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Control of flexible structures with spatially varying disturbance: spatial $\mathcal{H}_\infty$ approach

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Abstract— **This paper introduces a new optimal $\mathcal{H}_\infty$ control framework for controlling vibration of flexible structures with spatially varying disturbance. The spatially distributed nature of disturbance is taken into account in control design by introducing the concept of spatial $\mathcal{H}_\infty$ norm for systems with spatially distributed input. A control design approach is then introduced to allow a systematic control design for controlling structural vibration caused by spatially distributed disturbance. Simulation studies on a flexible beam demonstrate that the proposed spatial control can minimise vibration for spatially varying bending moment disturbance.**

Index Terms— **vibration; disturbance rejection; control design; flexible structures; distributed-parameter systems; $\mathcal{H}_\infty$ norm**

I. INTRODUCTION

FLEXIBLE structures are spatially distributed systems, which implies that the applied disturbance force/moment (the input) and the structural response (the output) are spatially distributed over the structure. An optimal control method, such as optimal $\mathcal{H}_\infty$ control [1], can be used to design a controller to minimise structural vibration. A particular approach is to consider a case where disturbances are only applied at known structural locations and vibration outputs are observed only at certain locations over the structure. In this case, vibrations at other points/locations are not explicitly accounted for in control design. To deal with such a problem, the concept of spatial $\mathcal{H}_\infty$ control was proposed [2], [3] by considering the spatial nature of the vibration output. Experiments performed in [4] demonstrate that such a spatial $\mathcal{H}_\infty$ controller can successfully minimise vibration of the entire structure.

However, the previous research only takes into account the system's spatial output, ignoring the spatial nature of the input. In practice, the vibratory disturbances may enter a structure at spatially varying locations across the structure. If a controller is designed with the assumption of point-wise disturbance inputs, it may not respond effectively when a disturbance input enters at a different location. Similarly, a disturbance input at a different location may excite vibration modes that are not controlled well by the controller. The question is how one can systematically design a vibration controller that can respond optimally to spatially varying disturbance. Thus, the work in this paper, motivated by the above question, attempts to introduce a spatial $\mathcal{H}_\infty$ control design concept that captures the spatial nature of the disturbance.

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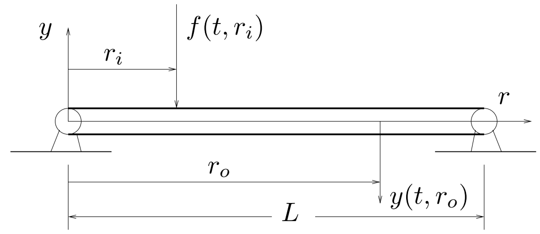

Fig. 1. A flexible beam structure

II. MODELS OF SPATIALLY DISTRIBUTED SYSTEMS

Consider a flexible beam with length L shown in Fig. 1, whose dynamics is governed by a partial differential equation (PDE) [5]:

$$EI\,\frac{\partial^4 y(t,r)}{\partial r^4} + \rho A_b \frac{\partial^2 y(t,r)}{\partial t^2} = f(t,r) + \frac{\partial m(t,r)}{\partial r} \qquad (1)$$

where the transverse deflection at point r is y. The Young's Modulus, moment of inertia, density, cross-sectional area, distributed force and bending moment are E, I, ρ, A_b, f and $\partial m/\partial r$ respectively.

Suppose a point (point-wise) force disturbance $f(t, r_i) = \delta(r - r_i)u(t)$ is applied to the beam at location $r = r_i$, where δ is the Dirac delta function and r belongs to a set $\mathcal{R}$ containing all possible locations along the structure. This point-wise force vibrates the beam and the vibration output y is monitored at a fixed location $r = r_o$. In the following, four different spatially distributed systems will be discussed:

A. point-wise input, point-wise output systems

Suppose that a point-wise disturbance input f is applied to the beam at $r = r_i$, and a point-wise output y is monitored at $r = r_o$. The system can be called as a *point-wise input, point-wise output* system.

B. point-wise input, spatial output systems

However, a flexible structure system has a spatially distributed output $y(t, r_o)$ where $r_o \in \mathcal{R}_o$ and $\mathcal{R}_o$ is a subset of $\mathcal{R}$ that contains all structural locations of interest for the output. For example, $\mathcal{R}_o = \mathcal{R} = [0, L]$. Such a system can be called a *point-wise input, spatial output* system that maps a point-wise input f at fixed $r = r_i$ to a spatial output

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$y(t,r_o), r_o \in \mathcal{R}_o$. For example, a system $G_o(s,r_o)$ maps n point-wise inputs to a spatial output $y(t,r_o) \in \mathbf{R} \times \mathcal{R}_o$.

