PROBLEMS IN PLASMA DYNAMICS AND FLUID MECHANICS

by

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May 14, 1968.

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Yours faithfully,

Tom M. L. Wigley
Assistant Professor

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CONTENTS

Abstract
Acknowledgements

PART I: PLASMA DYNAMICS - TRANSPORT PHENOMENA IN PLASMAS

Chapter 1	Introduction	1
1.1	Kinetic Theory of Gases	2
1.2	Kinetic Theory of Plasmas	6
1.3	Summary	16
Chapter 2	Scattering in a Debye Field	23
2.1	Scattering in a Cut-Off Coulomb Field	26
2.2	Scattering in a Debye Field	31
2.3	Transport Cross-sections in a Debye Field	37
2.4	Comparison with the Cut-Off Potential	41
Chapter 3	Kinetic Equation for a Plasma	43
3.1	The Hierachy of Equations	44
3.2	Terminating the Hierachy	46
3.3	The Kinetic Equation	50
Chapter 4	Transport Coefficients	58
4.1	The Landau form of the Kinetic Equation	58
4.2	The Collision Integral in Operator Form	65
4.3	Evaluation of F_{ab}	71
4.4	Transport Coefficients	79
4.5	An Exact Solution	102
4.6	Further Remarks	111
Chapter 5	Runaway Electrons	113

Appendix A
Appendix B
Bibliography

PART II; FLUID MECHANICS - POROUS FLOW AND CAVE BREATHING

Chapter 1	Introduction	1
Chapter 2	Cave Breathing	9
2.1	Porous Flow Theory	9
2.2	Cave Breathing	24
2.3	Discussion of other Results	45
Chapter 3	Extension to Hydrology	50
3.1	Flow into a Well with Arbitrary Discharge	51
3.2	Constant Discharge Case	54
3.3	Constant Drawdown Case	58
3.4	Partially Confined Aquifers	60

Bibliography

Appendix 'Non-Steady Flow through a Porous Medium and Cave Breathing', J. Geophys. Res. 72, 3199 (1967).

Explanatory Note:

Equations are numbered consecutively in each chapter with both the chapter number and the equation number. They are referred to within that chapter by the equation number only and in other chapters by both the chapter number and the equation number. References in the text are numbered in order of occurrence and these numbers are superscripted to distinguish them from equation numbers. Bibliography lists in numerical order are given separately at the end of each part of the thesis.

ABSTRACT

The thesis is divided into two major parts. In the first the problem of determining transport coefficients in a simple two-component plasma is approached from first principles and the Kinetic equation which is derived is solved using an operator technique. 'Runaway' electrons are also considered. In the second an unusual natural phenomenon, cave 'breathing' is explained theoretically by means of a diffusion equation and some experimental results are given which confirm the theory. An extension to other problems in Hydrology is made.

A Kinetic equation with Boltzmann-type collision integral is derived in which the interparticle potential is necessarily an exponentially shielded coulomb (Debye) potential. The explicit collision integral derived is of the Landau form, but with some minor differences due to the more rigorous manner in which approximations regarding the Coulomb logarithm term are made. The first order equation and corresponding auxiliary relations which are obtained on expanding the velocity distribution function are written in terms of an unknown operator which is related to the perturb-

ation from the equilibrium distribution function. The mass, momentum and particle fluxes are also written in terms of this operator. Thus, by solving the Kinetic equation in the operator form the transport coefficients can be obtained directly. The method of solution used is similar to the Chapman-Enskog technique although in this case the unknown is an operator rather than a function and the expansion is in powers of the Laplacian operator in velocity space rather than Sonine polynomials. An exact solution to a special form of the Kinetic equation is obtained using a Fourier transform method. This special form corresponds to the equation describing runaway electrons, although the solution obtained is not directly applicable to the runaway phenomenon. Runaway electrons are discussed further and an equation is derived which is equivalent to that used by other authors. A numerical method for solving the equation by expanding the distribution function in powers of a small field dependent parameter and spherical harmonics is given.

The 'breathing' of caves has been an unsolved geophysical problem for some time. In certain regions of the world, air is observed to move regularly into and out of caves and vents in the ground at rates which are inconsistent with the apparent physical dimensions of such features. It is conjectured that these air movements are caused by changes in atmospheric pressure and that the large volume of air which is moved originates from the porous limestone in which 'breathing' caves are found. Two models corresponding to different cave shapes are proposed and solved theoretically. The equation governing the changes in pressure in the surrounding limestone is found to be a diffusion equation and this is solved using the integral transform technique. The results for one model are in accord with the few published observations. Further experimental results are presented from caves which fit both models and these are discussed in detail. Because of the similarity of the problem to the hydrological problem of flow into a well the method of solution used is extended to cover this case. The results are in agreement with known results from the theory of heat conduction.

I, Tom Michael Lampe Wigley, certify that this thesis contains no material which has been accepted for the award of any other degrees or diplomas in any University, and that, to the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text.

Dec 1967

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PART I

CHAPTER ONE: INTRODUCTION

A Kinetic equation, describing the irreversible approach of a gas to equilibrium through a velocity or one-particle distribution function, is fundamental to most discussions of transport phenomena in gases and plasmas. Historically, the Boltzmann equation, a Kinetic equation for dilute gases (discussed in detail in the monograph of Chapman and Cowling[1]), was the first such equation. In general the explicit form and range of validity of a Kinetic equation is dependent on the interaction forces between the gas constituents. Although there are overall similarities between the theoretical treatment of gases (the word 'gas' is used here for the unionized gaseous phase of a fluid) and plasmas (completely ionized 'gases') there are also distinct and important differences. We will begin by discussing some of the fundamental concepts of the kinetic theory of gases and later extend this discussion to plasmas and point out the differences between the two.

1.1 KINETIC THEORY OF GASES

The most basic starting point for the derivation of a Kinetic equation is the Liouville equation which governs the time-evolution of the N-particle distribution function. The Liouville equation can be reduced to the equivalent BBGKY hierachy of equations (2,3,4,5). The first member of the hierachy, in which the one-particle and two-particle distribution functions are related is, in effect, an exact Kinetic equation. By making suitable approximations the hierachy can be made tractable; irreversibility (which is not a property of the hierachy) can be introduced, and an equation for the one-particle distribution function obtained.

There are certain time-scales which are important in following the approach of a gas to equilibrium: τ_o, the 'collision time' ($\tau_o \approx \frac{r_o}{u}$ where r_o is the range of the inter-particle forces and u is representative of the particle speed); t_o, the 'mean free time', representative of the time between collisions ($t_o \approx \frac{\lambda}{u}$ where λ is the mean free path); and T_o, a macroscopic time which is representative of the time a gas takes to relax from a non-equilibrium state to one close to equilibrium in which the gas has no 'local memory' of its initial configuration.

An important difference between a gas and a plasma is already apparent. In a plasma it is not clear how the collision time or the mean free time can be defined, since these times are dependent on the concept of a 'collision' being a well-defined event. With long-range interparticle forces a 'collision' becomes an ill-defined period of interaction.

Returning to a gas, where these times are meaningful, one finds that they differ considerably between each other ($\tau_o \ll t_o \ll T_o$). This fact can be used to advantage in deriving a kinetic equation from the BBGKY heirachy. In the method of Bogoliubov[3], for example, it is assumed that the behaviour of a gas can be adequately described by the one-particle distribution function for times $\gtrsim \tau_o$ and that, from an initial non-equilibrium condition at t = 0, after a time $\gtrsim \tau_o$ the explicit time dependence of the s-particle distribution functions (F_s, $s \geqslant 2$) becomes implicit and only arises through a functional dependence on the one-particle function (F_1). This is not necessarily equivalent to the statement of some authors that F_1 is assumed to be 'slowly' varying. Under these assumptions the hierachy can be solved by successive approximations by making a 'density expansion' of the F_s (i.e. an expansion in the small parameter $\frac{\tau_0}{t_0} \sim n r_0^3$ where n is the average number density). Irreversibility, for times long with respect to τ_0, is obtained through an 'initial' condition that the particles are uncorrelated in the 'past' (i.e. a long time back with respect to τ_0, but still $\lesssim t_0$). To first order in density a generalized form of Boltzmann's equation is obtained which reduced to the Boltzmann Equation if spatial inhomogeneities in F_1 over distances of order r_0 are neglected. This equation is necessarily valid for times long with respect to τ_0, of order t_0.

The assumptions of Bogoliubov's method are consistent with the original phenomenological derivation of the Boltzmann equation. The first order in density corresponds to the Boltzmann equation's describing only two-particle (binary) interactions; the neglect of spatial inhomogeneities over distances of order r_0 and the assumption leading to irreversibility correspond to the 'stosszahlansatz' of the Boltzmann equation which can therefore only describe the 'probable' behaviour of a gas where detailed microscopic processes (over times of order τ_o) are smoothed out. However, the Bogoliubov approach is not entirely satisfactory, and, to higher than third order in density it leads to divergences[6] which are most troublesome in the application of the theory to dense gases.

Alternative approaches have been offered by other authors[7,8,9,10,11,12] all of which can lead to a generalized Boltzmann equation provided certain restrictive assumptions, fundamentally similar to those of Bogoliubov, are made. These methods are usually restricted to repulsive interparticle forces, although the method of Hoffman and Green[10] is not, and Sandri[12] asserts that his technique is not restricted in this way (although this seems doubtful in view of some comments by Wu[13]).

If the BBGKY hierachy is written in a non-dimensional form (see, for example, Sandri 1966[12]) two-dimensionless parameters, nr_o^3 and $\frac{\varphi_o}{kT}$, the 'density' and 'strength' parameters, arise. Here φ_o is a value of the potential which is representative for an interaction, and k is Boltzmann's constant. In a gas φ_o can be taken as the depth of the potential. In a plasma this is not a meaningful step and the value of φ_0 in the strength parameter must be prescribed more carefully. The density expansion used in Bogoliubov's method corresponds to an expansion in the (small) parameter nr_0^3 which has been identified with the ratio of time-scales τ_0/t_0.

1.2 KINETIC THEORY OF PLASMAS

We now consider the application of these general aspects of gas kinetic theory to plasmas. It is apparent that, for dilute gases, the Boltzmann equation provides an adequate description of the gas for times of order t_0. In plasmas this simple description is no longer sufficient, and correlation effects (the effects of multiple, as opposed to binary, collisions) are important. In plasmas this is due primarily to the long-range nature of the coulomb forces which cause the Boltzmann collision integral to diverge. This divergence can be eliminated by the somewhat artificial expedient of modifying the coulomb potential on heuristic grounds (viz. by cutting-off the potential or by replacing it by a Debye potential). In view of the above discussion it is difficult to see exactly what the significance of such a step is, and to see in what way and how well multiple interaction effects are accounted for.

Most of the techniques which use the Liouville equation as a starting point for the derivation of a Kinetic equation can be applied to plasmas. Obviously some modifications are necessary, since, as has already been pointed out, the collision time τ_o and the mean free time t_o are difficult to define in a plasma. It has also been mentioned that most methods of deriving a Kinetic equation for a gas are restricted to repulsive interparticle forces. However, in applying these methods to plasmas, this presents only a formal restriction.

Bogoliubov proposed that the strength parameter could be used as an expansion parameter for plasmas in the same way as the density parameter is used in ordinary gas theory. Before discussing this possibility there is an apparent analogue in plasmas of the density parameter for gases which must be considered. In a plasma, r_o, the range of the potential, can be identified with the Debye screening length, a_D, since, on equilibrium considerations, a_D is effectively the range of the potential of a charged particle in the presence of other charged particles. It would thus appear that na_D^3 could be identified as the density parameter applicable to plasmas. This is not so, primarily for two reasons; firstly a_D itself is density dependent so that na_D^3 is proportional to $n^{-\frac{1}{2}}$, and secondly, one of the fundamental characteristics of a plasma is that there is a large number of particles inside a sphere of radius a_D so that $n^{-\frac{1}{3}} \ll a_D$ and $na_D^3 \gg 1$. Although na_D^3 does not correspond to the density parameter it is important. If ϵ is defined as

$$\epsilon = (na_D^3)^{-1}$$

then ϵ is very small and could be used as an expansion parameter. In fact an expansion in ϵ is equivalent to an expansion in the strength parameter since

$$\epsilon = \frac{1}{a_D}\frac{1}{na_D^2} \sim \frac{1}{a_D}\frac{e^2}{\varepsilon_0 kT} \sim \frac{\varphi_{r=a_D}}{kT}$$

and, for most interactions in a plasma, $\varphi_{r=a_D}$ is a representative value for the mutual potential energy of two particles.

Bogoliubov's suggestion that the strength parameter be used for plasmas in place of the density parameter used for gases is a useful one. This is one of the major differences between plasma kinetic theory and the theory for ordinary gases. Kinetic equations for the regime where the strength parameter (or ϵ) is small have been obtained independently by Balescu[14] (using the method of Prigogine and Balescu[11] in which the BBGKY hierachy is replaced by a system of coupled equations for the Fourier components of the distribution function and a diagram technique is used for the calculations) to first order in the strength parameter, and by Lenard[15] and Guernsey[16] (by making a double expansion, summing over all terms in one parameter, and retaining the first order term in the strength parameter). Further extensions of this work using the method of Bogoliubov have been made by Wu and Rosenberg[17] and by Wu[13]. Equivalent results have also been obtained using the cluster expansion technique of M.S. Green[8] by Rostoker and Rosenbluth[18] and others[19] in which the effects of the correlation functions, $G_{ab} = F_{2,ab} - F_{1,a} F_{1,b}$ etc., are assumed of order ϵ or smaller in relation to the uncorrelated product functions, $F_{1,a}F_{1,b}$, etc.

The Balescu-Lenard-Guernsey (BLG) Kinetic equation, being first order in the strength parameter,

$\frac{\varphi_{r=a_D}}{kT}$, diverges for small interparticle separations where $\varphi_{r=a_D}$ is not an adequate representation of the strength of the potential. The point where φ becomes equal to kT for a given temperature is called the distance of closest approach (σ_o). The BLG equation adequately accounts for only those interactions where particles are always separated by distances $\gg \sigma_o$. For these interactions only is the strength parameter both representative and small.

Vlasov[20] has proposed an equation in which the inherent approximations are similar to those of the BLG equation; it can be derived by completely neglecting the correlation functions in the collision integral. The Vlasov equation is, however, time-reversible and thus cannot be used in problems concerned with the approach of a plasma to equilibrium.

As an alternative to starting from the Liouville equation, a Kinetic equation can be derived phenomenono-logically using the theory of stochastic processes. Because of the long-range nature of coulomb forces most interactions between plasma particles cause only small changes in the particle trajectories. If it is assumed that the effect of the many simultaneous small deflections which a test particle in a plasma undergoes (under the many-body influence of surrounding field particles) is equivalent to a series of independent small deflections (a difficult assumption to justify and strictly true only if the mutual interactions among the field particles can be neglected[21]), then a kinetic equation can be derived by analogy with Brownian motion. This approach leads to a Fokker-Planck-type Kinetic equation which was first obtained by Landau[22] and later by other authors[23]. In its original form this equation fails to account properly for close 'collisions', and, since the assumption of small deflections implies that the strength parameter is small, must suffer from the same small-separation divergence difficulty as the BLG equation. Also, since the interparticle force is a coulomb force, and the many-particle interactions are considered as series of binary interactions, the divergence at large separation inherent in the use of such a potential occurs. These divergences can be eliminated by rather inadequate physical arguments similar to those used with the ordinary Boltzmann equation.

Sandri[12] has employed the method of Frieman and Sandri[9] to derive a kinetic equation which, in the appropriate limits, reduces to the Boltzmann equation, BLG equation and Fokker-Planck-Landau (FPL) equation. He uses an expansion of the independent variable, time, in terms of a hierachy of time scales together with an assumption regarding the order of magnitude of the correlation functions and states that the divergences of other approaches are eliminated. Wu[13] disputes this claim.

The ranges of applicability of the Boltzmann, BLG and FPL equations can be summarized in terms of the fundamental distances in plasma kinetic theory: a_D, the Debye length; $n^{-\frac{1}{3}}$, the mean particle separation; and σ_o, the distance of closest approach, which satisfy the relation $\sigma_o \ll n^{-\frac{1}{3}} \ll a_D$ in most plasmas. The Boltzmann equation satisfactorily describes only binary interactions, $r < n^{-\frac{1}{3}}$ (r = particle separation), and diverges for large r. The BLG equation describes such interactions poorly, is adequate only for small values of the strength parameter ($r > \sigma_o$), and gives divergent results at small r. The FPL equation suffers from both these defects and diverges for small and large r. By an appropriate combination of the collision terms of these three equations it is possible to eliminate the divergences and thus obtain a Kinetic equation having a greater range of validity (although this is not a direct logical consequence). This technique has been employed by Weinstock[24], Frieman and Books[25], and Wu[13]. All divergences can, of course, be eliminated by the choice of appropriate cut-offs in the coulomb potential or by the use of a Debye potential in the Boltzmann collision term.

An alternative procedure is an extension of the method of Born and Green[4] to plasmas which has been developed by Green and Leipnik[26]. These authors consider the time correlation functions rather than the distribution functions and use a hierachy of equations which is equivalent to the BBGKY hierachy. The procedure is to obtain a solution for the two-particle function by making an approximation to the three-particle function which is good for situations close to equilibrium. This approximation is called the 'disjunctive' approximation and it will be shown in chapter three of this part of the thesis to be closely related to a first order approximation in the strength parameter. The Kinetic equation which is obtained can be written in a form similar to the Boltzmann equation where the interparticle forces are described by a potential which, in the equilibrium limit, is a Debye potential. On making a further good approximation the usual Boltzmann form obtains (with Debye potential). This is a pleasing result since it justifies the intuitively based use of a Debye potential to eliminate divergences in the unmodified Boltzmann equation. The approach is well suited to the discussion of transport phenomena in plasmas which deviate only slightly from equilibrium and will be used in this thesis.

Throughout this thesis the validity of classical mechanics will be assumed. However, one quantum-mechanical aspect which is important in equilibrium theory must be mentioned since the Debye potential, a result of the equilibrium theory, is used extensively in this thesis. This is the fact that a lower bound exists to the energy of a system of charges of opposite sign. The existence of such a lower bound is fundamental to statistical mechanics. Statistical mechanics makes physical sense only if the thermodynamic quantities (Energy, Entropy, etc) are asymptotically proportional to the number of particles in a system. In a purely classical system no lower bound exists. The principals of both quantum mechanics and quantum statistics must be invoked in order to obtain the physically correct lower bound proportional to the number of particles in the system[46].

In a purely classical theory this can be taken into account by cutting off the attractive coulomb potential at small distances. This is done[27] by multiplying the mutual potential energy of two charges by the factor $(1 - e^{-r/R_1})$, where R_1 is the radius of the smallest Bohr orbit, if the charges are of opposite sign. This term has little effect for the great majority of interpaticle separations, but it does ensure the correct quantum-mechanical lower bound to the energy of a pair of opposite charges.

1.3 SUMMARY

In this part of the thesis a Kinetic equation is derived following the theory developed by Green and Leipnik[26] and a method for obtaining the transport coefficients from this equation is given. In chapter two for subsequent reference, a detailed discussion of scattering processes in a Debye potential field is given. The transport cross-sections, generalizations of the total scattering cross-section, which are needed to determine the Kinetic equation in explicit form, are evaluated using a cut-off coulomb field and a Debye field. Approximate analytic expressions are given for both the scattering angle and the transport cross-sections in the Debye field case. The techniques used are similar to those used by other authors[28,29], but the resulting expressions are a slight generalization of previous results. A rapid numerical procedure for evaluating the scattering angle is given in Appendix A.

In chapter three a Kinetic equation valid for small deviations from equilibrium, but restricted in no other way, is derived in a form closely resembling the Boltzmann equation, but in which the interparticle forces must be described by a Debye potential. Equations are developed from the BBGKY hierachy which is curtailed by the use of a disjunctive approximation. These are formally equivalent to those of Green and Leipnik[26], except that they involve the velocity distribution functions rather than the time correlation functions. The solution obtained by Green and Leipnik is used and is written in a Boltzmann equation form. The approximations employed in the derivation are discussed in reference to those used by other authors. It is demonstrated that the equation is accurate to first order in the strength parameter.

In chapter four the collision integral part of the Kinetic equation is evaluated explicitly using the results of chapter two. The form obtained is similar to that first given by Landau[30], but it is not subject to any restrictive conditions. An operator relating the non-equilibrium and the equilibrium velocity distribution functions is defined, and the collision integral is rewritten in terms of this operator. The Kinetic equation is thus reduced to a different type of equation where the unknown is an operator rather than a function.

Since the mass, momentum and energy fluxes can also be expressed in terms of this operator, a knowledge of its form makes it possible to estimate the transport coefficients which relate the fluxes to the gradients of hydrodynamic quantities.

Historically, Chapman, in 1916, and Enskog, more rigorously in 1917, were the first to derive transport coefficients from the Boltzmann equation and the method developed by them, the Chapman-Enskog method, is described in detail by Chapman and Cowling[1]. A functional ansatz, similar to that used by Bogoliubov in reducing the BBGKY hierachy to a Kinetic equation, is fundamental to the Chapman-Enskog method. The velocity distribution function is assumed to depend on time, only through its dependence on the macroscopic hydrodynamic variables, number density, mass average velocity and temperature, and the resulting equation is solved by expanding the deviation from the equilibrium distribution function in orthogonal (generally Sonine) polynomials. The functional ansatz ensures that rapidly varying solutions of the Boltzman equation are filtered out. Such solutions would describe processes which have characteristic times much less than T_o. Because the relaxation time for like particles in a plasma is much less than that for unlike particles, the Chapman-Enskog method can be used to describe situations where the plasma components have different temperatures.

Chapman and Cowling[1] briefly discuss the case of coulomb interparticle forces to first order in the Sonine polynomial expansion (although they overcome the divergences in an inadequate way), and also the case where external electric and magnetic fields are present. Marshall[31] has described the entire transport coefficient problem from the Chapman-Enskog point of view in some detail to first and second order using a variational procedure. Spitzer and Harm[23] have integrated the equations numerically for the zero magnetic field case and Landshoff[32] has calculated coefficients to third and fourth order in the Sonine polynomials when a weak magnetic field is present. Kaufman[33] has found transport coefficients for a large magnetic field, Robinson and Bernstein[34] have made similar, but more general calculations using a variational procedure, and Kaneko[35] has calculated thermal conduction and thermal diffusion coefficients in a magnetic field to sixth order in the Sonine polynomial expansion. Braginskii[36] has evaluated the transport coefficients for the case when the temperatures of the plasma components are not equal.

Other methods which are available for the discussion of transport coefficients are the mean-free-path method which was developed prior to the Chapman-Enskog method (see, for example, Jeans[37]), the method of Kubo[38] which is equivalent to the Chapman-Enskog method[39], and which has been used by Green and Leipnik[26] to discuss diffusion in a strong magnetic field, and the many-moment scheme devised by Grad[40] which is the only technique available which is not restricted to small deviations from equilibrium. In chapter four of this part of the thesis the Kinetic equation in operator form is solved by a method which parallels the Chapman-Enskog method. The unknown operator is expanded in powers of the Laplacian in velocity space. This expansion corresponds to the usual Sonine polynomial expansion and it similarly enables the transport coefficients to be obtained without completely determining the operator. The generalization of the method to plasmas in the presence of a magnetic field is briefly discussed.

The phenomenon of 'runaway' electrons is investigated in chapter five. The collisional drag on an electron in a plasma is a function of the speed of the particle. If a uniform electric field is present the resulting acceleration can exceed the collisional deceleration for particles moving at greater than a certain critical speed. Such particles thus accelerate continuously and they are called 'runaway' electrons. If the field is weak the critical speed is much greater than the average or thermal speed of the electrons and the flux of runaway electrons is small. The basic problem is to evaluate the distribution function for runaway electrons and the equation which governs the behaviour of the distribution function can be derived from the Kinetic equation by assuming that the electrons interact with an equilibrium background (valid if the number of runaways is small) and using the appropriate high speed forms for the collision integral.

An approximate form of the equation has been studied previously by Dreicer[41]. A more fundamental discussion of the problem, in which the correct form of the equation is used, has been given by Gurevich[42] who also has considered a generalization to the case when the background particles are not necessarily in equilibrium (Gurevich and Zhivlyuk[43]). The problem has also been discussed by Kruskal and Bernstein[44] who consider only a Lorentz gas (i.e. interactions of electrons with electrons are neglected). The expressions obtained by these authors are not in complete accord with one another and their estimates of the runaway flux are generally dependent on additional physical considerations. The approach of Gurevich[42], for instance, is valid only for particle speeds below a certain limit and his solution becomes unreal above this. Lebedev[45], however, has used a similar method which is not restricted in this way.

