Statistical errors in the estimation of time-averaged acoustic energy density using the two-microphone method

Justin Ghan,[a] Ben Cazzolato, and Scott Snyder
Department of Mechanical Engineering, University of Adelaide, SA 5005, Australia

(Received 31 August 2002; revised 31 October 2003; accepted 31 October 2003)

Time-averaged acoustic energy density can be estimated using the auto- and cross-spectral densities between two closely spaced microphones. In this paper, an analysis of the random errors that arise using two microphone measurements is undertaken. An expression for the normalized random error of the time-averaged acoustic energy density spectral density estimate is derived. This expression is verified numerically. The lower and upper bounds of the normalized random error are derived. It is shown that the normalized random error of the estimate is not a strong function of the sound field properties, and is chiefly dependent on the number of records averaged. © *2004 Acoustical Society of America.* [DOI: 10.1121/1.1639334]

PACS numbers: 43.58.-e, 43.60.Qv [JCB]

Pages: 1179–1184

I. INTRODUCTION

Acoustic energy density is defined as the sum of the acoustic potential energy density and the acoustic kinetic energy density at a point. It has been shown[1] that energy density provides a significantly better estimate of the total acoustic energy within an enclosure than does the acoustic potential energy (estimated by microphones). Subsequently, several authors have found energy density to be an effective sensor for active noise control applications, as it measures the total energy at a point, and generally outperforms microphones.[2–6]

The bias errors arising from the inherent and instrumentation errors in energy density sensing have been thoroughly investigated.[7–9] Additionally, the estimation of potential energy density and kinetic energy density in the frequency domain and the associated statistical errors have been discussed by Elko.[10] In this paper, the statistical errors associated with the estimation of total acoustic energy density are analyzed.

An expression for the estimation of time-averaged total acoustic energy density in the frequency domain has been previously derived.[11] In this paper, an expression for the normalized random error of this estimate is derived. The method here is similar to that used to derive the normalized random error in the estimation of sound intensity.[12]

A Simulink model will be used to simulate the estimation of the time-averaged acoustic energy density. The resulting measurements will be analyzed to verify the derived expression for the normalized random error.

It will be shown that the normalized random error does not depend heavily upon the nature of the sound field being measured. In fact, it is bounded between two functions dependent only on n_d, the number of records averaged to produce the "smooth" spectral density estimates used in the expression for the time-averaged acoustic energy density spectral density estimate. As a result, the averaging time required to attain a desired level of accuracy can be chosen without knowledge of any of the specifics of the experiment.

II. ANALYTICAL DERIVATION OF THE RANDOM ERROR

The instantaneous acoustic energy density, $E_D(t)$, at a point is defined as the sum of the acoustic potential energy density, $U(t)$, and the acoustic kinetic energy density, $T(t)$, at that point, given by[13]

$$U(t)=\frac{p^2(t)}{2\rho c^2}, \tag{1}$$

$$T(t)=\frac{\rho v^2(t)}{2}, \tag{2}$$

$$E_D(t)=U(t)+T(t)=\frac{p^2(t)}{2\rho c^2}+\frac{\rho v^2(t)}{2}, \tag{3}$$

where $p(t)$ and $v(t)$ are the instantaneous pressure and particle velocity magnitude, respectively, at that point, c is the speed of sound, and ρ is the mean density of the fluid.

It has been shown[11] that, using the two-microphone measurement method with a microphone separation distance of $2h$, the single-sided time-averaged acoustic energy density spectral density is approximated by

$$\overline{E_D}(\omega)\approx\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)(G_{11}(\omega)+G_{22}(\omega)) + \left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)(2\,\mathrm{Re}[G_{12}(\omega)]), \tag{4}$$

where $G_{11}(\omega)$ and $G_{22}(\omega)$ are the single-sided auto-spectral densities, and $G_{12}(\omega)$ is the single-sided cross-spectral density, of the two microphone pressure signals. If the cross-

[a]Author to whom all correspondence should be addressed; electronic mail: justin.ghan@adelaide.edu.au

spectral density is separated into real and imaginary components, $G_{12}(\omega)=C_{12}(\omega)+jQ_{12}(\omega)$, Eq. (4) can be rewritten as

$$\overline{E_D}(\omega)\approx\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)(G_{11}(\omega)+G_{22}(\omega))+\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)(2C_{12}(\omega)). \quad (5)$$

