{aligned}
\tilde{\Gamma}_1 &= \gamma s_1, \\
\tilde{\Gamma}_2 &= \frac{1}{2}(-\zeta(2)s_2 + \gamma^2 s_1^2) \\
\tilde{\Gamma}_3 &= \frac{1}{6}(2\zeta(3)s_3 - 3\gamma\zeta(2)s_2 s_1 + \gamma^3 s_1^3) \\
\tilde{\Gamma}_4 &= \frac{1}{24}(-6\zeta(4)s_4 + 8\gamma\zeta(3)s_3 s_1 + 3(\zeta(2))^2 s_2^2 - 6\gamma^2\zeta(2)s_2 s_1^2 + \gamma^4 s_1^4).
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
\frac{d}{dz}\ln\frac{1}{e^{\gamma z}\Gamma(1+z)} &= -\frac{d}{dz}\ln e^{\gamma z} - \frac{d}{dz}\ln\Gamma(1+z) \\
&= -\gamma + \gamma + \sum_{n=2}^{\infty}(-1)^{n-1}\zeta(n)z^{n-1} \\
&= \sum_{n=1}^{\infty}(-1)^n\zeta(n+1)z^n
\end{aligned}$$

Using the notation in the previous section, we have

$$\ell_j = \begin{cases} 0, & j = 1 \\ (-1)^{j-1}\zeta(j), & j \geq 2 \end{cases}$$

where we note that ℓ_j is the coefficient of the term z^{j+1}. Thus, since the constant term vanishes in the logarithmic derivative for $\hat{\Gamma}(z)$, ℓ_1 vanishes also. Applying the formula (4.3.2), we see that the first few polynomials of the $\hat{\Gamma}$-sequence are:

$$\begin{aligned}
\hat{\Gamma}_1 &= 0, \\
\hat{\Gamma}_2 &= -\frac{1}{2}\zeta(2)s_2 \\
\hat{\Gamma}_3 &= \frac{1}{3}\zeta(3)s_3 \\
\hat{\Gamma}_4 &= \frac{1}{8}(-2\zeta(4)s_4 + (\zeta(2))^2 s_2^2) \\
\hat{\Gamma}_5 &= \frac{1}{30}(6\zeta(5)s_5 - 5\zeta(3)\zeta(2)s_3 s_2) \\
\hat{\Gamma}_6 &= \frac{1}{144}(-24\zeta(6)s_6 + 18\zeta(4)\zeta(2)s_4 s_2 + 8(\zeta(3))^2 s_3^2 - 3(\zeta(2))^3 s_2^3).
\end{aligned}$$

Chapter 5

Equivariant de Rham Cohomology

In this chapter, we outline the theory of the ordinary equivariant de Rham cohomology of a G-manifold M. This is to fix notation and state results that will be used later. Readers who would like more details are referred to the book by Guillemin and Sternberg [24], or to the papers [3, 51].

We begin with a review of the topological definition of equivariant cohomology. Next, we consider the ingredients needed in defining a de Rham version of equivariant cohomology, before describing two models for equivariant de Rham cohomology: the Weil model and the Cartan model. The localization theorem and the corresponding localized theory are then reviewed, following which Jones–Petrack's completed equivariant cohomology for the free loop space [34] will be described. We conclude the chapter by relating the localization theorem to the equivariant Euler class.

5.1 The Borel construction

We begin by considering the action of a compact connected group G on a topological space M. We wish to study what information about the G-action can be obtained by studying the cohomology of the manifolds concerned. It turns out that this depends on the type of G-action we are studying.

If the G-action is free, then the quotient M/G is a manifold, and so we may define the G-equivariant cohomology of M to be $H_G(M) = H(M/G)$. However, if the action is not free, the quotient space M/G is usually not as well-behaved topologically. One solution is Borel's construction [11], which defines a G-equivariant cohomology for M by substituting a suitable space for the space M/G.

Borel's construction relies on Milnor's construction (see Husemoller [31, Chapter 4]) of a universal principal G-bundle, which exists for any G. Recall that

Definition 5.1.1. A *principal G-bundle* is a manifold E with a free G-action such that the projection $E \to B$ to the base manifold B (the *orbit space*) is a locally trivial bundle.

A universal principal G-bundle is then a bundle that induces any principal G-bundle.

Definition 5.1.2. A *classifying bundle* for G is a principal G bundle $EG \to BG$ with contractible total space EG, such that for any principal G-bundle $E \to B$, there is a map $f\colon B \to BG$, which is unique up to homotopy, such that E is isomorphic as a vector bundle to the pullback f^*EG.

There is a classical construction for the spaces EG and BG, due to Milnor.

Definition 5.1.3. The Milnor construction for EG is given by

$$EG = \varinjlim EG(n).$$

Here, $EG(n) = G^{n+1} \times \Delta^n / \sim$, where

$$G^{n+1} \times \Delta^n = \left\{ (x_0, t_0; x_1, t_1; \ldots ; x_n, t_n) |\ x_i \in G, t_i \in [0,1], \sum_{i=1}^{n} t_i = 1 \right\}$$

and $(x_0, t_0; x_1, t_1; \ldots ; x_n, t_n) \sim (x'_0, t'_0; x'_1, t'_1; \ldots ; x'_n, t'_n)$ if and only if for all i, $t_i = t'_i$ and $x_i = x'_i$ whenever $t_i = t'_i \neq 0$. We denote an equivalence class in $EG(n)$ by $[x_0, t_0; \ldots ; x_n, t_n]$.

Example 5.1.4. For $G = \mathbb{T}$, $EG(n) \cong S^{2n+1}$ via the map

$$\begin{array}{ccc} EG(n) & \to & S^{2n+1} \\ [x_0, t_0; \ldots ; x_n, t_n] & \mapsto & (\sqrt{t_0}x_0, \ldots, \sqrt{t_n}x_n) \end{array}$$

The inclusion maps that define EG are given by

$$\begin{array}{ccc} EG(n) & \subset & EG(n+1) \\ [x_0, t_0; \ldots ; x_n, t_n] & \mapsto & [x_0, t_0; \ldots ; x_n, t_n; x_{n+1}, 0] \end{array}$$

In this way, $BG(n) \cong \mathbb{C}\mathrm{P}^n$ and $BG \cong \mathbb{C}\mathrm{P}^\infty$, so that a classifying bundle for $G = \mathbb{T}$ is the principal $\mathbb{T}$-bundle $S^\infty \to \mathbb{C}\mathrm{P}^\infty$.

Example 5.1.5. Similarly, for $G = \mathbb{T}^2$, $EG \cong S^\infty \times S^\infty$ and $BG \cong \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$.

The Borel construction uses EG to construct a space which gives a cohomology theory that encodes more information about the G-action.

Definition 5.1.6. The *homotopy quotient* of a manifold M, with respect to the group G, is the space $M_G := (EG \times M)/G$.

Example 5.1.7. For any compact connected group G, the homotopy quotient of a point is $EG/G = BG$. In particular, for $G = \mathbb{T}$, the homotopy quotient of a point is $\mathbb{C}\mathrm{P}^\infty$, and $M_{\mathbb{T}} = (S^\infty \times M)/\mathbb{T}$. For $G = \mathbb{T}^2$, the homotopy quotient of a point is $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$.

Definition 5.1.8. The *G-equivariant cohomology* of M is the cohomology ring of its homotopy quotient

$$H_G(M) := H(M_G).$$

Note 5.1.9. At this point, we can already deduce some of the structure of the G-equivariant cohomology of M. Note that there is a map $M \to \mathrm{pt}$ from M to the point, which induces an algebra homomorphism

$$H_G(\mathrm{pt}) = H(BG) \to H_G(M),$$

making $H_G(M)$ a module over the ring $H(BG)$. In particular, for $G = \mathbb{T}$,

$$H(B\mathbb{T}) = H(\mathbb{C}\mathrm{P}^\infty) = \mathbb{C}[u],$$

where u is an indeterminate of degree 2, so that $H_{\mathbb{T}}(M)$ is a $\mathbb{C}[u]$-module. For $G = \mathbb{T}^2$, we have

$$H(B\mathbb{T}^2) = \mathbb{C}[u, v],$$

where v is another indeterminate of degree 2. Hence, the $\mathbb{T}^2$-equivariant cohomology of M is a $\mathbb{C}[u, v]$-module.

5.2 The de Rham complex

Consider the de Rham complex $(\Omega(M), d)$ of differential forms on a G-manifold M. The exterior derivative

$$d\colon \Omega^k(M) \to \Omega^{k+1}(M)$$

is a graded derivation of degree 1, satisfying

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^a \alpha(d\beta), \quad d^2 = 0$$

for $\alpha \in \Omega^a(M), \beta \in \Omega(M)$. If $\mathfrak{g}$ is the Lie algebra of G, then the action of $\mathfrak{g}$ on M gives two more graded derivations. For $X \in \mathfrak{g}$ and $\omega \in \Omega(M)$, the derivations are:

1. The Lie derivative $L_X\colon \Omega^k(M) \to \Omega^k(M)$ of degree 0, given by

   $$L_X\omega := \frac{d}{dt}(\omega \circ \exp(-tX))\bigg|_{t=0}.$$

   For $\alpha, \beta \in \Omega(M)$, it satisfies the identity

   $$L_X(\alpha\beta) = (L_X\alpha)\beta + \alpha(L_X\beta).$$

2. The contraction $\iota_X\colon \Omega^k(M) \to \Omega^{k-1}(M)$ of degree -1, satisfying, for $\alpha \in \Omega^a(M), \beta \in \Omega(M)$,

   $$\iota_X(\alpha\beta) = (\iota_X\alpha)\beta + (-1)^a\alpha(\iota_X\beta).$$

   Also, $\iota_X^2 = 0$.

Using these derivations, we can define the following sub-complex of $(\Omega(M), d)$.

Definition 5.2.1. The *basic* subcomplex is

$$(\Omega(M))_G = \{\omega \in \Omega(M) |\ L_X\omega = \iota_X\omega = 0 \text{ for all } X \in \mathfrak{g}\}\,.$$

This sub-complex will play a vital role in the definition of equivariant cohomology.

5.3 The Weil and Cartan models

We begin by introducing the Weil model of equivariant cohomology, before specializing to $G = \mathbb{T}$ and then introducing the Cartan model. First, we need to define the Weil algebra.

Definition 5.3.1. The *Weil algebra* of $\mathfrak{g}$ is the tensor product of the exterior algebra of the dual of $\mathfrak{g}$, $\Lambda\mathfrak{g}^*$, with the symmetric algebra of $\mathfrak{g}^*$, $S\mathfrak{g}^*$:

$$W\mathfrak{g} = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*.$$

It is a theorem that the Weil algebra is *acyclic*, i.e. the inclusion $\mathbb{R} \to W\mathfrak{g}$ is a homotopy equivalence. This condition can be viewed as the algebraic equivalence of the contractibility of EG [24, 51]. Thus, we are able to substitute the Weil algebra for EG in the definition of equivariant cohomology.

Definition 5.3.2. Under the *Weil model*, the equivariant cohomology of $(\Omega(M), d)$ is the cohomology defined by

$$H_G(\Omega(M)) := H((W\mathfrak{g} \otimes \Omega(M))_G).$$

The *Cartan model* of equivariant cohomology uses a different interpretation of the basic subcomplex and introduces a new differential. First, we note that for the case of $G = \mathbb{T}$, we can identify the $\mathbb{T}$-invariant subcomplex $(S\mathfrak{g}^* \otimes \Omega(M))^{\mathbb{T}}$ with $\Omega_{\mathbb{T}}(M)[u]$, the polynomial ring over the $\mathbb{T}$-invariant forms $\Omega_{\mathbb{T}}(M)$ that is generated by an indeterminate u of degree 2.