C. spatial input, point-wise output systems

Furthermore, since the input is also spatially distributed, the location of disturbance input $f(t,r_i)$ can also be varied across the structure. Here, $r_i \in \mathcal{R}_i$ and $\mathcal{R}_i$ is a sub-set of $\mathcal{R}$ that contains all structural locations of interest for the input. For example, $\mathcal{R}_i = \mathcal{R} = [0,L]$. Such a system can be called a *spatial input, point-wise output* system that maps a spatial input $f(t,r_i), r_i \in \mathcal{R}_i$ to a point-wise output $y(t,r_i)$ at fixed $r = r_o$. However, although y is also a function of spatial input parameter r_i, the output y is still considered a point-wise output since the output is observed from a point-wise location $r = r_o$ over a structure.

In many cases, it can be assumed that the disturbance input function is a separable function of its spatial and temporal parts, i.e. $f(t,r_i) = \bar{f}(r_i)u(t)$ where $\bar{f}(r_i) \in \mathcal{R}_i$ and $u(t) \in \mathbf{R}$. For instance, a system might map a spatial input $f(t,r_i) = \bar{f}(r_i)u(t) \in \mathbf{R} \times \mathcal{R}_i$ to m point-wise outputs. Since $f(t,r_i) = \bar{f}(r_i)u(t)$, the spatial characteristics of the disturbance input can be absorbed into the transfer function by including a spatial parameter $h_j(r_i)$. An example of a *spatial input, point-wise output* system is

$$G_i(s,r_i) \quad = \quad \sum_{j=1}^{J} \frac{F_j h_j(r_i)}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \tag{2}$$

where J, ζ_j and ω_j are the number of vibration modes, proportional damping ratio and natural frequency of vibration mode j respectively. Here, $F_j \in \mathbf{R}^m$ and $G_i(s,r_i)$ maps the temporal part of a spatial input $u(t) \in \mathbf{R}$ to m point-wise outputs. For instance, in modal analysis, suppose that the disturbance is a point-wise force $f(t,r_i) = \delta(r - r_i)u(t)$ and $\mathcal{R} = [0,L]$. Then, $\bar{f}(r_i) = \delta(r - r_i)$ and it is easy to show that $h_j(r_i) = \int_{\mathcal{R}} \phi_j(r)\bar{f}(r_i)dr = \phi_j(r_i)$, where $\phi_j(r)$ is the eigenfunction of the j-th vibration mode.

D. spatial input, spatial output systems

When the spatially distributed input and output are both considered, the system can be called as a *spatial input, spatial output* system. In this case, the disturbance input and the output are spatially distributed, i.e. $f(t,r_i) = \bar{f}(r_i)u(t) \in \mathbf{R} \times \mathcal{R}_i$ and $y(t,r_i,r_o) \in \mathbf{R} \times \mathcal{R}_i \times \mathcal{R}_o$. An example of the system is

$$G_{io}(s,r_i,r_o) \quad = \quad \sum_{j=1}^{J} \frac{\hat{g}_j(r_o) h_j(r_i) H_j}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \tag{3}$$

where $\hat{g}_j(r_o)$ and $h_j(r_i)$ are the spatial parameters, and $H_j \in \mathbf{R}$. Thus, $G_{io}(s,r_i,r_o)$ maps the temporal part of a spatial input $u(t) \in \mathbf{R}$ to a spatial output $y(t,r_i,r_o) \in \mathbf{R} \times \mathcal{R}_i \times \mathcal{R}_o$.

III. SPATIAL NORMS FOR SYSTEMS WITH SPATIAL INPUT/OUTPUT

This section discusses the spatial norm concept for spatially distributed systems, which can be used to obtain performance measures for such systems.

Definition 1: Weighted spatial $\mathcal{L}_2$ norm of a signal: Consider a signal $z(t,r) \in \mathbf{R}^\ell \times \mathcal{R}$. Then, the weighted spatial $\mathcal{L}_2$ norm of z is defined in [6] as

$$\ll z \gg_{2,Q}^2 = \int_0^\infty \int_{\mathcal{R}} z(t,r)^T Q(r) z(t,r) dr dt \tag{4}$$

where $Q(r)$ is a spatial weighting function.