An estimate of the time-averaged acoustic energy density spectral density is given by

$$\hat{\overline{E_D}}(\omega)\approx\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)(\hat{G}_{11}(\omega)+\hat{G}_{22}(\omega))+\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)(2\hat{C}_{12}(\omega)), \quad (6)$$

where $\hat{G}_{11}(\omega)$, $\hat{G}_{22}(\omega)$, and $\hat{C}_{12}(\omega)$ are "smooth" estimates of the auto-spectral densities and the real part of the cross-spectral density, obtained by averaging n_d statistically independent "raw" estimates.

Estimates of time-averaged potential energy density and time-averaged kinetic energy density are given by

$$\hat{\overline{U}}(\omega)\approx\frac{1}{8\rho c^2}(\hat{G}_{11}(\omega)+\hat{G}_{22}(\omega)+2\hat{C}_{12}(\omega)), \quad (7)$$

$$\hat{\overline{T}}(\omega)\approx\frac{1}{8\rho\omega^2h^2}(\hat{G}_{11}(\omega)+\hat{G}_{22}(\omega)-2\hat{C}_{12}(\omega)). \quad (8)$$

Elko[10] derived the normalized random error of these estimates. While the derivations were based upon incorrect expressions for the estimates, the method and results remain valid:

$$\epsilon(\hat{\overline{U}}(\omega))\approx\frac{1}{\sqrt{n_d}}, \quad (9)$$

$$\epsilon(\hat{\overline{T}}(\omega))\approx\frac{1}{\sqrt{n_d}}. \quad (10)$$

It can be shown that the variance of the estimate in Eq. (6) is, to a first-order approximation,

$$\begin{aligned}\mathrm{var}\{\hat{\overline{E_D}}(\omega)\}\approx&\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{11}}\right)^2\mathrm{var}\{\hat{G}_{11}(\omega)\}+\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{22}}\right)^2\mathrm{var}\{\hat{G}_{22}(\omega)\}+\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{12}}\right)^2\mathrm{var}\{\hat{C}_{12}(\omega)\}\\&+2\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{11}}\right)\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{22}}\right)\mathrm{cov}\{\hat{G}_{11}(\omega),\hat{G}_{22}(\omega)\}+2\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{11}}\right)\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{12}}\right)\mathrm{cov}\{\hat{G}_{11}(\omega),\hat{C}_{12}(\omega)\}\\&+2\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{22}}\right)\left(\frac{\partial\hat{\overline{E_D}}}{\partial\hat{G}_{12}}\right)\mathrm{cov}\{\hat{G}_{22}(\omega),\hat{C}_{12}(\omega)\},\end{aligned} \quad (11)$$

where the derivatives are evaluated at the true values $\hat{G}_{11}(\omega)=G_{11}(\omega)$, $\hat{G}_{22}(\omega)=G_{22}(\omega)$, and $\hat{C}_{12}(\omega)=C_{12}(\omega)$.

Assuming stationary Gaussian random signals, the relevant variances and covariances are[14]

$$\mathrm{var}\{\hat{G}_{11}(\omega)\}=\frac{G_{11}^2(\omega)}{n_d}, \quad (12)$$

$$\mathrm{var}\{\hat{G}_{22}(\omega)\}=\frac{G_{22}^2(\omega)}{n_d}, \quad (13)$$

$$\mathrm{var}\{\hat{C}_{12}(\omega)\}=\frac{G_{11}(\omega)G_{22}(\omega)+C_{12}^2(\omega)-Q_{12}^2(\omega)}{2n_d}, \quad (14)$$

$$\mathrm{cov}\{\hat{G}_{11}(\omega),\hat{G}_{22}(\omega)\}=\frac{|G_{12}(\omega)|^2}{n_d}, \quad (15)$$

$$\mathrm{cov}\{\hat{G}_{11}(\omega),\hat{C}_{12}(\omega)\}=\frac{G_{11}(\omega)C_{12}(\omega)}{n_d}, \quad (16)$$

$$\mathrm{cov}\{\hat{G}_{22}(\omega),\hat{C}_{12}(\omega)\}=\frac{G_{22}(\omega)C_{12}(\omega)}{n_d}. \quad (17)$$