The new differential, called the *equivariant differential*, is then given by the formula $d_{\mathbb{T}} = d + u\iota_X$. Note that it satisfies the identity

$$d_{\mathbb{T}}^2 = 0,$$

since

$$d^2 = 0, \quad \iota_X^2 = 0,$$

and $d\iota_X + \iota_X d = L_X = 0$ on the $\mathbb{T}$-invariant forms $\Omega_{\mathbb{T}}(M)$. Also, $d_{\mathbb{T}}$ is a derivation on $\Omega_{\mathbb{T}}(M)[u]$.

The following theorem of Cartan then gives an isomorphism between the Weil and Cartan model.

Theorem 5.3.3. [24, 51] *The projection*

$$W\mathfrak{g} \otimes \Omega(M) \to S\mathfrak{g}^* \otimes \Omega(M)$$

restricts to an isomorphism

$$(W\mathfrak{g} \otimes \Omega(M))_G \cong (S\mathfrak{g}^* \otimes \Omega(M))^G.$$

In particular, for $G = \mathbb{T}$, this gives an isomorphism on cohomology

$$H^*((W\mathfrak{t} \otimes \Omega(M))_{\mathbb{T}}, d) \cong H^*(\Omega_{\mathbb{T}}(M)[u], d_{\mathbb{T}}).$$

This yields the Cartan model

$$H_{\mathbb{T}}(M) := H(\Omega_{\mathbb{T}}(M)[u], d_{\mathbb{T}})$$

for the $\mathbb{T}$-equivariant cohomology for a manifold M. Putting $G = \mathbb{T}^2$ gives

$$H_{\mathbb{T}^2}(M) := H(\Omega_{\mathbb{T}^2}(M)[u, v], d_{\mathbb{T}^2}),$$

where $d_{\mathbb{T}^2} = d + u\iota_X + v\iota_Y$ is the corresponding $\mathbb{T}^2$-equivariant differential with corresponding fundamental vector fields X and Y generating the $\mathbb{T}^2$-action (cf. also [24, §4.2]).

5.4 Localized and completed cohomology theories

We have seen in the previous section that the $\mathbb{T}$-equivariant cohomology can be described as the cohomology of the complex $(\Omega_{\mathbb{T}}(M)[u], d_{\mathbb{T}})$, where u is an indeterminate of degree 2 and the operator is $d_{\mathbb{T}} = d + u\iota$.

It turns out that, by inverting (or *localizing*) u in the complex, we can create a very useful variant of equivariant cohomology.

Definition 5.4.1. The *localized equivariant cohomology* of M is the cohomology of the complex given by the space of Laurent polynomials in u:

$$u^{-1}H_{\mathbb{T}}(M) := H(\Omega_{\mathbb{T}}(M)[u, u^{-1}], d_{\mathbb{T}}).$$

A standard result in equivariant cohomology is the following theorem.

Theorem 5.4.2. *Let M be a closed oriented compact manifold, and suppose M has an action of the circle $\mathbb{T}$ with fixed point set F. Then the inclusion $i\colon F \hookrightarrow M$ induces an isomorphism on localized equivariant cohomology:*

$$u^{-1}H_{\mathbb{T}}(M) \cong u^{-1}H_{\mathbb{T}}(F) \cong H^*(F)[u, u^{-1}].$$

Remark 5.4.3. A very useful consequence is that the localized equivariant cohomology of M can be computed from the ordinary cohomology of its fixed point set, which is often easier to calculate.

By a result of Goodwillie [23], however, the localized equivariant cohomology of infinite-dimensional manifolds, like the free loop space LM, does not satisfy the localization theorem. In particular, Goodwillie's result asserts that $u^{-1}H_{\mathbb{T}}(LM)$ depends only on $\pi_1(M)$, so this implies that the fixed point theorem is not true for LM.

This motivated Jones and Petrack to construct a completed version of localized equivariant cohomology that satisfies a localization theorem.

Definition 5.4.4. The *completed equivariant cohomology* is defined to be the cohomology of the complex

$$\hat{H}_{\mathbb{T}}(M) := H(\Omega_{\mathbb{T}}(M)((u^{-1})), d_{\mathbb{T}}),$$

where $\Omega_{\mathbb{T}}(M)((u^{-1}))$ is the ring of formal Laurent series in u^{-1} bounded from above with coefficients in $\mathbb{T}$-invariant forms.

Remark 5.4.5. We note that a homogeneous element φ of total degree n in the complex $\Omega_{\mathbb{T}}(M)((u^{-1}))$ is of the form

$$\varphi = \sum_{-\infty<j\leq\frac{n}{2}} b_j u^j, \quad b_j \in \Omega_{\mathbb{T}}^{n-2j}(M).$$

Remark 5.4.6. Note that if M is a finite-dimensional manifold, then $\Omega_{\mathbb{T}}(M)((u^{-1})) = \Omega_{\mathbb{T}}(M)[u, u^{-1}]$, so that $\hat{H}_{\mathbb{T}}(M) = u^{-1}H_{\mathbb{T}}(M)$.

The completed equivariant cohomology satisfies the localization theorem of Jones–Petrack. We need to first define the technical condition – the regularity of the $\mathbb{T}$-action on the manifold – required in this theorem.

Definition 5.4.7. A $\mathbb{T}$-action on a smooth $\mathbb{T}$-manifold M is *regular* if the fixed point set F is a smooth sub-manifold with an invariant neighbourhood N such that the inclusion $i\colon F \to N$ is an equivariant map that is also a homotopy equivalence. We also say that a $\mathbb{T}$-manifold is regular if there exists a regular $\mathbb{T}$-action on it.

In particular, the free loop space LM of a manifold M satisfies the above condition, since it has an invariant tubular neighbourhood of M in LM (cf. [34, 57]). We thus have the following localization theorem:

Theorem 5.4.8. [34, Theorem 2.1] *If M is a regular $\mathbb{T}$-manifold, then the inclusion $i\colon F \to M$ of the fixed point set F in M induces an isomorphism*

$$i^*\colon \hat{H}^*_{\mathbb{T}}(M) \to \hat{H}^*_{\mathbb{T}}(F).$$

Remark 5.4.9. We omit the proof, since we shall not need it in the sequel. The interested reader is referred to [34] for the proof of this theorem.

5.5 Localization and the equivariant Euler class

A consequence of the localization in ordinary torus-equivariant cohomology is the integration formula of Duistermaat and Heckman [18, 19]. This was independently derived by Berline and Vergne [8], who also realized that the equivariant Euler class appears in the formula. In this section, we describe the set-up for this result for the case $G = \mathbb{T}$ and give a construction, due to Jones and Petrack [34], for the equivariant Euler class (cf. also [7], where the more general case of a compact Lie group G is treated). The case of $G = \mathbb{T}^2$ is similar (cf. §5.3) and will not be treated here.

Suppose M is a smooth manifold with an action of the circle $\mathbb{T}$. As we are working with de Rham cohomology, we would like an infinitesimal description of this action. We let X be the fundamental vector field generating the $\mathbb{T}$-action on M and write F for the fixed point set of this $\mathbb{T}$-action. This is embedded in M via the inclusion

$$i\colon F \hookrightarrow M.$$

Let ν_F be the normal bundle of F in M and endow ν_F with an orientation that is compatible with F. On ν_F, there is a skew-adjoint endomorphism, which we call L_{ν_F}, that is induced from the $\mathbb{T}$-action generated by X. We also have a $\mathbb{T}$-invariant metric connection on ν_F that is induced from the Riemannian connection on M. We denote the curvature of this $\mathbb{T}$-invariant connection by R_{ν_F}.

With these assumptions, we now state the localization formula

Theorem 5.5.1. [7, 19] *Let M be a smooth manifold with the above geometric data. Then, for a form $\alpha \in \Omega_{\mathbb{T}}(M)$ that is closed under $d_{\mathbb{T}}$,*

$$\int_M \alpha = \int_F i^*(\alpha)\left[\det\left(\frac{L_{\nu_F} + R_{\nu_F}}{2\pi i}\right)\right]^{-1}, \tag{5.5.1}$$

where L_{ν_F} and R_{ν_F} are considered to be complex endomorphisms when taking determinants. Furthermore, the denominator is the equivariant Euler class $e(\nu_F)$ of the normal bundle ν_F.

We now give a construction of an equivariant differential form that represents the equivariant Euler class $e(\nu_F)$ of the normal bundle of the fixed point set F. This is due to Jones and Petrack [34].

Proposition 5.5.2. *With the same hypotheses as in Theorem 5.5.1, let α be the differential form dual to X under the $\mathbb{T}$-invariant metric. Let $\tau \in \Omega_{\mathbb{T}}(M)[u, u^{-1}]$ be the $\mathbb{T}$-equivariant form given by*

$$\tau := e^{-d_{\mathbb{T}}\alpha}, \tag{5.5.2}$$

$\pi\colon M \to F$ be the projection from M to its fixed point set F, and $\pi_*\colon \Omega_{\mathbb{T}}(M)[u, u^{-1}] \to \Omega(F)[u, u^{-1}]$ be integration along the fibers of π. Then,

$$\pi_*(\tau) = \left[\det\left(\frac{uL_{\nu_F} + R_{\nu_F}}{2\pi i}\right)\right]^{-1}. \tag{5.5.3}$$

Remark 5.5.3. It is interesting to observe that τ is a factor in the Mathai–Quillen universal Thom form. The reader is invited to compare (5.5.2) with formula (6.9) of [49].

Proof. By construction, τ is a form closed under $d_{\mathbb{T}}$. We note that, since α vanishes on F, τ satisfies the identity $i^*(\tau) = 1$, where $i^*(\tau)$ is the pullback of τ by the inclusion of the fixed point set F in M. To see that (5.5.3) holds, recall that the equivariant Thom isomorphism states that, for an equivariant form $\beta \in \Omega_{\mathbb{T}}(M)[u, u^{-1}]$,

$$e(\nu_F)\pi_*(\beta) = i^*(\beta),$$

where $e(\nu_F)$ is the equivariant Euler class of the normal bundle ν_F of F in M. Since $i^*(\tau) = 1$, it follows that

$$\pi_*(\tau) = \frac{1}{e(\nu_F)}.$$

Formula (5.5.3) is then an immediate consequence of Theorem 5.5.1. $\square$

Chapter 6

Loop Spaces and Loop Bundles

In this chapter, we consider the free loop space LM of a manifold M, as well as loop bundles $LE \to LM$ over LM, formed by taking loops on vector bundles $E \to M$. Many of the results in this chapter have also appeared in the paper [46].

We begin by reviewing, in §6.1, the Atiyah–Witten regularization of the $\mathbb{T}$-equivariant Euler class of the normal bundle of M in LM, the key references here being [2, 63]. Next, in §6.2, we show how this regularization may be extended to a more general case. Our proposed regularization procedure then results in the derivation of the $\hat{\Gamma}$-genus.

In §6.3, we compare the result of our regularization procedure, which we call W-regularization, with that of the full zeta-regularization procedure, and show that W-regularization reduces to Atiyah–Witten regularization in the case of the complexification of a real bundle.

We devote §6.4 to a description of some properties of the $\hat{\Gamma}$-genus in low dimensions. The $\hat{\Gamma}$-genus is, to the best of our knowledge, a new genus and has certain interesting properties. We show, for example, that the $\hat{\Gamma}$-genus actually vanishes for certain classes of manifolds, despite being a smooth invariant for manifolds of certain low dimensions. Furthermore, for almost complex 6-dimensional manifolds, the dependence of the $\hat{\Gamma}$-genus on the choice of a complex structure becomes apparent. We use a construction of LeBrun [41] to illustrate this.