In chapter five of this part of the thesis the Kinetic equation appropriate for runaway electrons is derived from the results of earlier chapters and is found to be similar to that given by Gurevich[42] and Lebedev[45]. A numerical method for solving the equation is developed which employs an expansion in an electric field strength parameter and spherical harmonics.

CHAPTER TWO: SCATTERING IN A DEBYE FIELD

The dynamics of the interaction of an isolated pair of plasma particles can be determined using a coulomb potential. Because of the extreme long-range nature of this potential certain integrals basic to plasma kinetic theory are found to diverge for coulomb interactions. It is generally known however (see, for example, Green and Leipnik[26] and Green[27]), that, in any real plasma, the presence of other particles has the effect of modifying the interparticle forces in such a way that these divergences are eliminated. In a later chapter it will be shown, by using the results of Green and Leipnik, that the irreversible behaviour of a plasma can be described by means of a Boltzmann-type equation in which the interparticle forces are given by a Debye potential. In the explicit evaluation of the collision term of this equation it is necessary to determine the values of some integrals which occur quite frequently in plasma physics. These are the 'transport (or transfer) cross-sections'. The n-th cross-section is defined by

$$\sigma_n = \int (1-\cos^n\theta)\,\frac{\partial\sigma}{\partial\Omega}\,d\Omega = \int (1-\cos^n\theta)\,\underline{d\sigma}$$

where θ is the scattering angle, $\frac{\partial \sigma}{\partial \Omega}$ is the differential cross-section and $d\Omega$ is an element of solid angle. This may be written as

$$\sigma_n = 2\pi \int_0^\infty (1 - \cos^n \theta)\, b\, db \qquad \ldots\ldots (2.1)$$

in which b is the impact parameter, by using the familiar expression for the differential cross-section

$$\frac{\partial \sigma}{\partial \Omega} = -\frac{b\, db}{\sin\theta\, d\theta}$$

The case n = 0 yields the total cross-section: the cases n = 1 and n = 2 are the only others which occur in the evaluation of the collision integral and further discussion will be confined specifically to these.

If a coulomb potential is used to determine the dependence of scattering angle on impact parameter (Rutherford scattering) the transport cross-sections diverge. In plasma kinetic theory qualitative arguments are frequently used to modify the coulomb potential and make these cross-sections finite. For instance, it can be assumed that any pair of particles which are in a cloud of multiply interacting particles can be considered as a separate, isolated, pair interacting in a binary fashion. The effect of the other particles is assumed to be manifest as a modifying factor in the two-particle potential. From equilibrium theory the appropriate force between the two particles is expected to be that described by a Debye potential. There is evidence that this description can be rigorously justified as a good approximation and this justification will be considered further in a following chapter. Often in the literature a kind of cut-off coulomb potential is used (where the cut-off is in impact parameter rather than the range of the potential). This can be justified, either on similar, rather inadequate, phenomenological reasoning to the above, or by using such a potential as an approximation to a Debye potential which can itself, as stated above, be justified by more rigorous means. Many of the results presented in this chapter are comparatively well-known, although some of the techniques used in their derivation, and the notation, are new. A detailed comparison of the Debye and cut-off coulomb potentials appears to be lacking from the literature even though it is frequently stated that they give similar results.

2.1 SCATTERING IN A CUT-OFF COULOMB FIELD

If n_a and e_a are number density and charge for type 'a' particles in a plasma of temperature T, the Debye length, a_D, is given by

$$a_D^{-2} = \frac{\sum_a n_a e_a^2}{\varepsilon_o kT}$$

The Debye (or Debye-Hueckel) potential is

$$V = \frac{e_a e_b}{4\pi\varepsilon_o r} \exp(-r/a_D) \qquad \ldots\ldots (2.2)$$

where $\underline{r}$ is radius vector measured from the potential source. The Debye potential is thus an exponentially screened coulomb potential; for small $r(\ll a_D)$ it is almost the same as a coulomb potential, while for large $r(>a_D)$ it is much weaker. A test particle moving in a Debye field is thus effectively screened from the field particle for large r and sees the field as a coulomb one for small r. Because of this a cut-off coulomb potential could be used to approximate the Debye potential; in such a field the potential would be coulomb up to $r \sim a_D$ and zero beyond this. By considering the dynamics of a two particle interaction with this type of potential the dependence of scattering angle on impact parameter could be found and the transport cross-sections evaluated. This is still a difficult task, however, and usually a further approximation is made to simplify the problem. A cut-off can be made in the impact parameter rather than the field so that particles with impact parameter greater than a_D are assumed to be unaffected by the field particle, while those with $b \leqslant a_D$ are assumed to behave as if they had moved in a coulomb potential for all time, rather than only during the time when the interparticle distance $|\underline{r}_{ab}|$ (or r) is less than a_D.

Suppose that the impact parameter cut-off is made at $b = b_o$. Then

$$\begin{aligned} \theta &= 0 \qquad , \quad b > b_o \\ \cot \tfrac{1}{2}\theta &= \frac{b \rho_o^2}{\lambda} \qquad , \quad b \leqslant b_o . \end{aligned} \qquad \text{.....} (2.3)$$

Here ρ is the relative velocity of the interacting particles and the suffix zero indicates initial value.

λ is defined by

$$\lambda = \frac{e_a e_b}{4\pi \varepsilon_o M}$$

where $M \left(= \frac{m_a m_b}{m_a + m_b} \right)$ is the reduced mass. It is convenient to measure all lengths in terms of a_D and write accordingly

$$B = \frac{b}{a_D} \quad , B_o = \frac{b_o}{a_D} \quad ,$$

and to introduce the parameter, Q, defined by

$$Q = \frac{2\lambda}{a_D \rho_o^2}$$

Q is thus the ratio of the potential energy of a test particle in a coulomb field at r = a to the total kinetic energy in the centre of mass frame. It should be noted that $Q \sim \frac{\varphi_{r=a_D}}{kT}$ (φ denoting potential energy) and Q is thus closely related to the strength parameter mentioned in the previous chapter. In terms of B and Q equation (3) may be written

$$\theta = 0 \qquad , B > B_o$$

$$\cos \theta = \frac{4B^2 - Q^2}{4B^2 + Q^2} \qquad , B \leqslant B_o \ .$$

Using these expressions the first three transport cross-sections are easily evaluated to give

$$\sigma_0 = \pi a_D^2 B_0^2$$

$$\sigma_1 = \pi a_D^2 B_0^2 \frac{Q^2}{2B_0^2} \ln\left(\frac{4B_0^2 + Q^2}{Q^2}\right) \qquad \ldots\ldots (2.4)$$

$$\sigma_2 = \pi a_D^2 \left\{ Q^2 \ln\left(\frac{4B_0^2+Q^2}{Q^2}\right) - \frac{4B_0^2 Q^2}{4B_0^2 + Q^2} \right\}$$

If the impact parameter cut-off is made at the Debye length (i.e. $B_0 = 1$), then, since Q is generally small ($\sim 10^{-3}$ to 10^{-8}), the cross-sections are approximately

$$\sigma_0 = \pi a_D^2$$

$$\sigma_1 = \pi a_D^2 Q^2 \ln\left(\frac{2}{|Q|}\right) \qquad \ldots\ldots (2.5)$$

$$\sigma_2 = 2\pi a_D^2 Q^2 \left\{ \ln\left(\frac{2}{|Q|}\right) - \frac{1}{2} \right\}$$

The term $\frac{1}{2}\ln\left(\frac{4+Q^2}{Q^2}\right)$ (or $\ln\left(\frac{2}{|Q|}\right)$) is often called the 'Coulomb logarithm'. In this thesis a more general expression will be referred to as the Coulomb logarithm. This will be defined by

$$\Lambda = \frac{\sigma_1}{\pi a_D^2 Q^2} \qquad \ldots\ldots (2.6)$$

Thus, for the impact-parameter-cut-off approximation the Coulomb logarithm is

$$\Lambda_c \approx \ln\left(\frac{2}{|Q|}\right) = \ln\left(\frac{a_D \rho_0^2 4\pi\varepsilon_0}{|e_a e_b|}\right)$$

A further approximation to Λ_c, where $\frac{1}{2}M\rho_0^2$ is replaced by the mean kinetic energy $\frac{3}{2}$ kT, is also called 'Coulomb logarithm' in the literature, and this form appears extensively in plasma kinetic theory calculations. The justification for using both this approximation and the cut-off potential is essentially qualitative. It is defined by (6) which arises naturally from the theory used in this thesis.

Sivukhin[47] presents an interesting attempt at a more quantitative justification for the use of and the impact-parameter-cut-off potential. He supposes that the interaction time is finite and considers a problem more closely allied to a coulomb potential which is cut-off at a certain distance from the source. He obtains a term essentially the same as Λ_c and thus demonstrates the close approximation of the cut-off in impact parameter to the cut-off in the range of the potential. This latter must still be justified by a phenomenological argument.

2.2 SCATTERING IN A DEBYE FIELD

Scattering in a Debye potential has been studied by Everhart et. al., Lane and Everhart, and Firsov[28]. Their work has been extended to more general exponentially screened repulsive potentials by Baroody[29]. This can be extended to cover attractive potentials and general expressions for the transport cross-sections obtained.

Consider the scattering of two particles (of types 'a' and 'b') in the centre of mass system. If ϑ is the coordinate angle representing the displacement of the relative position vector $\underline{r}$ with respect to some fixed direction, then the solution to the orbit equation (see, for example, Goldstein[48], or any other book on classical mechanics) is

$$\vartheta = \vartheta_0 - \int_{u_0}^{u} du \left\{ \frac{2E}{ML^2} - \frac{2V}{ML^2} - u^2 \right\}^{-\frac{1}{2}}$$

where $u = \frac{1}{r}$, $\underline{L} = \underline{r} \times \underline{\rho}$, M is the reduced mass, V is the Debye potential (equation (2)) and E is the total energy. If r_m is the minimum value of r (the distance of closest approach) and $u_m = 1/r_m$, the scattering angle, θ, is given by

$$\frac{\pi}{2} - \frac{\theta}{2} = \int_0^{u_m} du \left\{ \frac{\rho_0^2}{L^2} - \frac{2V}{ML^2} - u^2 \right\}^{-1/2} .$$

u_m can be found using conservation of energy,

$$u_m^2 = \frac{1}{L^2}\left(\rho_0^2 - \frac{2V_m}{M}\right) ,$$

where $V_m = \frac{\lambda}{r_m} e^{-r_m/a_D}$ is the value of the Debye potential at $r = r_m$. On using this expression for u_m and transforming the above integral by a change of variable to x $\left(= \frac{u}{u_m}\right)$ it becomes

$$\frac{\pi}{2} - \frac{\theta}{2} = \int_0^1 dx \left\{ \frac{\rho_0^2 - \frac{2V}{M}}{\rho_0^2 - \frac{2V_m}{M}} - x^2 \right\}^{-1/2} .$$

It is convenient to rewrite this expression in terms of Q and a new parameter, y, defined by $y = \frac{a}{r_m}$ (thus y is the inverse of the distance of closest approach measured in units of Debye length). Hence

$$\frac{\pi}{2} - \frac{\theta}{2} = \int_0^1 dx \left\{ \frac{1 - Qxy\, e^{-1/xy}}{1 - Qy\, e^{-1/y}} - x^2 \right\}^{-1/2} \qquad \text{.....(2.7)}$$

In this form only one energy dependent parameter (Q) is involved. In previous calculations two parameters are used which are both energy dependent. They are 'c' the distance of closest approach in head-on collision and

$-\left[\frac{d \ln V}{d \ln r}\right]_{r=c}$, parameters which are meaningful only for repulsive forces.

The impact parameter can be defined as the length of the vector which is the perpendicular drawn from one particle to the initial line of motion of the other (i.e. b = $\rho_0^{-2} \, |\underline{\rho_0} \times \underline{L}|$). This is related to the distance of closest approach and the energy parameter, Q, by

$$B^2 y^2 + Q y e^{-1/y} - 1 = 0 \qquad \text{..... (2.8)}$$

Equations (7) and (8), by eliminating y between them, are sufficient to determine the scattering angle as a function of impact parameter for any Q. It is not possible to determine the exact nature of this dependence analytically, although a good approximate solution can be found for the case of small Q (which is relevant here). For the more general case one must resort to numerical methods. This has been done by Everhart et. al.[28]. An improved method, in which the difficulty due to the divergence of the integrand in (7) is overcome, and the results can be obtained rapidly to as great an accuracy as desired, is presented in Appendix A. Returning to an approximate evaluation, equation (7) can be written

$$\frac{\theta}{2} = -\int_0^1 dx\left[\left(\frac{1-Qxy\,e^{-1/xy}}{1-Qy\,e^{-1/y}} - x^2\right)^{-1/2} - (1-x^2)^{-1/2}\right]$$

$$= \int_0^1 (1-x^2)^{-1/2}\left[1-\left\{1+\frac{Qye^{-1/y}(1-x\,e^{-1/xy}\,e^{1/y})}{(1-x^2)(1-Qy\,e^{-1/y})}\right\}^{-1/2}\right]dx\,.$$

Suppose that $\Delta = \left|\frac{Qy\,e^{-1/y}(1-xe^{-1/xy}e^{1/y})}{(1-x^2)(1-Qye^{-1/y})}\right|$. Then, if Δ is small in the interval $x = \langle 0,1\rangle$, the second part of the integrand can be expanded to give

$$\frac{\theta}{2} \approx \frac{Qy\,e^{-1/y}}{2(1-Qye^{-1/y})}\int_0^1 (1-x^2)^{-3/2}\;(1-xe^{-1/xy}\,e^{1/y})\,dx\,.$$

Separating the integrand into two parts and integrating by parts, changing variable to $^1/x$ and integrating by parts again reduces this to a known integral representation of the first order modified Bessel function of the second kind,

$$\theta = \frac{Q}{y(1-Qye^{-1/y})}\int_1^\infty e^{-X/y}\,(X^2-1)^{1/2}\,dX$$

$$= \frac{Q}{1-Qye^{-1/y}}\,K_1\!\left(\frac{1}{y}\right).$$

The approximation that Δ is small is valid provided $\left|\frac{Qye^{-1/y}}{1-Qye^{-1/y}}\right| \ll 1$ which is true provided $|Q|ye^{-1/y}$ is small.

It can also be seen that, for small Δ, $|\theta| \sim \Delta \frac{\pi}{2}$ which indicates that the small Δ approximation is valid only for small scattering angles. In view of these facts it is meaningful only to write the solution, valid for small θ, as

$$\theta = Q K_1\left(\frac{1}{y}\right) \qquad \text{..... (2.9)}$$

This solution has been obtained by Baroody (1962)[29] for repulsive forces using a similar method. It is valid not only for small θ, but for relatively small y ($y \ll |Q|^{-1}$).

It is possible to find upper and lower bounds for θ which are valid for all y by using the inequality

$$1 + y^{-1}(1 - x^{-1}) \leqslant e^{-1/xy} \cdot e^{1/y} \leqslant 1$$

so that

$$Qxye^{-1/y}\left[1 + y^{-1}(1-x^{-1})\right] \underset{\geqslant}{\leqslant} Qxye^{-1/xy} \underset{\geqslant}{\leqslant} Qye^{-1/y} \quad ; \quad Q \gtrless 0 .$$

The left- and right-hand sides of these inequalities when substituted into the expression for the scattering angle (7) give upper and lower bounds for $\frac{\pi}{2} - \frac{\theta}{2}$. On integrating we find

$$\frac{Qye^{-1/y}}{2-Qye^{-1/y}} \underset{\geqslant}{\leqslant} \sin\frac{\theta}{2} \underset{\geqslant}{\leqslant} \frac{Qy(y+1)\,e^{-1/y}}{(2-Qye^{-1/y})y+Qye^{-1/y}} \quad ; \quad Q \underset{<}{>} 0 ,$$

which may be written as

$$\left|\frac{Qye^{-1/y}}{2-Qye^{-1/y}}\right| \leqslant \sin\frac{\theta}{2} \leqslant \left|\frac{Qy(y+1)\,e^{-1/y}}{(2-Qye^{-1/y})y+Qye^{-1/y}}\right| \quad . \qquad \text{.....} \ (2.10)$$

Provided $|2(Qye^{-1/y} - 1)| \ll |(2 - Qye^{-1/y})y + Qye^{-1/y}|$, a relation which holds for large y (say $\gtrsim 10$) independent of Q, either side of the inequality (10) is a good approximation to $\sin\frac{\theta}{2}$. Also, since for large y, $Qye^{-1/y} \rightarrow Qy$, using equation (8)

$$\frac{Q^2(1-Qye^{-1/y})}{Q^2y^2e^{-2/y}} \rightarrow B^2 .$$

Hence

$$\frac{Qye^{-1/y}}{2-Qye^{-1/y}} = \left\{1 + 4\left(\frac{1-Qye^{-1/y}}{Q^2y^2e^{-2/y}}\right)\right\}^{-1} \rightarrow \frac{Q^2}{4B^2+Q^2}$$

This shows that, in the large y limit, $\sin\frac{\theta}{2} \rightarrow \frac{Q^2}{4B^2+Q^2}$. This is the result obtained using a coulomb potential and is expected since large y corresponds to the case when the test particle penetrates well into the regions where the Debye and coulomb potentials are close together in value.

2.3 TRANSPORT CROSS-SECTIONS IN A DEBYE FIELD

The results of the previous section give the scattering angle as a function of y for specified Q. Together with equation (8) this is sufficient to determine the transport cross-sections. The zero-th cross-section (i.e. the total cross-section) is infinite, although this is so only in the purely classical limit, since $\sigma_0 = -2\pi\left[b^2(\theta)\right]_0^\pi$. To evaluate σ_1 and σ_2 we must use the analytic results corresponding to large and small y. To do this it is necessary to divide the integration over impact parameter into two parts; zero to b_1 (or B_1) and b_1 to infinity, where the choice of b_1 will be made later.

For large y, which corresponds to small B, it has been shown that the scattering approximates to coulomb scattering. If $\sigma_n(B_1)$ denotes the contribution to σ_n from integration up to B_1, then, for small B_1, equation (4) can be used to determine $\sigma_1(B_1)$ and $\sigma_2(B_1)$. Hence

$$\begin{aligned} \sigma_1(B_1) &= \tfrac{1}{2}\pi Q^2 a_D^2 \ln\left(\frac{4B_1^2+Q^2}{Q^2}\right) \\ \sigma_2(B_1) &= 2\sigma_1(B_1) - \pi a_D^2 Q^2\left(\frac{4B_1^2}{4B_1^2+Q^2}\right) \end{aligned} \qquad \text{..... (2.11)}$$

The remaining contributions to σ_1 and σ_2 come from integration from B_1 to infinity. In this range the approximate solution $\theta = QK_1(\frac{1}{y})$ (equation (9)) can be used provided B_1 is such that this solution is valid at $B = B_1$. In other words the value of y corresponding to $B = B_1$, y_1 say, must be such that $|Q|\ y_1\ e^{-1/y_1}$ is small since this condition is sufficient for the validity of (9). The use of the coulomb approximation in the lower range of integration also places a restriction on B_1 (and hence y_1) since it has already been shown that this approximation holds well only for y (i.e. y_1) $\gtrsim 10$. Thus B_1 must be chosen to make $y_1 \gtrsim 10$ and $|Q|\,y_1\,e^{-1/y_1}$ small and it can be seen that this is only possible if $|Q|$ is small ($\ll \frac{1}{10}$). This is generally the case.

Returning to the integration from B_1 to infinity we use the fact that $\theta = QK_1(\frac{1}{y})$ is a small angle approximation to replace $1 - \cos\theta = 2\sin^2\frac{\theta}{2} \approx \frac{\theta^2}{2} \approx \frac{1}{2}Q^2(K_1(\frac{1}{y}))^2$ in the integrand for σ_1 obtaining

$$\sigma_1 - \sigma_1(B_1) = \pi Q^2 a_D^2 \int_{B_1}^{\infty} \left\{K_1\left(\tfrac{1}{y}\right)\right\}^2 B\,dB .$$

Since B_1 must be chosen so that $|Q|ye^{-1/y}$ is small for

$B > B_1$, in the range B_1 to infinity $B \approx \frac{1}{y}$. Making this substitution and integrating gives

$$\sigma_1 - \sigma_1(B_1) = \pi a_D^2 Q^2 \left[B_1 K_0(B_1) K_1(B_1) - \frac{B_1^2}{2} \left\{ (K_1(B_1))^2 - (K_0(B_1))^2 \right\} \right]$$

Now $y_1 \gtrsim 10$ and $B_1 \approx \frac{1}{y_1}$ so that $B_1 \lesssim \frac{1}{10}$ and the limiting (small B_1) forms for K_0 and K_1 may be used.

$$\sigma_1 - \sigma_1(B_1) = -\pi a_D^2 Q^2 \left[\ln \frac{B_1}{2} + \gamma + \frac{1}{2} \right] \qquad \text{..... (2.12)}$$

where γ is Euler's constant (0.577216). Also since $|Q| y_1 e^{-1/y_1} \ll 1$ and $y_1 \gtrsim 10$, $|Q| y_1 \ll 1$ or, using $B_1 \approx \frac{1}{y_1}$, $|Q| \ll B_1$. Using this fact to simplify equation (11) and combining this with (12) gives

$$\sigma_1 = -\pi a_D^2 Q^2 \left[\ln|Q| + \gamma + \frac{1}{2} - \ln 4 \right]$$

$$\therefore \quad \sigma_1 = -\pi a_D^2 Q^2 \left[\ln|Q| - 0.3091 \right] \qquad \text{..... (2.13)}$$

a result similar to that obtained by Baroody for repulsive potentials.

To evaluate σ_2 the large B contribution remains to be found. This is

$$\sigma_2 - \sigma_2(B_1) = 2\pi \int_{b_1}^{\infty} (1 - \cos^2\theta)\, b\, db$$

$$= 8\pi a_D^2 \int_{B_1}^{\infty} \left(\sin^2\frac{\theta}{2} - \sin^4\frac{\theta}{2} \right) B\, dB .$$

In this range of integration $\sin^4\frac{\theta}{2} \ll \sin^2\frac{\theta}{2}$ and the integral reduces to

$$\sigma_2 - \sigma_2(B_1) \approx 8\pi a_D^2 \int_{B_1}^{\infty} \sin^2\frac{\theta}{2}\, B\,dB \;=\; 2(\sigma_1 - \sigma_1(B_1)) \qquad \text{..... (2.14)}$$

To estimate the error caused by neglecting the $\sin^4\frac{\theta}{2}$ term we can use the inequality (10) modified by the known condition that $|Q|\,ye^{-1/y}$ is small to obtain

$$\int_{B_1}^{\infty} \left\{\frac{Q\,ye^{-1/y}}{2}\right\}^4 B\,dB < \int_{B_1}^{\infty} \sin^4\frac{\theta}{2}\, B\,dB < \int_{B_1}^{\infty} \left\{\frac{Q\,ye^{-1/y}(y+1)}{2y}\right\}^4 B\,dB.$$

Making the change of variable to $x = \frac{4}{y} \approx 4B$ this becomes, since $|Q|$ is small,

$$2Q^4 \int_{4B_1}^{\infty} x^{-3} e^{-x}\,dx < \int_{B_1}^{\infty} \sin^4\frac{\theta}{2}\, B\,dB < 2Q^4 \int_{4B_1}^{\infty} x\left(\frac{1}{x} + \frac{1}{4}\right)^4 e^{-x}\,dx$$

which integrates to

$$\frac{Q^4}{16B_1^2}(1-3B_1) < \int_{B_1}^{\infty} \sin^4\frac{\theta}{2}\, B\,dB < \frac{Q^4}{16B_1^2}(1+3B_1)$$

where small terms have been neglected. Since B_1 is small we have

$$\int_{B_1}^{\infty} \sin^4\frac{\theta}{2}\, B\,dB \approx \frac{Q^4}{16B_1^2} \qquad \text{..... (2.15)}$$

Since the conditions $|Q| \ll B_1$ and $|Q| \ll \frac{1}{10}$ this term can be neglected and on combining (11) and (14) σ_2 obtains as

$$\sigma_2 = 2\sigma_1 - \pi a_D^2 Q^2 \qquad \text{..... (2.16)}$$

The inequality (10) can be used to determine $\int_{B_1}^{\infty} \sin^2\frac{\theta}{2}\, B\, dB$ and hence σ_1 in the same way as it was used to evaluate the $\sin^4\frac{\theta}{2}$ term. This results in the equality

$$\pi Q^2 a_D^2\, E_1(2B_1) < \sigma_1 - \sigma_1(B_1) < \pi Q^2 a_D^2 \left[E_1(2B_1) + \frac{5+2B_1}{4} e^{-2B_1}\right]$$

where $E_1(2B_1)$ is the exponential integral $\int_{2B_1}^{\infty} e^{-x} x^{-1}\, dx$. Using the small B_1 form gives

$$-\pi Q^2 a_D^2 \left(\ln\frac{B_1}{2} + \gamma + \ln 4\right) < \sigma_1 - \sigma_1(B_1) < -\pi Q^2 a_D^2 \left(\ln\frac{B_1}{2} + \gamma + \ln 4 - \frac{5}{4}\right)$$

a result which is consistent with (12) since $\ln 4 - \frac{5}{4} < \frac{1}{2} < \ln 4$.