The spatial norm of z can be regarded as the energy of a spatially distributed signal z. Reference [2] describes the definition of a weighted spatial $\mathcal{H}_\infty$ norm with respect to spatial output. In this work, the concept of spatial $\mathcal{H}_\infty$ norm will be extended further to allow systems with spatially distributed input to be considered.

A. Spatial input, point-wise output systems

Definition 2: Weighted spatial $\mathcal{H}_\infty$ norm with respect to spatial input: Consider a stable spatially distributed LTI system $G_i(s,r_i)$, such as in (2). The weighted spatial $\mathcal{H}_\infty$ norm of this system with respect to spatial input is

$$\ll G_i \gg_{\infty,Q_i}^2 = \sup_{\omega \in \mathbf{R}} \lambda_{\max} \left(\int_{\mathcal{R}_i} G_i(j\omega,r_i)^* Q_i(r_i) G_i(j\omega,r_i) dr_i \right) \tag{5}$$

where $\lambda_{\max}\{\beta\}$ is the largest eigenvalue of a matrix β and $Q_i(r_i)$ is a spatial weighting function that can be used for emphasizing the spatial disturbance region that is more important.

This spatial norm considers every possible disturbance location over the structure in an averaged sense. It will be shown later how this spatial norm can be incorporated for designing an optimal controller for systems with spatially varying disturbance input. Next, consider a linear operator $\mathcal{G}_i$ that maps inputs u of system $G_i(s,r_i)$ in (2) to its outputs y. The following induced norm is defined, using the spatial norm description in Definition 1:

Definition 3: Weighted spatial induced norm of a system with spatial input: The weighted spatial induced norm of $\mathcal{G}_i$ is defined as

$$\ll \mathcal{G}_i \gg_{Q_i}^2 = \sup_{0 \neq u \in \mathcal{L}_{2[0,\infty)}} \frac{\ll y \gg_{2,Q_i}^2}{\|u\|_2^2}. \tag{6}$$

This induced norm has a similar form to the spatial induced norm discussed in [2]. However, the difference is that the above definition considers a spatially distributed input $f(t,r_i) = \bar{f}(r_i)u(t)$ with a fixed point-wise output, while the definition in [2] considers a fixed point-wise input with a spatial output instead.

For *spatial input, point-wise output* systems, there is an equivalent between the spatial $\mathcal{H}_\infty$ norm of the system and the induced norm of its input and output. This property resembles the case for *point-wise input, spatial output* systems discussed in [2]. The following theorem describes this spatial $\mathcal{H}_\infty$ norm property:

Theorem 1: Consider a stable linear system $G_i(s, r_i)$ and let $\mathcal{G}_i$ to be the linear operator that maps its input to its infinite-dimensional output. Then

$$\ll \mathcal{G}_i \gg_{Q_i} = \ll G_i \gg_{\infty, Q_i} .$$

Proof: The proof is omitted for brevity since it follows the same approach used in Theorem 3 in Section III-B for *spatial input, spatial output* systems. ■

Next, consider a spatially distributed system in the following state-space form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(r_i)u(t) \\ y(t, r_o) &= C(r_o)x(t) \end{aligned} \quad (7)$$

The next theorem allows one to find the spatial $\mathcal{H}_\infty$ norm of a system by finding an equivalent finite-dimensional system representation. A standard $\mathcal{H}_\infty$ norm computational method can then be used to find the spatial norm. In this theorem, a disturbance input of the form

$$f(t, r_i) = \left(\sum_{l=1}^{R} \bar{f}(r_i^l) \right) u(t) \quad (8)$$

is considered, where $u(t) \in \mathbf{R}$ is a single temporal input, $r_i^l \in \mathcal{R}_i^l \subset \mathcal{R}$ and $r_i \in \mathcal{R}_i = \mathcal{R}_i^1 \times ... \times \mathcal{R}_i^R$. For instance, two ($R = 2$) point-wise forces can be applied to a beam structure in the form of $f(t, r_i) = \{\delta(r - r_i^1) + \delta(r - r_i^2)\}u(t)$.