The $\hat{\Gamma}$-genus may well be novel, but it turns out that its generating function, the $\hat{\Gamma}$-function, has appeared in the study of multiple zeta values (MZVs). In §6.5, we show how W-regularization can be viewed in the light of Hoffman's formalism. Furthermore, this allows us to treat the Γ-genus of Libgober [42] on the same footing and leads us to view the $\hat{\Gamma}$-genus as a truncated version of the Γ-genus.

To simplify the exposition, we shall assume throughout this chapter that M is an almost complex compact connected manifold of complex dimension m (here, "complex dimension m" means that M is $2m$-dimensional as a real manifold). In addition, for simplicity, we require that $\pi_1(M) = 0$, or that M is simply connected, so as to ensure that LM is connected.

6.1 The Atiyah–Witten regularization

In this section, we review Atiyah's formal derivation of the $\hat{A}$-genus using an idea that originated from Witten (cf. also [58]).

Let M be an almost complex compact connected spin manifold of complex dimension m and $LM = C^\infty(\mathbb{T}, M)$ be its free loop space. The free loop space LM has a natural $\mathbb{T}$-action given by the rotation of loops, i.e. the translation of the angular parameter on $\mathbb{T}$. Thus, the fixed points of such an action are given by the constant maps from $\mathbb{T}$ to M, which is the image of the embedding of M in LM. We shall thus identify M with its embedding in LM.

We consider the normal bundle $N \to M$ of M in LM, following the approach outlined by Taubes [58]. Over $x \in M$, the fiber of N is a subspace of $LTM|_x$. We note that N can be decomposed in the following way. Restrict TM to an open set U such that the space $TM|_U$ has an orthonormal basis of coordinate functions $\{e_a\}$. If $x \in U$ is a point in U, then an element $y \in LTM|_x$ is of the form $\gamma(t) \cdot e(x)$, where $\gamma(t) = \{\gamma^a(t)\}_{a=1}^m$ is a map $\gamma(t)\colon \mathbb{T} \to \mathbb{R}^m$. By representation theory (cf. [1]) and Fourier analysis, this map can be decomposed into its Fourier components $\{f_n\}_{n=1}^\infty$, where each $f_n \in \mathbb{C}^m$ is an element in $\mathbb{C}^m$, so that

$$y = \sum_{n=1}^{\infty} (f_n \exp(-int) + \bar{f}_n \exp(int)) \cdot e(x) \in LTM|_U \,,$$

where $\bar{f}_n$ is the complex conjugate of f_n. Thus, we can decompose the normal bundle as in [63]:

$$N = \bigoplus_{\substack{n\in\mathbb{Z}\\ n\neq 0}} TM_{[n]},$$

where $TM_{[n]}$ is a copy of TM carrying a $\mathbb{T}$-action of weight n. Alternatively, there is a Fourier decomposition of N (in the sense of Atiyah [2] and Cohen–Stacey [16]) given by

$$N = \bigoplus_{n=1}^{\infty} (TM \otimes \mathbb{C})_{[n]} \,,$$

where $(TM \otimes \mathbb{C})_{[n]}$ is a copy of $TM \otimes \mathbb{C}$ carrying a $\mathbb{T}$-action of weight n.

Next, we approximate the $\mathbb{T}$-equivariant Euler class of N. We note that there are finite-dimensional subbundles

$$N_k = \bigoplus_{n=1}^{k} (TM \otimes \mathbb{C})_{[n]}$$

with inclusions $j_k\colon N_k \hookrightarrow N$ into N and projections $\pi_k\colon N_k \to M$ onto M. Let τ_k denote the $\mathbb{T}$-equivariant form on N_k as constructed in Proposition 5.5.2. The base manifold M is now the fixed point set of the $\mathbb{T}$-action on N_k. Noting that the $\mathbb{T}$-action of weight n induces multiplication by n on $H^2(B\mathbb{T})$ (see [4, p.149]), we can apply Proposition 5.5.2 to see that the $\mathbb{T}$-equivariant Euler class of the bundle N_k is given by

$$e(N_k) = \frac{1}{(\pi_k)_*(\tau_k)} = \prod_{n=1}^{k} \det\left(\frac{nuL_{TM\otimes\mathbb{C}} + R_{TM\otimes\mathbb{C}}}{2\pi i}\right). \tag{6.1.1}$$

Taking the limit as $k \to \infty$ then gives an approximation of the $\mathbb{T}$-equivariant Euler class of N.

Finally, we wish to regularize $e(N)$, since the infinite product turns out to be divergent. According to Atiyah [2] and Duistermaat and Heckman [19], this class can be re-written in terms of Chern classes in the following way. Recalling that $L_{TM\otimes\mathbb{C}}$ is just $i \cdot \mathrm{Id}$, where Id is the identity endomorphism, we factorize out this term:

$$e(N) = \left[\prod_{n=1}^{\infty}\left(\frac{nu}{2\pi}\right)^{2m}\right]\left[\prod_{n=1}^{\infty}\det\left(I + \frac{R_{TM\otimes\mathbb{C}}}{inu}\right)\right].$$

We see that the determinant can be expressed in terms of the Chern classes of $TM \otimes \mathbb{C} = TM \oplus \overline{TM}$ by interpreting it as a "total Chern class" and formally factorizing it into the product

$$\det\left(I + \frac{R_{TM\otimes\mathbb{C}}}{inu}\right) = \prod_{n=1}^{m}\left(1 + \frac{2\pi x_j}{inu}\right)\left(1 - \frac{2\pi x_j}{inu}\right).$$

Here, we have used the splitting principle to decompose

$$TM \oplus \overline{TM} = \bigoplus_{j=1}^{m} L_j \oplus \overline{L_j}$$

into a formal direct sum of line bundles, and $x_j = c_1(L_j)$ are the first Chern classes of the formal line bundles. Thus, we can express $e(N)$ as

$$e(N) = \left[\prod_{n=1}^{\infty}\left(\frac{nu}{2\pi}\right)^{2m}\right]\left[\prod_{n=1}^{\infty}\prod_{n=1}^{m} 1 + \left(\frac{2\pi x_j}{nu}\right)^2\right].$$

The second infinite product yields the $\hat{A}$-class (cf. Definition 4.2.2), up to normalization, while the first product can be zeta-regularized.

6.2 Derivation of the $\hat{\Gamma}$-genus

In this section, we derive the $\hat{\Gamma}$-genus using our proposed regularization procedure. Our point of departure is the paper by Jones and Petrack [34], but we take a slightly different approach and work in a broader setting.

We start with a complex vector bundle $\pi : E \to M$ of complex rank $m \geq 2$ (see Remark 6.2.4 for a justification for this condition). Note that we do not require M to be a spin manifold, but we endow the vector bundle with a spin structure, a smooth $\mathbb{T}$-action and a $\mathbb{T}$-invariant metric.

Next, we take loops to obtain a *rank m loop bundle* (in the sense of Cohen–Stacey [16]) $\pi_\ell : LE \to LM$ over LM. This has $LU(m)$ as its structural group. The original spaces are embedded via the inclusions $j\colon E \hookrightarrow LE$ and $i\colon M \hookrightarrow LM$ in their corresponding free loop spaces:

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d] & LE \arrow[d] \\ M \arrow[r, hook, "i"] & LM \end{tikzcd}$$

By analogy with the normal bundle construction for TM, we define the following.

Definition 6.2.1. Let $E \to M$ be a complex vector bundle of complex rank $m \geq 2$. The *E-normal bundle* $\nu(E) \to M$ is the bundle defined by $\nu(E) = i^*(LE)/E$.

We now analyze the structure of $\nu(E)$ in more detail. Note that $\nu(E)$ inherits a complex vector bundle structure. From §6.1, we see that it has a Fourier decomposition (cf. also the definition of Cohen–Stacey in [16])

$$\nu(E) = \bigoplus_{n=1}^{\infty} E_n,$$

where each of the E_n is a copy of E with a $\mathbb{T}$-action of weight n. There are finite-dimensional subbundles

$$\nu_k(E) = \bigoplus_{n=1}^{k} E_n$$

with inclusions $j_k\colon \nu_k(E) \hookrightarrow \nu(E)$ into $\nu(E)$ and projections $\pi_k\colon \nu_k(E) \to M$ onto M.

Within this setup, let τ_k denote the $\mathbb{T}$-equivariant form on $\nu_k(E)$ as constructed in Proposition 5.5.2. The base manifold M is now the fixed point set of the $\mathbb{T}$-action on $\nu_k(E)$, so we can apply Proposition 5.5.2 to see that

Lemma 6.2.2. *The equivariant cohomology class*

$$(\pi_k)_*(\tau_k) = \left[\prod_{n=1}^{k} \det\left(\frac{nuL_E + R_E}{2\pi i}\right)\right]^{-1} \tag{6.2.1}$$

is the inverse of the $\mathbb{T}$-equivariant Euler class of the bundle $\nu_k(E)$.

This suggests that a $\mathbb{T}$-equivariant Euler class for $\nu(E)$ may be approximated using the formula for the $\mathbb{T}$-equivariant Euler class of its sub-bundles $\nu_k(E)$. We can then use some form of regularization to deal with any divergence that may occur in constructing such an infinite product.

Before we begin, however, we need to first consider the notion of orientability for $\nu(E)$. From physical grounds, Witten [62] has argued that LM is orientable if and only if M is spin. Atiyah [2] and Segal [56] have shown that, provided $\pi_1(M) = 0$, Witten's statement is true, since the obstruction to M being spin transgresses to the obstruction to LM being orientable. McLaughlin [50] has proven the following result for real vector bundles:

Theorem 6.2.3. *Let $\pi_1(M) = 0$ and $E \to M$ be a real vector bundle with structural group $SO(n)$, where $n \geq 4$. Then the following conditions are equivalent:*

1. *$E \to M$ is a vector bundle with a spin structure.*

2. *The structural group of the real loop bundle $LE \to LM$ can be reduced to $L_0SO(n)$, the connected component of the identity of $LSO(n)$. (This is the condition for the orientability of a loop space [56].)*

3. *The structural group of $LE \to LM$ has a lifting to $L\,\mathrm{Spin}(n)$.*

Remark 6.2.4. We can now explain the reason for the hypothesis on the rank of the complex vector bundle $\pi : E \to M$. By considering the underlying real bundle, we note that the condition on the structural group is equivalent to requiring that $E \to M$ has structural group $U(m) \subset SO(2m)$ for $m \geq 2$, i.e. $E \to M$ has to be of rank $m \geq 2$. It then follows that $i^*(LE) \to M$, and therefore $\nu(E) \to M$, is orientable if and only if $E \to M$ is spin.

With the conditions for orientability for $\nu(E)$ now determined, we can now make the following definition.

Definition 6.2.5. The $\mathbb{T}$-*equivariant Euler class of the E-normal bundle* $\nu(E)$ is defined to be

$$e(\nu(E)) := \lim_{k\to\infty} \frac{1}{(\pi_k)_*(\tau_k)} = \lim_{k\to\infty} \prod_{n=1}^{k} \det\left(\frac{nuL_E + R_E}{2\pi i}\right). \tag{6.2.2}$$

We show how this class can be written in terms of characteristic classes.