2.4 COMPARISON WITH THE CUT-OFF POTENTIAL

The coulomb potential with impact parameter cut-off at the Debye length gives results which are in remarkably good agreement with the analysis using a

Debye potential. The cut-off potential produces a more realistic finite result for σ_0. The relation between σ_1 and σ_2 (equation (16)), which holds approximately for the Debye potential, can be seen to hold exactly for the cut-off potential. In comparing the results for σ_1 it is convenient to consider the Coulomb logarithm defined by $\Lambda = \frac{\sigma_1}{\pi Q^2 a_D^2}$. The cut-off and Debye field values for Λ are

$$\begin{aligned} \Lambda_{\text{cut-off}} &\approx \Lambda_c \approx \ln\frac{2}{|Q|} \approx -\ln|Q| + 0.7 \\ \Lambda_{\text{Debye}} &\approx -\ln|Q| + 0.3 \approx \ln\frac{1.36}{|Q|}. \end{aligned} \qquad \text{.....} \; (2.17)$$

The agreement is very good since, as $|Q|$ is small, the logarithm term is the dominant one in each expression. Exact correspondence is obtained when the cut-off is made at about $0.7a_D$, a value which must be considered very close to a_D due to the comparative insensitivity of Λ to the exact position of the impact parameter cut-off.

CHAPTER THREE: KINETIC EQUATION FOR A PLASMA

The rigorous derivation of a Kinetic Equation is not the main aim of this section of the thesis. However, since a Kinetic equation is a necessary starting point in most discussions, in particular the present discussion, of plasma transport theory, and since most other discussions lack rigor in this respect, an outline of the derivation of a Kinetic equation from first principles will be given in this chapter. The method used is an extension of the technique developed by Born and Green(4) as summarized by Green(49). Much of the mathematical detail follows that of Green and Leipnik(26) and will not be repeated here. However, sufficient detail will be retained to keep track of the magnitude of the approximations made so that the accuracy of the final equation can be compared with those of other authors (considered at some length in chapter one).

3.1 THE HIERACHY OF EQUATIONS

We will number the particles (electrons or ions) of the plasma 1,2,3...., and denote the mass, charge, position and velocity of the i-th particle by m_i, e_i, $\underline{x}_i$, and ξ_i respectively. The phase-space distribution function F_N, is defined such that, if

$$d\Omega_N = \frac{\prod_i \underline{dx}_i\, \underline{dp}_i}{\prod_a (N_a)!} ,$$

then $F_N\, d\Omega_N$ is the probability that the a-th constituent of the plasma consists of N_a particles (a = 1,2,...) and the i-th particle will be found in the element of actual space, $\underline{dx}_i$, and the element of momentum space, $\underline{dp}_i$. The average value of any microscopic quantity G is

$$\langle G \rangle = \sum_N \int G\, F_N\, d\Omega_N \qquad \text{..... (3.1)}$$

where the summation $\sum_N$ is over all possible values of the N_a. F_N is a constant of the motion and satisfies the Liouville equation.

The one-particle distribution function, f (or f_a), is the first of a sequence of functions, f, f_2, f_3... (or f_a, f_{ab}, f_{abc} ...), which characterize the velocity distributions of groups of 1,2,3.... particles in a plasma. These functions may be defined quite generally by

$$f_a = \left\langle \sum_i \delta_{ia}\, \delta(\underline{x} - \underline{x}_i)\, \delta(\underline{\xi} - \underline{\xi}_i) \right\rangle$$

$$f_{ab} = \left\langle \sum_i \delta_{ia}\, \delta(\underline{x} - \underline{x}_i)\, \delta(\underline{\xi} - \underline{\xi}_i) \sum_j \delta_{jb}\, \delta(\underline{x} - \underline{x}_j)\, \delta(\underline{\xi} - \underline{\xi}_j) \right\rangle , \qquad \text{.....} (3.2)$$

f_{abc} , etc., similarly. These definitions written in the form (1) in conjunction with the Liouville equation lead to the BBGKY hierachy of equations, the first two being

$$\frac{\partial f_a}{\partial t} + \underline{\xi}_a \cdot \frac{\partial f_a}{\partial \underline{x}_a} + \frac{\underline{F}_a^{ext}}{m_a} \cdot \frac{\partial f_a}{\partial \underline{\xi}_a} - \sum_b \iint \frac{1}{m_a} \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} \cdot \frac{\partial f_{ab}}{\partial \underline{\xi}_a}\, d\underline{x}_b\, d\underline{\xi}_b = 0$$

$$\frac{\partial f_{ab}}{\partial t} + \underline{\xi}_a \cdot \frac{\partial f_{ab}}{\partial \underline{x}_a} + \underline{\xi}_b \cdot \frac{\partial f_{ab}}{\partial \underline{x}_b} + \frac{\underline{F}_a^{ext}}{m_a} \cdot \frac{\partial f_{ab}}{\partial \underline{\xi}_a} + \frac{\underline{F}_b^{ext}}{m_b} \cdot \frac{\partial f_{ab}}{\partial \underline{\xi}_b} - \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} \cdot \left[\frac{1}{m_a} \frac{\partial f_{ab}}{\partial \underline{\xi}_a} - \frac{1}{m_b} \frac{\partial f_{ab}}{\partial \underline{\xi}_b} \right] \qquad \text{.....} (3.3)$$

$$- \sum_c \iint \left[\frac{1}{m_a} \frac{\partial \varphi_{ac}}{\partial \underline{x}_a} \cdot \frac{\partial f_{abc}}{\partial \underline{\xi}_a} + \frac{1}{m_b} \frac{\partial \varphi_{bc}}{\partial \underline{x}_b} \cdot \frac{\partial f_{abc}}{\partial \underline{\xi}_b} \right] d\underline{x}_c\, d\underline{\xi}_c = 0$$

where $\underline{F}_a^{ext}$ is the external force acting on a particle of the a-th type and φ_{ab} is the mutual potential energy of two particles of types a and b and is thus, in a plasma, the coulomb potential, $\varphi_{ab} = \dfrac{e_a e_b}{4\pi\epsilon_0 r}$, $\underline{r} = \underline{x}_a - \underline{x}_b$.

The hierachy is insoluble until terminated by the introduction of a suitable approximation.

3.2 TERMINATING THE HIERACHY

In order to make the hierachy (3) soluble we follow Green[26,49] in borrowing a technique from equilibrium theory, the 'superposition approximation' of Kirkwood[50].

$$n_{abc} = \frac{n_{ab}\, n_{bc}\, n_{ac}}{n_a n_b n_c}$$

where $n_a = \int f_a \, d\underline{\xi}_a$ (n_{ab}, n_{abc} similarly) is the number density of particles of the a-th kind at the point $\underline{x}_a$. Green[27] has shown that the superposition approximation is a good approximation for use with plasmas and is possibly better with plasmas than with ordinary fluids. An expansion in powers of the parameter $\frac{\Phi_{ab}}{kT}$, the 'strength parameter' of chapter one, leads to the superposition approximation when higher than first order terms are neglected. A further consequence of this is that the Debye-Hueckel approximation of equilibrium theory is rather a better approximation than is generally suggested and the range of validity of the Debye-Hueckel theory extends much further than the approximations used by Debye and Hueckel.

The non-equilibrium generalization of the superposition approximation, the 'generalized superposition approximation'[49] is

$$f_{abc} = \frac{f_{ab}\, f_{bc}\, f_{ac}}{f_a\, f_b\, f_c} \qquad \text{.....} \; (3.4)$$

It can reasonably be expected that this approximation is as good as the superposition approximation and is equivalent to a first order approximation in powers of the strength parameter. That this is so can be illustrated in the following way. The distribution functions f_a, f_{ab} etc. are 'normalized' to the number densities n_a, n_{ab} etc. As a consequence, in equilibrium, the generalized superposition approximation reduces to the superposition approximation. An alternative generalization of the superposition approximation, which is independent of it, is

$$F_{abc} = \frac{F_{ab}\, F_{bc}\, F_{ac}}{F_a\, F_b\, F_c} \qquad \text{.....} \; (3.5)$$

where the F_a, F_{ab} etc. are 'normalized' to unity (i.e. $f_a = n_a F_a$ etc.). Equation (5) holds trivially in equilibrium, and can be seen to be accurate to first order in the strength parameter by comparison with the cluster expansions used by Rostoker and Rosenbluth[18] and others[19]. The cluster expansions may be written

$$F_{ab} = F_a F_b + G_{ab} \qquad \text{..... (3.6)}$$

$$F_{abc} = F_a F_b F_c + G_{ab} F_c + G_{bc} F_a + G_{ac} F_b + g_{abc} \qquad \text{..... (3.7)}$$

To obtain a Kinetic equation accurate to first order in the strength parameter one assumes $G_{ab} \sim \epsilon$, $g_{abc} \sim \epsilon^2$ where $\epsilon = (na_D^3)^{-1}$ can be identified with the strength parameter (see chapter one). Hence

$$\frac{G_{ab} G_{bc}}{F_b} + \frac{G_{ab} G_{ac}}{F_a} + \frac{G_{ab} G_{bc}}{F_c} + \frac{G_{ab} G_{bc} G_{ac}}{F_a F_b F_c} \sim \epsilon^2$$

This expression can be added to the right-hand side of equation (7) to obtain an alternative expression valid to the same degree of approximation since $g_{abc} \sim \epsilon^2$. This can be factorized to obtain the generalized superposition approximation (5) plus terms of order ϵ^2 as required.

Instead of using the generalized superposition approximation to terminate the BBGKY hierachy, a further refinement, the 'disjunctive approximation' (see Green[49]), which is considered also to be an improvement, will be used. If $f_a^{(0)}$, $f_{ab}^{(0)}$, ... are the local equilibrium distribution functions ('local' in the sense that the macroscopic variables n, $\underline{u}$ and T depend on position and time) and if f_a', f_{ab}', ... are defined by

$$f_a' = f_a - f_a^{(0)}$$

etc., then the disjunctive approximation is

$$\frac{f_{abc}'}{f_{abc}^{(0)}} = \frac{f_{ab}'}{f_{ab}^{(0)}} + \frac{f_{bc}'}{f_{bc}^{(0)}} + \frac{f_{ac}'}{f_{ac}^{(0)}} - \frac{f_a'}{f_a^{(0)}} - \frac{f_b'}{f_b^{(0)}} - \frac{f_c'}{f_c^{(0)}} \qquad \text{..... (3.8)}$$

This is closely related to the generalized superposition approximate (5) and to the cluster expansion (7), and these relationships indicate that it is at least as good as a first order approximation in powers of the strength parameter. On making the substitutions $f_a = f_a^{(0)} + f_a'$ etc. in equation (5) and neglecting terms higher than first order in the deviations from equilibrium, f_a', f_{ab}' etc., one obtains the disjunctive approximation. To compare the disjunctive approximation with the cluster expansion it is convenient to rewrite equation (7) as

$$F_{abc} = F_{ab}F_c + F_{bc}F_a + F_{ac}F_b - 2F_aF_bF_c + g_{abc}$$

Equation (8) can be rewritten as

$$F_{abc} = F_{ab}F_{c} + F_{bc}F_{a} + F_{ac}F_{b} - 2F_{a}F_{b}F_{c} + \frac{2f'_a f'_b f'_c}{n_a n_b n_c}$$

and the correspondence between the two expressions is obvious. These results indicate that the disjunctive approximation is at least as good as first order in ϵ. It should be noted that the disjunctive approximation holds trivially in equilibrium and is a most suitable approximation for discussions of transport phenomena where deviations from equilibrium are small.

3.3 THE KINETIC EQUATION

The hierachy (3) can be simplified to

$$\frac{\partial f_a}{\partial t} + \frac{\partial}{\partial \underline{x}_a}\cdot(f_a \underline{\xi}_a) + \frac{\partial}{\partial \underline{\xi}_a}\cdot(f_a \underline{\eta}_a) = 0$$

$$\frac{\partial f_{ab}}{\partial t} + \frac{\partial}{\partial \underline{x}_a}\cdot(f_{ab}\underline{\xi}_a) + \frac{\partial}{\partial \underline{x}_b}\cdot(f_{ab}\underline{\xi}_b) + \frac{\partial}{\partial \underline{\xi}_a}\cdot(f_{ab}\underline{\eta}_{ab}) + \frac{\partial}{\partial \underline{\xi}_b}\cdot(f_{ab}\underline{\eta}_{ba}) = 0 \qquad \text{.....(3.9)}$$

where $\underline{\eta}_a$ is the mean acceleration of type-a particles and $\underline{\eta}_{ab}$ is the mean acceleration of type-a particles conditional on a particle of type b being at $(\underline{x}_b, \underline{\xi}_b)$.

$f_a \underline{\xi}_a$ is the particle flux in actual space and $f_a \underline{\eta}_a$ is the particle flux in velocity space. We have, exactly, the relation

$$m_a \underline{\eta}_a = \underline{F}_a^{ext} + \langle \Sigma' \underline{F} \rangle$$

where $\underline{F}_a^{ext}$ $(= e_a(\underline{E}^{ext} + \underline{\xi}_a \times \underline{B}^{ext}))$ is the external force on a particle of type a and $\langle \Sigma' \underline{F} \rangle$ is the mean force on particle a due to all other particles (Σ' denotes summation over all particles except the one in question). $\langle \Sigma' \underline{F} \rangle$ can be expressed as

$$f_a \langle \Sigma' \underline{F} \rangle = -\sum_b \iint (f_{ab} - f_a f_b) \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} d\underline{x}_b \, d\underline{\xi}_b - f_a \sum_b \iint f_b \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} d\underline{x}_b \, d\underline{\xi}_b$$

where an amount $f_a f_b$ has been added and subtracted from the integrand to separate the macroscopic contribution, $-\sum_b \iint f_b \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} d\underline{x}_b \, d\underline{\xi}_b$. This term can be added to the external force to obtain the total macroscopic force and it corresponds to the Vlasov term contribution to the Kinetic equation (see chapter one). If the macroscopic force, $\underline{F}_a$, is defined by

$$\underline{F}_a = \underline{F}_a^{ext} - \sum_b \iint f_b \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} d\underline{x}_b \, d\underline{\xi}_b$$

then $\underline{\eta}_a$ becomes

$$\underline{\eta}_a = \frac{1}{m_a}\underline{F}_a - \frac{1}{f_a}\sum_b \iint \frac{1}{m_a}\frac{\partial \varphi_{ab}}{\partial \underline{x}_a}\left(f_{ab} - f_a f_b\right) d\underline{x}_b \, d\underline{\xi}_b \qquad ..... (3.10)$$

Similarly we obtain

$$\underline{\eta}_{ab} = \frac{1}{m_a}\underline{F}_a - \frac{1}{f_{ab}}\sum_c \iint \frac{1}{m_a}\frac{\partial \varphi_{ac}}{\partial \underline{x}_a}\left(f_{abc} - f_{ab} f_c\right) d\underline{x}_c \, d\underline{\xi}_c - \frac{1}{m_a}\frac{\partial \varphi_{ab}}{\partial \underline{x}_a} \qquad ..... (3.11)$$

This procedure separates long-range and short-range effects. The former are grouped with the external forces and the latter will contribute to the collision integral which must therefore automatically include the effects of shielding. If a shielded potential ψ_{ab} is defined by

$$\frac{\partial \psi_{ab}}{\partial \underline{x}_a} = \frac{\partial \varphi_{ab}}{\partial \underline{x}_a} + \sum_c \iint \frac{1}{m_a}\frac{\partial \varphi_{ac}}{\partial \underline{x}_a}\left(\frac{f_{abc}}{f_{ab}} - f_c\right) d\underline{x}_c \, d\underline{\xi}_c$$

then equation (11) becomes

$$\underline{\eta}_{ab} = \frac{1}{m_a}\underline{F}_a - \frac{1}{m_a}\frac{\partial \psi_{ab}}{\partial \underline{x}_a} \qquad ..... (3.12)$$

In equilibrium ψ_{ab} reduces to the Debye potential (equation (2.2)).

Equations (10) and (12) can now be substituted back into the hierachy (9). To simplify the resulting expressions we introduce the functions γ_a, γ_{ab}, ...etc. and γ'_a, γ'_{ab}, ... etc. defined by

$$\gamma_a = \frac{f_a}{f_a^{(0)}} = \gamma'_a + 1 \quad , \quad \gamma_{ab} = \frac{f_{ab}}{f_{ab}^{(0)}} = \gamma'_{ab} + 1 \quad , \quad \text{etc..}$$

(In this notation the disjunctive approximation is $\gamma'_{abc} = \gamma'_{ab} + \gamma'_{bc} + \gamma'_{ac} - \gamma'_a - \gamma'_b - \gamma'_c$). After some manipulation, and using the fact that $f_a^{(0)}$ and $f_{ab}^{(0)}$ satisfy the same equations as f_a and f_{ab}, the first two members of the hierachy reduce to

$$D_a(f_a^{(0)}\gamma_a) = \sum_b \iint \frac{1}{m_a}\frac{\partial \varphi_{ab}}{\partial \underline{x}_a} \cdot \left[\frac{\partial}{\partial \underline{\xi}_a}(f_{ab}^{(0)}\gamma_{ab}) - f_a^{(0)} f_b^{(0)} \frac{\partial \gamma_a}{\partial \underline{\xi}_a} - \gamma_a \frac{\partial f_{ab}^{(0)}}{\partial \underline{\xi}_a}\right] d\underline{x}_b \, d\underline{\xi}_b \quad \text{.....(3.13)}$$

$$D_{ab}\gamma_{ab} = \frac{1}{m_a}\frac{\partial \psi_{ab}}{\partial \underline{x}_a} \cdot \frac{\partial \gamma_{ab}}{\partial \underline{\xi}_a} + \frac{1}{m_b}\frac{\partial \psi_{ab}}{\partial \underline{x}_b} \cdot \frac{\partial \gamma_{ab}}{\partial \underline{\xi}_b} - \Delta_{ab} - \Delta_{ba} \quad \text{..... (3.14)}$$

where
$$D_a \equiv \frac{\partial}{\partial t} + \underline{\xi}_a \cdot \frac{\partial}{\partial \underline{x}_a} + \frac{1}{m_a}\underline{F}_a \cdot \frac{\partial}{\partial \underline{\xi}_a}$$

$$D_{ab} \equiv D_a + \underline{\xi}_b \cdot \frac{\partial}{\partial \underline{x}_b} + \frac{1}{m_b}\underline{F}_b \cdot \frac{\partial}{\partial \underline{\xi}_b}$$

and

$$\Delta_{ab} = \sum_c \iint \frac{1}{m_a}\frac{\partial \varphi_{ac}}{\partial \underline{x}_a} \cdot \left[\frac{f_{abc}^{(0)}}{f_{ab}^{(0)} f_a^{(0)}} \frac{\partial}{\partial \underline{\xi}_a}\left(f_a^{(0)}\gamma'_{abc} - f_a^{(0)}\gamma'_{ab}\right) - f_c^{(0)} \frac{\partial \gamma_{ab}}{\partial \underline{\xi}_a}\right] d\underline{x}_c \, d\underline{\xi}_c$$

with Δ_{ba} defined similarly.

The disjunctive approximation is now used to eliminate γ'_{abc} from the expressions for Δ_{ab} and Δ_{ba}. The resulting equations are identical in form to those obtained by Green and Leipnik[26] except that the equations of these authors describe the behaviour of the time correlation functions which, in the above expressions, are replaced by the velocity distribution functions. The method of solution can now follow Green and Leipnik exactly and need not be repeated in detail here.

Equation (13) and the assumption that the particle densities are effectively constant within a sphere of radius of order a_D, the Debye length, can be used to simplify the expressions for Δ_{ab} and Δ_{ba}. If the terms in these expressions which are small for small deviations from equilibrium are neglected equation (14) becomes

$$D_{ab}\gamma_{ab} - \frac{1}{m_a}\frac{\partial \psi_{ab}}{\partial \underline{x}_a}\cdot\frac{\partial \gamma_{ab}}{\partial \underline{\xi}_a} - \frac{1}{m_b}\frac{\partial \psi_{ab}}{\partial \underline{x}_b}\cdot\frac{\partial \gamma_{ab}}{\partial \underline{\xi}_b} = D_{ab}(\gamma_a + \gamma_b)$$

which, apart from the inhomogeneous term $D_{ab}(\gamma_a + \gamma_b)$ is the Liouville equation for a pair of particles in a potential ψ_{ab}. This equation can be solved. By defining the operator O (acting on an arbitrary function

H) as

$$O(H) = \int_{-\infty}^{t} H(t-t_o)\, dt_o \quad ,$$

and using the boundary condition that particles are uncorrelated at some initial time t_o (this introduces irreversibility into the problem in the same way as does the Bogoliubov technique described in chapter one) the following expression can be obtained after correcting some minor errors in Green and Leipnik.

$$O(D_a(f_a^{(o)}\gamma_a)) = \sum_b \iint f_a^{(o)} f_b^{(o)} \left[O(\gamma_{ao} + \gamma_{bo} - \gamma_a - \gamma_b)\right] \rho \, \underline{d\sigma} \, \underline{d\xi_b} \quad .....(3.15)$$

where $\underline{\rho} = \underline{\xi_a} - \underline{\xi_b}$, $\underline{d\sigma}$ is an element of the cross-section for collisions in a potential ψ_{ab} (see chapter two), $\gamma_{ao} = \gamma_a(\underline{\xi_a} = \underline{\xi_{ao}})$, γ_{bo} similarly, and $\underline{\xi_{ao}}$ and $\underline{\xi_{bo}}$ are initial ('pre-collision') velocities at time $t = t_o$. Here 'pre-collision' and 'initially' mean 'effectively outside the range of the potential ψ_{ab}' (which is of order a_D).

In deriving this equation two approximations have been made. The first is that strong external forces are absent. If strong external forces are present they will have an appreciable affect on particle trajectories over distances of the order of the range of ψ_{ab} and must appear inside the collision integral. This case is considered by Green and Leipnik. The second approximation is that $\textcircled{H}(r) \approx 1$ where $\textcircled{H}(r)$ is defined by ($\underline{r} = \underline{x}_a - \underline{x}_b$)

$$\frac{\partial \varphi_{ab}}{\partial r} = \textcircled{H}(r) \frac{\partial \psi_{ab}}{\partial r}$$

and relates the coulomb potential to the potential ψ_{ab}. Since $\textcircled{H}(r)$ appears only as a factor multiplying terms which are deviations from equilibrium the equilibrium form of this function may be used. In equilibrium is the Debye potential so that $\textcircled{H}(r) \approx 1 + \frac{1}{2}(\frac{r}{a_D})^2$. The approximation $\textcircled{H}(r) = 1$ is satisfactory since the range of the potential ψ_{ab} is only of order a_D. For similar reasons the equilibrium form of ψ_{ab} may be used to determine the equation (15) more explicitly provided we consider only phenomena where deviations from equilibrium are small.

Eliminating the operator 0 from (15) we obtain

$$D_a f_a = \sum_b \iint f_a^{(o)} f_b^{(o)} (\gamma'_{ao} + \gamma'_{bo} - \gamma'_a - \gamma'_b) \rho \, d\sigma \, d\underline{\xi}_b \quad \ldots\ldots (3.16)$$

where the two-particle potential which describes the particle interactions is a Debye potential. This is the same form as the linearized Boltzmann equation.

This equation is valid to first order in the strength parameter and to first order in deviations from equilibrium. Because of its similarity to the Boltzmann equation it obviously does not suffer from the divergence difficulties at small separations ($\underline{x}_a - \underline{x}_b$ small) inherent in equations which rely on an expansion in powers of the strength parameter for their derivation (the BLG equation of chapter one). This is partly due to the neglect of spatial inhomogeneities over distances of order a_D or less; the approximation used to reduce the generalized Boltzmann equation to the Boltzmann equation. Irreversibility has been introduced by an initial condition when particles were uncorrelated. That this initial time corresponds to when particle separation $\underline{x}_a - \underline{x}_b$ was large compared with a_D introduces a time-scale into the problem. The initial time t_o is a long time in the 'past' compared with the 'collision time' $(= \frac{a_D}{u}$, see chapter one), but short compared with the time between collisions so that the equation (16) cannot describe phenomena which have characteristic times of order τ_o or less.