Theorem 2: Consider a state-space system for a *spatial input, point-wise output* system in (7), whose system can be described as $(A, B(r_i), C, 0)$. Then

$$\ll G_i \gg_{\infty, Q_i} = \|g_i\|_\infty = \sup_{\omega \in \mathbf{R}} g_i(j\omega)$$

where $g_i(j\omega)^2 = \sum_k \lambda_k(\tilde{G}_i(j\omega)^* \tilde{G}_i(j\omega))$ and $\tilde{G}_i(j\omega) = C(j\omega I - A)^{-1}\Omega$ is a finite-dimensional system with

$$\Omega\, \Omega^T = \int_{\mathcal{R}_i} B(r_i) Q_i(r_i) B(r_i)^T dr_i. \quad (9)$$

Proof: The proof follows a similar approach used in Theorem 4 in the following Section III-B. The proof is omitted for brevity. ■

B. Spatial input, spatial output systems

Definition 4: Weighted spatial $\mathcal{H}_\infty$ norm with respect to spatial input and spatial output: Consider a stable spatially distributed LTI system $G_{io}(s, r_i, r_o)$, such as in (3). The weighted spatial $\mathcal{H}_\infty$ norm of G_{io} with respect to spatial input and spatial output is

$$\ll G_{io} \gg_{\infty, Q_i, Q_o}^2 = \sup_{\omega \in \mathbf{R}} \lambda_{\max} \left(\int_{\mathcal{R}_o} \int_{\mathcal{R}_i} \alpha(j\omega, r_i, r_o)\, dr_i\, dr_o \right) \quad (10)$$

with

$$\alpha = G_{io}(j\omega, r_i, r_o)^* Q_i(r_i) Q_o(r_o) G_{io}(j\omega, r_i, r_o)$$

where $Q_i(r_i)$ and $Q_o(r_o)$ are spatial weighting functions for emphasizing the spatial input and output regions that are more important.

The spatial $\mathcal{H}_\infty$ norm with respect to spatial input can be conveniently used as a measure of maximum energy transfer from the disturbance input, applied at all possible spatial locations, to the (point-wise/spatial) output. Consider a linear operator $\mathcal{G}_{io}$ that maps inputs u of a stable system $G_{io}(s, r_i, r_o)$ in (3) to its outputs y. The following induced norm is introduced in this work:

Definition 5: Weighted spatial induced norm of a system with spatial input and spatial output: The weighted spatial induced norm of $\mathcal{G}_{io}$ is defined as

$$\ll \mathcal{G}_{io} \gg_{Q_i, Q_o}^2 = \sup_{0 \neq u \in \mathcal{L}_{2[0,\infty)}} \frac{\ll y \gg_{2, Q_i, Q_o}^2}{\|u\|_2^2} \quad (11)$$

where

$$\ll y \gg_{2, Q_i, Q_o}^2 = \int_0^\infty \int_{\mathcal{R}_o} \int_{\mathcal{R}_i} y^T Q_i(r_i) Q_o(r_o) y dr_i\, dr_o\, dt.$$

Let the Fourier transform of $y(t)$ and $u(t)$ to be $\hat{y}(j\omega)$ and $\hat{u}(j\omega)$. The following theorem relates the spatial $\mathcal{H}_\infty$ norm of a system with spatial input and spatial output, to the induced norm of system input and output:

Theorem 3: Consider a stable linear system $G_{io}(s, r_i, r_o)$ and let $\mathcal{G}_{io}$ to be the linear operator that maps its input to its infinite-dimensional output. Its weighted induced norm $\ll \mathcal{G}_{io} \gg_{Q_i, Q_o}$ satisfies

$$\ll \mathcal{G}_{io} \gg_{Q_i, Q_o} = \ll G_{io} \gg_{\infty, Q_i, Q_o} .$$

Proof: Consider the case where $Q_o(r_o) = Q_i(r_i) = 1$. Since $y = G_{io}u$ and using Parseval relation,

$$\begin{aligned} \ll y \gg_{2,i,o}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathcal{R}_o} \int_{\mathcal{R}_i} \hat{u}(j\omega)^* G_{io}(j\omega, r_i, r_o)^* \\ &\quad \times G_{io}(j\omega, r_i, r_o) \hat{u}(j\omega) dr_i\, dr_o d\omega \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda_{\max}\left(S(j\omega)\right) \hat{u}(j\omega)^* \hat{u}(j\omega) d\omega \end{aligned}$$

where

$$S(j\omega) = \int_{\mathcal{R}_o} \int_{\mathcal{R}_i} G_{io}(j\omega, r_i, r_o)^* G_{io}(j\omega, r_i, r_o) dr_i\, dr_o.$$

Using the description of spatial $\mathcal{H}_\infty$ norm in Definition 4, this implies

$$\ll y \gg_{2,i,o}^2 \leq \ll G_{io} \gg_{\infty,i,o}^2 \|u\|_2^2.$$

The above result shows that $\ll G_{io} \gg_{\infty,i,o}$ is an upper bound for $\ll \mathcal{G}_{io} \gg_{i,o}$ defined in Definition 5, i.e.