Lemma 6.2.6. *Let $E \to M$ be a complex spin bundle of complex rank $m \geq 2$ with a formal splitting*

$$E = \bigoplus_{j=1}^{m} L_j,$$

such that x_j is the first Chern class of L_j. Then, the $\mathbb{T}$-equivariant Euler class of $\nu(E)$ can be expressed as

$$e(\nu(E)) = \lim_{k\to\infty} \prod_{n=1}^{k} \left(\frac{nu}{2\pi}\right)^m \cdot \lim_{k\to\infty} \left[\prod_{n=1}^{k}\prod_{j=1}^{m} \left(1 + \frac{2\pi x_j}{nu}\right)\right]. \tag{6.2.3}$$

Proof. We begin with the observation that the endomorphism L_E is just i times the identity. Thus, we find that we can simplify as follows:

$$\begin{aligned}\lim_{k\to\infty} \prod_{n=1}^{k} \det\left(\frac{nuL_E + R_E}{2\pi i}\right) &= \lim_{k\to\infty} \prod_{n=1}^{k} \det\left(\frac{nuL_E}{2\pi i}\right) \det\left(I + \frac{L_E^{-1}R_E}{nu}\right)\\ &= \lim_{k\to\infty} \prod_{n=1}^{k} \left(\frac{nu}{2\pi}\right)^m \cdot \lim_{k\to\infty} \prod_{n=1}^{k} \det\left(I + \frac{R_E}{inu}\right).\end{aligned}$$

Our next step is an observation, made by Duistermaat and Heckman [19], that the determinant in the second product can be expressed in terms of characteristic classes. Recall that the total Chern class of a complex vector bundle E may be written as

$$c(E) = \det\left(I + \frac{R_E}{2\pi i}\right) = 1 + c_1(E) + \cdots + c_n(E).$$

By the splitting principle, this determinant can be formally factorized into the product

$$\det\left(I + \frac{R_E}{2\pi i}\right) = \prod_{j=1}^{m} (1 + x_j),$$

where the x_j's are the so-called *Chern roots*, i.e. the first Chern classes of the respective formal line bundles L_j. Applying this factorization then yields equation (6.2.3) and completes the proof of the lemma. □

Note that both infinite products in formula (6.2.3) are divergent. We now propose a regularization procedure for $e(\nu(E))$.

From the theory of zeta-regularization of infinite products, which we reviewed in Chapter 3, we see that the first infinite product in (6.2.3) can be handled easily. In fact, zeta-regularization of the first infinite product in (6.2.3) gives

$$\prod_{n=1}^{\infty}{}_z \left(\frac{nu}{2\pi}\right)^m = \left[\left(\frac{u}{2\pi}\right)^{\zeta(0)} \prod_{n=1}^{\infty}{}_z\, n\right]^m = \left(\frac{2\pi}{\sqrt{u}}\right)^m, \tag{6.2.4}$$

where the associated zeta function is the Riemann zeta function, with $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\log\sqrt{2\pi}$ (see also Example 3.5.1).

Next, we implement the regularization of the second infinite product in (6.2.3) using the following regularization map ψ_{reg}.

Definition 6.2.7. The *regularization map* ψ_{reg} is the operator defined by extending the map

$$\begin{aligned}\psi_{\text{reg}}\colon H^\bullet(M)[u,u^{-1}] &\to H^\bullet(M)[u,u^{-1}]\\ (1+A) &\mapsto (1+A)e^{-A}\end{aligned}$$

multiplicatively to a finite product of factors of this form. Here, A is a linear rational expression in terms of the Chern roots and the indeterminate u.

Remark 6.2.8. In the classical theory of infinite products, what ψ_{reg} achieves is to append a convergence factor to each factor of the form $(1+A)$. Thus, when the number of factors tends to infinity, the resultant infinite product becomes uniformly convergent in every bounded set.

Applying our regularization procedure to the formula for the $\mathbb{T}$-equivariant Euler class of $\nu(E)$ gives us the following:

Definition 6.2.9. The *regularized* $\mathbb{T}$*-equivariant Euler class* of $\nu(E)$ is given by

$$e_{\text{reg}}(\nu(E)) := \prod_{n=1}^{\infty}{}_{z} \left(\frac{nu}{2\pi}\right)^m \cdot \lim_{k\to\infty} \psi_{\text{reg}}\left[\prod_{n=1}^{k}\prod_{j=1}^{m}\left(1+\frac{2\pi x_j}{nu}\right)\right]. \tag{6.2.5}$$

Proposition 6.2.10. *The regularized equivariant Euler class of* $\nu(E)$ *evaluates to*

$$e_{\text{reg}}(\nu(E)) = \left(\frac{2\pi}{\sqrt{u}}\right)^m \prod_{j=1}^{m}\left[\hat{\Gamma}\left(\frac{2\pi x_j}{u}\right)\right]^{-1}. \tag{6.2.6}$$

Proof. Observe that ψ_{reg} acts on the second product to give

$$\psi_{\text{reg}}\left[\prod_{n=1}^{k}\prod_{j=1}^{m}\left(1+\frac{2\pi x_j}{nu}\right)\right] = \prod_{n=1}^{k}\prod_{j=1}^{m}\left[\left(1+\frac{2\pi x_j}{nu}\right)e^{-2\pi x_j/nu}\right]. \tag{6.2.7}$$

It follows from Remark 6.2.8, together with (6.2.4) and (6.2.7), that

$$\begin{aligned} e_{\text{reg}}(\nu(E)) &= \left(\frac{2\pi}{\sqrt{u}}\right)^m \prod_{n=1}^{\infty}\prod_{j=1}^{m}\left(1+\frac{2\pi x_j}{nu}\right)e^{-2\pi x_j/nu}\\ &= \left(\frac{2\pi}{\sqrt{u}}\right)^m \prod_{j=1}^{m}\left[\hat{\Gamma}\left(\frac{2\pi x_j}{u}\right)\right]^{-1}. \end{aligned}$$

This completes the proof. ∎

6.3 Comparison of regularizations

In this section, we place our regularization procedure in the context of the theory of zeta-regularization, as described in [53, 59]. We first show that our regularization procedure reduces to the Atiyah–Witten regularization when $E = TM \otimes \mathbb{C}$ (see Proposition 6.3.3). We then show that the theory of zeta-regularized products give a different genus in the more general case: the Γ-genus of Libgober [42].

The reader may observe that the form of the regularized product (6.2.5) closely resembles a zeta-regularized product. We clarify this by making the following definition.

Definition 6.3.1. Let $L = \{\lambda_k\}$ be a zeta-regularizable sequence of non-zero complex numbers with indices in a countable set K. The *W-regularized product* is defined to be

$$\prod_{k\in K}{}_W (\lambda_k - z) = \left[\prod_{k\in K}{}_z \lambda_k\right] W_L(z),$$

where $W_L(z)$ is the Weierstrass canonical product (3.4.1) associated to L.

Proposition 6.3.2. *The regularized $\mathbb{T}$-equivariant Euler class $e_{\mathrm{reg}}(\nu(E))$ is a W-regularized product:*

$$e_{\mathrm{reg}}(\nu(E)) = \prod_{j=1}^{m} \prod_{n=1}^{\infty}{}_W \left(\frac{nu}{2\pi} + x_j\right).$$

Proof. This follows from (6.2.5) and the proof of Proposition 6.2.10. □

This helps us to show how our proposed regularization behaves when $E = \eta \otimes \mathbb{C}$ is the complexification of a vector bundle $\pi_R \colon \eta \to M$ of real rank $2m$. Note that since E is now the complexification of a real vector bundle, R_E is skew-symmetric, so that

$$c(E) = \det\left(I + \frac{R_E}{2\pi i}\right) = \det\left(I - \frac{R_E}{2\pi i}\right).$$

In particular, since we are working over the complex numbers, the odd Chern classes vanish. Observe also that $c(E)$ can now be formally factorized into

$$c(E) = \prod_{j=1}^{m} (1 + x_j)(1 - x_j),$$

where the x_j's are the Chern roots coming from the formal splitting of E described in Lemma 6.2.6. The $\mathbb{T}$-equivariant Euler class of $\nu(E)$ is then given by the formula

$$e(\nu(E)) = \lim_{k\to\infty} \prod_{n=1}^{k} \left(\frac{nu}{2\pi}\right)^{2m} \cdot \lim_{k\to\infty} \left[\prod_{n=1}^{k} \prod_{j=1}^{m} \left(1 + \frac{2\pi x_j}{inu}\right)\left(1 - \frac{2\pi x_j}{inu}\right)\right]. \qquad (6.3.1)$$

The regularization procedure in this case then defines $e_{\mathrm{reg}}(\nu(E))$ to be

$$e_{\mathrm{reg}}(\nu(E)) := \prod_{n=1}^{\infty}{}_{z} \left(\frac{nu}{2\pi}\right)^{2m} \cdot \lim_{k\to\infty} \psi_{\mathrm{reg}} \left[\prod_{n=1}^{k}\prod_{j=1}^{m}\left(1+\frac{2\pi x_j}{inu}\right)\left(1-\frac{2\pi x_j}{inu}\right)\right]. \quad (6.3.2)$$

Proposition 6.3.3. *Let $\pi\colon E \to M$ be the complexification $E = \eta \otimes \mathbb{C}$ of a vector bundle η over M of real rank $2m$, such that E has a spin structure. Then the regularized $\mathbb{T}$-equivariant Euler class of $\nu(E)$ evaluates to*

$$e_{\mathrm{reg}}(\nu(E)) = \left(\frac{4\pi^2}{u}\right)^m \prod_{j=1}^{m}\left[\hat{A}\left(\frac{4\pi^2 x_j}{u}\right)\right]^{-1}. \quad (6.3.3)$$

In particular, if $\eta = TM$ is the tangent bundle of M, then our regularization procedure reduces to the Atiyah–Witten regularization, up to scaling of the $\hat{A}$-genus.

Proof. We consider the action of the map ψ_{reg} on the product in (6.3.2). Observe that

$$\begin{aligned}
&\psi_{\mathrm{reg}}\left[\prod_{n=1}^{k}\prod_{j=1}^{m}\left(1+\frac{2\pi x_j}{inu}\right)\left(1-\frac{2\pi x_j}{inu}\right)\right] \\
&= \prod_{n=1}^{k}\prod_{j=1}^{m}\left[\left(1+\frac{2\pi x_j}{inu}\right)e^{-2\pi x_j/inu}\left(1-\frac{2\pi x_j}{inu}\right)e^{2\pi x_j/inu}\right] \\
&= \prod_{n=1}^{k}\prod_{j=1}^{m}\left[1+\left(\frac{2\pi x_j}{nu}\right)^2\right],
\end{aligned}$$

where the last equality is possible because of the absolute convergence of Weierstrass canonical products. We also note that

$$\frac{\sinh(2\pi^2 x/u)}{(2\pi^2 x/u)} = \prod_{n=1}^{\infty}\left[1+\frac{4\pi^2 x^2}{(nu)^2}\right].$$

It follows that the regularized $\mathbb{T}$-equivariant Euler class is given by

$$e_{\mathrm{reg}}(\nu(E)) = \left(\frac{4\pi^2}{u}\right)^m \prod_{j=1}^{m}\frac{\sinh(2\pi^2 x_j/u)}{2\pi^2 x_j/u} = \left(\frac{4\pi^2}{u}\right)^m \prod_{j=1}^{m}\left[\hat{A}\left(\frac{4\pi^2 x_j}{u}\right)\right]^{-1}.$$

In particular, if $\eta = TM$ is the tangent bundle of M, then the evaluation of $e_{\mathrm{reg}}(\nu(E))$ against the fundamental class of M gives the inverse of the $\hat{A}$-genus of M, up to normalization. We thus recover the Atiyah–Witten regularization. $\square$

Returning now to the general case where E is a complex spin vector bundle of complex rank $m \geq 2$, we make the following definition.