CHAPTER FOUR: TRANSPORT COEFFICIENTS

4.1 THE LANDAU FORM OF THE KINETIC EQUATION

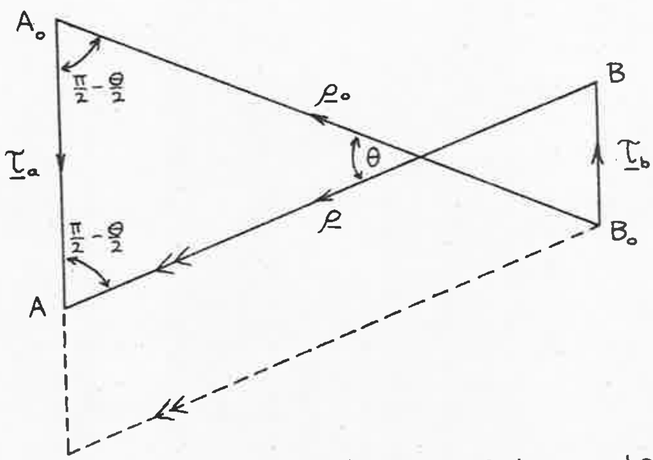
In chapter three a Kinetic equation valid to first order in deviations from equilibrium has been derived.

$$D_a f_a = \sum_b C_{ab} = \sum_b \iint f_a^{(0)} f_b^{(0)} (\gamma_{ao}^{`} + \gamma_{bo}^{`} - \gamma_a^{`} - \gamma_b^{`}) \rho \, d\underline{\delta} \, d\underline{\xi}_b \quad \ldots\ldots (4.1)$$

This equation can be simplified. To do so, first consider the dynamics of a two particle interaction involving a particle of type a and one of type b (Debye potential). The 'initial' and 'final' velocities are $\underline{\xi}_a$, $\underline{\xi}_b$, $\underline{\xi}_{ao}$, $\underline{\xi}_{bo}$ where 'initial' and 'final' correspond to pre- and post-collision times when the particles are separated by amounts much greater than the range of the interparticle (Debye) potential. From simple considerations (which do not involve the explicit form of the potential) a number of relationships can be established. These are made clearer by reference to the following diagram in which the points A_o, B_o, A and B are the end-points of the vectors $\underline{\xi}_{ao}$, $\underline{\xi}_{bo}$, $\underline{\xi}_a$ and $\underline{\xi}_b$. These points are necessarily coplanar. The velocity vectors themselves, which do not, in general, lie in the plane of the diagram, are not shown. The vectors $\underline{\tau}_a$ and $\underline{\tau}_b$ are defined by

$$\underline{\tau}_a = \underline{\xi}_a - \underline{\xi}_{ao} \quad ; \quad \underline{\tau}_b = \underline{\xi}_b - \underline{\xi}_{bo} \quad ,$$

$\underline{\rho} = \underline{\xi}_a - \underline{\xi}_b$ ($\underline{\rho}_o = \underline{\xi}_{ao} - \underline{\xi}_{bo}$), as previously, and θ is the scattering angle. In constructing the diagram $|\underline{\rho}_o| = |\underline{\rho}|$, which follows from conservation of momentum, has been used.



From conservation of momentum $m_b|\underline{\tau}_b| = m_a|\underline{\tau}_a|$. Also $\underline{\tau}_a$ and $\underline{\tau}_b$ are anti-parallel. From the diagram $\tau_a + \tau_b = 2\rho \sin \frac{\theta}{2}$ so that

$$\frac{m_a}{M} \tau_a = \frac{m_b}{M} \tau_b = 2\rho \sin\frac{\theta}{2} \qquad \text{..... (4.2)}$$

where $M = \dfrac{m_a m_b}{m_a+m_b}$ is the reduced mass.

Since most interactions are such that the change of velocity of both particles is relatively small the terms $\gamma'_{ao}-\gamma'_a$ and $\gamma'_{bo}-\gamma'_b$ in the integrand in equation (1) can be expanded as Taylor series and third and higher order terms neglected. Hence the collision integral C_{ab} becomes

$$C_{ab} = \iint f_a^{(0)} f_b^{(0)} \left[-\underline{\tau}_a \cdot \frac{\partial \phi_a}{\partial \underline{\xi}_a} + \frac{1}{2} \underline{\tau}_a \underline{\tau}_a : \frac{\partial}{\partial \underline{\xi}_a} \frac{\partial \phi_a}{\partial \underline{\xi}_a} - \underline{\tau}_b \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} + \frac{1}{2} \underline{\tau}_b \underline{\tau}_b : \frac{\partial}{\partial \underline{\xi}_b} \frac{\partial \phi_b}{\partial \underline{\xi}_b} \right] \rho \, d\underline{\sigma} \, d\underline{\xi}_b$$

where, to comply with a more commonly used notation, γ'_a and γ'_b have been replaced by ϕ_a and ϕ_b. The γ'_a and ϕ_a are related by

$$f_a = f_a^{(0)} + f_a^{(1)} = f_a^{(0)} + \gamma'_a f_a^{(0)} = f_a^{(0)} + \phi_a f_a^{(0)}$$

Using the relation (2) C_{ab} can be written

$$C_{ab} = \iint f_a^{(0)} f_b^{(0)} \left[\left\{ \frac{m_b}{m_a} \underline{\tau}_b \cdot \frac{\partial \phi_a}{\partial \underline{\xi}_a} + \frac{m_b^2}{2m_a^2} \underline{\tau}_b \underline{\tau}_b : \frac{\partial}{\partial \underline{\xi}_a} \frac{\partial \phi_a}{\partial \underline{\xi}_a} \right\} + \left\{ -\underline{\tau}_b \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} + \frac{1}{2} \underline{\tau}_b \underline{\tau}_b : \frac{\partial}{\partial \underline{\xi}_b} \frac{\partial \phi_b}{\partial \underline{\xi}_b} \right\} \right] \rho \, d\underline{\sigma} \, d\underline{\xi}_b$$

so that

$$C_{ab} = I_a + I_b \; ; \; I_b = \iint f_a^{(0)} f_b^{(0)} \left[-\underline{\tau}_b \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} + \frac{1}{2} \underline{\tau}_b \underline{\tau}_b : \frac{\partial}{\partial \underline{\xi}_b} \frac{\partial \phi_b}{\partial \underline{\xi}_b} \right] \rho \, d\underline{\sigma} \, d\underline{\xi}_b \quad \ldots\ldots(4.3)$$

where I_a can be obtained from I_b by substituting $-\frac{m_b}{m_a}\frac{\partial\phi_a}{\partial\underline{\xi}_a}$ for $\frac{\partial\phi_b}{\partial\underline{\xi}_b}$.

To evaluate I_b we notice that, if $\underline{\tau}_b$ is split up into components parallel and perpendicular to $\underline{\rho}$

$$-\underline{\tau}_b = \tau_b \sin\frac{\theta}{2}\ \hat{\underline{\rho}} + \tau_b \cos\frac{\theta}{2}\ \hat{\underline{\rho}}_\perp$$

(the circumflex denotes unit vector), the contribution to the integral (3) from the perpendicular part will be zero. Thus $\underline{\tau}_b$ can be replaced by $-\frac{\tau_b}{\rho}\sin\frac{\theta}{2}\underline{\rho} = -2\sin^2\frac{\theta}{2}\ \frac{M}{m_b}\underline{\rho}$ (using equation (2)) under the integral. Similarly $\underline{\tau}_b\underline{\tau}_b$ can be replaced by

$$\underline{\rho}\underline{\rho}\left[\frac{M^2}{m_b^2}\left(2\sin^2\frac{\theta}{2}\left(1-3\sin^2\frac{\theta}{2}\right)\right)\right] - \underline{\underline{\delta}}\left[\frac{M^2}{m_b^2}\left(2\rho^2\sin^2\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right)\right)\right]$$

so that I_b becomes

$$I_b = \iint f_a^{(0)} f_b^{(0)}\Big[\underline{\rho}\cdot\frac{\partial\phi_b}{\partial\underline{\xi}_b}\left\{\frac{M}{m_b}(1-\cos\theta)\right\} - \frac{1}{2}\underline{\rho}\underline{\rho}:\frac{\partial}{\partial\underline{\xi}_b}\frac{\partial\phi_b}{\partial\underline{\xi}_b}\left\{\frac{M^2}{2m_b^2}(1-\cos\theta)(1-3\cos\theta)\right\}$$
$$-\frac{1}{2}\rho^2\frac{\partial}{\partial\underline{\xi}_b}\cdot\frac{\partial\phi_b}{\partial\underline{\xi}_b}\left\{\frac{M^2}{2m_b^2}(1-\cos\theta)(1+\cos\theta)\right\}\Big]\rho\, d\underline{\sigma}\, d\underline{\xi}_b\ .$$

I_b can now be expressed in terms of the transport cross-sections σ_1 and σ_2 defined (and evaluated for the case of a Debye potential) in chapter one. Using the relation $\sigma_2 = 2\sigma_1 - \pi a_b^2 Q^2$ (equation (2.16)) we find

$$I_b = \int f_a^{(0)} f_b^{(0)} \Big[\rho \cdot \frac{\partial \phi_b}{\partial \xi_b} \Big(\frac{M}{m_b} \sigma_1 \Big) + \tfrac{1}{2} \rho\rho : \frac{\partial}{\partial \xi_b} \frac{\partial \phi_b}{\partial \xi_b} \Big(\frac{M^2}{m_b^2} \big(\sigma_1 - \tfrac{3}{2}\pi a_D^2 Q^2\big)\Big)$$
$$- \tfrac{1}{2} \rho^2 \frac{\partial}{\partial \xi_b} \cdot \frac{\partial \phi_b}{\partial \xi_b} \Big(\frac{M^2}{m_b^2} \big(\sigma_1 - \tfrac{1}{2}\pi a_D^2 Q^2\big)\Big)\Big] \rho \, d\xi_b \quad \text{..... (4.4)}$$

Equation (4) can be further simplified by using the following relation (in which $R(\rho)$ is an unspecified function of ρ).

$$\frac{\partial}{\partial \xi_a} \cdot \int R(\rho) \exp\{-\tfrac{1}{2}\beta m_a \xi_a^2 - \tfrac{1}{2}\beta m_b \xi_b^2\} \; \frac{\rho^2 \underline{\delta} - \rho\rho}{\rho^3} \cdot \frac{\partial \phi_b}{\partial \xi_b} \, d\xi_b$$
$$= \int R(\rho) \exp\{-\tfrac{1}{2}\beta m_a \xi_a^2 - \tfrac{1}{2}\beta m_b \xi_b^2\} \Big(\frac{\partial}{\partial \xi_a} - \frac{m_a}{m_b} \frac{\partial}{\partial \xi_b} \Big) \cdot \frac{\rho^2 \underline{\delta} - \rho\rho}{\rho^3} \cdot \frac{\partial \phi_b}{\partial \xi_b} \, d\xi_b$$
$$= -\frac{2 m_a m_b}{M^2} \int \frac{R(\rho)}{\rho^3} \exp\{-\tfrac{1}{2}\beta m_a \xi_a^2 - \tfrac{1}{2}\beta m_b \xi_b^2\} \Big[-\frac{M}{m_b} \rho \cdot \frac{\partial \phi_b}{\partial \xi_b}$$
$$- \tfrac{1}{2}\rho\rho : \frac{M^2}{m_b^2} \frac{\partial}{\partial \xi_b} \frac{\partial \phi_b}{\partial \xi_b} + \tfrac{1}{2}\rho^2 \frac{M^2}{m_b^2} \frac{\partial}{\partial \xi_b} \cdot \frac{\partial \phi_b}{\partial \xi_b} \Big] d\xi_b \quad \text{..... (4.5)}$$

The detail derivation of (5) is quite straight-forward and is given in Appendix B. It can be seen that, if $\pi a_D^2 Q^2 \ll |\sigma_1|$, I (equation (4)) can be written in the form (5) with

$$R(\rho) = \tfrac{1}{16} \pi^{-3} n_a n_b \beta^3 \sqrt{m_a m_b} \, M^2 \rho^4 \sigma_1 \; ; \; \beta = \frac{1}{kT} .$$

From equation (2.13) (viz. $\sigma_1 = -\pi a_D^2 Q^2 (\ln|Q| - .309) \approx -\pi a_D^2 Q^2 \ln|Q|$), since $\ln|Q| \gg 1$ for plasmas of interest, it follows that $|\sigma_1|$ is much greater than $\pi a_D^2 Q^2$. Using the above expression for σ_1 and the explicit form $Q = \frac{2\lambda}{a_D \rho_0^2} = \frac{2\lambda}{a_D \rho^2}$

(where $\lambda = \frac{e_a e_b}{4\pi\varepsilon_o M}$ has been defined in chapter two) I_b, with the aid of equation (5) becomes,

$$I_b = -\frac{\partial}{\partial \underline{\xi}_a} \cdot \left[\int f_a^{(o)} f_b^{(o)} \frac{2\pi M^2 \lambda^2}{m_a m_b} (\ln|Q| - .309) \left(\frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \right) d\underline{\xi}_b \right] .$$

Using this result, obtaining I_a from it by putting $-\frac{m_b}{m_a} \frac{\partial \phi_a}{\partial \underline{\xi}_a}$ for $\frac{\partial \phi_b}{\partial \underline{\xi}_b}$, and defining the tensor $\underline{\underline{K}}_{ab}$ as

$$\underline{\underline{K}}_{ab} = -\frac{2\pi M^2 \lambda^2}{m_a m_b} (\ln|Q| - .309) \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} , \qquad \text{.....(4.6)}$$

equation (3) can be written

$$C_{ab} = \frac{\partial}{\partial \underline{\xi}_a} \cdot \left[\int f_a^{(o)} f_b^{(o)} \underline{\underline{K}}_{ab} \cdot \left(\frac{\partial \phi_b}{\partial \underline{\xi}_b} - \frac{m_b}{m_a} \frac{\partial \phi_a}{\partial \underline{\xi}_a} \right) d\underline{\xi}_b \right] \qquad \text{.....(4.7)}$$

This result is similar to Landau's form of the collision integral for plasmas, differing in the exact form of the logarithm term. In the original derivation of Landau[30], the inherent divergences are eliminated by introducing a quantitatively justified shielded potential. The form (7) can also be deduced from the Fokker-Planck equation (see, for example, Robinson and Bernstein[34]) a derivation which suffers from the same short-comings as Landau's (see chapter one). In the present derivation the Debye potential occurs quite naturally as part of the theory. Equation (7) differs from that of Landau in that it is strictly limited to situations where deviations from equilibrium are small. Equation (7) also retains an additional ρ-dependent term in the integrand through the dependence of $\underline{\underline{K}}_{ab}$ on Q. The derivation presented introduces no restrictions on the range of validity of the equation.

To simplify this Kinetic equation further the term $\ln|Q|$ $(= \ln\left|\frac{2\lambda}{a_D\rho^2}\right|)$ in $\underline{\underline{K}}_{ab}$ would appear to create some difficulty. A procedure, equivalent to that which has been used in similar situations by most other authors (see, for example, Rosenbluth et. al.[23], Sivukhin[47]), would be to replace $|Q|$ by some average value independent of ρ. Such a step is usually justified by the statement that $\ln|Q|$ is comparatively insensitive to variations in ρ. However, this procedure is not only unnecessary, but by its use some insight into the problem is lost. The arguments for the choice of a particular average value for $|Q|$ are generally unsatisfactory. Robinson and Bernstein[34] have estimated the error in making this step. They show that it is small, but their result is quite sensitive to their choice of a trial form for part of the non-equilibrium distribution function. In the following section we will derive a more general result which eliminates the necessity for this approximation and shows that it is a good approximation without having to assume any trial form for the distribution function.

4.2 THE COLLISION INTEGRAL IN OPERATOR FORM

In the previous section (the application of equation (5)) and in chapter three the equilibrium value for the velocity distribution function has been assumed to be a Maxwellian distribution of the form

$$f_a^{(0)} = n_a\left(\frac{\beta m_a}{2\pi}\right)^{3/2} \exp\left(-\tfrac{1}{2}\beta m_a \xi_a^2\right)$$

although this has not been stated explicitly. The derivations given, however, do not depend on the mass average velocity being zero and the equilibrium distribution

$$f_a^{(0)} = n_a\left(\frac{\beta m_a}{2\pi}\right)^{3/2} \exp\left\{-\tfrac{1}{2}\beta m_a\left(\underline{\xi}_a - \underline{u}\right)^2\right\} ,$$

where $\underline{u} = \frac{1}{\rho}\sum_a m_a \int \underline{\xi}_a f_a d\underline{\xi}_a$ $(\rho = \sum_a m_a n_a)$ is the mass average velocity, can be used equally well. We will use the notation $\underline{v}_a = \underline{\xi}_a - \underline{u}$ for the peculiar velocity. The following discussion will be restricted to small deviations from equilibrium where the plasma components have the same temperature. Some authors(36) have studied the details of the relaxation to this stage during which the plasma components are assumed to be individually approaching equilibrium, but not to be near collective equilibrium; the components thus have different temperatures. Such a situation may be important in astrophysical applications of plasma kinetic theory where particle number densities are very low. Much of the work here could be generalized in the same way that other discussions have been extended to cover this possibility.

In the determination of transport coefficients the most commonly used technique is the Chapman-Enskog method, discussed at length in Chapman and Cowling(1). In this method the functional ansatz, that the distribution function depends on time only through a functional dependence on the (time-dependent) macroscopic variables, n, $\underline{u}$ and T, is made. The Kinetic equation is solved for ϕ, the deviation from equilibrium, by an expansion procedure. It is then found that the transport coefficients can be evaluated without determining the form of ϕ explicitly. This method of solving the Kinetic equation will not be used here. Instead an operator which determines ϕ will be defined and the Kinetic equation will be solved for the operator rather than for the function ϕ. The basis of this method is the same as the Chapman-Enskog method in that a functional ansatz must be made. The details of the solution for the operator, although fundamentally different to solving for an unknown function, bear a degree of similiarity to the more usual method. The Chapman-Enskog method can be generalized for the many-moment scheme proposed by Grad[40], but the calculations, when carried out in this way, are very cumbersome. In plasma applications a many-moment scheme is often desirable and the operator formalism outlined below could be used in this application to some advantage. However, only the usual Chapman-Enskog method will be considered here.

In terms of the peculiar velocities the collision integral (7) becomes

$$C_{ab} = \frac{\partial}{\partial \underline{v}_a} \cdot \left[\int f_a^{(0)} f_b^{(0)} \underline{\underline{K}}_{ab} \cdot \left(\frac{\partial \phi_b}{\partial \underline{v}_b} - \frac{m_b}{m_a} \frac{\partial \phi_a}{\partial \underline{v}_a} \right) d\underline{v}_b \right] \qquad \ldots\ldots (4.8)$$

We define an operator $\underline{J}_a$ ($\underline{J}_b$ similarly), which relates the deviation from equilibrium to the equilibrium distribution function, by

$$m_a \underline{J}_a f_a^{(0)} = f_a^{(0)} \frac{\partial \phi_a}{\partial \underline{v}_a} \qquad \ldots\ldots (4.9)$$

$\underline{J}_a$ can depend only on $\frac{\partial}{\partial \underline{v}_a}$ since this is the only vector operator in the problem. Strictly $\underline{J}_a$ should be written $\underline{J}_a\left(\frac{\partial}{\partial \underline{v}_a}\right)$, a notation which will sometimes be necessary. Later we will solve for $\underline{J}_a$ by making an expansion in powers of the Laplacian operator in a-velocity space, $\frac{\partial}{\partial \underline{v}_a} \cdot \frac{\partial}{\partial \underline{v}_a}$. The collision integral (8) can be expressed in terms of $\underline{J}_a$ and $\underline{J}_b$. Because of the intimate relationship between the $\underline{J}$ and the ϕ, the mass, momentum and energy fluxes can also be expressed in operator form, so that a knowledge of $\underline{J}$ will, in turn, enable the transport coefficients to be determined.

Equation (8) becomes

$$C_{ab} = \frac{\partial}{\partial \underline{v}_a} \cdot \int m_b \underline{\underline{K}}_{ab} \cdot \left[f_a^{(0)} \underline{J}_b f_b^{(0)} - f_b^{(0)} \underline{J}_a f_a^{(0)} \right] d\underline{v}_b . \qquad \ldots\ldots (4.10)$$

If the tensor $\underline{\underline{F}}_{ab}$ is defined by

$$\underline{\underline{F}}_{ab} = \int \underline{\underline{K}}_{ab} f_b^{(0)} d\underline{v}_b \qquad \ldots\ldots (4.11)$$

equation (10) can be written in terms of $\underline{\underline{F}}_{ab}$ ($\underline{\underline{F}}_{ab}$ will be evaluated explicitly in the next section). The left-hand term inside the integral is

$$\int m_b \underline{\underline{K}}_{ab} \cdot f_a^{(0)} \underline{J}_b f_b^{(0)} d\underline{v}_b = m_b f_a^{(0)} \int \underline{\underline{K}}_{ab} \cdot \underline{J}_b\left(\frac{\partial}{\partial \underline{v}_b}\right) f_b^{(0)} d\underline{v}_b .$$

Repeated integration by parts of the powers of $\frac{\partial}{\partial \underline{v}_b}$ in $\underline{J}_b$ reduces this term to

$$m_b f_a^{(0)} \int f_b^{(0)} \underline{J}_b\left(\frac{\partial}{\partial \underline{v}_b}\right) \cdot \underline{\underline{K}}_{ab} d\underline{v}_b \qquad \ldots\ldots (4.12)$$

Since $\underline{J}_b\left(\frac{\partial}{\partial \underline{v}_b}\right)$ will be expanded in powers of the Laplacian, $\left(\frac{\partial}{\partial \underline{v}_b}\right)^2$, this step in effect, involves a repeated application of integral identities similar to

$$\int \Phi \frac{\partial}{\partial \underline{v}} \cdot \frac{\partial}{\partial \underline{v}} \Psi \, d\underline{v} = \int \Psi \frac{\partial}{\partial \underline{v}} \cdot \frac{\partial}{\partial \underline{v}} \Phi \, d\underline{v}$$

a relation which holds when Φ and Ψ vanish on the infinite velocity surface. Equation (12) can be written as

$$m_b f_a^{(0)} \int f_b^{(0)} \underline{J}_b\left(\frac{\partial}{\partial \underline{v}_a}\right) \cdot \underline{\underline{K}}_{ab} d\underline{v}_b$$

(where $\underline{J}_b\left(\frac{\partial}{\partial v_a}\right)$ is the operator $\underline{J}_b\left(\frac{\partial}{\partial v_b}\right)$ with all the v_b replaced by v_a) since $\underline{\underline{K}}_{ab}$ is a function only of the velocity difference, $\underline{v}_a - \underline{v}_b$. This expression is simply

$$m_b f_a^{(0)} \underline{J}_b\left(\frac{\partial}{\partial v_a}\right) \cdot \underline{\underline{F}}_{ab}$$

because $\underline{J}_b$ contains terms in velocity only as derivatives, $\frac{\partial}{\partial \underline{v}}$. Using the definition (11) the right-hand term under the integral in (10) may be written directly as

$$m_b \underline{\underline{F}}_{ab} \cdot \underline{J}_a\left(\frac{\partial}{\partial v_a}\right) f_a^{(0)} \; .$$

Hence the collision integral becomes

$$C_{ab} = \frac{\partial}{\partial \underline{v}_a} \cdot \left[m_b f_a^{(0)} \underline{J}_b\left(\frac{\partial}{\partial v_a}\right) \cdot \underline{\underline{F}}_{ab} - m_b \underline{\underline{F}}_{ab} \cdot \underline{J}_a\left(\frac{\partial}{\partial v_a}\right) f_a^{(0)} \right] \quad \ldots\ldots (4.13)$$

This is the required form for the collision integral. Before proceding further it is necessary to evaluate the tensor $\underline{\underline{F}}_{ab}$.