$$\ll \mathcal{G}_{io} \gg_{i,o} \leq \ll G \gg_{\infty,i,o} .$$

To show that $\ll G_{io} \gg_{\infty,i,o}$ is the least upper bound, a frequency ω_o is selected where the eigenvalue of $\int_{\mathcal{R}_o}\int_{\mathcal{R}_i} G_{io}(j\omega, r_i, r_o)^* G_{io}(j\omega, r_i, r_o) dr_i\, dr_o$ is the largest. An input $\hat{u}(j\omega)$ is chosen such that the input is non-zero in frequency ranges $[\omega_o - \epsilon, \omega_o + \epsilon]$ and $[-\omega_o - \epsilon, -\omega_o + \epsilon]$ and zero otherwise, where ϵ is a small positive number. Then it can be shown that

$$\ll y \gg_{2,i,o}^2 \quad \approx \quad \ll G_{io} \gg_{\infty,i,o}^2 \|u\|_2^2$$

This completes the proof. The proof for general $Q_o(r_o)$ and $Q_i(r_i)$ follows the same approach. ∎

The next theorem allows one to find the spatial $\mathcal{H}_\infty$ norm of a system by finding an equivalent finite-dimensional system representation.

Theorem 4: Consider a state-space system in (7), so that $G_{io}(s, r_i, r_o) = C(r_o)(sI - A)^{-1}B(r_i)$. Then

$$\ll G_{io} \gg_{\infty,Q_i,Q_o} = \|g_{io}\|_\infty = \sup_{\omega \in \mathbf{R}} g_{io}(j\omega)$$

where $g_{io}(j\omega)^2 = \sum_k \lambda_k(\tilde{G}_{io}(j\omega)^* \tilde{G}_{io}(j\omega))$ and $\tilde{G}_{io}(j\omega) = \Gamma(j\omega I - A)^{-1}\Omega$ is a finite-dimensional system and

$$\Omega\,\Omega^T = \int_{\mathcal{R}_i} B(r_i)Q_i(r_i)B(r_i)^T dr_i$$
$$\Gamma^T\Gamma = \int_{\mathcal{R}_o} C(r_o)^T Q_o(r_o)C(r_o)dr_o. \qquad (12)$$

Proof: Consider the case where $Q_o(r_o) = Q_i(r_i) = 1$ and let $N(j\omega) = (j\omega I - A)^{-1}$. Since $\hat{u}(j\omega)$ is a scalar,

$$\begin{aligned}
\ll y \gg_{2,i,o}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathcal{R}_o}\int_{\mathcal{R}_i} \hat{u}(-j\omega)G_{io}^* \\
&\quad \times G_{io}(j\omega, r_i, r_o)\hat{u}(j\omega)dr_i\, dr_o d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathcal{R}_i} \hat{u}(-j\omega)B(r_i)^T N(-j\omega)^T \\
&\quad \times \left(\int_{\mathcal{R}_o} C(r_o)^T C(r_o) dr_o\right) N(j\omega) \\
&\quad \times B(r_i)\hat{u}(j\omega)dr_i\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(-j\omega)\text{tr}\{\Gamma N(j\omega) \\
&\quad \times \left(\int_{\mathcal{R}_i} B(r_i)B(r_i)^T dr_i\right) N(-j\omega)^T \\
&\quad \times \Gamma^T\}\hat{u}(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(j\omega)^* g_{io}(j\omega)^2 \hat{u}(j\omega)d\omega.
\end{aligned}$$

Here, $\text{tr}\{\tilde{G}_{io}(j\omega)\tilde{G}_{io}(j\omega)^*\} = \sum_k \lambda_k(\tilde{G}_{io}^*\tilde{G}_{io}) = g_{io}^2$, a positive real valued number. This implies $\ll y \gg_{2,i,o} = \|\tilde{y}\|_2$ where $\tilde{y} = g_{io}u$. It can be shown that the standard $\mathcal{H}_\infty$ norm of $\tilde{G}_{io}$ has the following property [1]:

$$\|g_{io}\|_\infty^2 = \sup_{0 \neq u \in \mathcal{L}_{2[0,\infty)}} \frac{\|\tilde{y}\|_2^2}{\|u\|_2^2}.$$

This completes the proof of the theorem. The proof for general scalar spatial functions, $Q_o(r_o)$ and $Q_i(r_i)$, is straightforward using a similar approach. ∎