Therefore,

$$\text{var}\{\hat{\overline{E_D}}(\omega)\} \approx \left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)^2\frac{G_{11}^2(\omega)}{n_d}+\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)^2\frac{G_{22}^2(\omega)}{n_d}+4\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)^2$$
$$\times\frac{G_{11}(\omega)G_{22}(\omega)+C_{12}^2(\omega)-Q_{12}^2(\omega)}{2n_d}+2\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)^2\frac{|G_{12}(\omega)|^2}{n_d}+4\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)$$
$$\times\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)\frac{G_{11}(\omega)C_{12}(\omega)}{n_d}+4\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)\frac{G_{22}(\omega)C_{12}(\omega)}{n_d}. \quad (18)$$

Noting that

$$[\overline{E_D}(\omega)]^2 \approx \left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)^2(G_{11}^2(\omega)+G_{22}^2(\omega)+2G_{11}(\omega)G_{22}(\omega))$$
$$+\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)^2(4C_{12}^2(\omega))+2\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)(G_{11}(\omega)+G_{22}(\omega))(2C_{12}(\omega)), \quad (19)$$

it can be seen that

$$\text{var}\{\hat{\overline{E_D}}(\omega)\} \approx \frac{1}{n_d}\left[[\overline{E_D}(\omega)]^2+2\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)^2(G_{11}(\omega)G_{22}(\omega)-C_{12}^2(\omega)-Q_{12}^2(\omega))\right.$$
$$\left.+2\left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)^2(|G_{12}(\omega)|^2-G_{11}(\omega)G_{22}(\omega))\right]. \quad (20)$$

Since $C_{12}^2(\omega)+Q_{12}^2(\omega)=|G_{12}(\omega)|^2=\gamma_{12}^2(\omega)G_{11}(\omega)G_{22}(\omega)$, where $\gamma_{12}^2(\omega)$ is the coherence between the two microphone pressure signals,

$$\text{var}\{\hat{\overline{E_D}}(\omega)\} \approx \frac{1}{n_d}\left[[\overline{E_D}(\omega)]^2-8\left(\frac{1}{8\rho c^2}\right)\left(\frac{1}{8\rho\omega^2h^2}\right)(1-\gamma_{12}^2(\omega))G_{11}(\omega)G_{22}(\omega)\right]. \quad (21)$$

Therefore, the normalized random error of the estimate $\hat{\overline{E_D}}(\omega)$, defined as

$$\epsilon(\hat{\overline{E_D}}(\omega))=\frac{\sqrt{\text{var}\{\hat{\overline{E_D}}(\omega)\}}}{\overline{E_D}(\omega)}, \quad (22)$$

is

$$\epsilon(\hat{\overline{E_D}}(\omega)) \approx \frac{1}{\sqrt{n_d}}\left[1-\frac{8\left(\frac{1}{8\rho c^2}\right)\left(\frac{1}{8\rho\omega^2h^2}\right)(1-\gamma_{12}^2(\omega))G_{11}(\omega)G_{22}(\omega)}{[\overline{E_D}(\omega)]^2}\right]^{1/2}. \quad (23)$$

Since $\text{Re}[G_{12}(\omega)]=|G_{12}(\omega)|\cos\phi_{12}(\omega)=\gamma_{12}(\omega)\sqrt{G_{11}(\omega)G_{22}(\omega)}\cos\phi_{12}(\omega)$, where $\phi_{12}(\omega)$ is the phase angle between the two microphone pressure signals and $\gamma_{12}(\omega)$ is the positive square root of $\gamma_{12}^2(\omega)$, Eq. (4) can be rewritten as

$$\overline{E_D}(\omega) \approx \left(\frac{1}{8\rho c^2}+\frac{1}{8\rho\omega^2h^2}\right)(G_{11}(\omega)+G_{22}(\omega))+\left(\frac{1}{8\rho c^2}-\frac{1}{8\rho\omega^2h^2}\right)(2\gamma_{12}(\omega)\sqrt{G_{11}(\omega)G_{22}(\omega)}\cos\phi_{12}(\omega)). \quad (24)$$