4.3 EVALUATION OF $\underline{\underline{F}}_{ab}$

From equations (11) and (6) and equation (2.17)

$$\underline{\underline{F}}_{ab} = \frac{2\pi M^2 \lambda^2}{m_a m_b} \int \Lambda \, \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \, f_b^{(0)} \, d\underline{v}_b \qquad \text{..... (4.14)}$$

where

$$\Lambda = \Lambda_{Debye} = -\ln|Q| + .309 \approx \ln \frac{1.36}{|Q|}$$

can be written

$$\Lambda = 2 \ln \frac{\rho}{\eta} \quad , \quad \eta^2 = \frac{2|\lambda|}{1.36\, a_D} \qquad \text{..... (4.15)}$$

If $\bar{\Lambda}$ is defined by

$$\underline{\underline{F}}_{ab} = \frac{2\pi M^2 \lambda^2}{m_a m_b} \bar{\Lambda} \int \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \, f_b^{(0)} \, d\underline{v}_b \qquad \text{..... (4.14a)}$$

then $\bar{\Lambda}$ is equivalent to what is often called the Coulomb logarithm in the literature,(see, for example, Rosenbluth et. al.[23] and Sivukhin[47], and the discussion after equation (2.5)). To find the tensor $\underline{\underline{F}}_{ab}$ explicitly it is necessary to evaluate the integrals

$$\int \ln \frac{\eta}{\rho} \, \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \, f_b^{(0)} \, d\underline{v}_b \quad , \quad \int \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \, f_b^{(0)} \, d\underline{v}_b \, .$$

To simplify the calculations a, b, $\underline{\mathcal{J}}_{ab}$, x and z are introduced and defined by

$$\left.\begin{aligned} a^2 &= \frac{\beta m_a}{2} \\ b^2 &= \frac{\beta m_b}{2} \\ \underline{\gamma}_{ab} &= b\,\underline{v}_a = \sqrt{\tfrac{1}{2}\beta m_b}\;\underline{v}_a \\ x &= \underline{\gamma}_{ab}\cdot\hat{\underline{\rho}} \\ z &= b\rho = \sqrt{\tfrac{1}{2}\beta m_b}\;\rho\,. \end{aligned}\right\} \quad (4.16)$$

$\underline{\gamma}_{ab}$ is a non-dimensional velocity. x and z could be written as suffixed variables, but the omission of suffixes should create no confusion. Indeed, the suffixes on $\underline{\gamma}_{ab}$ will frequently be omitted, but will always be used when a distinction between different particle interactions must be made. The symbol 'b' has already been used for the impact parameter, but no confusion should arise here. The integrals can be evaluated using a spherical polar coordinate system with $\underline{v}_a$ as polar axis and $\underline{\rho}$ as radius vector. In such a system

$$\int \ldots\ldots\, d\underline{v}_b = -\int \ldots\ldots\, d\underline{\rho} = -\frac{2\pi}{\gamma b^3}\int_0^\infty \int_{-\gamma}^{\gamma} \ldots\ldots\, dx\, z^2\, dz\,.$$

Since ρ is replaced by $z = b\rho$, η will be replaced by $b\eta$ which will be denoted by N (strictly N_{ab}).

$$N = b\eta = \left[\frac{\beta m_b}{2}\;\frac{2|\lambda|}{1{\cdot}36\, a_D}\right]^{1/2} \qquad \ldots\ldots (4.17)$$

Integrals of the form (where h is an arbitrary function)

$$\int (\underline{\underline{\delta}} - \hat{\underline{\rho}}\hat{\underline{\rho}}) \; h(\underline{\rho}, \underline{v}_a) \; d\underline{\rho} \qquad \text{..... (4.18)}$$

can be evaluated by dividing the tensor $\underline{\underline{\delta}} - \hat{\underline{\rho}}\hat{\underline{\rho}}$ into unit and traceless parts and using the fact that the contribution from the traceless part must be a tensor parallel to $\underline{\underline{\delta}} - 3\hat{\underline{v}}_a\hat{\underline{v}}_a$. The integral (18) then becomes

$$\tfrac{2}{3}\underline{\underline{\delta}} \int h \, d\underline{\rho} \;+\; \left(\tfrac{1}{3}\underline{\underline{\delta}} - \hat{\underline{v}}_a\hat{\underline{v}}_a\right) \int \tfrac{1}{2}\left[3(\hat{\underline{v}}_a\cdot\hat{\underline{\rho}})^2 - 1\right] h \, d\underline{\rho} \qquad \text{..... (4.18a)}$$

Using (18) and (18a) gives

$$\int \ln\left(\frac{n}{\rho}\right) \frac{\rho^2\underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} f_b^{(0)} d\underline{v}_b = \frac{-n_b b^3}{\pi^{3/2}}\left[\tfrac{2}{3}\underline{\underline{\delta}}\, I_3 + \left(\tfrac{1}{3}\underline{\underline{\delta}} - \hat{\underline{v}}_a\hat{\underline{v}}_a\right)\left(\tfrac{3}{2}I_4 - \tfrac{1}{2}I_3\right)\right] \text{.....(4.19)}$$

$$\int \frac{\rho^2\underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} f_b^{(0)} d\underline{v}_b = \frac{-n_b b^3}{\pi^{3/2}}\left[\tfrac{2}{3}\underline{\underline{\delta}}\, I_1 + \left(\tfrac{1}{3}\underline{\underline{\delta}} - \hat{\underline{v}}_a\hat{\underline{v}}_a\right)\left(\tfrac{3}{2}I_2 - \tfrac{1}{2}I_1\right)\right] \text{.....(4.20)}$$

where

$$I_1 = \int \frac{1}{\rho} e^{-b^2 v_b^2} d\underline{\rho} = \frac{2\pi}{y b^3}\int_0^\infty \int_{-y}^{y} \frac{b}{z} \exp(-z^2 - 2xz - y^2)\, z^2 \, dx\, dz$$

$$I_2 = \int (\hat{\underline{v}}_a\cdot\hat{\underline{\rho}})^2 \frac{1}{\rho} e^{-b^2 v_b^2} d\underline{\rho} = \frac{2\pi}{y b^3}\int_0^\infty \int_{-y}^{y} \frac{b}{z}\frac{x^2}{y^2} \exp(-z^2 - 2xz - y^2)\, z^2 \, dx\, dz$$

$$I_3 = \int \frac{1}{\rho} \ln\left(\frac{n}{\rho}\right) e^{-b^2 v_b^2} d\underline{\rho} = \frac{2\pi}{y b^3}\int_0^\infty \int_{-y}^{y} \frac{b}{z} \ln\left(\frac{N}{z}\right) \exp(-z^2 - 2xz - y^2)\, z^2 \, dx\, dz$$

$$I_4 = \int (\hat{\underline{v}}_a\cdot\hat{\underline{\rho}})^2 \frac{1}{\rho} \ln\left(\frac{n}{\rho}\right) e^{-b^2 v_b^2} d\underline{\rho} = \frac{2\pi}{y b^3}\int_0^\infty \int_{-y}^{y} \frac{x^2}{y^2}\frac{b}{z} \ln\left(\frac{N}{z}\right) \exp(-z^2 - 2xz - y^2)\, z^2 \, dx\, dz.$$

These integrals can be evaluated in terms of known functions and the two integrals $\mathcal{G}_1$ and $\mathcal{G}_2$ given by

$$\mathcal{G}_1 = \int_0^\infty \ln z \left[\exp\{-(z+\mathcal{Y})^2\} - \exp\{-(z-\mathcal{Y})^2\}\right] dz$$

$$\mathcal{G}_2 = \int_0^\infty z^{-1}\, e^{-z^2} (\cosh 2\mathcal{Y}z \; - \; 1)\, dz \; .$$

The details are rather tedious and are given in Appendix B. $\mathcal{G}_1$ and $\mathcal{G}_2$ cannot be described by simple functions for all values of $\mathcal{Y}$ and the limiting forms for small and large $\mathcal{Y}$ are also given in this Appendix. Finally the corresponding small and large $\mathcal{Y}$ forms for I_1, I_2, I_3 and I_4 are given in equations (3) (6) (16) (17) (18) and (19) of Appendix B. Further simplification gives the following results which are best presented in the form of a table.

	Small $\mathcal{Y}$	Large $\mathcal{Y}$
I_1	$\frac{2\pi}{b^2}$	$\frac{\pi^{3/2}}{\mathcal{Y} b^2}$
I_2	$\frac{2\pi}{b^2}$	$\frac{\pi^{3/2}}{\mathcal{Y} b^2}(\mathcal{Y}^2 - 1)$
I_3	$\frac{2\pi}{b^2}\left(\ln N + \frac{\gamma}{2}\right)$	$\frac{\pi^{3/2}}{\mathcal{Y} b^2}(\ln N - \ln \mathcal{Y})$
I_4	$\frac{2\pi}{b^2}\left(\ln N + \frac{\gamma}{2} - 1\right)$	$\frac{\pi^{3/2}}{\mathcal{Y} b^2}(\mathcal{Y}^2 - 1)(\ln N - \ln \mathcal{Y})$

The results

$$I_1 = \frac{\pi^{3/2}}{\mathcal{Y} b^2} \, \mathrm{Erf}\, \mathcal{Y} \qquad \text{..... (4.21)}$$

$$I_2 = \frac{\pi}{\mathcal{Y}^3 b^2} \left[\sqrt{\pi} (\mathcal{Y}^2 - 1) \, \mathrm{Erf}\, \mathcal{Y} + 2\mathcal{Y} e^{-\mathcal{Y}^2} \right] \qquad \text{..... (4.22)}$$

(equations (3) and (6) of Appendix B) are exact and valid for all $\mathcal{Y}$. The term lnN is large for all plasmas of interest. Because of this, from the tabulated results above it can be seen that

$$\frac{I_3}{I_1} \approx \frac{I_4}{I_2} \approx \ln N \quad , \text{ small } \mathcal{Y} \qquad \text{..... (4.23)}$$

and

$$\frac{I_3}{I_1} \approx \frac{I_4}{I_2} \approx \ln N - \ln \mathcal{Y} \quad , \text{ large } \mathcal{Y} \, . \qquad \text{..... (4.24)}$$

$\underline{\underline{F}}_{ab}$ (as given by equation (14)) can now be determined using (19), (23) and (24), and (21) and (22). We find

$$\underline{\underline{F}}_{ab} = 2A \, \frac{2\sqrt{\pi} M^2 \lambda^2 b n_b}{m_a m_b \mathcal{Y}} \left[\frac{2}{3} \underline{\underline{\delta}} \sqrt{\pi} \, \mathrm{Erf}\, \mathcal{Y} + \left(\frac{\underline{\underline{\delta}}}{3} - \hat{\underline{v}}_a \hat{\underline{v}}_a \right) \left(\sqrt{\pi} \, \mathrm{Erf}\, \mathcal{Y} \left(1 - \frac{3}{2\mathcal{Y}^2}\right) + \frac{3}{\mathcal{Y}} e^{-\mathcal{Y}^2} \right) \right] \text{..... (4.25)}$$

where A = lnN for small $\mathcal{Y}$ and A = lnN-ln$\mathcal{Y}$ for large $\mathcal{Y}$. The form (14a) for $\underline{\underline{F}}_{ab}$ can be found from (20), (21) and (22). On comparing this with (24), $\overline{\Lambda}$ is seen to be the same as -2A. To simplify the expression for $\underline{\underline{F}}_{ab}$ the following terms are defined.

$$P_{ab} = \frac{2}{3}\sqrt{\pi}\,\xi^{-1}\,\mathrm{Erf}\,\xi \qquad \text{..... (4.26)}$$

$$Q_{ab} = \frac{3-2\xi^2}{\xi^5}\,\frac{\sqrt{\pi}}{2}\mathrm{Erf}\,\xi \; - \; \frac{3}{\xi^4}\,e^{-\xi^2} \qquad \text{..... (4.27)}$$

$$k_{ab} = -\,\frac{2\sqrt{\pi}\,b\,n_b\,M^2\lambda^2}{m_a m_b}\,\bar{\Lambda} \qquad \text{..... (4.28)}$$

and the traceless, dimensionless velocity tensor is denoted by

$$\underline{\underline{U}}_{ab} = \underline{\xi}\,\underline{\xi} - \frac{1}{3}\,\underline{\underline{\delta}}\,\xi^2 \qquad \text{..... (4.29)}$$

With these expressions

$$\underline{\underline{F}}_{ab} = k_{ab}\left[\underline{\underline{\delta}}\,P_{ab} + \underline{\underline{U}}_{ab}\,Q_{ab}\right] \qquad \text{..... (4.30)}$$

The suffix 'ab' has been omitted from $\underline{\xi}_{ab}$ in the above and may also, on occasions, be omitted from P_{ab} , Q_{ab} and $\underline{\underline{U}}_{ab}$.

Before proceeding further, it is necessary to discuss the Coulomb logarithm term, Λ, which has been found to be -2lnN for small ξ and -2ln$(\frac{N}{\xi})$ for large ξ . It will be recognized that $\underline{\xi}_{ab}$ is a non-dimensionalized a-particle velocity where the dimensions have been removed by referring to the average speed of the b-particles. In a two-component plasma made up of electrons and (relatively heavy) ions the order of magnitude of $\mathcal{J}_{ab}$ falls into distinct and widely differing ranges depending on the types of particles which are interacting (i.e. whether 'a' and/or 'b' is an electron and/or an ion). For 'average' particles (those moving at speeds near the thermal or root-mean-square speed)

$$\mathcal{J}_{ab}^2 \sim \frac{m_b}{m_a} \qquad \text{..... (4.31)}$$

Hence, if suffix 'e' stands for electron and 'i' for ion,

$$\mathcal{J}_{ei} \sim 40 \,, \quad \mathcal{J}_{ee} \sim \mathcal{J}_{ii} \sim 1 \,, \quad \mathcal{J}_{ie} \sim \frac{1}{40} \,.$$

Of these only $\mathcal{J}_{ei}$ is large so that $\bar{\Lambda} = -2\ln\frac{N}{\mathcal{J}}$ is valid. The extra $\ln\mathcal{J}$ term is precisely the correction which is needed to bring $\bar{\Lambda}_{ei}$ into line with the values for ion-electron and like-particle interactions.

Expanding the expression for N (equation (17))

$$N^2 = N_{ab}^2 = \frac{\beta\,|e_a e_b|}{1{\cdot}36\,(4\pi\varepsilon_0 a_D)}\;\frac{m_a + m_b}{m_a} = \text{constant}\left(\frac{m_a + m_b}{m_a}\right) . \;\text{.....(4.32)}$$

Hence, if the constant is exp(-W), $W = -\ln\left(\frac{\beta\,|e_a e_b|}{1{\cdot}36(4\pi\varepsilon_0 a_D)}\right)$,

$$\bar{\Lambda}_{ie} = W - \ln\frac{m_i + m_e}{m_i} \approx W \,. \qquad \text{..... (4.33)}$$

For like-particle interactions $\mathcal{J}_{ab}$ is between the 'large' and 'small' ranges. However for $\mathcal{J} \sim 1$, $\ln \mathcal{J} \ll \ln N$, so that

$$\bar{\Lambda}_{ee} = \bar{\Lambda}_{ii} \approx W - \ln 2 \,. \qquad \text{..... (4.34)}$$

For electron-ion interactions, using the approximation (31),

$$\bar{\Lambda}_{ei} \approx W - \ln \frac{m_i + m_e}{m_e} + \ln \frac{m_i}{m_e} = W - \ln \frac{m_i + m_e}{m_i} = \bar{\Lambda}_{ie} \qquad \text{..... (4.35)}$$

Since $W \gg \ln 2$, $\ln 2$ may be neglected in equation (34), giving

$$\bar{\Lambda}_{ie} = \bar{\Lambda}_{ee} = \bar{\Lambda}_{ii} = \bar{\Lambda}_{ei} = W = -\ln\left[\frac{\beta |e_a e_b|}{1.36\,(4\pi\varepsilon_0 a_D)}\right]. \qquad \text{..... (4.36)}$$

This result verifies the expression frequently called the Coulomb logarithm in plasma physics literature[23,47,51] which is

$$-\ln\left[\frac{\beta |e_a e_b|}{3\,(4\pi\varepsilon_0 a_D)}\right].$$

4.4 TRANSPORT COEFFICIENTS

The Kinetic equation for particles of type a is, from equations (1) and (13),

$$D_a f_a = \sum_b m_b \frac{\partial}{\partial \underline{v}_a} \cdot \left(f_a^{(0)} \underline{J}_b \cdot \underline{\underline{F}}_{ab} - \underline{\underline{F}}_{ab} \cdot \underline{J}_a f_a^{(0)} \right) = \sum_b C_{ab} = C_a \quad \text{.....} (4.37)$$

The macroscopic equations of change are found by multiplying through by 1, $m_a \underline{v}_a$ and $\frac{1}{2} m_a \underline{v}_a^2$, integrating over a-velocity space and summing over particle types. These are (where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}$),

Conservation of Mass:

$$\frac{\partial n_a}{\partial t} + \underline{\nabla} \cdot n_a \underline{u}_a = \frac{d n_a}{dt} + n_a \underline{\nabla} \cdot \underline{u} + \underline{\nabla} \cdot n_a \underline{V}_a = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \underline{\nabla} \cdot \underline{u} = 0$$

Conservation of Momentum:

$$\rho_a \frac{d u_a}{dt} - \underline{V}_a \underline{\nabla} \cdot \rho_a \underline{V}_a + \rho_a \underline{V}_a \cdot \underline{\nabla} \underline{u} + \underline{\nabla} \cdot \underline{\underline{p}}_a - n_a \overline{\underline{F}}_a = m_a \int \underline{v}_a C_a d\underline{\xi}_a = \underline{R}_a$$

$$\rho \frac{du}{dt} + \underline{\nabla} \cdot \underline{\underline{p}} - \sum_a n_a \overline{\underline{F}}_a = 0$$

Energy equation:

$$\frac{3}{2} \frac{d p_a^o}{dt} + \frac{3}{2} p_a^o \underline{\nabla} \cdot \underline{u} + \underline{\nabla} \cdot \underline{q}_a - n_a \underline{V}_a \cdot \overline{\underline{F}}_a + \rho_a \underline{V}_a \cdot \frac{du}{dt} + \underline{\underline{p}}_a : \underline{\nabla} \underline{u} = \frac{1}{2} m_a \int v_a^2 C_a d\underline{\xi}_a = Q_a$$

$$\frac{3}{2} \frac{dp^o}{dt} + \frac{3}{2} p^o \underline{\nabla} \cdot \underline{u} + \underline{\nabla} \cdot \underline{q} + \underline{\underline{p}} : \underline{\nabla} \underline{u} - \sum_a n_a \underline{V}_a \cdot \underline{F}_a = 0$$

where the variables are defined as follows

$$n_a = \int f_a \, d\underline{\xi}_a \quad , \quad n = \sum_a n_a$$

$$\rho_a = m_a n_a \quad , \quad \rho = \sum_a \rho_a$$

$$n_a \underline{u}_a = \int f_a \underline{\xi}_a \, d\underline{\xi}_a \quad , \quad \rho \underline{u} = \sum_a \rho_a \underline{u}_a \quad , \quad \underline{v}_a = \underline{\xi}_a - \underline{u}$$

$$n_a \underline{V}_a = \int f_a \underline{v}_a \, d\underline{\xi}_a = n_a(\underline{u}_a - \underline{u})$$

$$\underline{\underline{p}}_a = m_a \int \underline{v}_a \underline{v}_a f_a \, d\underline{\xi}_a \quad , \quad \underline{\underline{p}} = \sum_a \underline{\underline{p}}_a \quad \text{(pressure tensor)}$$

$$\frac{3}{2} k T_a = \frac{1}{n_a} \int \frac{1}{2} m_a v_a^2 f_a \, d\underline{\xi}_a \quad , \quad nkT = \sum_a n_a k T_a$$

$$p_a^o = \frac{1}{3} \mathrm{Tr}(\underline{\underline{p}}_a) = n_a k T_a$$

$$p^o = \frac{1}{3} \mathrm{Tr}(\underline{\underline{p}}) = \sum_a p_a^{(o)} = nkT \quad \text{(hydrostatic pressure)}$$

$$\underline{q}_a = \frac{1}{2} m_a \int v_a^2 \underline{v}_a f_a \, d\underline{\xi}_a \quad , \quad \underline{q} = \sum_a \underline{q}_a \quad \text{(heat flux)}$$

and $\underline{\bar{F}}_a$ is the external force averaged over a-velocity space.

The first step in the Chapman-Enskog method for determination of transport coefficients is to expand the distribution function in terms of an ordering parameter

$$f_a = f_a^{(0)} + \varepsilon f_a^{(1)} + \varepsilon^2 f_a^{(2)} + \ldots\ldots\ldots$$

and to assume that the collision term is of order ε^{-1}. Hilbert[52] has shown that the use of such an expansion enables one to determine solutions which are completely determined by the number density, mass velocity and temperature. Thus the Chapman-Enskog method can be used if it is assumed that the gas has relaxed to a state close to equilibrium in which the time dependence of the distribution function appears only through a functional dependence on the slowly varying macroscopic variables n, $\underline{u}$ and T. This does not necessarily mean that the distribution function is slowly varying, but it does ensure that rapid fluctuations are 'smoothed' out. Near to equilibrium the behaviour of the gas is adequately described by such a solution for the distribution function.

The Kinetic equation is now solved by successive approximations. The expansion of the distribution function in terms of ε is substituted into the Kinetic equation and terms of the same order are collected. Differentiated terms are supposed to be of one higher order than the corresponding undifferentiated terms and the equations of change are used to eliminate the time derivatives of the macroscopic variables.

Using equation (37) there is no zero-order equation. The usual zero-order equation is obtained by setting the collision term (with f_a replaced by $f_a^{(0)}$ etc.) equal to zero. The zero order solution is then found to be the distribution function for local equilibrium, $f_a^{(0)} = n_a\left(\frac{\beta m_a}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}\beta m_a(\underline{\xi}_a - \underline{u})^2\right\}$, where n, $\underline{u}$ and $\beta\left(=\frac{1}{kT}\right)$ are functions of $\underline{r}$ and t. This fact has already been used in the present approach and equation (37) is strictly applicable only for small deviations from the equilibrium situation. Consequently, in equation (37) there is no zero-order term in the collision integral; this part of the equation contains only first order terms since higher order terms have been neglected in the deriving of the equation. To first order in ε the left-hand side of (37) is $D_a f_a^{(0)}$ so that, to find a solution to first order, the equation to be solved is

$$D_a f_a^{(0)} = C_a$$

Eliminating the time derivatives from $D_a f_a^{(0)}$ by use of the equations of change gives

$$f_a^{(0)}\Big[\underline{v}_a\cdot\underline{\nabla}\ln\beta\left(\tfrac{5}{2}-\tfrac{1}{2}\beta m_a v_a^2\right) + \frac{\beta}{n_a}\underline{v}_a\cdot\underline{R}_a^{(1)} + \beta m_a\left(\underline{v}_a\underline{v}_a - \tfrac{1}{3}\underline{\underline{\delta}}\, v_a^2\right):\underline{\nabla}\,\underline{u}$$
$$+ \frac{e_a}{m_a}(\underline{v}_a\times\underline{B})\cdot\frac{\partial\phi_a}{\partial\underline{v}_a} + \frac{2}{nkT}(\underline{j}\times\underline{B})\cdot\underline{v}_a\Big] = C_a \quad \text{..... (4.38)}$$

where $\underline{j}$ is the conduction current,

$$\underline{j} = \sum_a e_a n_a \underline{V}_a = \sum_a e_a \int f_a^{(0)}\phi_a\,\underline{v}_a\,d\underline{\xi}_a$$

since the zero order contribution to $\underline{j}$ is zero (assuming a neutral plasma, $\sum_a e_a = 0$). $\underline{R}_a^{(1)}$ is the first order term corresponding to $\underline{R}_a$ ($\underline{R}_a^{(1)}$ corresponds to $nkT\underline{d}_{ab}$ in the notation of Chapman and Cowling[1]).

Even though the zero-order magnetic force term vanishes, the magnetic field has a strong effect on the flow and this term is weighted in comparison to other forces by considering it to be of order ε^{-1} compared with other force terms. The extra term introduced into the zero-order equation in this way does not affect the form of the zero-order solution $f_a^{(0)}$.