Definition 6.3.4. The *zeta-regularized* $\mathbb{T}$*-equivariant Euler class* $e_\zeta(\nu(E))$ is the zeta-regularized product

$$e_\zeta(\nu(E)) = \prod_{j=1}^{m} \prod_{n=1}^{\infty}{}_{z} \left(\frac{nu}{2\pi} + x_j\right).$$

Proposition 6.3.5. *Let* $\pi\colon E \to M$ *be a complex spin vector bundle of complex rank* $m \geq 2$*. The zeta-regularized* $\mathbb{T}$*-equivariant Euler class then evaluates to*

$$e_\zeta(\nu(E)) = \left(\frac{2\pi}{\sqrt{u}}\right)^m \prod_{j=1}^{m} \left[\Gamma\left(1 + \frac{2\pi x_j}{u}\right)\right]^{-1} \tag{6.3.4}$$

Proof. This follows from Example 3.5.1 and the proof of Proposition 6.2.10. □

6.4 The $\hat{\Gamma}$-genus in low dimensions

In this section, we consider the behaviour of the $\hat{\Gamma}$-genus for almost complex manifolds of low (complex) dimensions. At these dimensions, the $\hat{\Gamma}$-genus is already exhibiting fairly curious properties, but it is also fairly well-behaved here.

We begin with connected compact Riemann surfaces, i.e. almost complex manifolds of complex dimension 1. In the course of computing the first few polynomials of the $\hat{\Gamma}$-sequence (cf. Section A.4 of Appendix A), we observe that $\hat{\Gamma}_1(c_1) = 0$. This gives the $\hat{\Gamma}$-genus the following curious property:

Proposition 6.4.1. *Let* Σ *be a connected compact Riemann surface.*

1. $\hat{\Gamma}(\Sigma) = 0$.
2. *If* $M = \Sigma \times N$ *is a product of an almost complex connected compact manifold* N *and* Σ*, and if* M *has the product almost complex structure (coming from the complex structures of* Σ *and* N*), then* $\hat{\Gamma}(M) = 0$.

Proof. Recall that if Σ is a Riemann surface, then

$$\hat{\Gamma}(\Sigma) := \hat{\Gamma}_1(c_1(\Sigma))[\Sigma],$$

but $\hat{\Gamma}_1$ vanishes identically, so that $\hat{\Gamma}(\Sigma) = 0$.

For the second statement, we first recall (see Example 4.2.7) that $\hat{\Gamma}(z)$ defines a multiplicative genus, i.e. if M_1 and M_2 are two almost complex manifolds, then

$$\hat{\Gamma}(M_1 \times M_2) = \hat{\Gamma}(M_1)\hat{\Gamma}(M_2),$$

where $M_1 \times M_2$ has the almost complex structure induced from the almost complex structures of M_1 and M_2. It follows that if $M = \Sigma \times N$ by the given hypothesis, then the multiplicativity of the $\hat{\Gamma}$-genus, together with the first statement, implies that $\hat{\Gamma}(M) = 0$. □

Remark 6.4.2. We remark that it is important to impose the hypothesis that M has the almost complex structure coming from the product. We shall see why this hypothesis is needed when we consider almost complex manifolds of complex dimension 3.

Remark 6.4.3. This interesting property of the $\hat{\Gamma}$-genus sets it apart from the classical genera in Hirzebruch's theory. The latter are generated by multiplicative sequence of polynomials that either came from even generating functions (like the $\hat{A}$- and L-genera), and so vanish at odd degrees, or from odd generating functions (like the Todd genus) that do not vanish identically at any degree. The multiplicative sequence generated by $\hat{\Gamma}(z)$ then has the rather unusual property of vanishing only at the first degree.

Next, we consider the case where M is a manifold of complex dimension 2. Here, we can deduce a necessary and sufficient condition for the $\hat{\Gamma}$-genus to vanish.

Proposition 6.4.4. *Let M be an almost complex compact connected manifold of complex dimension 2. Then $\hat{\Gamma}(M) = 0$ if and only if $\frac{1}{2}p_1(M) = 0$.*

Proof. From Table A.4.2, we see that

$$\begin{aligned}\hat{\Gamma}_2(c_1(M), c_2(M)) &= -\zeta(2)\left(\frac{1}{2}(c_1^2(M) - 2c_2(M))\right)\\ &= -\zeta(2)\left(\frac{1}{2}p_1(M)\right).\end{aligned}$$

Since $\zeta(2) \neq 0$, it follows that $\hat{\Gamma}(M) = \hat{\Gamma}_2(c_1(M), c_2(M))[M]$ vanishes if and only if $\frac{1}{2}p_1(M)$ vanishes. □

Finally, we look at an almost complex manifold M of complex dimension 4. There is also a necessary and sufficient vanishing condition, which is only slightly more complicated.

Proposition 6.4.5. *Let M be an almost complex compact connected manifold of complex dimension 4. Then $\hat{\Gamma}(M) = 0$ if and only if $8p_2(M) + p_1^2(M) = 0$.*

Proof. From Table A.4.2, we see that

$$\hat{\Gamma}_4(p_1(M), p_2(M)) = \frac{1}{8}(4\zeta(4)p_2 + ((\zeta(2))^2 - 2\zeta(4))p_1^2).$$

Using the well-known identities

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90},$$

we can simplify $\hat{\Gamma}_4$

$$\begin{aligned}\hat{\Gamma}_4(p_1(M), p_2(M)) &= \frac{1}{8}\left(\frac{4\pi^4}{90}p_2(M) + \left(\frac{\pi^4}{36} - \frac{2\pi^4}{90}\right)p_1^2(M)\right)\\ &= \frac{\pi^4}{8}\left(\frac{8p_2(M) + p_1^2(M)}{180}\right).\end{aligned}$$

Thus, $\hat{\Gamma}_4$ vanishes whenever $8p_2(M) + p_1^2(M) = 0$, so we have that

$$\hat{\Gamma}(M) = \hat{\Gamma}_4(p_1(M), p_2(M))[M] = 0$$

if and only if $8p_2(M) + p_1^2(M) = 0$. □

However, these are probably the only nice properties of $\hat{\Gamma}$-genus. This is because, as Hirzebruch noted [25], the Chern numbers are defined using the almost complex structure of a manifold, and are therefore dependent *a priori* on the almost complex structure. In fact they do depend on the complex structure. To illustrate this, we recall the example of a compact connected manifold of complex dimension 3 that was considered by LeBrun [41].

Example 6.4.6. Let $M = K3 \times S^2$ be the product of a $K3$ surface and the 2-sphere. Then, for each positive integer m, LeBrun has shown that there is a complex structure J_m on M such that

$$c_2c_1(M, J_m)[M] = 48m, \quad c_1^3(M, J_m) = 0.$$

Note that the complex structure on M is the product complex structure when $m = 1$. However, when $m = 2$, the complex structure comes from M considered as a twistor space, in which case the structure can never be of Kähler type (see, for example, [27]).

We recall that the Euler characteristic of M is $\chi(M) = c_3(M)[M] = 48$. Thus, we observe that

Proposition 6.4.7. *Let $M = K3 \times S^2$. Then, for m a positive integer, we have*

$$\hat{\Gamma}(M, J_m) = 16\zeta(3)(1 - m),$$

so that $\hat{\Gamma}(M, J_m)$ vanishes if and only if $m = 1$, i.e. the complex structure on M is the product complex structure.

Proof. From Table A.4.2, we see that

$$\hat{\Gamma}_3(c_1(M), c_2(M), c_3(M)) = \frac{1}{3}\zeta(3)(c_1^3 - 3c_2c_1 + 3c_3).$$

From the above discussion, we observe that for the pair $(M = K3 \times S^2, J_m)$,

$$\hat{\Gamma}(M, J_m) = \frac{1}{3}\zeta(3)(0 - 48m + 48) = 16\zeta(3)(1 - m),$$

and this vanishes exactly when $m = 1$. □

Corollary 6.4.8. *The $\hat{\Gamma}$-genus of an almost complex manifold M depends on the choice of the almost complex structure when M is a compact manifold of complex dimension 3.*

6.5 The Γ-genera and Hoffman's formalism

In this section, we describe an algebraic formalism, due to Hoffman [28] and arising from his study of MZVs, that allows us to give an alternative interpretation of the map ψ_{reg} in our proposed regularization of the inverse equivariant Euler class. Hoffman's formalism requires some elementary theory of symmetric functions, which we have recalled in Appendix A.

In his study of MZVs, Hoffman [28] has defined a homomorphism $Z\colon \mathrm{Sym} \to \mathbb{R}$, such that on the power sum symmetric polynomials s_i,

$$Z(s_1) = \gamma, \quad Z(s_i) = \zeta(i) \text{ for } i \geq 2.$$

In particular, Z acts on the generating function $S(t)$ to give

$$Z(S(t)) = \gamma + \sum_{i=2}^{\infty} \zeta(i) t^{i-1} = -\psi(1-t),$$

where $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$. It follows from (A.1.1) that

$$Z(C(t)) = \frac{1}{\Gamma(1+t)}$$

We now observe that a similar map $\hat{Z}\colon \mathrm{Sym} \to \mathbb{R}$ can be defined to yield the $\hat{\Gamma}$-function. Essentially, $\hat{Z}$ is a truncated version of Z and acts on the power sum symmetric polynomials in the following way:

$$\hat{Z}(s_1) = 0, \quad \hat{Z}(s_i) = \zeta(i) \text{ for } i \geq 2.$$

It follows that

$$\hat{Z}(C(t)) = \frac{1}{\hat{\Gamma}(t)}. \tag{6.5.1}$$

We use this formalism to deduce the following

Proposition 6.5.1. *Let E be a complex vector bundle over M and x be one of its Chern roots. Let ψ_{reg} be the regularization map. Then the following identity holds:*

$$\begin{aligned} \lim_{k\to\infty} \psi_{\mathrm{reg}} \left(\prod_{n=1}^{k} \left(1 + \frac{2\pi x}{nu} \right) \right) &= \hat{Z} \left(\lim_{k\to\infty} \prod_{n=1}^{k} \left(1 + \frac{2\pi x}{nu} \right) \right) \\ &= \left(\hat{\Gamma} \left(\frac{2\pi x}{u} \right) \right)^{-1}. \end{aligned}$$

Proof. Recall that the left-hand side gives the infinite product expansion of $1/\hat{\Gamma}(\frac{2\pi x}{u})$. It follows from (6.5.1) that the right-hand side also yields the same expression. ☐

We now state a straightforward variation of a result of Hoffman, which gives a rather elegant description of the coefficients of the multiplicative $\hat{\Gamma}$-sequence. We omit the proof, since it is identical to the one given in [29].

Proposition 6.5.2. *Let λ be a partition of n. Then $\hat{Z}(m_\lambda)$ is the coefficient of c_λ in the polynomial $\hat{\Gamma}_n(c_1, \ldots, c_n)$.*

Chapter 7

Double Loop Spaces

In this chapter, we consider the double free loop space $LLM = C^\infty(\mathbb{T}^2, M)$ of an almost complex compact 2-connected smooth manifold M of complex dimension d. These assumptions guarantee, in particular, that LLM is connected, which will simplify our exposition here. We shall also assume that LLM is orientable.

In §7.1, we consider the normal bundle $\nu(M)$ of M in LLM. M is here viewed as an embedded submanifold of LLM. In contrast to the case of LM, we find that all the variants of regularization procedures that we have considered previously turn out to produce different characteristic classes. The framework provided by the theory of zeta-regularized products then becomes crucial to a better understanding of the situation.