Substituting Eq. (24) into Eq. (23) and multiplying the numerator and denominator of the fraction by $(8\rho\omega^2h^2)^2/[4G_{11}(\omega)G_{22}(\omega)]$ yields

$$\epsilon(\hat{\overline{E_D}}(\omega)) \approx \frac{1}{\sqrt{n_d}}\left[1-\frac{2(kh)^2(1-\gamma_{12}^2(\omega))}{[((kh)^2+1)X+((kh)^2-1)\gamma_{12}(\omega)\cos\phi_{12}(\omega)]^2}\right]^{1/2}, \quad (25)$$

where the wave number is given by $k=\omega/c$, and

$$X=\frac{1}{2}\left(\sqrt{\frac{G_{11}(\omega)}{G_{22}(\omega)}}+\sqrt{\frac{G_{22}(\omega)}{G_{11}(\omega)}}\right). \quad (26)$$

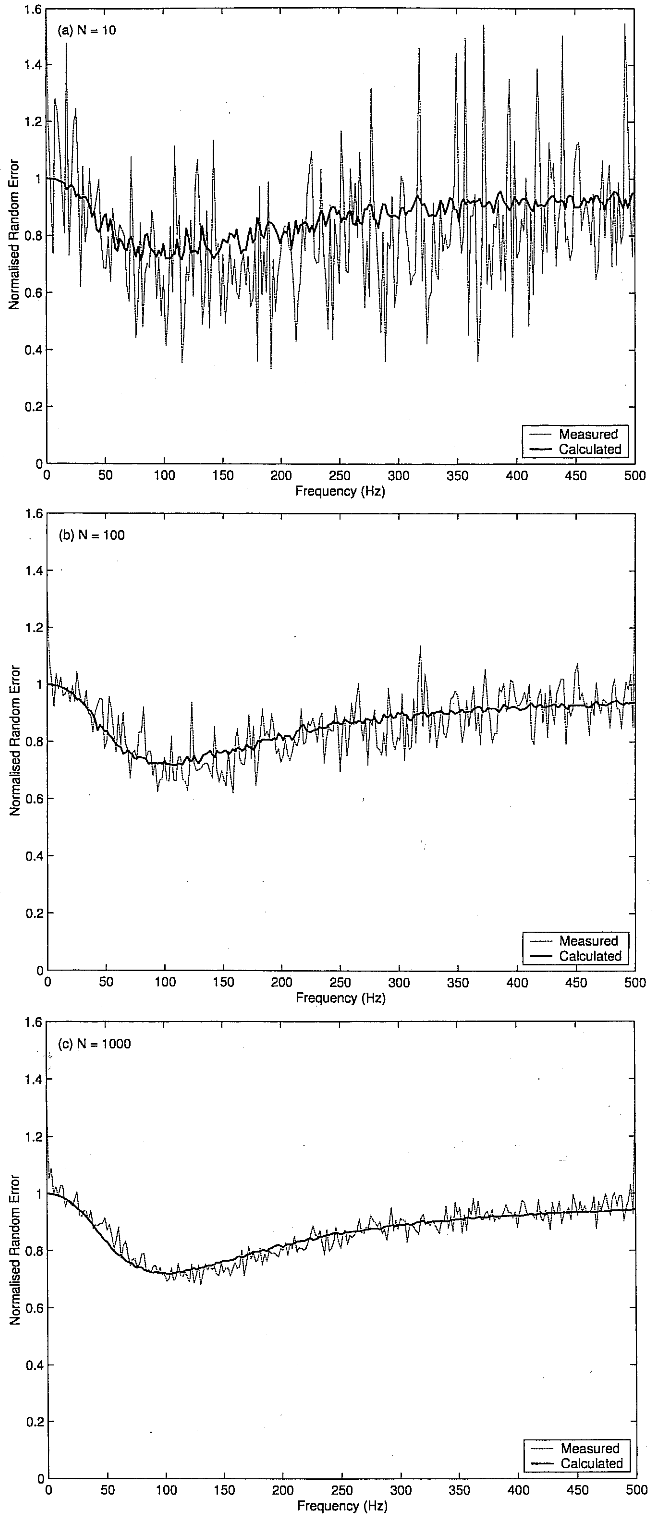

FIG. 4. Estimates of the normalized random error of the acoustic energy density spectral density estimates, obtained from the standard deviation of measurements, compared with the calculated normalized random error as predicted by the derived theoretical expression. Results are shown for (a) 10, (b) 100, and (c) 1000 acoustic energy density spectral density estimates.