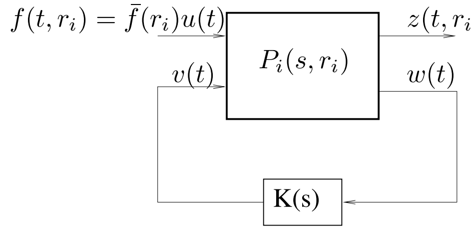

Fig. 2. Spatial $\mathcal{H}_\infty$ control problem: spatial input, point-wise output

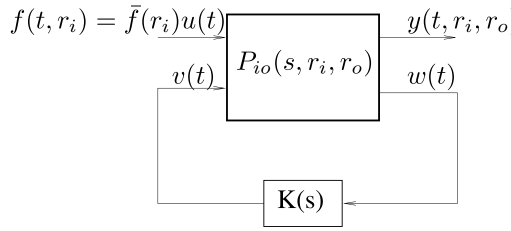

Fig. 3. Spatial $\mathcal{H}_\infty$ control problem: spatial input, spatial output

IV. SPATIAL $\mathcal{H}_\infty$ CONTROL DESIGN

A. Control of spatial input, point-wise output systems

Consider a *spatial input, point-wise output* system $G_i(s, r_i)$ in (2), whose state-space realization is $(A, B(r_i), C, 0)$. Fig. 2 shows the control problem with a generalized plant $P_i(s, r_i)$, where z, w and v are the point-wise performance output, measured sensor output and control input respectively. Then the spatial $\mathcal{H}_\infty$ control problem for a *spatial input, point-wise output* system is to design a stabilizing controller $K(s)$ with a state-space realization of (A_k, B_k, C_k, D_k) such that the closed-loop system satisfies:

$$\inf_{K \in V} \sup_{u \in \mathcal{L}_2[0,\infty)} J_{\infty,i} < \gamma^2 \qquad (13)$$

where V is the set of all stabilizing controllers and

$$J_{\infty,i} = \frac{\int_0^\infty \int_{\mathcal{R}_i} y(t, r_i)^T Q(r_i) y(t, r_i) dr_i dt}{\int_0^\infty u(t)^T u(t) dt}. \qquad (14)$$

The numerator in (14) is the weighted spatial $\mathcal{L}_2$ norm of signal $y(t, r_i)$ as in Definition 1, i.e. $\ll y \gg_{2,Q_i}$. Thus, $J_{\infty,i}$ can be considered as the ratio between the input energy and the point-wise output energy due to spatially varying input. From Theorem 2, $g_i(j\omega)$ is the sum of all eigenvalues of $\tilde{G}_i^*\tilde{G}_i$. In this work, an approximation of g_i is used by considering the largest eigenvalue of $\tilde{G}_i^*\tilde{G}_i$ instead. In this case, the simplified control problem is to find an optimal controller that minimises the $\mathcal{H}_\infty$ norm of the finite-dimensional system, $\tilde{G}_i \equiv (A, \Omega, C, 0)$ with Ω obtained from (9). A standard computational tool for an optimal $\mathcal{H}_\infty$ control problem [1] can then be used to solve for the optimal spatial $\mathcal{H}_\infty$ controller.

B. Control of spatial input, spatial output systems

Consider a *spatial input, spatial output* system $G_{io}(s, r_i, r_o)$ in (3), with a state-space realization of $(A, B(r_i), C(r_o), 0)$. The control problem with a generalized plant $P_{io}(s, r_i, r_o)$ is depicted in Fig. 3. The spatial $\mathcal{H}_\infty$ control problem for a *spatial input, spatial output* system is to design a stabilizing controller $K(s) \equiv (A_k, B_k, C_k, D_k)$ such that the closed-loop system satisfies:

$$\inf_{K \in V} \sup_{u \in \mathcal{L}_2[0,\infty)} J_{\infty,i,o} < \gamma^2 \tag{15}$$

where V is the set of all stabilizing controllers and

$$J_{\infty,i,o} = \frac{\ll y \gg^2_{2,Q_i,Q_o}}{\|u\|_2^2} \tag{16}$$

where

$$\ll y \gg^2_{2,Q_i,Q_o} = \int_0^\infty \int_{\mathcal{R}_o} \int_{\mathcal{R}_i} y(t, r_i, r_o)^T Q_i(r_i) \times Q_o(r_o) y(t, r_i, r_o) dr_i \, dr_o \, dt. \tag{17}$$

Here, $J_{\infty,i,o}$ can be considered as the ratio between the input energy and the spatial output energy due to spatially varying input. Similar to the *spatial input, point-wise output* case, the simplified control problem is solved using a finite-dimensional system $\tilde{G}_{io} \equiv (A, \Omega, \Gamma, 0)$, where Ω, Γ are obtained from (12). It will be shown in the simulations that the control design simplification also performs well.