If suffix 'e' is used for electrons and 'p' is used for ions (or protons in a hydrogenous plasma) then $m_e \ll m_p$. Using this we find

$$\underline{R}_e^{(1)} = -\underline{R}_p^{(1)} = \tfrac{1}{2} ne\, \underline{E}' = \tfrac{1}{2} ne \left\{ \underline{E} + \underline{u} \times \underline{B} + \frac{1}{ne} \underline{\nabla}(nkT) \right\}$$

which defines $\underline{E}'$, the 'generalized electric field'. Further use of $m_e \ll m_p$ reduces the Kinetic equations for electrons and protons to

$$f_e^{(0)}\left[\underline{v}_e \cdot \underline{\nabla} \ln\beta\left(\tfrac{5}{2} - \tfrac{1}{2}\beta m v_e^2\right) + \beta m\left(\underline{v}_e \underline{v}_e - \tfrac{1}{3}\underline{\underline{\delta}} v_e^2\right) : \underline{\nabla}\underline{u} + \beta e\, \underline{E}' \cdot \underline{v}_e + \frac{e}{m}\underline{B} \times \underline{v}_e \cdot \frac{\partial \phi_e}{\partial \underline{v}_e}\right] = C_e \quad \text{..... (4.39)}$$

$$f_p^{(0)}\left[\underline{v}_p \cdot \underline{\nabla} \ln\beta\left(\tfrac{5}{2} - \tfrac{1}{2}\beta m_p v_p^2\right) + \beta m_p\left(\underline{v}_p \underline{v}_p - \tfrac{1}{3}\underline{\underline{\delta}} v_p^2\right) : \underline{\nabla}\underline{u} - \frac{e_p}{m_p}\underline{B} \times \underline{v}_p \cdot \frac{\partial \phi_p}{\partial \underline{v}_p}\right] - C_{pp}$$

$$= C_{pe} - \frac{1}{nkT}(\underline{j} \times \underline{B}) \cdot \underline{v}_p f_p^{(0)} + \beta e_p\, \underline{E}' \cdot \underline{v}_p f_p^{(0)} = 0 \quad \text{..... (4.40)}$$

where m (= m_e) is the electron mass, e = $(-e_e)$ is the magnitude of the electron charge. The ion equation exhibits the familiar decoupling.

If the magnetic field is zero the Kinetic equation is

$$f_a^{(0)}\left[\underline{v}_a\cdot\underline{\nabla}\ln\beta\left(\tfrac{5}{2}-\tfrac{1}{2}\beta m_a v_a^2\right)-\beta e_a\,\underline{v}_a\cdot\underline{E}'+\beta m_a\left(\underline{v}_a\underline{v}_a-\tfrac{1}{3}\underline{\underline{\delta}}\,v_a^2\right):\underline{\nabla}\,\underline{u}\right]=C_{aa}+C_{ab} \quad \text{.....} (4.41)$$

Replacing a by e in equation (41) gives the electron equation and replacing a by p, putting $\underline{E}' = 0$ and $C_{pe} = 0$ gives the ion Kinetic equation. The left-hand side can be expressed as a perfect divergence so that

$$\frac{\partial}{\partial\underline{v}_a}\cdot\left\{f_a^{(0)}\left[\frac{1}{\beta m_a}\underline{\nabla}\ln\beta\left(\tfrac{1}{2}\beta m_a v_a^2-\tfrac{3}{2}\right)+\frac{e_a}{m_a}\underline{E}'-\tfrac{1}{2}\underline{S}_a\right]\right\}=C_{aa}+C_{ab}$$

where $\underline{S}_a = \underline{v}_a\cdot\underline{\nabla}^{(s)}\underline{u} + \underline{\nabla}^{(s)}\underline{u}\cdot\underline{v}_a$; $\underline{\nabla}^{(s)}\underline{u} = \underline{\nabla}\,\underline{u} - \frac{1}{3}\underline{\underline{\delta}}\,\underline{\nabla}\cdot\underline{u}$. This follows on noticing that $(\underline{v}_a\underline{v}_a - \frac{1}{3}\underline{\underline{\delta}}\,v_a^2):\underline{\nabla}\,\underline{u} = \frac{1}{2}\underline{v}_a\cdot\underline{S}_a$ and that $\frac{\partial}{\partial\underline{v}_a}\cdot(f_a^{(0)}\underline{S}_a) = -\beta m_a f_a^{(0)}\underline{v}_a\cdot\underline{S}_a$. Hence

$$f_a^{(0)}\left[\frac{1}{\beta m_a}\underline{\nabla}\ln\beta\left(\tfrac{1}{2}\beta m_a v_a^2-\tfrac{3}{2}\right)+\frac{e_a}{m_a}\underline{E}'-\tfrac{1}{2}\underline{S}_a\right]=\sum_b m_b\left[f_a^{(0)}\underline{J}_b\cdot\underline{\underline{F}}_{ab}-\underline{\underline{F}}_{ab}\cdot\underline{J}_a f_a^{(0)}\right]. \quad \text{.....} (4.42)$$

The solution is subject to the auxiliary relations which result from choosing the arbitrary constants of the zero-order solution to correspond to the macroscopic variables which are therefore determined solely by the zero-order distribution function. These can be written

$$\int f_a^{(0)} \phi_a \, d\underline{\xi}_a = 0$$

$$\sum_a \int f_a^{(0)} \phi_a \, m_a \underline{v}_a \, d\underline{\xi}_a = 0$$

$$\sum_a \int f_a^{(0)} \phi_a \tfrac{1}{2} m_a v_a^2 \, d\underline{\xi}_a = 0 .$$

These conditions on ϕ_a can be expressed in terms of the operator $\underline{J}_a$ which is related to ϕ_a by equation (9). Since $f_a^{(0)}$ can be written as a perfect divergence,

$$\exp\{-\tfrac{1}{2}\beta m_a v_a^2\} = e^{-\mathcal{J}_{aa}^2} = \frac{\partial}{\partial \underline{v}_a}\cdot\left[\underline{v}_a\left\{\frac{\sqrt{\pi}\,\mathrm{Erf}\,\mathcal{J}_{aa}}{4\,\mathcal{J}_{aa}^3} - \frac{e^{-\mathcal{J}_{aa}^2}}{2\,\mathcal{J}_{aa}^2}\right\}\right] = \frac{\partial}{\partial \underline{v}_a}\cdot\left[\underline{v}_a \, g_a\right],$$

the first condition becomes

$$\int \frac{\partial}{\partial \underline{v}_a}\cdot(\underline{v}_a g_a) \; \phi_a \, d\underline{v}_a = 0 .$$

Integrating this by parts gives

$$\int \frac{g_a}{f_a^{(0)}} \underline{v}_a \cdot \underline{J}_a f_a^{(0)} \, d\underline{v}_a = 0. \qquad \text{..... (4.43)}$$

The function g_a is similar to the functions P_{ab} and Q_{ab} which make up $\underline{\underline{F}}_{ab}$. In fact g_a can be written $4\underline{v}_a g_a = \underline{v}_a\cdot[\underline{\underline{\delta}} P_{aa} + \underline{\underline{U}}_{aa} Q_{aa}] = \frac{1}{k_{aa}} \underline{v}_a \cdot \underline{\underline{F}}_{aa}$ so that (39) becomes

$$\int \frac{1}{f_a^{(0)}} \underline{v}_a \cdot \underline{\underline{F}}_{aa} \cdot \underline{J}_a f_a^{(0)} \, d\underline{v}_a = 0. \qquad \text{..... (4.43a)}$$

The second of the auxiliary relations can be expressed as

$$\sum_a \int \frac{\partial f_a^{(0)}}{\partial \underline{v}_a} \phi_a \, d\underline{v}_a = 0$$

and, on integrating by parts, this gives

$$\sum_a \int m_a \underline{J}_a f_a^{(0)} \, d\underline{v}_a = 0. \qquad \text{..... (4.44)}$$

The third condition is

$$\sum_a \int \underline{v}_a \cdot \frac{\partial f_a^{(0)}}{\partial \underline{v}_a} \phi_a \, d\underline{v}_a = 0$$

so that, integrating by parts and using the first condition,

$$\sum_a \int m_a \underline{v}_a \cdot \underline{J}_a f_a^{(0)} \, d\underline{v}_a = 0. \qquad \text{..... (4.45)}$$

Since the equation (42) is linear, and since the solutions to the corresponding homogeneous first-order equation do not contribute to the required solution, we look for a solution which is a linear combination of the inhomogeneous terms. Consider first the part of the general solution which depends on the temperature gradient.

This can be written $\underline{\underline{J}}_a^{(T)} \cdot \underline{\nabla} \ln\beta$ and the most general possible form for $\underline{\underline{J}}_a^{(T)}$ must be used. Hence

$$\underline{J}_a^{(T)} = \underline{\underline{J}}_a^{(T)} \cdot \underline{\nabla} \ln\beta \qquad \text{..... (4.46)}$$

where

$$\underline{\underline{J}}_a^{(T)} = \underline{\underline{\delta}}\, J_a^{(T1)} + \left(\frac{\partial}{\partial \underline{v}_a} \frac{\partial}{\partial \underline{v}_a} - \frac{1}{3} \underline{\underline{\delta}} \frac{\partial}{\partial \underline{v}_a} \cdot \frac{\partial}{\partial \underline{v}_a} \right) J_a^{(T2)}, \qquad \text{..... (4.47)}$$

$J_a^{(T1)}$ and $J_a^{(T2)}$ being scalar operators depending only on $\frac{\partial}{\partial \underline{v}_a}$. Before using this expression, some tensor identities must be established. The tensor $\underline{\underline{U}}_{ab}$ has been defined as $\underline{\mathcal{J}}_{ab}\underline{\mathcal{J}}_{ab} - \frac{1}{3}\underline{\underline{\delta}}\, \mathcal{J}_{ab}^2$, so that

$$\underline{\underline{U}}_{ab} = \frac{\beta m_b}{2} \left(\underline{v}_a \underline{v}_a - \frac{1}{3} \underline{\underline{\delta}}\, v_a^2 \right) = b^2 \underline{\underline{U}}_a \qquad \text{..... (4.48)}$$

This defines the tensor $\underline{\underline{U}}_a = \underline{v}_a \underline{v}_a - \frac{1}{3} \underline{\underline{\delta}}\, v_a^2$. The operator which corresponds closely to this tensor is that multiplying $J_a^{(T2)}$ in equation (47). It will be denoted by

$$\tilde{\underline{\underline{U}}}_a = \frac{\partial}{\partial \underline{v}_a} \frac{\partial}{\partial \underline{v}_a} - \frac{1}{3} \underline{\underline{\delta}} \frac{\partial}{\partial \underline{v}_a} \cdot \frac{\partial}{\partial \underline{v}_a} .$$

Hence, if $h(v_a)$ is any well-behaved function of $\underline{v}_a$ then a function $H(v_a)$ exists such that

$$\underline{\underline{U}}_a h = \tilde{\underline{\underline{U}}}_a H . \qquad \text{..... (4.49)}$$

This is easily demonstrated by expanding the right-hand side to give $\tilde{\underline{\underline{U}}}_a H = 4\frac{\partial^2 H}{\partial \chi^2}\underline{\underline{U}}_a$, $\chi = v_a^2$. Using this it can be shown that, for arbitrary $h(v_a)$, there exist functions $H_1(v_a)$ and $H_2(v_a)$ such that

$$\underline{\underline{J}}_a \cdot \underline{\underline{U}}_a h = \tilde{\underline{\underline{U}}}_a \cdot \underline{\underline{J}}_a H_1 = \underline{\underline{U}}_a \cdot \underline{\underline{J}}_a H_2 . \qquad \text{.....} \ (4.50)$$

Now, substituting the form (46) into the collisional part of equation (42), and using the identity (48) and equation (30) for $\underline{\underline{F}}_{ab}$ gives

$$\sum_b m_b k_{ab} \underline{\nabla} \ln\beta \cdot \left[f_a^{(0)} \left(\underline{\underline{J}}_b^{(T)} P_{ab} + b^2 \underline{\underline{J}}_b^{(T)} \cdot \underline{\underline{U}}_a Q_{ab} \right) - \left(P_{ab} \underline{\underline{J}}_a^{(T)} + b^2 Q_{ab} \underline{\underline{U}}_a \cdot \underline{\underline{J}}_a^{(T)} \right) f_a^{(0)} \right] .$$

Using (49) and (50) and the expression (47) for $\underline{\underline{J}}_a^{(T)}$, and noticing that $\underline{\underline{U}}_a \cdot \underline{\underline{U}}_a = \frac{1}{3} v_a^2 \underline{\underline{U}}_a$ and that $\tilde{\underline{\underline{U}}}_a f_a^{(0)} = 4a^4 f_a^{(0)} \underline{\underline{U}}_a$, this becomes

$$\sum_b m_b k_{ab} \underline{\nabla} \ln\beta \cdot \Big[f_a^{(0)} \Big\{ \underline{\underline{\delta}} J_b^{(T1)} P_{ab} + \underline{\underline{U}}_a J_b^{(T2)} P_{ab} + b^2 \underline{\underline{U}}_a J_b^{(T1)} Q_{ab,1} + \tfrac{1}{3} v_a^2 \underline{\underline{U}}_a J_b^{(T2)} Q_{ab,2} \Big\}$$

$$- \Big\{ \underline{\underline{\delta}} P_{ab} J_a^{(T1)} f_a^{(0)} + \underline{\underline{U}}_a P_{ab} J_a^{(T2)} f_a^{(0)} + b^2 \underline{\underline{U}}_a Q_{ab} J_a^{(T1)} f_a^{(0)} + \tfrac{1}{3} v_a^2 4a^4 \underline{\underline{U}}_a \left(J_a^{(T2)} f_a^{(0)} \right) Q_{ab} \Big\} \Big]$$

where $\underline{\underline{J}}_b^{(T)} \cdot \underline{\underline{U}}_a Q_{ab} = \underline{\underline{U}}_a \cdot \underline{\underline{J}}_b^{(T)} Q_{ab,1}$; $\tilde{\underline{\underline{U}}}_a Q_{ab,1} = \underline{\underline{U}}_a Q_{ab,2}$. This can be considerably simplified by collecting all the terms parallel to $\underline{\nabla} \ln\beta \cdot \underline{\underline{U}}_a$ to give

$$\sum_b m_b k_{ab} \left[\underline{\nabla} \ln\beta \left(f_a^{(0)} J_b^{(T1)} P_{ab} - P_{ab} J_a^{(T1)} f_a^{(0)} \right) + \underline{\underline{U}}_a \cdot \underline{\nabla} \ln\beta \, f_a^{(0)} \, \mathcal{P}(v_a) \right]$$

$$= f_a^{(0)} \left[\frac{1}{\beta m_a} \underline{\nabla} \ln\beta \left(\tfrac{1}{2} \beta m_a v_a^2 - \tfrac{3}{2} \right) \right] \qquad \text{.....} \ (4.51)$$

In exactly the same way the diffusion part becomes

$$\sum_b m_b k_{ab}\left[\underline{E}'\left(f_a^{(0)} J_b^{(E1)} P_{ab} - P_{ab} J_a^{(E1)} f_a^{(0)}\right) + \underline{\underline{U}}_a \cdot \underline{E}' f_a^{(0)} \mathcal{P}\right] = f_a^{(0)}\left[\frac{e_a}{m_a}\underline{E}'\right] \qquad \text{.....(4.52)}$$

on using $\underline{J}_a^{(E)} = \underline{\underline{J}}_a^{(E)} \cdot \underline{E}'$, $\underline{\underline{J}}_a^{(E)} = \underline{\underline{\delta}} J_a^{(E1)} + \tilde{\underline{\underline{U}}}_a J_a^{(E2)}$.

Consider now the auxiliary relations (43), (44) and (45) which must be satisfied by the solution for $\underline{J}_a$. The expression $\underline{A}\cdot(\underline{\underline{\delta}} J_a^{(1)} + \tilde{\underline{\underline{U}}}_a J_a^{(2)})$ can be used to cover $\underline{\nabla}\ln\beta \cdot \underline{\underline{J}}_a^{(T)}$ and $\underline{E}' \cdot \underline{\underline{J}}_a^{(E)}$. If this form is substituted into the relation (43b) we find

$$\int \frac{1}{f_a^{(0)}} \underline{v}_a \underline{A} : \left[\left(\underline{\underline{\delta}} P_{aa} + b^2 \underline{\underline{U}}_a Q_{aa}\right)\cdot\left(\underline{\underline{\delta}}\, J_a^{(1)} f_a^{(0)} + \tilde{\underline{\underline{U}}}_a J_a^{(2)} f_a^{(0)}\right)\right] d\underline{v}_a = 0 .$$

Now $J_a^{(i)} f_a^{(0)}$ can be written as $H_i f_a^{(0)}$, where H_i is a function of v_a, and using (49) $\tilde{\underline{\underline{U}}}_a H_2 f_a^{(0)}$ can be replaced by $\underline{\underline{U}}_a h_2 f_a^{(0)}$ H_1, H_2 and h_2 need not be determined explicitly. On multiplying out the terms inside $[......]$ in the above expression and using $\underline{\underline{\delta}}\cdot\underline{\underline{\delta}} = \underline{\underline{\delta}}$, $\underline{\underline{\delta}}\cdot\underline{\underline{U}} = \underline{\underline{U}}\cdot\underline{\underline{\delta}} = \underline{\underline{U}}$ and $\underline{\underline{U}}\cdot\underline{\underline{U}} = \frac{1}{3}v^2\underline{\underline{U}}$ it becomes

$$\int \underline{v}_a \underline{A} : \left\{\underline{\underline{\delta}} P_{aa} H_1 + \underline{\underline{U}}_a\left(b^2 Q_{aa} H_1 + P_{aa} h_2\right) + \tfrac{1}{3} v_a^2\, \underline{\underline{U}}_a b^2 Q_{aa} h_2\right\} d\underline{v}_a = 0$$

$$\therefore \int \underline{A}\cdot\underline{v}_a\left\{P_{aa} H_1 + \tfrac{2}{3} v_a^2\left(b^2 Q_{aa} H_1 + P_{aa} h_2\right) + \tfrac{2}{9} v_a^4\, b^2 Q_{aa} h_2\right\} d\underline{v}_a = 0$$

since $\underline{v}_a \cdot \underline{\underline{U}}_a = \frac{2}{3} v_a^2 \underline{v}_a$. This relationship is satisfied identically.

In a similar manner it can be shown that the relation (45) is also satisfied identically and that (44) reduces to

$$\sum_a m_a \int J_a^{(1)} f_a^{(0)} d\underline{v}_a = 0 \qquad \text{..... (4.53)}$$

We now solve for $J_a^{(T1)}$ and $J_a^{(E1)}$ (and $J_b^{(T1)}$ and $J_b^{(E1)}$, remembering that these are parts of the operator $\underline{J}_b$ with $\frac{\partial}{\partial \underline{v}_b}$ replaced by $\frac{\partial}{\partial \underline{v}_a}$) by using equations (51), (52) and (53) and expanding $J_a^{(1)}$ in powers of the Laplacian operator in a-velocity space. Since $J_a^{(1)}$ is a scalar and $\frac{\partial}{\partial \underline{v}_a} \cdot \frac{\partial}{\partial \underline{v}_a}$ (and powers of $\left(\frac{\partial}{\partial \underline{v}_a}\right)^2$) is the only scalar operator in the problem this is the only possible expansion of $J_a^{(1)}$ which can be made. The expansions are written

$$\left.\begin{aligned}
J_a^{(T1)} &= \sum_n \left(\frac{2}{\beta m_a}\right)^n \alpha_{an}^{(T)} \left(\frac{\partial}{\partial \underline{v}_a}\right)^{2n} \\
J_a^{(E1)} &= \sum_n \left(\frac{2}{\beta m_a}\right)^n \alpha_{an}^{(E)} \left(\frac{\partial}{\partial \underline{v}_a}\right)^{2n} \\
J_b^{(T1)} &= \sum_n \left(\frac{2}{\beta m_b}\right) \alpha_{bn}^{(T)} \left(\frac{\partial}{\partial \underline{v}_a}\right)^{2n} \\
J_b^{(E1)} &= \sum_n \left(\frac{2}{\beta m_b}\right) \alpha_{bn}^{(E)} \left(\frac{\partial}{\partial \underline{v}_a}\right)^{2n}
\end{aligned}\right\} \qquad (4.54)$$

Before proceeding, certain properties of $\left(\frac{\partial}{\partial \underline{v}_a}\right)^2$ must be determined. The new variable x_a is introduced and defined by

$$x_a = \mathcal{J}_{aa}^2 = \tfrac{1}{2}\beta m_a v_a^2 .$$

The scalar operator Δ_a is defined by

$$\Delta_a = \left(\frac{\partial}{\partial \underline{\varsigma}_{aa}}\right)^2 = \frac{2}{\beta m_a}\left(\frac{\partial}{\partial \underline{v}_a}\right)^2$$

so that the expansions (54) become

$$\left.\begin{aligned} J_a^{(1)} &= \sum_n \alpha_{an}(\Delta_a)^n \\ J_b^{(1)} &= J_b^{(1)}\left(\frac{\partial}{\partial \underline{v}_a}\right) = \sum_n \left(\frac{m_a}{m_b}\right)^n \alpha_{bn}(\Delta_a)^n \end{aligned}\right\} \quad \text{..... (4.55)}$$

where the superscripts E and T have been omitted. The subscript a will also be omitted in cases where this creates no ambiguity.

If H(x) is an arbitrary function of x then

$$\left.\begin{aligned} \Delta(H(x)) &= 4\left\{x\frac{\partial^2 H}{\partial x^2} + \frac{3}{2}\frac{\partial H}{\partial x}\right\} \\ \Delta(f^{(0)}H(x)) &= 4f^{(0)}\left\{x\frac{\partial^2 H}{\partial x^2} + \left(\frac{3}{2}-2x\right)\frac{\partial H}{\partial x} - \left(\frac{3}{2}-x\right)H\right\}. \end{aligned}\right\} \quad \text{..... (4.56)}$$

These relations can be compared with the following properties of the Sonine polynomials $S_n^{(m)}(x)$ (which are defined as the coefficient of S^m in the power series expansion of $(1-s)^{-n-1}\exp\left[-\frac{xs}{1-s}\right]$);

$$x\frac{\partial}{\partial x}(S_n^{(m)}) + (n+m-1-x)S_n^{(m)} = (m+1)S_n^{(m+1)}$$

$$x\frac{\partial^2}{\partial x^2}(S_n^{(m)}) + (n+1-x)\frac{\partial}{\partial x}(S_n^{(m)}) + mS_n^{(m)} = 0$$

from which it follows that

$$x \frac{\partial^2}{\partial x^2}\left(S_n^{(m)}\right) + (n+1-2x)\frac{\partial}{\partial x}\left(S_n^{(m)}\right) - (n+1-x)\, S_n^{(m)} = -(m+1)\, S_n^{(m+1)} \quad \text{.....} (4.57)$$

On comparing this with equation (56) it can be seen that

$$\Delta\, f^{(0)} S_{1/2}^{(m)} = -4 f^{(0)} (m+1)\, S_{1/2}^{(m+1)}$$

and, using this, it is easily proved by induction that

$$(\Delta)^n\, f^{(0)} S_{1/2}^{(m)} = (-4)^n \frac{(m+n)!}{m!}\, S_{1/2}^{(m+n)}\, f^{(0)}$$

and, in particular,

$$(\Delta)^n\, f^{(0)} = (-4)^n\, n!\, f^{(0)} S_{1/2}^{(n)} \quad \text{.....} (4.58)$$

Using the expansion (55) and equation (58), the auxiliary relation (53) becomes

$$\sum_a m_a \sum_n \alpha_{an} (-4)^n\, n! \int S_{1/2}^{(n)}\, f_a^{(0)}\, d\underline{v}_a = 0. \quad \text{.....} (4.53a)$$

This can be evaluated using the normalization integral for Sonine polynomials;

$$\int_0^\infty S_k^{(p)} S_k^{(q)}\, x^k\, e^{-x}\, dx = \frac{\Gamma(k+p+1)}{p!}\, \delta_{pq}$$

which can be written

$$\int S_k^{(p)} S_k^{(q)} x^{k-\frac{1}{2}} f_a^{(0)} d\underline{v}_a = \frac{2n_a}{\sqrt{\pi}} \frac{\Gamma(k+p+1)}{p!} \delta_{pq}$$

or, for $k = \frac{1}{2}$,

$$\int S_{\frac{1}{2}}^{(p)} S_{\frac{1}{2}}^{(q)} f_a^{(0)} d\underline{v}_a = n_a \frac{(2p+1)!}{(2^p p!)^2} \delta_{pq} \qquad \text{.....(4.59)}$$

Equation (53a) now becomes

$$\sum_a m_a n_a \sum_n \alpha_{an} (-4)^n n! \, \delta_{on} = 0$$

$$\therefore \sum_a m_a \alpha_{ao} = 0 \qquad \text{.....(4.60)}$$

If the expansion (55) is substituted into equations (51) and (52) the coefficients α_{an} and α_{bn} can be extracted by multiplying both sides of the equations by the Sonine polynomial $S_{\frac{1}{2}}^{(m)}$ and integrating over a-velocity space. By doing so the function $\mathcal{P}(v_a)$ is eliminated and we find

$$\sum_b m_b k_{ab} \underline{\nabla} \ln\beta \int \left[\left[f_a^{(0)} \sum_n \alpha_{bn}^{(T)} \left(\frac{m_a \Delta_a}{m_b}\right)^n P_{ab} - P_{ab} \sum_n \alpha_{an}^{(T)} (\Delta_a)^n f_a^{(0)} \right] S_{\frac{1}{2}}^{(m)} d\underline{v}_a \right.$$

$$= \int \frac{1}{\beta m_a} \underline{\nabla} \ln\beta \; f_a^{(0)} (x_a - \tfrac{3}{2}) S_{\frac{1}{2}}^{(m)} d\underline{v}_a \; , \qquad \text{..... (4.61)}$$

and a similar equation for the diffusion contribution. Hence we have

$$\beta m_a \sum_b m_b k_{ab}\Big[\sum_n \alpha_{bn}^{(T)} \Big(\frac{m_a}{m_b}\Big)^n \int f_a^{(0)} S_{1/2}^{(m)}(\Delta_a)^n P_{ab}\, d\underline{v}_a - \sum_n \alpha_{an}^{(T)} \int P_{ab} S_{1/2}^{(m)}(\Delta_a)^n f_a^{(0)} d\underline{v}_a\Big] = -\tfrac{3}{2} n_a \delta_{1m} \quad \ldots\ldots (4.62)$$

$$\beta m_a \sum_b m_b k_{ab}\Big[\sum_n \alpha_{bn}^{(E)} \Big(\frac{m_a}{m_b}\Big)^n \int f_a^{(0)} S_{1/2}^{(m)}(\Delta_a)^n P_{ab}\, d\underline{v}_a - \sum_n \alpha_{an}^{(E)} \int P_{ab} S_{1/2}^{(m)}(\Delta_a)^n f_a^{(0)} d\underline{v}_a\Big] = \beta n_a e_a \delta_{0m} \quad \ldots\ldots (4.63)$$

These equations hold both when the a-particle is an electron and, after an obvious modification (see discussion after equation (41)), when the a-particle is an ion.