The lower bound in Eq. (38) shows that the averaging time resulting from this approach will exceed the minimum necessary averaging time at most by a factor of $\sqrt{2}$.

VI. CONCLUSIONS

An expression for the normalized random error of the time-averaged acoustic energy density spectral density estimates has been derived. A simulation in Simulink was used to validate the expression numerically. The estimates of the normalized random error resulting from the simulated measurements agree with the error predicted by the expression.

Because the normalized random error is only heavily dependent on n_d, an experiment can be designed to achieve a desired normalized random error without measuring the local properties of the sound field first.

[1] R. K. Cook and P. A. Schade, "New method for the measurement of the total energy density of sound waves," in Proceedings of Inter-Noise 74, 1974, pp. 101–106.

[2] P. J. Nashif and S. D. Sommerfeldt, "An active control strategy for minimising the energy density in enclosures," in Proceedings of Inter Noise 92, 1992, pp. 357–361.

[3] W. Shen and J. Q. Sun, "A study of shell interior noise control," Proc. SPIE **3041**, 812–818 (1997).

[4] Y. C. Park and S. D. Sommerfeldt, "Global attenuation of broadband noise fields using energy density control," J. Acoust. Soc. Am. **101**, 350–359 (1997).

[5] S. D. Sommerfeldt and P. J. Nashif, "A comparision of control strategies for minimising the sound field in enclosures," in Proceedings of Noise-Con 91, 1991, pp. 299–306.

[6] S. D. Sommerfeldt and J. Parkins, "Active control of energy density in three dimensional enclosures," J. Acoust. Soc. Am. **95**, 2989 (1994).

[7] B. S. Cazzolato and C. H. Hansen, "Errors arising from three-dimensional energy density sensing in one-dimensional sound fields," J. Sound Vib. **236**, 375–400 (2000).

[8] J. W. Parkins, S. D. Sommerfeldt, and J. Tichy, "Error analysis of a practical energy density sensor," J. Acoust. Soc. Am. **108**, 211–222 (2000).

[9] B. S. Cazzolato and C. H. Hansen, "Errors in the measurement of acoustic energy density in one-dimensional sound fields," J. Sound Vib. **236**, 800–831 (2000).

[10] G. W. Elko, "Frequency domain estimation of the complex acoustic intensity and acoustic energy density," PhD thesis, The Pennsylvania State University, 1984.

[11] J. Ghan, B. S. Cazzolato, and S. D. Snyder, "Expression for the estimation of time-averaged acoustic energy density using the two-microphone method," J. Acoust. Soc. Am. **113**, 2404–2407 (2003).

[12] A. F. Seybert, "Statistical errors in acoustic intensity measurements," J. Sound Vib. **75**, 519–526 (1981).

[13] F. Fahy, *Sound Intensity*, 2nd ed. (E&FN Spon, London, 1995).

[14] J. S. Bendat and A. G. Piersol, *Random Data—Analysis and Measurement Procedures*, 2nd ed. (Wiley, New York, 1986).

If the auto-spectral densities of the two microphone pressure signals are approximately equal, $G_{11}(\omega)\approx G_{22}(\omega)$, then $X\approx 1$, so

$$\epsilon(\hat{\overline{E_D}}(\omega))\approx\frac{1}{\sqrt{n_d}}\left[1-\frac{2(kh)^2(1-\gamma_{12}^2(\omega))}{[((kh)^2+1)+((kh)^2-1)\gamma_{12}(\omega)\cos\phi_{12}(\omega)]^2}\right]^{1/2}. \quad (27)$$

The normalized random error for this case is plotted in Fig. 1, assuming that the phase between the two signals is solely due to the propagation delay so that $\phi_{12}=2kh$. It is important to note that while the bias errors that occur when using the two-microphone method are affected by spatial aliasing, the random errors are not.