On a more interesting note, we also show that we recover the Witten genus (Definition 7.1.5), as well as the $\hat{\Theta}$-genus (Definition 7.1.6) described in [44], which appear in the study of elliptic cohomology.

In §7.2, we consider a sub-bundle of $\nu(M)$ and show that its regularized $\mathbb{T}^2$-equivariant Euler class yields a class generated by an expression based on Barnes' double gamma function. This new class — the $\hat{\Gamma}_2$-class — can also be considered as a multiplicative genus.

7.1 Θ-genera and double loop spaces

In this section, we consider some expressions approximating the $\mathbb{T}^2$-equivariant Euler class of the normal bundle $\nu(M)$ of M in LLM, where M is considered as the space of constant maps in LLM. We show that we retrieve multiplicative genera which are generated by expressions involving the Jacobi theta function. This gives a detailed proof of a (corrected) assertion of Liu [44, p. 244].

We identify M with the image of its embedding as the space of constant maps in LLM. Then M is the fixed-point set of the natural action of $\mathbb{T}^2$ on LLM and we consider the $\mathbb{T}^2$-equivariant Euler class of the normal bundle $\nu(M)$ of M in LLM, following the approach taken in Chapter 6.

The normal bundle $\nu(M)$ can be decomposed according to the $\mathbb{T}^2$-action. From representation theory, we recall that the irreducible representations of $\mathbb{T}^2$ are indexed by a pair of integers [1]. Thus, we see that the normal bundle splits in the following way

$$\nu(M) = \bigoplus_{m,n\in\mathbb{Z}}{}' TM_{[m,n]},$$

where the prime denotes that we exclude the summand corresponding to $(m,n) = (0,0)$. Again, we can consider finite-dimensional sub-bundles

$$\nu_k(M) = \bigoplus_{m,n=-k}^{k}{}' TM_{[m,n]}$$

with inclusions $j_k\colon \nu_k(M) \hookrightarrow \nu(M)$ and projections $\pi_k\colon \nu_k(M) \to M$.

We can construct a $\mathbb{T}^2$-equivariant Euler class of $\nu_k(M)$ in the same way as in Chapter 6. Recall from Chapter 5 that $\mathbb{T}^2$-equivariant cohomology is a $\mathbb{C}[u,v]$-module, where u and v are indeterminates of degree 2. Hence, we see that

$$e(\nu_k(M)) = \prod_{m,n=-k}^{k}{}' \det\left(\frac{(mu+nv)L_{TM} + R_{TM}}{2\pi i}\right).$$

To apply the theory of zeta-regularized products, we shall also view u and v as complex parameters, and require that $\operatorname{Im} v/u = \operatorname{Im}\tau > 0$ and $\tau \notin (-\infty, 0]$. For convenience of notation, we henceforth set $u = 1$ and $v = \tau$.

Definition 7.1.1. The $\mathbb{T}^2$-*equivariant Euler class* $e(\nu(M))$ of $\nu(M)$ is defined to be

$$\begin{aligned} e(\nu(M)) &= \left[\prod_{m,n\in\mathbb{Z}}{}' \left(\frac{m+n\tau}{2\pi}\right)^d\right]\left[\prod_{m,n\in\mathbb{Z}}{}' \det\left(I + \frac{R_{TM}}{i(m+n\tau)}\right)\right] \\ &= \left[\prod_{m,n\in\mathbb{Z}}{}' \left(\frac{m+n\tau}{2\pi}\right)^d\right]\left[\prod_{m,n\in\mathbb{Z}}{}' \prod_{j=1}^{d}\left(1 + \frac{2\pi x_j}{i(m+n\tau)}\right)\right]. \end{aligned} \tag{7.1.1}$$

Here, we have made use of the splitting principle to split TM into a formal direct sum of line bundles

$$TM = \bigoplus_{j=1}^{d} L_j,$$

where the line bundle L_j has first Chern class x_j.

We shall see that the situation for LLM is different from that of LM. First, we consider the following definition of Liu [44], which we can motivate by observing, from §3.2, that the second infinite product in (7.1.1) is convergent.

Definition 7.1.2. The *normalized* $\mathbb{T}^2$*-equivariant Euler class* $e_{\mathrm{norm}}(\nu(M))$ of $\nu(M)$ is defined to be

$$e_{\mathrm{norm}}(\nu(M)) = \prod_{m,n\in\mathbb{Z}}' \prod_{j=1}^{d} \left(1 + \frac{2\pi x_j}{i(m+n\tau)}\right). \tag{7.1.2}$$

However, we can also keep the first product in the expression (7.1.1), after performing zeta regularization on it. This is the Atiyah–Witten approach, as applied to the case of LLM:

Definition 7.1.3. The *AW-regularized* $\mathbb{T}^2$*-equivariant Euler class* $e_{\mathrm{AW}}(\nu(M))$ of $\nu(M)$ is defined to be

$$e_{\mathrm{AW}}(\nu(M)) = \left[\prod_{m,n\in\mathbb{Z}}{}'_{z} \left(\frac{m+n\tau}{2\pi}\right)^{d}\right] \left[\prod_{m,n\in\mathbb{Z}}' \prod_{j=1}^{d} \left(1 + \frac{2\pi x_j}{i(m+n\tau)}\right)\right]. \tag{7.1.3}$$

Finally, we have W-regularization, its feature being the replacement of the infinite product coming from the determinant by the corresponding Weierstrass product. In this case, the convergence factors do not vanish.

Definition 7.1.4. The *W-regularized* $\mathbb{T}^2$*-equivariant Euler class* $e_W(\nu(M))$ of $\nu(M)$ is defined to be

$$\begin{aligned} e_W(\nu(M)) = &\left[\prod_{m,n\in\mathbb{Z}}{}'_{z} \left(\frac{m+n\tau}{2\pi}\right)^{d}\right] \times \\ &\left[\prod_{m,n\in\mathbb{Z}}' \prod_{j=1}^{d} \left(1 + \frac{2\pi x_j}{i(m+n\tau)}\right) \exp\left(\frac{2\pi x_j}{i(m+n\tau)} + \frac{1}{2}\left(\frac{2\pi x_j}{i(m+n\tau)}\right)^2\right)\right]. \end{aligned} \tag{7.1.4}$$

As we shall be proving an assertion of Liu in [44], we recall some of the definitions made there.

Definition 7.1.5. The *Witten class* of M is defined to be the class

$$W(x_j; M) = \prod_{j=1}^{d} x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} = \prod_{j=1}^{d} 2\pi x_j \frac{\eta(\tau)^3}{\theta(x_j,\tau)}$$

where the second equality comes from the identity $\theta'(0,\tau) = 2\pi\eta(\tau)^3$ [14, (6.9)].

Definition 7.1.6. The $\hat{\Theta}$*-class* of M is given by

$$\hat{\Theta}(x_j; M) = \prod_{j=1}^{d} x_j \frac{\eta(\tau)}{\theta(x_j,\tau)}.$$

We have the following proposition.

Proposition 7.1.7. *Let M be an almost complex compact 2-connected smooth manifold of complex dimension d and $\nu(M)$ be the normal bundle of M in LLM. Then*

1. *The normalized $\mathbb{T}^2$-equivariant Euler class is given by*

$$e_{\text{norm}}(\nu(M)) = [W(2\pi x_j/i; M)]^{-1}.$$

2. *The AW-regularized $\mathbb{T}^2$-equivariant Euler class is given by*

$$e_{AW}(\nu(M)) = [\hat{\Theta}(2\pi x_j/i; M)]^{-1} \prod_{j=1}^{d} \left[2\pi i x_j \exp\left(-\frac{\pi i \tau}{6}\right)\right].$$

3. *The W-regularized $\mathbb{T}^2$-equivariant Euler class is given by*

$$e_W(\nu(M)) = \prod_{j=1}^{d} -\frac{\eta(\tau)^2}{x_j}\sigma(2\pi x_j/i)\exp\left(-\frac{\pi i \tau}{6}\right),$$

where $\sigma(z)$ is the Weierstrass sigma function in Definition 3.2.3.

Remark 7.1.8. The first statement in Proposition 7.1.7 corrects the assertion made in Liu [44, p. 244] that the normalized $\mathbb{T}^2$-equivariant Euler class would give the inverse $\hat{\Theta}$-class. It is actually the Witten-regularized $\mathbb{T}^2$-equivariant Euler class that yields the $\hat{\Theta}$-class, up to factors introduced by the zeta-regularized product in 7.1.3.

Proof. Statement 1 follows from Proposition 3.2.7 once we observe that

$$e_{\text{norm}}(\nu(M)) = \prod_{j=1}^{d} \left[\frac{1}{\left(\frac{2\pi x_j}{i}\right)} \cdot \varphi\left(\frac{2\pi x_j}{i}\right)\right].$$

To prove statement 2, recall equation (3.5.3):

$$\prod_{m,n\in\mathbb{Z}}{}'_{z}(m+n\tau) = 2\pi i \prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})^2 = 2\pi i \eta(\tau)^2 \exp\left(-\frac{\pi i\tau}{6}\right).$$

Thus, we have

$$\prod_{m,n\in\mathbb{Z}}{}'_{z}\left(\frac{m+n\tau}{2\pi}\right)^d = \left[(2\pi)^{-d}\right]^{-1}\left[2\pi i\eta(\tau)^2\exp\left(-\frac{\pi i\tau}{6}\right)\right]^d,$$

where we have used the identity that the zeta function associated to this sequence,

$$Z(s) = \zeta(s) + (e^{-\pi i})^{-s}\zeta(s) + \sum_{n\neq 0}(m+n\tau)^{-s},$$

has the value $Z(0) = -1$ at $s = 0$. Thus, we obtain the identity in statement 2.

Statement 3 follows by comparing the Weierstrass product in (7.1.4) with the infinite product representation for the Weierstrass sigma function. □

7.2 The $\hat{\Gamma}_2$-genus

In this section, we consider a sub-bundle of $\nu(M)$ and an expression approximating its $\mathbb{T}^2$-equivariant Euler class. We find that we obtain a new genus, which we call the $\hat{\Gamma}_2$-genus, since its generating function involves Barnes' double gamma function $\Gamma_2(z;1,\tau)$.

First, we define the generating function.

Definition 7.2.1. The $\hat{\Gamma}_2$-function is defined to be

$$\hat{\Gamma}_2(z;u,v) = ze^{\gamma_{22}z+\gamma_{21}\frac{z^2}{2}}\Gamma_2(z;u,v),$$

where γ_{21} and γ_{22} are the double modular constants.

Remark 7.2.2. By (3.5.10), we see that

$$\left[\hat{\Gamma}_2(z;u,v)\right]^{-1} = \prod_{m,n=0}^{\infty}{}' \left\{\left(1+\frac{z}{mu+nv}\right)\exp\left[-\frac{z}{mu+nv}+\frac{1}{2}\left(\frac{z}{mu+nv}\right)^2\right]\right\}.$$

Next, we describe the sub-bundle we wish to consider. This is the bundle

$$\nu^+(M) = \bigoplus_{m,n=1}^{\infty} TM_{[m,n]} \subset \nu(M).$$

There are sub-bundles

$$\nu_k^+(M) = \bigoplus_{m,n=1}^{k} TM_{[m,n]}$$

with the corresponding inclusions $j_k\colon \nu_k^+(M) \hookrightarrow \nu^+(M)$ into $\nu^+(M)$ and projections $\pi_k\colon \nu_k^+(M) \to M$. The $\mathbb{T}^2$-equivariant Euler class of $\nu_k^+(M)$ is then given by the formula

$$e(\nu_k^+(M)) = \prod_{m,n=1}^{k} \det\left(\frac{(mu+nv)L_{TM}+R_{TM}}{2\pi i}\right).$$

Again, we set $u=1$ and $v=\tau$ for convenient notation.