V. SIMULATION STUDIES

A flexible beam with simply-supported/pinned boundary conditions is considered for simulation studies (see Fig. 1). The structure is a $L = 500$ mm long uniform aluminum beam of a rectangular cross section (35 mm $\times$ 3 mm). Consider a point-wise displacement sensor that is located at $r = 225$mm. A piezoelectric ceramic element PIC151 with dimensions of (50 mm $\times$ 25 mm $\times$ 0.25 mm) is attached $r = 200$ mm away from one end of the beam (refer to [4] for the physical properties of PIC151). A spatially varying point-wise bending moment $\partial m(t, r_i)/\partial r = \delta'(r - r_i)u(t)$, instead of a point-wise force, is considered. A state-space model of the modal model of the beam (using its first 7 vibration modes with damping ratio $\zeta_j = 0.7\%$) is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1(r_i)u(t) + B_2 v(t) \\ y(t, r_o) &= C_1(r_o)x(t) \\ w(t) &= C_2 x(t) \end{aligned} \tag{18}$$

where x, w, u and v are the state vector, measured sensor output, disturbance input and control input respectively.

The *point-wise input, point-wise output* control problem is solved using a standard $\mathcal{H}_\infty$ control method (refer to [1], for example). The controller minimises the $\mathcal{H}_\infty$ norm of the closed-loop system that maps its point-wise bending moment input $\partial m/\partial r$ at fixed $r = r_i$ to its point-wise output y at $r = r_o$. Note that the bending moment input is in the form of $\partial m(t, r_i)/\partial r = \delta'(r - r_i)u(t)$.

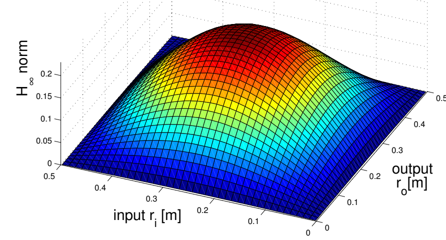

Fig. 4. Open loop $\mathcal{H}_\infty$ norm plot

Next, the *point-wise input, spatial output* control problem is solved using the approach described in [4]. The controller is designed to minimise the spatial $\mathcal{H}_\infty$ norm of the closed-loop system that maps its point-wise bending moment input $\partial m/\partial r$ at fixed $r = r_i$ to its spatially distributed output y with $r_o \in [0, L]$. It is assumed that all structural outputs across the beam are weighted equally, i.e. $Q_o = 1$.

For the *spatial input, point-wise output* case, the bending moment input $\partial m/\partial r$ is spatially varied with $r_i \in [0, L]$ and the performance output y is fixed at $r = r_o$. All possible disturbance inputs across the beam are weighted equally, $Q_i = 1$. The equivalent finite-dimensional system with a new performance output $\hat{y}$ with input $\tilde{u}$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \Omega\tilde{u}(t) + B_2 v(t) \\ \hat{y}(t) &= \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \epsilon_1 \end{bmatrix} v(t) \\ w(t) &= C_2 x(t) + \epsilon_2 \tilde{u}(t) \end{aligned} \tag{19}$$

where ϵ_1 and ϵ_2 are the control design parameters.

Finally, for the *spatial input, spatial output* system, the bending moment input $\partial m/\partial r$ and performance output y are both spatially varied with $r_i, r_o \in [0, L]$. All disturbance inputs and performance outputs are weighted equally across the beam, i.e. $Q_i = Q_o = 1$. The equivalent finite-dimensional system with a performance output $\tilde{y}$ with input $\tilde{u}$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \Omega\tilde{u}(t) + B_2 v(t) \\ \tilde{y}(t) &= \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \epsilon_1 \end{bmatrix} v(t) \\ w(t) &= C_2 x(t) + \epsilon_2 \tilde{u}(t). \end{aligned} \tag{20}$$