III. SIMULINK SIMULATION

A Simulink model (see Fig. 2) was created to simulate the two-microphone method for estimating acoustic energy density.

The two microphone pressure signals were calculated by simulating two points in a free space pressure field generated by a white noise point source. The transfer function from a monopole point source to a sensor at distance r is

$$G(s)=\frac{1}{r}e^{-(r/c)s}, \quad (28)$$

where c is the speed of sound. These transfer functions were implemented using a second-order Padé approximation in series with a gain. In order to vary the coherence between the two pressure readings, an amount of incoherent noise was introduced to one of the signals.

To compute a single estimate, a frequency spectrum of each of the microphone pressure readings was obtained by performing a 512 point fast Fourier transform (FFT) (see Fig. 3). The auto- and the cross-spectral densities were then calculated, and Eq. (4) was applied to produce an acoustic energy density spectral density estimate with $n_d=1$ (see Fig. 3). A number, N, of acoustic energy density spectral density estimates were calculated, and the standard deviation of each spectral line was divided by the mean value to produce an estimate of the normalized random error.

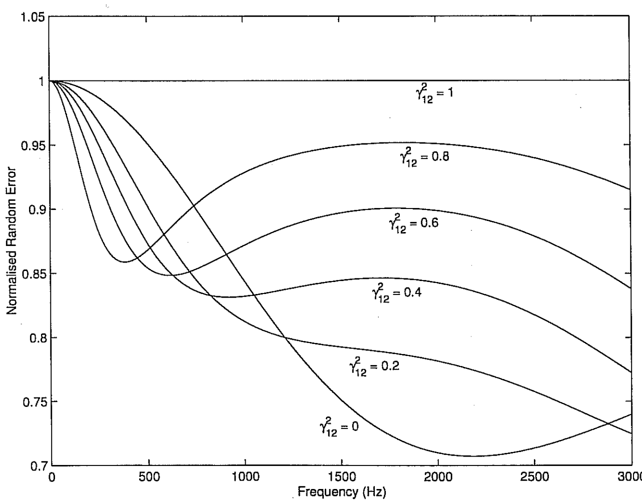

FIG. 1. Normalized random error of the acoustic energy density estimate ($n_d=1$), for $2h=50$ mm, $c=343$ m s^{-1}. It is assumed that $G_{11}(\omega)\approx G_{22}(\omega)$ and $\phi_{12}=2kh$. When the microphone separation distance exceeds λ/2, half the wavelength, the bias errors become very large, so the random error is unimportant. In this case, this occurs at frequencies above 3430 Hz.

The microphone pressure readings, p_1 and p_2, were also sampled in the time domain. Each time signal was divided into overlapping 512 point sections, each of which was windowed. The FFTs of these windowed sections were averaged to obtain spectra estimates, which were then used to calculate the auto- and cross-spectral densities, coherence and phase angle. Equation (25) was applied to calculate the normalized random error.

Comparisons of the results between the two methods for estimating the normalized random error for various values of N are shown in Fig. 4. As the number, N, of acoustic energy density estimates used in the average increases, the estimate of the normalized random error converges to that predicted by the derived equation (25). This would seem to confirm the validity of the expression for the normalized random error of the acoustic energy density spectral density estimate.

IV. BOUNDS OF THE RANDOM ERROR

Since the fraction in Eq. (25) is always non-negative, the normalized random error is bounded above by

$$\epsilon(\hat{\overline{E_D}}(\omega))\leqslant\frac{1}{\sqrt{n_d}}, \quad (29)$$

and equality occurs when $\gamma_{12}^2(\omega)=1$.