When the integrals are evaluated equations (62) and (63) together constitute an infinite set of coupled linear equations for the coefficients $\alpha_{an}^{(T)}$, $\alpha_{an}^{(E)}$, $\alpha_{bn}^{(T)}$ and $\alpha_{bn}^{(E)}$ which, together with the auxiliary relation (60) is equivalent to the usual set of equations which arises from an expansion of the deviation from equilibrium, ϕ_a, in Sonine polynomials.

The integrals in (62) and (63) are

$$I_{1,ab}^{mn} = \int f_a^{(0)} S_{1/2}^{(m)} (\Delta_a)^n P_{ab}\, d\underline{v}_a \quad \text{, and} \qquad \ldots\ldots (4.64)$$

$$I_{2,ab}^{mn} = \int P_{ab} S_{1/2}^{(m)} (\Delta_a)^n f_a^{(0)}\, d\underline{v}_a \, . \qquad \ldots\ldots (4.65)$$

They can be evaluated by the standard method developed in Chapman and Cowling[1]. We first note that

$$\frac{m_a}{m_b} \Delta_a P_{ab} = -\frac{8}{3} \exp\left(-\tfrac{1}{2}\beta m_b v_a^2\right) \text{, so that, using (58),}$$

$$\left(\frac{m_a}{m_b}\Delta_a\right)^n P_{ab} = \frac{2}{3}(-4)^n (n-1)!\, \exp\left(-\frac{m_b}{m_a}x\right) S_{1/2}^{(n-1)}\left(\frac{m_b}{m_a}x\right).$$

Hence, using the definition for Sonine polynomials given earlier, the integral (64) is, for $n > 0$, just the coefficient of $S^{n-1} t^m$ in the power series expansion of

$$\frac{4n_a}{3\sqrt{\pi}}\left(\frac{-4m_b}{m_a}\right)^n (n-1)!\,(1-s)^{-3/2}(1-t)^{-3/2}\int_0^\infty \exp\left(-\frac{xsm_b}{(1-s)m_a}\right)\exp\left(\frac{-xt}{1-t}\right)\exp\left(\frac{-m_b x}{m_a}\right)e^{-x}x^{1/2}\,dx \quad \text{.....(4.64a)}$$

For $n = 0$ the integral (64) is the same as (65). Integral (65) is the coefficient of $S^n t^m$ in the power series expansion of

$$\frac{4n_a}{3}\left(\frac{m_a}{m_b}\right)^{1/2}(-4)^n\, n!\,(1-s)^{-3/2}(1-t)^{-3/2}\int_0^\infty \mathrm{Erf}\left(\sqrt{\frac{m_b x}{m_a}}\right)\exp\left(\frac{-xs}{1-s}\right)\exp\left(\frac{-xt}{1-t}\right)e^{-x}\,dx \quad \text{.....(4.65a)}$$

The solution of the Kinetic equation in operator form thus reduces to a problem equivalent to that which obtains on solving the more usual form of the Kinetic equation. It remains only to show that the transport coefficients can be expressed in terms of the coefficients α_a. Consider the diffusion coefficients first. By definition

$$\underline{V}_a - \underline{V}_b = \frac{1}{n_a}\int f_a^{(0)}\phi_a \underline{v}_a\, d\underline{v}_a - \frac{1}{n_b}\int f_b^{(0)}\phi_b \underline{v}_b\, d\underline{v}_b \, .$$

The integrals can be written in terms of the operators

$\underline{J}_a$ and $\underline{J}_b$ in exactly the same way that the auxiliary relations were written in operator form to give

$$\underline{V}_a - \underline{V}_b = \frac{1}{\beta n_a}\int \underline{J}_a f_a^{(0)} d\underline{v}_a - \frac{1}{\beta n_b}\int \underline{J}_b f_b^{(0)} d\underline{v}_b$$

Using $\underline{J}_a = \underline{\underline{J}}_a^{(T)} \cdot \underline{\nabla} \ln\beta + \underline{\underline{J}}_a^{(E)} \cdot \underline{E}'$ and $\underline{\underline{J}}_a = \underline{\underline{\delta}} J_a^{(1)} + \underline{\underline{\tilde{U}}}_a J_a^{(2)}$ this becomes

$$\underline{V}_a - \underline{V}_b = \frac{1}{\beta n_a}\underline{\nabla}\ln\beta \int J_a^{(T1)} f_a^{(0)} d\underline{v}_a + \frac{1}{\beta n_a}\underline{E}'\int J_a^{(E1)} f_a^{(0)} d\underline{v}_a$$

$$- \frac{1}{\beta n_b}\underline{\nabla}\ln\beta \int J_b^{(T1)} f_b^{(0)} d\underline{v}_b - \frac{1}{\beta n_b}\underline{E}'\int J_b^{(E1)} f_b^{(0)} d\underline{v}_b$$

where $J_b = J_b\left(\frac{\partial}{\partial v_b}\right)$. Substituting the expansion (55) gives

$$\underline{V}_a - \underline{V}_b = \frac{1}{\beta}\underline{\nabla}\ln\beta \left(\alpha_{a0}^{(T)} - \alpha_{b0}^{(T)}\right) + \frac{1}{\beta}\underline{E}'\left(\alpha_{a0}^{(E)} - \alpha_{b0}^{(E)}\right) \qquad \text{..... (4.66)}$$

where the zero-th coefficients are related by equation (60).

Similarly the heat flux becomes

$$\underline{q} = \sum_a \frac{1}{\beta^2}\int \left(\tfrac{1}{2}\beta m_a v_a^2 + 1\right)\underline{J}_a f_a^{(0)} d\underline{v}_a$$

so that

$$\underline{q} - \frac{5}{2\beta}\sum_a n_a \underline{V}_a = \sum_a \frac{1}{\beta^2}\int \left(x_a - \tfrac{3}{2}\right)\underline{J}_a f_a^{(0)} d\underline{v}_a \; .$$

On expanding $\underline{J}_a$ this becomes

$$\underline{q} - \frac{5}{2\beta}\sum_a n_a \underline{V}_a = \frac{3n}{\beta^2}\left\{\underline{\nabla}\ln\beta\left(\alpha_{a1}^{(T)} + \alpha_{b1}^{(T)}\right) + \underline{E}'\left(\alpha_{a1}^{(E)} + \alpha_{b1}^{(E)}\right)\right\}$$

which, eliminating $\underline{E}'$ by using (66), reduces to

$$\underline{q} - \frac{5}{2\beta}\sum_a n_a \underline{V}_a = \frac{3n}{\beta^2}\left\{\underline{\nabla}\ln\beta\left[\frac{\alpha_{ao}^{(E)}\left(\alpha_{a1}^{(T)} + \alpha_{b1}^{(T)}\right) - \alpha_{ao}^{(T)}\left(\alpha_{a1}^{(E)} + \alpha_{b1}^{(E)}\right)}{\alpha_{ao}^{(E)}}\right] + \left(\underline{V}_a - \underline{V}_b\right)\left[\frac{\beta\left(\alpha_{a1}^{(E)} + \alpha_{b1}^{(E)}\right)}{\alpha_{ao}^{(E)} - \alpha_{bo}^{(E)}}\right]\right\} \qquad \text{..... (4.67)}$$

Hence, as is to be expected, the coefficients of diffusion, thermal diffusion and thermal conduction are completely determined by the zero-th and first coefficients of the expansion of the operator $\underline{J}$.

$$D_{ab} = -\frac{1}{\beta^2 e}\alpha_{ao}^{(E)} \qquad \text{..... (4.68)}$$

$$D_T = \frac{1}{2\beta}\alpha_{ao}^{(T)} \qquad \text{..... (4.69)}$$

$$\lambda = \frac{3nk}{\beta}\left[\alpha_{a1}^{(T)} + \alpha_{b1}^{(T)} - \frac{\alpha_{ao}^{(T)}}{\alpha_{ao}^{(E)}}\left(\alpha_{a1}^{(E)} + \alpha_{b1}^{(E)}\right)\right] \qquad \text{..... (4.70)}$$

To evaluate the α_a the integrals (64a) and (65a) must be evaluated, in order to determine $I_{1,ab}^{mn}$ and $I_{2,ab}^{mn}$. Performing the integrations and defining w by $w = \frac{m_b}{m_a}$ shows that $I_{1,ab}^{mn}$ $(n \neq 0)$ is the coefficient of $S^{n-1}\, t^m$ in the expansion of

$$\frac{2n_a}{3}(-4w)^n (n-1)!\,(w+1)^{-3/2}\left[1 - \frac{wt+s}{w+1}\right]^{-3/2}$$

and $I^{mn}_{2,ab}$ is the coefficient of $S^n t^m$ in the expansion of

$$\frac{4n_a}{3}(-4)^n\, n!\,(w+1)^{-1/2}(1-St)^{-1}\left[1-\frac{wS+wt+(w-1)St}{w+1}\right]^{-1/2}.$$

For $n = 0$ $I^{mn}_{1,ab} = I^{mn}_{2,ab}$. The results, up to $n = 2$, $m = 2$, are presented in table form below where a factor $\frac{4n_a}{3\sqrt{1+w}}$ has been omitted for simplicity.

Table of $I^{mn}_{1,ab}\left(\frac{3\sqrt{1+w}}{4n_a}\right)$

m \ n	0	1	2
0	1	$-\frac{2w}{1+w}$	$\frac{12w^2}{(1+w)^2}$
1	$\frac{w}{2(1+w)}$	$-\frac{3w^2}{(1+w)^2}$	$\frac{30w^3}{(1+w)^3}$
2	$\frac{3w^2}{8(1+w)^2}$	$-\frac{15w^3}{4(1+w)^3}$	$\frac{105w^4}{2(1+w)^4}$

Table of $I^{mn}_{2,ab}\left(\frac{3\sqrt{1+w}}{4n_a}\right)$

m \ n	0	1	2
0	1	$-\frac{2w}{1+w}$	$\frac{12w^2}{(1+w)^2}$
1	$\frac{w}{2(1+w)}$	$-\frac{9w^2+8w+2}{(1+w)^2}$	$\frac{2w(35w^2+16w-4)}{(1+w)^3}$
2	$\frac{3w^2}{8(1+w)^2}$	$-\frac{w(35w^2+16w-4)}{4(1+w)^3}$	$\frac{2(76w^3+32w^2+34w+14)}{(1+w)^3}$

The usual procedure is to curtail the expansion (in this instance the expansion of $\underline{J}$) after a finite number of terms. The transport coefficients can be obtained to 'first order' by considering only the first two terms of the expansion. The linear equations to solve follow when the tabled results are substituted into equations (62) and (63). The case when the a-particle is an ion is solved first to give

$$\alpha_{pi}^{(E)} = 0 \qquad \text{..... (4.71)}$$

$$\alpha_{pi}^{(T)} = \frac{3\sqrt{2}}{8e\beta\sqrt{Z}}\, C \qquad \text{..... (4.72)}$$

where $Z = \frac{m_p}{m_e} \approx 1800$ and $C = \frac{-3e}{4m^2 k_{ee}}$. In the expression for C, e and m are the charge (magnitude) and mass of an electron and k_{ee} is the electron-electron value of k_{ab} (equation (28)). The electron equation, using equation (60) and the results (71) and (72), gives

$$\alpha_{ei}^{(E)} = \frac{1}{9+2\sqrt{2}}\alpha_{eo}^{(E)} = -\frac{\beta e}{3}\alpha_{ei}^{(T)} = -\frac{\beta e}{6}\alpha_{eo}^{(T)} = \frac{-C}{8+2\sqrt{2}} \qquad \text{..... (4.73)}$$

Using these results we find for the diffusion coefficient (equation (68))

$$D_{ep} = 1.09\,(4\pi\varepsilon_o)^2 \frac{3}{16n}\left(\frac{2}{\pi\beta m}\right)^{1/2}\left(\frac{2}{\beta e}\right)^2 (\bar{\Lambda})^{-1}$$

$$\bar{\Lambda} = \ln\left[\frac{1.36\,(4\pi\varepsilon_o a_D)}{\beta e^2}\right]$$

which is the usual first order result. The thermal-diffusion ratio

$$k_T = \frac{D_T}{D_{ab}}$$

can be found either directly from (68) and (69) or from the coefficient of $\underline{V}_a - \underline{V}_b$ in equation (67). Using (60) and (71) we find

$$k_T = 3\frac{\alpha_{e1}^{(E)}}{\alpha_{eo}^{(E)}} = -\frac{\beta e}{2}\frac{\alpha_{eo}^{(T)}}{\alpha_{eo}^{(E)}} .$$

The two results can be seen to be equal on inspection and they give

$$k_T = 0.13 .$$

In verifying the equality of the two expressions for k_T terms of order Z^{-1} have been neglected since they have also been neglected in the values for the α_a given by equation (73). However, a more detailed calculation shows that the identity holds when these terms are retained. In a similar manner the thermal conductivity is found to be

$$\lambda = 1.07\,(4\pi\varepsilon_o)^2\,\frac{75k}{16}\left(\frac{1}{\pi\beta m}\right)^{1/2}\left(\frac{1}{\beta e^2}\right)^2(\Lambda)^{-1}$$

which agrees with other estimates to first order. Thus, although there is not an exact correspondence between the expansion of the operator $\underline{J}$ and the Sonine polynomial expansion of ϕ, the resulting values of the transport coefficients are the same to first order and this equality should hold to higher orders.

4.5 AN EXACT SOLUTION

In the previous section an integro-differential equation similar to the Boltzmann equation has been reduced to a differential equation in which the unknown is an operator. If the tensor $\underline{\underline{F}}_a$ is defined by

$$\underline{\underline{F}}_a = \sum_b m_b \underline{\underline{F}}_{ab} \quad , \qquad \text{..... (4.74)}$$

this equation can be written (using equation (37)

$$D_a f_a = \frac{\partial}{\partial \underline{v}_a} \cdot \left[f_a^{(0)} \left(\sum_b m_b \underline{J}_b \cdot \underline{\underline{F}}_{ab} \right) - \underline{\underline{F}}_a \cdot \underline{J}_a f_a^{(0)} \right] . \qquad \text{..... (4.75)}$$

Since the operator $\underline{J}$ depends only on $\frac{\partial}{\partial \underline{v}}$, on taking the Fourier transform this differential operator becomes an algebraic function in the transform space. The possibility therefore arises of reducing equation (75) to an algebraic equation by the use of Fourier transforms. Because there are two terms on the right-hand side it is not immediately obvious how this could be done. However, if these terms are considered separately, their form indicates that, by taking the Fourier transform, $\underline{J}$ could be reduced to an algebraic function.

Although it has not been possible to obtain an exact solution to equation (75) it is possible to obtain an exact solution for a particular case in which one of the parts of the collision term vanishes. If the background distribution of the b-particles is in equilibrium the term $\sum_b m_b \underline{J}_b \cdot \underline{\underline{F}}_{ab}$ vanishes and the Kinetic equation becomes

$$D_a f_a = -\frac{\partial}{\partial \underline{v}_a} \cdot \left[\underline{\underline{F}}_a \cdot \underline{J}_a f_a^{(0)} \right] . \qquad \text{..... (4.76)}$$

This equation describes the behaviour of a special group of type-a particles. The particles themselves are not in an equilibrium state. However, the total number of particles in the group is so small that they collide only with particles outside the group and these particles, external to the special group, are in equilibrium. Runaway electrons are an example of such a group. Runaway electrons are a small group of fast-moving particles which interact only with an equilibrium background of ions and electrons and do not interact among themselves.

Before proceeding the tensor $\underline{\underline{F}}_a$ must be evaluated explicitly. From equation (40)

$$\underline{\underline{F}}_a = \sum_b m_b k_{ab} \left(\underline{\underline{\delta}} P_{ab} + \underline{\underline{\hat{U}}} \mathcal{Y}_{ab}^2 Q_{ab} \right) \qquad \text{..... (4.77)}$$

where $\hat{\underline{\underline{U}}}_a = \gamma_{ab}^{-2} \underline{\underline{U}}_{ab} = \hat{\underline{\gamma}}_{ab} \hat{\underline{\gamma}}_{ab} - \frac{1}{3} \underline{\underline{\delta}}$, the circumflex denoting a unit vector (or a tensor composed of unit vectors). The functional forms of P_{ab} and Q_{ab} are given by equations (26) and (27). We will consider the a-particle to be an electron and write

$$\gamma = \gamma_{ee} \quad , \quad k_e = k_{ee} \quad , \quad m = m_e \ .$$

Substituting explicit expressions for P_{ab} and Q_{ab}. and using the facts that $k_{ep} = k_e Z^{-1/2}$, $m_p = Zm$, $\gamma_{ep} = \sqrt{Z}\,\gamma \gg 1$, equation (77) becomes after some manipulation and ignoring small terms of order Z^{-1}

$$\underline{\underline{F}}_a = \underline{\underline{F}}_e = mk_e\sqrt{\pi}\left(\underline{\underline{\delta}} P_e + \hat{\underline{\underline{U}}}_e Q_e\right) \qquad \text{.....}(4.78)$$

where

$$P_e = \tfrac{2}{3}\gamma^{-1}(\mathrm{Erf}\,\gamma + 1) \qquad \text{.....}(4.79)$$

$$Q_e = \tfrac{2}{3}\gamma^{-3}\,\mathrm{Erf}\,\gamma - \gamma^{-1}(\mathrm{Erf}\,\gamma + 1) - \tfrac{3}{\sqrt{\pi}}\gamma^{-2}e^{-\gamma^2} \qquad \text{.....}(4.80)$$

Equation (76) therefore becomes, on dropping the suffix 'e' for simplicity,

$$DF = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left[\left\{\frac{\partial}{\partial\underline{\gamma}}\cdot(\underline{\underline{\delta}}P + \hat{\underline{\underline{U}}}Q)\right\}\cdot\underline{J}\,f^{(0)} + (\underline{\underline{\delta}}P + \hat{\underline{\underline{U}}}Q):\frac{\partial}{\partial\underline{\gamma}}\underline{J}\,f^{(0)}\right]$$

which, on differentiating the left-hand term of the collision part, expanding and collecting terms reduces to

$$Df = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left[\left(P-\tfrac{1}{3}Q\right)\frac{\partial}{\partial \underline{g}}\cdot\underline{J}f^{(0)} - \frac{3P}{g^2}\underline{g}\cdot\underline{J}f^{(0)} + Q\hat{\underline{g}}\hat{\underline{g}}:\frac{\partial}{\partial \underline{g}}\underline{J}f^{(0)}\right] \quad \ldots\ldots(4.81)$$

This relation is still not amen able to the elimination of the differential form of $\underline{J}$ by the use of a Fourier transformation. However, the possibility of such an elimination can be demonstrated by considering an oversimplification of the problem. Suppose that the distribution function depends only on the magnitude of $\underline{g}$. In this case the deviation from equilibrium, ϕ, will have the following property;

$$\frac{\partial\phi}{\partial\underline{g}} = \hat{\underline{g}}\,\frac{\partial\phi}{\partial g}\,.$$

This is not a completely unphysical assumption. Volume viscosity effects contribute to the distribution function in this way. Volume viscosity is a second order (in density) effect and may be relevant in plasmas in view of the long-range nature of the interparticle forces. In this instance, however, the assumption is made only in the interests of mathematical simplification.

It means that $\underline{J}f^{(0)}$ is parallel to $\underline{\mathcal{Y}}$ and may therefore be written

$$\underline{J}f^{(0)} = \underline{\mathcal{Y}}Jf^{(0)}$$

Using this, and commuting $\underline{J}$ and $\frac{\partial}{\partial \underline{\mathcal{Y}}}$ where necessary, equation (81) becomes

$$Df = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left[\left(P+\tfrac{2}{3}Q\right)\underline{\mathcal{Y}}\cdot J\frac{\partial}{\partial \underline{\mathcal{Y}}}f^{(0)}\right] \qquad \text{.....} (4.82)$$

Now if $\underline{H}$ is any vector function

$$\underline{\mathcal{Y}}\cdot J\underline{H} = \underline{\underline{\delta}}:\underline{\mathcal{Y}}J\underline{H} = \underline{\underline{\delta}}:\underline{J}\underline{H}$$

so that

$$\underline{\mathcal{Y}}\cdot J\frac{\partial}{\partial \underline{\mathcal{Y}}}f^{(0)} = \underline{\underline{\delta}}:\underline{J}\frac{\partial}{\partial \underline{\mathcal{Y}}}f^{(0)} = \underline{\underline{\delta}}:\frac{\partial}{\partial \underline{\mathcal{Y}}}\underline{J}f^{(0)} = \frac{\partial}{\partial \underline{\mathcal{Y}}}\cdot\underline{J}f^{(0)}.$$

The Kinetic equation therefore becomes, replacing Df by the appropriate first order term

$$Df^{(0)} = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left(P+\tfrac{2}{3}Q\right)\frac{\partial}{\partial \underline{\mathcal{Y}}}\cdot\underline{J}f^{(0)}, \qquad \text{.....} (4.83)$$

where

$$P+\tfrac{2}{3}Q = \mathcal{Y}^{-3}\left(\operatorname{Erf}\mathcal{Y} - \tfrac{2}{\sqrt{\pi}}\mathcal{Y}e^{-\mathcal{Y}^2}\right).$$

It can now be seen that by dividing equation (83) through by $(P + \frac{2}{3}Q)$ the equation is reduced to one in which the left-hand side is a known function of $\mathfrak{y}$ $\left(-mk_e\sqrt{\frac{\beta m\pi}{2}}\,L(\mathfrak{y})\right.$, say)

$$L(\mathfrak{y}) = \frac{\partial}{\partial \underline{\mathfrak{y}}}\cdot \underline{J}\, f^{(0)} \qquad \ldots\ldots (4.84)$$

If $\mathfrak{F}$ denotes the Fourier transform operator

$$\mathfrak{F}\{H(\underline{\mathfrak{y}})\} = \frac{1}{(2\pi)^3}\int H(\underline{\mathfrak{y}})\, e^{i(\underline{p}\cdot\underline{\mathfrak{y}})}\, d\underline{\mathfrak{y}}$$

taking the transform of both sides of (84) gives

$$\mathfrak{F}\{L\} = -i\underline{p}\cdot \underline{J}(-i\underline{p})\, \mathfrak{F}\{f^{(0)}\}$$

which is simply an algebraic equation for $\underline{J}$.