MATLAB μ-Analysis and Synthesis Toolbox is used for control design and the controllers have gain margin at least 6 dB and phase margin larger than 55^o, to ensure sufficient stability robustness. The $\mathcal{H}_\infty$ norm plot for the uncontrolled system is shown in Fig. 4. In this case, the $\mathcal{H}_\infty$ norm is plotted for all possible combinations of input and output locations (r_i, r_o). It is shown that the largest norm occurs at $(r_i, r_o) = (250\text{mm}, 250\text{mm})$, mainly contributed by the first mode. Fig. 5 shows the performance of a *point-wise*

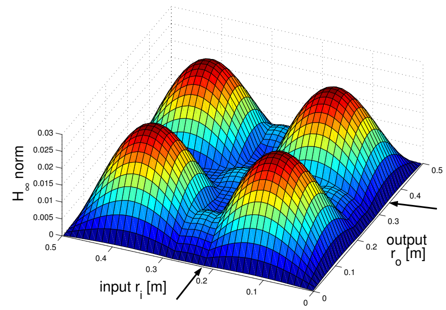

Fig. 5. $\mathcal{H}_\infty$ norm: point-wise input ($r_i = 220$ mm), point-wise output ($r_o = 350$ mm) control

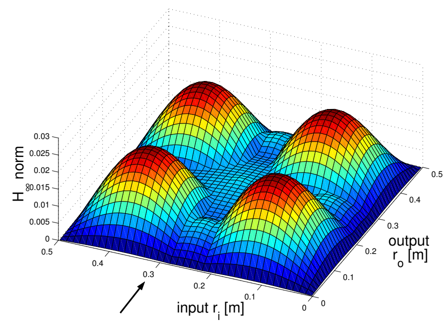

Fig. 6. $\mathcal{H}_\infty$ norm: point-wise input ($r_i = 300$ mm), spatial output control

input, point-wise output controller in minimising vibration at $(r_i, r_o) = (220\text{mm}, 350\text{mm})$, where the $\mathcal{H}_\infty$ norm at that location is reduced. When the disturbance enters from a different location, say $r_i = 350$ mm, the vibration across the beam is not reduced effectively.

Next, Fig. 6 describes the performance of a *point-wise input, spatial output* controller with a bending moment input $\partial m/\partial r$ at fixed $r_i = 300$ mm. The cost function used is reflected by the area under the plot at constant $r_i = 300$ mm. Fig. 7 shows the performance of a *spatial input, point-wise output* controller, designed with an output y at a fixed location $r_o = 190$ mm. At location r_o, the vibration level is minimised when the bending moment input is spatially varied across the beam $r_i \in [0, L]$, as expected.

Finally, the performance of a *spatial input, spatial output* controller is described in Fig. 8. The controller takes into account the disturbance that may enter at different locations across the beam, where the cost function is reflected by the volume under the $\mathcal{H}_\infty$ norm plot. The controller is able to minimise vibration at various output locations $r_o \in [0, L]$, when the bending moment is applied at various locations $r_i \in [0, L]$.

In general, other control design methods can also be used for controlling structural vibration. It can be argued that as long as the controller is designed to increase the closed-loop damping of selected vibration modes, the controller

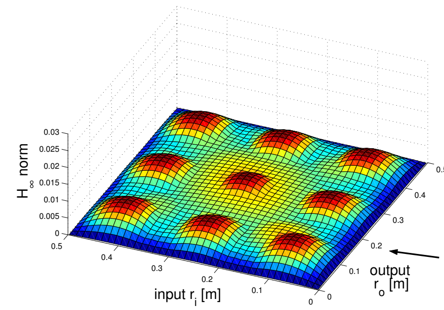

Fig. 7. $\mathcal{H}_\infty$ norm: spatial input, point-wise output ($r_o = 190$ mm) control

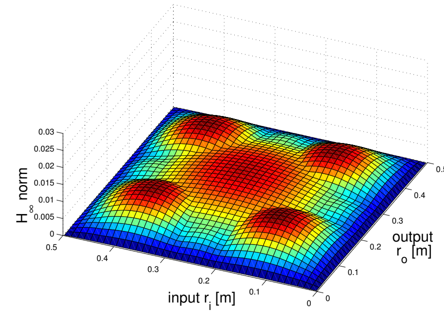

Fig. 8. $\mathcal{H}_\infty$ norm: spatial input, spatial output control

can reduce the structural vibration caused by disturbance applied at various spatial locations. However, the proposed spatial $\mathcal{H}_\infty$ control method has the advantage in taking into account explicitly the spatial nature of the disturbance input.

VI. Conclusion

An optimal $\mathcal{H}_\infty$ control framework for systems with spatially distributed input has been proposed. This allows one to design a vibration controller that performs well against spatially varying disturbance.

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