The denominator of the fraction in Eq. (25) can be rearranged as

$$\begin{aligned}&[((kh)^2+1)X+((kh)^2-1)\gamma_{12}(\omega)\cos\phi_{12}(\omega)]^2\\&\quad=[(X+\gamma_{12}(\omega)\cos\phi_{12}(\omega))(kh)^2\\&\qquad+(X-\gamma_{12}(\omega)\cos\phi_{12}(\omega))]^2.\end{aligned} \quad (30)$$

Since the arithmetic mean of

$$(X+\gamma_{12}(\omega)\cos\phi_{12}(\omega))(kh)^2$$

and

$$(X-\gamma_{12}(\omega)\cos\phi_{12}(\omega))$$

is greater than or equal to their geometric mean,

$$\begin{aligned}&[(X+\gamma_{12}(\omega)\cos\phi_{12}(\omega))(kh)^2+(X-\gamma_{12}(\omega)\cos\phi_{12}(\omega))]^2\\&\quad\geqslant 4(X^2-\gamma_{12}^2(\omega)\cos^2\phi_{12}(\omega))(kh)^2.\end{aligned} \quad (31)$$

Since $\cos^2\phi_{12}(\omega)\leqslant 1$,

$$4(X^2-\gamma_{12}^2(\omega)\cos^2\phi_{12}(\omega))(kh)^2\geqslant 4(X^2-\gamma_{12}^2(\omega))(kh)^2. \quad (32)$$

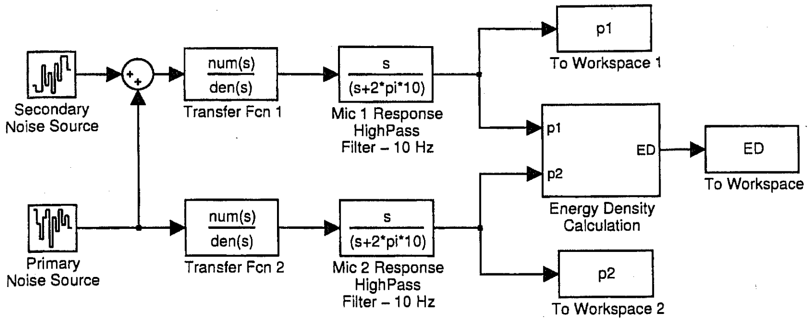

FIG. 2. The simulated sound field. The microphones (separation distance, $2h=50$ mm) are placed at distances $r_1=5.00$ m and $r_2=5.05$ m from the noise sources. The primary noise source produces continuous white noise of 1.0 Pa at 1 m (94 dB *re* 20 μPa). The secondary (contaminating) noise source produces continuous white noise of 0.010 Pa at 1 m (54 dB *re* 20 μPa). Speed of sound, $c=343$ m s^{-1}, and density of air, $\rho=1.21$ kg m^{-3}. The "energy density calculation" submodel is shown in Fig. 3.

Equation (30) and inequalities (31) and (32) yield

$$[((kh)^2+1)X+((kh)^2-1)\gamma_{12}(\omega)\cos\phi_{12}(\omega)]^2 \geqslant 4(X^2-\gamma_{12}^2(\omega))(kh)^2. \tag{33}$$

Therefore,

$$\frac{1-\gamma_{12}^2(\omega)}{2(X^2-\gamma_{12}^2(\omega))} \geqslant \frac{2(kh)^2(1-\gamma_{12}^2(\omega))}{[((kh)^2+1)X+((kh)^2-1)\gamma_{12}(\omega)\cos\phi_{12}(\omega)]^2}, \tag{34}$$

so that the normalized random error is bounded below by

$$\epsilon(\hat{E}_D(\omega)) \geqslant \frac{1}{\sqrt{n_d}}\left[1-\frac{1-\gamma_{12}^2(\omega)}{2(X^2-\gamma_{12}^2(\omega))}\right]^{1/2}. \tag{35}$$

Equality occurs in inequality (31) when $(X+\gamma_{12}(\omega)\cos\phi_{12}(\omega))(kh)^2=(X-\gamma_{12}(\omega)\cos\phi_{12}(\omega))$, and equality occurs in inequality (32) when $\cos^2\phi_{12}(\omega)=1$. Therefore, equality occurs in inequality (35) either when $\phi_{12}(\omega)=0$ and $(X+\gamma_{12}(\omega))(kh)^2=(X-\gamma_{12}(\omega))$, or when $\phi_{12}(\omega)=\pi$ and $(X-\gamma_{12}(\omega))(kh)^2=(X+\gamma_{12}(\omega))$.