Definition 7.2.3. The $\mathbb{T}^2$-equivariant Euler class $e(\nu^+(M))$ of $\nu(M)$ is defined to be

$$e(\nu^+(M)) = \left[\prod_{m,n=1}^{\infty}\left(\frac{m+n\tau}{2\pi}\right)^d\right]\left[\prod_{m,n=1}^{\infty}\det\left(I+\frac{R_{TM}}{i(m+n\tau)}\right)\right]. \tag{7.2.1}$$

Remark 7.2.4. By the splitting principle, we can express $e(\nu^+(M))$ as

$$e(\nu^+(M)) = \left[\prod_{m,n=1}^{\infty}\left(\frac{m+n\tau}{2\pi}\right)^d\right]\left[\prod_{m,n=1}^{\infty}\prod_{j=1}^{d}\left(1+\frac{2\pi x_j}{i(m+n\tau)}\right)\right], \tag{7.2.2}$$

where x_j is the first Chern class of the formal line bundle L_j in the formal splitting

$$TM = \bigoplus_{j=1}^{d} L_j$$

of the tangent bundle of M.

We now consider the result of W-regularization on this class.

Definition 7.2.5. The W-regularized $\mathbb{T}^2$-equivariant Euler class of $\nu^+(M)$ is defined to be

$$e_W(\nu^+(M)) = \left[\prod_{m,n=1}^{\infty}{}_z\left(\frac{m+n\tau}{2\pi}\right)^d\right]\left[\prod_{m,n=1}^{\infty}\prod_{j=1}^{d}\left(1+\frac{2\pi x_j}{i(m+n\tau)}\right)e^{-\frac{2\pi x_j}{i(m+n\tau)}+\frac{1}{2}\left(\frac{2\pi x_j}{i(m+n\tau)}\right)^2}\right]$$

Proposition 7.2.6. *The W-regularized $\mathbb{T}^2$-equivariant Euler class $e_W(\nu^+(M))$ evaluates to the formula*

$$e_W(\nu^+(M)) = \left(\frac{1}{2\pi}\right)^{d\left(\frac{5}{4}+\frac{1}{12}\left(\tau+\frac{1}{\tau}\right)\right)}\tau^{\frac{1}{2}d}\prod_{j=1}^{d}\frac{\rho_2(1,\tau)}{\hat{\Gamma}_2\left(\frac{2\pi x_j}{i};1,\tau\right)}. \tag{7.2.3}$$

Proof. We note that

$$\prod_{m,n=1}^{\infty}\prod_{j=1}^{d}\left(1+\frac{2\pi x_j}{i(m+n\tau)}\right)e^{-\frac{2\pi x_j}{i(m+n\tau)}+\frac{1}{2}\left(\frac{2\pi x_j}{i(m+n\tau)}\right)^2} = \prod_{j=1}^{d}\left[\hat{\Gamma}_2\left(\frac{2\pi x_j}{i};1,\tau\right)\right]^{-1}.$$

It remains to evaluate the zeta-regularized product

$$\prod_{m,n=1}^{\infty}{}_z\left(\frac{m+n\tau}{2\pi}\right)^d = \left(\frac{1}{2\pi^d}\right)^{Z(0)}\left[\prod_{m,n=1}^{\infty}{}_z(m+n\tau)\right]^d,$$

where $Z(0)$ is the value of the zeta function, evaluated at $s=0$, of the sequence $\{m+n\tau\}$:

$$Z(s) = \sum_{m,n=1}^{\infty}(m+n\tau)^{-s}.$$

From Example 3.5.3, we recall that $Z(0) = \frac{1}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})$ and that

$$\prod_{m,n=1}^{\infty}{}_z(m+n\tau) = (2\pi)^{-1}\tau^{\frac{1}{2}}\rho_2(1,\tau).$$

Hence, we see that

$$\prod_{m,n=1}^{\infty}{}_{z}\left(\frac{m+n\tau}{2\pi}\right)^{d} = \left(\frac{1}{2\pi}\right)^{\frac{5}{4}+\frac{1}{12}(\tau+\frac{1}{\tau})} \tau^{\frac{1}{2}d}[\rho_2(1,\tau)]^d,$$

from which (7.2.3) follows. $\square$

Chapter 8

Conclusion

In this chapter, we summarize the results obtained in this thesis and discuss some interesting connections and questions that have arisen during this investigation. Our results can be viewed as coming out of the consideration of variations on the zeta-regularization of infinite products, as we shall highlight in §8.1.

From §8.2 onwards, our discussion takes on a more speculative nature. We discuss, in that section, the mysterious connections that the $\hat{\Gamma}$-function and the Γ-genera have with other areas of mathematics. All these naturally inspire some questions that remain to be answered.

Finally, in §8.3, we discuss our results for the case of the double loop space LLM and indicate what future developments may flow from the work done here.

8.1 Variations on zeta-regularization

The results achieved in this thesis revolved around the application of the theory of zeta-regularized products to the evaluation of infinite products of equivariant characteristic classes. These classes of interest are characteristic classes of infinite-dimensional bundles over a finite-dimensional base manifold M, e.g. the normal bundle of M in LM. Our investigations led us to consider variations that are possible within the framework of zeta-regularized products, as presented in the works of Jorgenson–Lang [35], Quine et al. [53] and Voros [59], which are themselves a refinement of the theory of zeta-regularization.

Our first result was a derivation of the $\hat{\Gamma}$-genus, a new multiplicative genus generated by the function

$$\frac{1}{\hat{\Gamma}(z)} = \frac{1}{e^{\gamma z}\Gamma(1+z)},$$

using a variant of zeta-regularization, which we called W-regularization. We then showed that W-regularization reduces to Atiyah–Witten regularization when applied to the $\mathbb{T}$-equivariant Euler class of the normal bundle of M in LM. We also investigated some interesting properties of the $\hat{\Gamma}$-genus, which turned out to exhibit some rather wild behaviour, as shown in §6.4.

An interesting aspect of our investigation was the serendipitous discovery that the function $\hat{\Gamma}(z)$ has appeared in the study of multiple zeta values (MZVs), e.g. in the work of Cartier [13] and Ihara, Kaneko and Zagier [32]. Indeed, the coefficients of the $\hat{\Gamma}$-sequence turn out to be linear combinations of products of zeta values (see Proposition 6.5.2), as we learnt when we considered Hoffman's work on MZVs [28, 29].

We then took our investigations to the double loop space LLM. In this case, we find that Atiyah–Witten regularization (or *AW-regularization*), as extended to this context, turns out to be different from W-regularization. We have seen that both these regularizations are different from the zeta-regularization that yields zeta-regularized products (see Theorem 3.4.3 for the structure of zeta-regularized products). In doing so, we also recovered some interesting genera from elliptic cohomology, which involves considering the normal bundle of M in LLM (see Proposition 7.1.7): the Witten genus as its normalized $\mathbb{T}^2$-equivariant Euler class and the $\hat{\Theta}$-genus as its AW-regularized $\mathbb{T}^2$-equivariant Euler class.

Finally, we applied W-regularization to a sub-bundle $\nu^+(M)$ of the normal bundle of M in LLM in §7.2. This is similar to the context in which the $\hat{\Gamma}$-genus was derived, and it turns out that we obtain a multiplicative genus generated by the function

$$[\hat{\Gamma}_2(z;u,v)]^{-1} = [ze^{\gamma_{22}z+\gamma_{21}\frac{z^2}{2}}\Gamma_2(z;u,v)]^{-1},$$

where $\Gamma_2(z;u,v)$ is Barnes' double gamma function. This is yet another exciting discovery, as we shall see later in §8.3.

8.2 The $\hat{\Gamma}$-function and the Γ-genera

In this thesis, we have considered generating functions of multiplicative genera that are based on the Γ-function. It is interesting to note that the Γ-function has appeared in the same guise in the work of Libgober [42] on mirror symmetry. More specifically, Libgober was looking to generalize the work of Hosono et al. [30] in the study of the mirror symmetry of Calabi-Yau hypersurfaces.

Very recently, Katzarkov, Kontsevich and Pantev [36] have continued this story. For a compact symplectic manifold M of complex dimension d and with the Chern roots of TM being c_j, they introduced what is essentially the Γ-class of Libgober

$$\Gamma(TM) = \prod_{j=1}^{d} \frac{1}{\Gamma(1+c_j)},$$

which they called the $\hat{\Gamma}$-class, in their recent work on **nc**-Hodge theory in mirror symmetry. In [36, Remark 3.3], they noted the various other appearances of this class, particularly in the work of Kontsevich [38, §4.6] on deformation quantization. Kontsevich has argued in that work that the functions $\hat{A}(z)$ and $\hat{\Gamma}(z)$ — we note here that the $\hat{\Gamma}$-function appears implicitly in the argument — lie in the same orbit of the action of the Grothendieck–Teichmüller group on deformation quantizations. It is interesting to note that our results give the simultaneous occurrence of these two functions in a different context: that of the generating functions of multiplicative genera derived using zeta-regularization.

The construction of Katzarkov et al., in which the Γ-class appears, is also interesting. We have seen that the Γ-class, up to some factors, can be obtained as the zeta-regularized (as opposed to W-regularized) $\mathbb{T}$-equivariant Euler class of $\nu(M)$, the normal bundle of M in LM. Katzarkov et al. made use of the Γ-class in a map

$$\begin{aligned}\Gamma_{\mathrm{rtl}}\colon H^{\bullet}(M,\mathbb{C}) &\to H^{\bullet}(M,\mathbb{C})\\ \beta &\mapsto \Gamma(TM)\wedge\beta.\end{aligned}$$

One may speculate that the Γ-class acts as a pseudo-Euler class in this construction, although one obvious objection is that the Γ-class here is not a $\mathbb{T}$-equivariant class, but a cohomology class in ordinary cohomology.

In fact, it is not even clear what a natural receptacle for the W-regularized Euler class would be. As this regularized class contains a factor of $1/\sqrt{u}$, it would not be an element of the Jones–Petrack completed $\mathbb{T}$-equivariant cohomology. The theory of this new — and, at present, still hypothetical — cohomology is one development that may be best left to future work.

There is a further connection to the study of MZVs, which comes again from Hoffman's formalism. The Z-map of Hoffman can be considered as an evaluation map Z_* that is a homomorphism with respect to one of two products on the MZVs: the *harmonic product* $*$ defined by Hoffman [28]. There is another product on the MZVs — the *shuffle product* $\circ$ — and the Z-map is again a homomorphism with respect to this product. In this case, we write $Z_\circ$ to highlight this. The *double shuffle relations* between MZVs follow from the fact that the Z-homomorphism is the same map that happens to be a homomorphism with respect to two different products.

The $\hat{\Gamma}$-function then appears in the context of a regularization formula in this context. It turns out that the double shuffle relations are not enough to account for all possible relations between MZVs. By defining a linear map using the $\hat{\Gamma}$-function, it appears that almost all of these "missing" double shuffle relations between MZVs can be recovered [13, 32]. This appears to parallel its appearance here — as a result of our proposed regularization procedure — in the guise of the $\hat{\Gamma}$-genus.

All of these may perhaps shed a little light on some speculative remarks of Morava [52], who has proposed a sketch of a theory of motivic Thom isomorphisms in which the Γ-genus may make an appearance. Morava's work was, not surprisingly, inspired by Kontsevich's remarks mentioned above. The results in our work may perhaps shed some light on these conjectural remarks.