Inequalities (29) and (35) together yield

$$\frac{1}{\sqrt{n_d}}\left[\frac{2X^2-1-\gamma_{12}^2(\omega)}{2X^2-2\gamma_{12}^2(\omega)}\right]^{1/2} \leqslant \epsilon(\hat{E}_D(\omega)) \leqslant \frac{1}{\sqrt{n_d}}. \tag{36}$$

When $G_{11}(\omega)=G_{22}(\omega)$, $X=1$ so that the lower bound in Eq. (36) becomes $1/\sqrt{2n_d}$. The definition of X (26) shows that $X\geqslant 1$ always, since the arithmetic mean of $\sqrt{G_{11}(\omega)/G_{22}(\omega)}$ and $\sqrt{G_{22}(\omega)/G_{11}(\omega)}$ is greater than or equal to their geometric mean. Therefore,

$$\frac{2X^2-1-\gamma_{12}^2(\omega)}{2X^2-2\gamma_{12}^2(\omega)} = 1-\frac{1-\gamma_{12}^2(\omega)}{2(X^2-\gamma_{12}^2(\omega))} \geqslant 1-\frac{1-\gamma_{12}^2(\omega)}{2(1-\gamma_{12}^2(\omega))} = \frac{1}{2}. \tag{37}$$

This shows that the lower bound for the normalized random error is minimum at $1/\sqrt{2n_d}$ when $G_{11}(\omega)=G_{22}(\omega)$. Indeed,

$$\frac{1}{\sqrt{2n_d}} \leqslant \epsilon(\hat{E}_D(\omega)) \leqslant \frac{1}{\sqrt{n_d}}. \tag{38}$$

V. DISCUSSION

The expression (36) shows that the normalized random error of time-averaged acoustic energy density spectral density estimates is always bounded within a narrow tolerance. As the upper bound in Eq. (29) is dependent only on $\sqrt{n_d}$, the number of records averaged, it can be ensured that the normalized random error is no greater than a specified tolerance by an appropriate choice of n_d, without the need to know any other information about the measurement. Thus it is not in fact necessary to first estimate the coherence or phase angle.

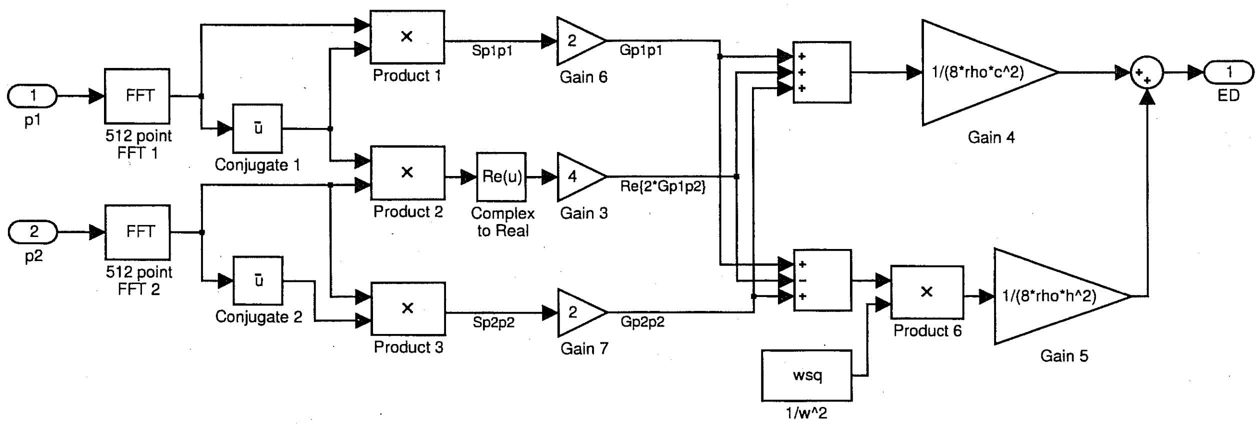

FIG. 3. Model for the calculation of the time-averaged acoustic energy density estimate using the frequency domain expression in terms of the auto- and cross-spectral densities of the two pressure readings.