8.3 The case of LLM and beyond

The case of the double loop space presents its own tantalizing hints for future developments. In this thesis, we have already recovered the Witten genus, as well as clarify and prove an assertion of Liu [44], which gave us the $\hat{\Theta}$-genus. These are genera that have appeared in elliptic cohomology.

What is more interesting is that we are able to derive another new multiplicative genus, this time based on Barnes' double gamma function. The study of the double gamma function has already been revived in recent years and a useful bibliography may be found in [15]. Multiple gamma functions have also been studied and it is reasonable to speculate that the construction in this thesis can be taken to higher loop spaces and should yield genera based on these multiple gamma functions.

What is unclear at present, and which is therefore not treated in this thesis, is the question of the orientability of these higher loop spaces. It is reasonable to expect that the answer should come from homotopy theory: one may approach it by considering the obstruction to reducing the structure group of the higher loop bundle to its connected component of the identity and finding equivalent conditions. This is the approach taken by McLaughlin for principal loop bundles [50] and a similar method may work for double loop bundles, although one may need to take into account the fact that for a Lie group G, $\pi_2(G)$ is trivial. Future developments may have to rest upon the hope that the orientability conditions are not too severe as to render the future results valid only for a very restricted class of spaces.

What is exciting, however, is that the application of the theory of zeta-regularized products to the study of characteristic classes has yielded many intriguing results. The framework that it provides has certainly led to the incarnation of many more functions as multiplicative genera and the sighting of new connections between mathematics. Perhaps more surprises are yet to come from this theory.

Appendix A

Polynomial Sequences

In this appendix, we collect some of the multiplicative sequences that have appeared in the preceding chapters, together with their generating function (or characteristic power series) and the first few polynomials in each sequence. Since the computation of such sequences requires some basic theory of symmetric functions, we review this in the first section of this appendix. Note that our notation in this areas is different from the traditional one to avoid conflict with notations used for the characteristic classes.

A.1 Symmetric Functions

In this section, we recall some basic theory of symmetric functions from [47] and set up some notation, which differs in some cases from the usual convention. This theory is needed for the computation of multiplicative sequences, which was discussed in §4.3.

Definition A.1.1. A *partition* λ of an integer m is a sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \ldots$$

with finitely many nonzero entries, such that the sum of the nonzero entries equals m. This sum is also known as the *weight* of the partition λ, while the number of nonzero entries is called the *length* of λ.

Remark A.1.2. For convenience, we sometimes write $\lambda = (3^r, 2^s, \ldots)$ to indicate that, for example, the number 3 occurs r times in λ, and the number 2 occurs s times.

Let $x_1, x_2, \ldots$ be indeterminates. We write Sym for the algebra of symmetric functions in countably many indeterminates $x_1, x_2, \ldots$.

Definition A.1.3. The *monomial symmetric function* m_λ of a partition

$$\lambda = (\lambda_1, \ldots, \lambda_n)$$

of length $\leq n$ is the polynomial

$$m_\lambda(x_1, \ldots, x_n) = \sum_\alpha x^\alpha,$$

where the sum is over all distinct permutations of λ.

Definition A.1.4. The nth *elementary symmetric function* c_n is the polynomial

$$c_n = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

so that c_n corresponds to m_λ, where $\lambda = (1^n)$ The generating function is given by

$$C(t) := \prod_{i=1}^{\infty} (1 + x_i t) = 1 + \sum_{n=1}^{\infty} c_n t^n.$$

Remark A.1.5. We have chosen to depart from the usual convention of using e_n and $E(t)$ because, in treating Hirzebruch's theory, we shall need to view the Chern classes c_n as being the elementary symmetric functions of the indeterminates x_i.

Definition A.1.6. The nth *power sum symmetric function* s_n is the sum

$$s_n = m_n = \sum_i x_i^n$$

The generating function is given by

$$S(t) := \frac{d}{dt} \ln \prod_{i=1}^{\infty} \frac{1}{(1 - x_i t)} = \frac{d}{dt} \log C(-t)^{-1} \tag{A.1.1}$$

Remark A.1.7. Again, we have departed from the usual convention of using p_n and $P(t)$ for the power sum symmetric function. This is to avoid conflict with the notation for the Pontrjagin classes that occur in topology.

The following result is well-known and the reader is referred to texts on symmetric functions (e.g. MacDonald [47]) for the proof.

Theorem A.1.8. *If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition with entries as given, we shall write*

$$c_\lambda = c_{\lambda_1} c_{\lambda_2} \ldots, \quad s_\lambda = s_{\lambda_1} s_{\lambda_2} \ldots.$$

Then the following sets form bases for Sym *as a vector space over $\mathbb{Q}$:*

1. $\{c_\lambda : \lambda$ *is a partition*$\}$.
2. $\{m_\lambda : \lambda$ *is a partition*$\}$.
3. $\{s_\lambda : \lambda$ *is a partition*$\}$.

In the computation of multiplicative sequences, it is useful to know the relation between two of these bases.

Example A.1.9. The power sum symmetric polynomials s_i and the elementary symmetric polynomials c_i are related by Newton's identities [26, 47]:

$$s_n = -(-1)^n n c_n - \sum_{j=1}^{n-1} (-1)^j c_j s_{n-j}. \tag{A.1.2}$$

The first few s_i's are given in Table A.1.1.

Table A.1.1: The first few s_i's in terms of c_i's.

n	s_n
1	c_1
2	$c_1^2 - 2c_2$
3	$c_1^3 - 3c_2c_1 + 3c_3$
4	$c_1^4 - 4c_2c_1^2 + 2c_2^2 + 4c_3c_1 - 4c_4$
5	$c_1^5 - 5c_2c_1^3 + 5c_2^2c_1 - 5c_3c_2 + 5c_3c_1^2 - 5c_4c_1 + 5c_5$
6	$c_1^6 - 6c_2c_1^4 + 9c_2^2c_1^2 - 2c_2^3 + 6c_3c_1^3 - 12c_3c_2c_1 + 3c_3^2 - 6c_4c_1^2 + 6c_4c_2 + 6c_5c_1 - 6c_6$

A.2 The Pontrjagin Sequence

This is the sequence generated by the function

$$p(z) := 1 + z^2.$$

As Hirzebruch observed [26], the sequence expresses the Pontrjagin classes in terms of the Chern classes. The first few polynomials are:

$$\begin{aligned} p_1 &= c_1^2 - 2c_2, \\ p_2 &= c_2^2 - 2c_3c_1 + 2c_4, \\ p_3 &= c_3^2 - 2c_4c_2 + 2c_5c_1 - 2c_6. \end{aligned} \tag{A.2.1}$$

By comparing (A.2.1) with Table A.1.1, we obtain the following identities, which relate the power sum symmetric polynomials in the c_i's to Pontrjagin classes:

$$\begin{aligned} s_2 &= p_1, \\ s_4 &= p_1^2 - 2p_2, \\ s_6 &= p_1^3 - 3p_2p_1 + 3p_3. \end{aligned} \tag{A.2.2}$$

A.3 The Γ-sequence

The Γ-sequence studied by Libgober in the paper [42] is generated by the function

$$\frac{1}{\Gamma(1+z)}.$$

Table A.3.1 gives the first few polynomials in terms of the power sum symmetric polynomials, while Table A.3.2 gives the same polynomials in terms of the Chern and Pontrjagin classes.

Table A.3.1: The first few polynomials of $\{\Gamma_n\}$ in terms of s_i's.

n	Γ_n
1	γs_1
2	$-\frac{1}{2}\left[\zeta(2)s_2+\gamma^2 s_1^2\right]$
3	$\frac{1}{6}\left[2\zeta(3)s_3-3\gamma\zeta(2)s_2s_1+\gamma^3 s_1^3\right]$
4	$\frac{1}{24}\left[-6\zeta(4)s_4+8\gamma\zeta(3)s_3s_1+3(\zeta(2))^2s_2^2-6\gamma^2\zeta(2)s_2s_1^2+\gamma^4 s_1^4\right]$

Table A.3.2: The first few polynomials of $\{\Gamma_n\}$ in Chern and Pontrjagin classes.

n	Γ_n
1	γc_1
2	$\frac{1}{2}\left[-\zeta(2)p_1+\gamma^2 c_1^2\right]$
3	$\frac{1}{6}\left[(\gamma^3-3\gamma\zeta(2)+2\zeta(3))c_1^3+(6\gamma\zeta(2)-6\zeta(3))c_2c_1+6\zeta(3)c_3\right]$
4	$\frac{1}{24}[(3(\zeta(2))^2-6\zeta(4))p_1^2-6\gamma^2\zeta(2)p_1c_1^2+12\zeta(4)p_2-24\gamma\zeta(3)c_2c_1^2+24\gamma\zeta(3)c_3c_1+(8\gamma\zeta(3)+\gamma^4)c_1^4]$

A.4 The $\hat{\Gamma}$-sequence

This first few polynomials in the sequence $\{\hat{\Gamma}_n\}$ generated by the function

$$\hat{\Gamma}(z):=\frac{1}{e^{\gamma z}\Gamma(1+z)}$$

are given in Table A.4.1 in terms of the power sum symmetric polynomials s_i.

In Table A.4.2, we make use of the relations (A.2.2), which allow us to express the polynomials in terms of a mixture of Chern and Pontrjagin classes. These expressions are somewhat less cumbersome than if they were expressed purely in terms of Chern classes. If so desired, (A.2.1) may be used to convert Pontrjagin classes into Chern classes.

Table A.4.1: The first few polynomials of $\{\hat{\Gamma}_n\}$ in terms of s_i's.

n	$\hat{\Gamma}_n$
1	0
2	$-\frac{1}{2}\zeta(2)s_2$
3	$\frac{1}{3}\zeta(3)s_3$
4	$\frac{1}{8}\left[-2\zeta(4)s_4+(\zeta(2))^2s_2^2\right]$
5	$\frac{1}{30}\left[6\zeta(5)s_5-5\zeta(3)\zeta(2)s_3s_2\right]$
6	$\frac{1}{144}\left[-24\zeta(6)s_6+18\zeta(4)\zeta(2)s_4s_2+8(\zeta(3))^2s_3^2-3(\zeta(2))^3s_2^3\right]$

Table A.4.2: The first few polynomials of $\{\hat{\Gamma}_n\}$ in Chern and Pontrjagin classes.

n	$\hat{\Gamma}_n$
1	0
2	$-\frac{1}{2}\zeta(2)p_1$
3	$\frac{1}{3}\zeta(3)(c_1^3-3c_2c_1+3c_3)$
4	$\frac{1}{8}\left[4\zeta(4)p_2+((\zeta(2))^2-2\zeta(4))p_1^2\right]$
5	$\zeta(5)(c_5-c_4c_1)+(\zeta(2)\zeta(3)-\zeta(5))c_3c_2+(\zeta(5)-\frac{1}{2}\zeta(2)\zeta(3))c_3c_1^2+(\zeta(5)-\zeta(2)\zeta(3))c_2^2c_1+(\frac{5}{6}\zeta(2)\zeta(3)-\zeta(5))c_2c_1^3+(\frac{1}{5}\zeta(5)-\frac{1}{6}\zeta(2)\zeta(3))c_1^5$
6	$\frac{1}{144}[-72\zeta(6)p_3+(72\zeta(6)-36\zeta(4)\zeta(2))p_2p_1-(24\zeta(6)-18\zeta(4)\zeta(2)+3(\zeta(2))^3)p_1^3+8(\zeta(3))^2(c_1^3-3c_2c_1+3c_3)^2]$

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