This example serves to illustrate the approach which is necessary in order to obtain an exact solution to the special Kinetic equation, equation (76). We will now consider a more realistic example, the problem of the diffusion of the special group of particles described by (76). It has already been noted that the diffusion coefficients depend only on the function P_{ab} in the tensor $\underline{\underline{F}}_{ab}$ (see discussion leading to equation (61)). Hence equation (81) becomes

$$Df = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left[P\frac{\partial}{\partial \underline{\gamma}}\cdot\underline{J}f^{(0)} - \frac{P}{\gamma^2}\underline{\gamma}\cdot\underline{J}f^{(0)}\right].$$

Here the coefficient of $\underline{\gamma}\cdot\underline{J}f^{(0)}$ is $-\frac{P}{\gamma^2}$ rather than $-\frac{3P}{\gamma^2}$ as might be expected on inspection of equation (81). This is because P and Q are related and a contribution $-\frac{2P}{\gamma^2}$ actually comes from Q on differentiation of the original equation.

It is more convenient to retain the form

$$-mk_e\sqrt{\frac{\beta m\pi}{2}}\ \frac{\partial}{\partial \underline{\gamma}}\cdot(P\underline{J}f^{(0)})$$

for the right-hand side. Following the usual Chapman-Enskog procedure and considering only the diffusion part of the operator $\underline{J}$ ($\underline{J}^{(E)}$) we find (see, for example, equation (42))

$$f^{(0)}\frac{e}{m}\underline{E}' = -mk_e\sqrt{\pi}\,P\underline{J}^{(E)}f^{(0)} \qquad \text{.....(4.85)}$$

Dividing through by P and taking the Fourier transform of both sides of this equation gives

$$\frac{e}{m}\underline{E}'\,\mathcal{F}\left\{\frac{f^{(0)}}{P}\right\} = -mk_e\sqrt{\pi}\,\underline{J}^{(E)}(-i\underline{p})\,\mathcal{F}\left\{f^{(0)}\right\} \qquad \text{.....(4.86)}$$

The diffusion part of $\underline{J}$ can be written $J^{(E)}\underline{E}'$ so that

$$J^{(E)} = \frac{-e}{m^2k_e\sqrt{\pi}}\ \frac{\mathcal{F}\left\{\frac{f^{(0)}}{P}\right\}}{\mathcal{F}\left\{f^{(0)}\right\}} \qquad \text{.....(4.87)}$$

This is an exact solution for the diffusion part of the operator $\underline{J}$.

To evaluate the diffusion coefficient one has only to know the zero-th coefficient in the expansion of $\underline{J}^{(E)}$ in powers of $\left(\frac{\partial}{\partial \underline{v}}\right)^2$. This expansion corresponds to an expansion of the transform $\underline{J}(-i\underline{p})$ in powers of p^2 so that, to find the diffusion coefficient, D_{ab}, it is only necessary to find the coefficient of p^0 in the solution of (87).

For near-thermal and higher speeds Erf $\mathcal{V} \approx 1$ so that we may approximate P by $\frac{2}{3}\mathcal{V}^{-1}$. Using the following Fourier transforms

$$\mathcal{F}\{e^{-\mathcal{V}^2}\} = (4\pi)^{-3/2} \exp\left(-\tfrac{1}{4}p^2\right)$$

$$\mathcal{F}\{\mathcal{V} e^{-\mathcal{V}^2}\} = \frac{1}{2p}(2\pi)^{-3}\left[(p^2-2)D\left(\tfrac{p}{2}\right) - 2p\right]$$

where $D(\frac{p}{2})$ is Dawson's Integral

$$D(y) = e^{-y^2}\int_0^y e^{x^2}dx \;,$$

and collecting coefficients of p^0 ($J^{(E)} = \alpha_0^{(E)} + \ldots\ldots\ldots$) in equation (87) gives

$$\alpha_0^{(E)} = \frac{8e}{m^2 k_e \sqrt{\pi}\,(4\pi)^{3/2}} \qquad \ldots\ldots(4.88)$$

Hence, using (68) for the diffusion coefficient and substituting for k we find

$$D_{ep} = \frac{4}{3\pi^2}(4\pi\varepsilon_0)^2 \frac{3}{16n}\left(\frac{2}{\pi\beta m}\right)^{1/2}\left(\frac{2}{\beta e}\right)^2(\bar{\Lambda})^{-1} \qquad .....(4.89)$$

This expression is of the same form as the usual diffusion coefficient, but the numerical factor is somewhat smaller. It is the coefficient for diffusion of a small group of non-equilibrium electrons in an equilibrium plasma. As such it would be expected to be smaller in value than the full diffusion coefficient. Although this is a rather special problem the solution (89) is exact and the corresponding exact solution to the Kinetic equation does not rely on any of the usual expansion procedures.

Although the original equation (76) is appropriate for runaway electrons the solutions presented, ((84) and (87)), are not. The distribution function for runaway electrons must be non-isotropic so that the form of solution, $\frac{\partial \phi}{\partial \underline{\mathfrak{J}}}$ parallel to $\hat{\underline{\mathfrak{J}}}$, is not valid. The diffusion solution (87) is more relevant, but only for the early stages in the development of runaway particles when the distribution function for these particles is close to equilibrium. It is a much later stage, when a steady-state has been reached which is of greatest interest and this problem will be discussed further in the next chapter.

4.6 FURTHER REMARKS

Viscosity has not been considered in the preceding sections. In determining the coefficient of viscosity the method outlined for the diffusion and thermal coefficients in 4.4 can be used in a completely analogous fashion. The appropriate trial form for $\underline{J}_a$ is

$$\underline{J}_a = \left[\frac{\partial}{\partial \underline{v}_a} \cdot \underline{\nabla}^{(s)} \underline{u} + \underline{\nabla}^{(s)} \underline{u} \cdot \frac{\partial}{\partial \underline{v}_a} \right] J_a^{(V1)} + \frac{\partial}{\partial \underline{v}_a} \tilde{\underline{\underline{U}}}_a : \underline{\nabla}\, \underline{u}\, J_a^{(V2)} .$$

For diffusion the function Q_{ab} was found to be unimportant. In discussing viscosity the Q_{ab} term cannot be neglected. Equation (58) is a key relation in the determination of diffusion coefficients. The equivalent equation which must be used for viscosity is

$$(\Delta_a)^n \frac{\partial}{\partial \underline{v}_a} f_a^{(0)} = -\beta m_a \underline{v}_a (-4)^n\, n!\, S_{3/2}^{(n)}(x_a)\, f_a^{(0)} .$$

Transport coefficients in the presence of a magnetic field can also be discussed using the methods of section 4.4 since the magnetic term in the Kinetic equation can be written directly in operator form as

$$e_a (\underline{v}_a \times \underline{B}) \cdot \underline{J}_a f_a^{(0)} .$$

The exact solution given in section 4.5 to a specialized problem indicates that it may be possible, by the use of Fourier transforms, to obtain an exact solution the general Kinetic equation. Although the usual Chapman-Enskog postulates have been used in this chapter, the operator method could also be used in conjunction with many-moment schemes. The possibility that a Fourier transform procedure following the lines of section 4.5 may lead to an exact solution of the Kinetic equation is a most important aspect of the operator form presented in this thesis.

CHAPTER FIVE: RUNAWAY ELECTRONS

Runaway electrons occur when a plasma is situated in an electric field. The retardation of an electron by collisions with other particles decreases as the speed of the electron increases. In the presence of an electric field all particles experience an acceleration, the electrons in the direction of $-\underline{E}$. If an electron is moving sufficiently fast the collisional drag can be less than the acceleration caused by the field. Such a particle will experience a net positive acceleration and its speed will increase indefinitely: these particles are said to 'runaway'. No matter what the magnitude of the field is, there will always be some electrons moving fast enough to runaway. The larger the field the more runaway electrons there will be, and, in fact, if the field strength is sufficiently large even average electrons can runaway; for this to happen the field must be such that the acceleration due to the field is the same as the collisional drag on thermal electrons.

Some insight into this phenomenon can be gained using simple semi-quantitative arguments. It is of interest to determine the particle speed above which the field acceleration exceeds the collisional deceleration (this is the 'critical' speed). Dreicer[41] and Spitzer[51], for example, determine the changes in the velocity component parallel to the field due to the field and due to collisions and, by equating the two, show that the critical speed is proportional to the inverse of the magnitude of the field strength

$$v_c \propto E^{-1} \qquad \text{.....(5.1)}$$

However this result is rather misleading since, if the runaway problem is considered from a kinetic theory point of view, the critical speed is actually proportional to $E^{-\frac{1}{2}}$ [44,45]. For a weak field this critical speed is quite large, certainly well in excess of thermal speeds.

The basic problem is to determine the distribution function for runaway electrons. We will restrict ourselves here to the case of a weak electric field and a homogeneous plasma and look for a steady-state solution to the relevant Kinetic equation. If a plasma is in equilibrium and a weak field is 'turned on' a small number of electrons, those with sufficiently high speeds (i.e. in the 'runaway region') will immediately become runaway electrons. Since the critical speed is high the number of such electrons must be small and they will tend to collide only with the equilibrium background of slower moving particles. The effect of the field is manifest predominantly in the runaway electrons and these will soon become removed appreciably from equilibrium. The early stages of this development, during which the runaway group is still close to equilibrium, has been discussed in the previous chapter. Here we are interested in the more general solution. As time progresses some particles will continually be scattered by random collisions into the runaway region and particles in the runaway region will accelerate rapidly and escape from the confines of the plasma. A steady-state solution could therefore be maintained by introducing into the problem a high-speed sink and a balancing source at thermal speeds to fit this qualitative description.

The fundamental Kinetic equation for runaway electrons is equation (4.81) (or (4.76));

$$Df = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left[\left(P-\frac{Q}{3}\right)\frac{\partial}{\partial \underline{\gamma}}\cdot\underline{J}F^{(0)} - \frac{3P}{\gamma^2}\underline{\gamma}\cdot\underline{J}F^{(0)} + Q\,\hat{\underline{\gamma}}\hat{\underline{\gamma}}:\frac{\partial}{\partial\underline{\gamma}}\underline{J}F^{(0)}\right] \quad .....(5.2)$$

where $\gamma = \gamma_{ee} = \frac{1}{2}\beta m v_e^2$. Since the runaway particles are moving at above thermal speeds the limiting forms for

P and Q can be used,

$$P = \tfrac{4}{3}\gamma^{-1} \quad , \quad Q = -2\gamma^{-1} + \tfrac{3}{2}\gamma^{-3} \; .$$

The small term $\frac{3}{2\gamma^3}$ must be retained for reasons which will be explained below. Equation (2) becomes

$$\frac{\partial f}{\partial t} - \sqrt{\frac{\beta m}{2}}\,\frac{e}{m}\underline{E}\cdot\frac{\partial f}{\partial \underline{\gamma}} = -mk_e\sqrt{\frac{\beta m\pi}{2}}\left[\left(\frac{2}{\gamma} - \frac{1}{2\gamma^3}\right)\frac{\partial}{\partial \underline{\gamma}}\cdot\underline{J}f^{(0)} - \frac{4}{\gamma^3}\underline{\gamma}\cdot\underline{J}f^{(0)} + \left(\frac{3}{2\gamma^3} - \frac{2}{\gamma}\right)\hat{\underline{\gamma}}\hat{\underline{\gamma}}:\frac{\partial}{\partial \underline{\gamma}}\underline{J}f^{(0)}\right] \quad \text{.....(5.3)}$$

It has been shown in section 4.3 that, for large γ, k_e depends on γ. The Coulomb logarithm term in $k_e\,(\bar{\Lambda})$ is $-2\ln N$ for small γ and $-2\ln(\frac{N}{\gamma})$ for large γ. This is a feature which has not been noticed by previous authors. However, it is of only minor importance. Since $\ln\gamma$ is such a slowly varying function of γ, and since the correction is appreciable only for $\gamma \gtrsim N^{-1}$ (which is generally larger even than the particle speeds we are considering in this instance), we are justified in omitting this γ-dependence.

Now, from the definition of $\underline{J}$, $m\underline{J}f^{(0)} = f^{(0)}\frac{\partial \phi}{\partial \underline{v}}$, we find

$$\underline{J}f^{(0)} = \frac{1}{m}\sqrt{\frac{\beta m}{2}}\left(\frac{\partial f}{\partial \underline{\gamma}} + 2\underline{\gamma}f\right)$$

so that equation (3) becomes

$$-\sqrt{\frac{2}{\beta m\pi}}\,\frac{1}{mk_e}\frac{\partial f}{\partial t}-\alpha^2\hat{\underline{E}}\cdot\frac{\partial f}{\partial \underline{y}}=\frac{1}{y}\left[-\frac{2}{y^2}\underline{y}\cdot\;+\left(\frac{\partial}{\partial \underline{y}}\cdot-\hat{\underline{y}}\hat{\underline{y}}:\frac{\partial}{\partial \underline{y}}\right)-\frac{1}{4y^2}\left(\frac{\partial}{\partial \underline{y}}\cdot-3\hat{\underline{y}}\hat{\underline{y}}:\frac{\partial}{\partial \underline{y}}\right)\right]\left(\frac{\partial f}{\partial \underline{y}}+2\underline{y}f\right), \quad \text{.....(5.4)}$$

where

$$\alpha^2=\frac{-eE}{2m\sqrt{\pi}\,k_e}\sqrt{\frac{2}{\beta m}}=(4\pi\varepsilon_0)^2\frac{E}{\pi\beta e^3 n\Lambda}$$

follows the notation of Lebedev[45]. Some authors[41,42] have used the 'critical field', E_c, as a parameter which is related to α by

$$E_c=\alpha^{-2}E.$$

In the weak field case α^2 is a small parameter.

Equation (4) becomes

$$-\sqrt{\frac{2}{\beta m\pi}}\,\frac{1}{mk_e}\frac{\partial f}{\partial t}-\alpha^2\hat{\underline{E}}\cdot\frac{\partial f}{\partial \underline{y}}=$$

$$\frac{1}{y}\left[\left\{-\frac{2}{y}\hat{\underline{y}}\cdot\frac{\partial f}{\partial \underline{y}}+\frac{\partial}{\partial \underline{y}}\cdot\frac{\partial f}{\partial \underline{y}}-\hat{\underline{y}}\hat{\underline{y}}:\frac{\partial}{\partial \underline{y}}\frac{\partial f}{\partial \underline{y}}\right\}-\frac{1}{4y^2}\left\{\frac{\partial}{\partial \underline{y}}\cdot\frac{\partial f}{\partial \underline{y}}-3\hat{\underline{y}}\hat{\underline{y}}:\frac{\partial}{\partial \underline{y}}\frac{\partial f}{\partial \underline{y}}\right\}+\left\{\frac{1}{y}\hat{\underline{y}}\cdot\frac{\partial f}{\partial \underline{y}}\right\}\right]. \quad \text{.....(5.4a)}$$

It can now be seen why it was important to retain the small term in Q. It is this term which leads to the last two bracket-ed terms in equation (4a). If the small term has been ignored only the first bracket-ed term would remain. This term, on expansion in spherical polar coordinates, contains no derivatives with respect to the speed $\mathfrak{J}$ and thus represents only the diffusion of particles in a plane normal to $\underline{\mathfrak{J}}$. The last term on the left-hand side of (4a) is the most important one since it describes the effect of collisional retardation in the direction of motion. The second term contains only small contributions to both the other terms. In summary then, the left-hand side of equation (4a) contains a diffusion term, a small term and a drag term. It is permissible to neglect the small term and, of the remaining terms to consider only the drag term since it is this term which is responsible for the essential effects of collisions on runaway electrons. The diffusion term will have only an overall blurring effect on the solution. Equation (4a) becomes

$$-\sqrt{\frac{2}{\beta m \pi}}\,\frac{1}{mk_e}\,\frac{\partial f}{\partial t} \;-\alpha^2\,\hat{\underline{E}}\cdot\frac{\partial f}{\partial \underline{y}} \;-\frac{1}{y}\,\hat{\underline{y}}\cdot\frac{\partial f}{\partial \underline{y}} \;= 0$$

Since k_e is negative we can write $\lambda^{-1} = -\sqrt{\frac{2}{\beta m \pi}}\,\frac{1}{mk_e}$ where λ is a positive constant. Hence

$$\frac{\partial f}{\partial t} + \left(-\lambda\alpha^2\hat{\underline{E}} - \frac{\lambda \underline{y}}{y^3}\right)\cdot\frac{\partial f}{\partial \underline{y}} = 0. \qquad \text{.....(5.5)}$$

In this equation $-\lambda\alpha^2\hat{\underline{E}}$ describes the effect of the field in accelerating electrons in the $-\hat{\underline{E}}$ direction and $-\frac{\lambda \underline{y}}{y^3}$

describes the drag on an electron due to collisions tending to reduce the magnitude of $\underline{v}$ in the direction of $\underline{v}$. It is evident from this equation that the critical speed is when $|-\lambda\alpha^2\underline{\hat{E}}| \sim |\lambda \frac{\underline{v}}{v^3}|$ so that $v_c \sim \alpha^{-1} \sim E^{-1/2}$ as stated previously. Equation (5), being only a first order differential equation could be solved exactly and a solution has been obtained by Green (pers. comm.). However, this equation does not have any diffusion term and so will not be used here.

Instead, using a spherical polar coordinate system with $-\underline{\hat{E}}$ as polar axis, $\underline{v}$ as radius vector and with $\mu = \cos\theta$ where θ is the latitudinal angle the complete equation (4a′) becomes

$$v\lambda^{-1}\frac{\partial f}{\partial t} + \alpha^2\left(v\mu\frac{\partial f}{\partial v} + (1-\mu^2)\frac{\partial f}{\partial \mu}\right) - \frac{1}{2v^2}\left(\frac{\partial^2 f}{\partial v^2} + 2v\left(1-\frac{1}{2v^2}\right)\frac{\partial f}{\partial v} + \left(2-\frac{1}{2v^2}\right)\frac{\partial}{\partial\mu}\left(1-\mu^2\frac{\partial f}{\partial \mu}\right)\right) = 0 \quad \ldots\ldots(5.6)$$

which can be written

$$\frac{\partial f}{\partial t} + O(f) = 0$$

(where O is a complicated differential operator) for simplicity.

In order for a steady-state solution to exist we need to introduce a sink and a source into equation (6).

The sink can be assumed to be such that particles moving at higher than a fixed (large) speed are removed from the system. The source can be taken to be Maxwellian and of strength S so that a term $Se^{-\xi^2}$ must be added to the right-hand side of (6) and (6a). Equation (6a) is thus

$$\frac{\partial f}{\partial t} + O(f) = Se^{-x}, \qquad .....(5.7)$$

where we have introduced the variable x, $x = \xi^2$.

The source term can be troublesome. It is not immediately obvious that the asymptotic solution $(t\to\infty)$ of (7), which is physically the required solution, is the same as the steady-state ($\frac{\partial}{\partial t} \equiv 0$) solution. If $f = f_o$ when $t = 0$ the formal solution to (7) is

$$f = e^{-tO} f_o + \int_o^t e^{-\tau O} (Se^{-x})\, d\tau.$$

Assuming that as $t\to\infty$, $f\to g$ and $e^{-tO} f_o \to f_\infty$ then the function g satisfies

$$g = f_\infty + \int_o^\infty e^{-\tau O} (Se^{-x})\, d\tau.$$

We must verify that $O(g) = Se^{-x}$. Applying the operator O to the above equation and integrating gives

$$O(g) = O(f_\infty) - S\left[e^{-\tau O} e^{-x}\right]_o^\infty .$$

Since the initial distribution is Maxwellian (Ae^{-x}, say) we have

$$O(g) = O(f_\infty) + Se^{-x} - \frac{S}{A} f_\infty \qquad .....(5.8)$$

On inspection of equation (6) it can be seen that $O(f_o) = 0$ so that

$$e^{-tO} f_o = f_o$$

This means that $f_\infty = f_o = Ae^{-x}$ and equation (8) becomes

$$O(g) = 0 \, .$$

This is not the required form of the equation.

However, if we replace the operator O by $O + \epsilon$, where ϵ is a small positive number, in the above argument and ultimately allow ϵ to vanish we find $f_\infty = 0$. The equation for g is therefore

$$O(g) = Se^{-x} \qquad .....(5.9)$$

which is the required steady-state equation.

On rewriting (9) with the operator expanded in terms of x we find

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}(1-\alpha^2\mu x) + \frac{1}{2x}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial f}{\partial\mu}\right] - \frac{1}{2}\alpha^2(1-\mu^2)\frac{\partial f}{\partial\mu} = -\frac{S}{2\lambda}\sqrt{x}\, e^{-x} \quad .....(5.10)$$

Except for the source term, this is the same equation as that given by Gurevich(42) and Lebedev(45). These authors justify the omission of the source term by saying that the source can be considered to be located wholly in the small-Υ region and can be omitted because the solution is required only for large Υ. Near thermal speeds the solution is assumed to be the equilibrium (Maxwellian) distribution function and the large-Υ solution must match this in an intermediate range. The solution of Gurevich is an unacceptable one since it is functionally dependent on $(1-\alpha^2 x)^{1/2}$ which is unreal for large values of x. Gurevich also has to assume a particular trial form

$$f = f_{(0)} \exp\{\varphi_1(x) + \varphi_2(x)(\mu-1)\}$$

for the solution, without any quantitative justification. Lebedev has attempted to reformulate the approach of Gurevich to eliminate the unreal behaviour of the solution. He assumes the same trial form and obtains a solution which he claims is valid for all x (i.e. all speeds) by using an expansion in α^2. However, it is not clear how his solution is obtained, nor how he employs the matching condition for small $\mathcal{J}$. As Lebedev has pointed out, there is a wide variation from author to author in the solutions to the runaway electron problem. Also, as illustrated above, the necessity for a source term introduces some complications in to the problem, and its omission should be justified by a more complete discussion than that given by most authors. Equation (10) has a greater range of validity in veloci ty-space than generally attributed to it. The approximations made are good even close to thermal speeds and the equilibrium distribution function is a solution for $\alpha^2 = 0$ and S = 0 which is valid for all speeds.

Since α^2 is a small parameter one would expect to be able to solve equation (10) by an expansion in powers of this parameter. Before attempting this, an idea of the form of the solution can be obtained by considering a simplification of the equation (Green, pers. comm.),

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\left(1 - \alpha^2 x\right) = -\frac{S}{2\lambda}\sqrt{x}\, e^{-x} .$$

This is a simple first order differential equation in $\frac{\partial f}{\partial x}$ corresponding to (10) near $\mu = 1$ and with the μ—dependence neglected. The solution is

$$\exp(x - \tfrac{1}{2}\alpha^2 x^2)\ \frac{\partial f}{\partial x} = -\frac{S}{2\lambda}\int_{\infty}^{x} \sqrt{t}\ e^{-\frac{1}{2}\alpha^2 t^2}\, dt.$$

Changing variable to $y = \frac{1}{2}\alpha^2 t^2$ gives

$$\exp(x - \tfrac{1}{2}\alpha^2 x^2)\,\frac{\partial f}{\partial x} = -\frac{S}{2\lambda}\,2^{-1/4}\,\alpha^{-3/2}\int_{\infty}^{1/2\,\alpha x^2} y^{-1/4}\, e^{-y}\, dy\,.$$

The boundary conditions will affect the explicit form of this solution. Further consideration is unnecessary since it can be seen that a solution, convergent for all x, can be obtained as an expansion is powers of α^2 which will be proportional to strength of the source, S, (an expected result) and will contain a Maxwellian term, e^{-x}, and a term $\alpha^{-3/2}$. For S = 0 there can be no steady-state solution.

We look for a solution of the form

$$f(\mu, x) = \frac{S}{2\lambda}\,\alpha^{-3/2}\, e^{-x}\, g(\mu, x)\,. \qquad \text{.....(5.11)}$$

Equation (10) becomes

$$\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial x}(1+\alpha^2\mu x) + \alpha^2\mu x g + \frac{1}{2x}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial g}{\partial\mu}\right] - \frac{\alpha^2}{2}(1-\mu^2)\frac{\partial g}{\partial\mu} = -\alpha^{3/2}\sqrt{x} \quad \text{.....(5.12)}$$

We now suppose that the solution is $g = \sum_{n=0}^{\infty}\alpha^{2n}\, g_n$ so that

$$\sum_{n=0}^{\infty}\alpha^{2n}\left\{\frac{\partial^2 g_n}{\partial x^2} - \frac{\partial g_n}{\partial x} + \frac{1}{2x}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial g_n}{\partial\mu}\right] - \mu x\left[\frac{\partial g_{n-1}}{\partial\mu} - g_{n-1}\right] - \frac{1}{2}(1-\mu^2)\frac{\partial g_{n-1}}{\partial\mu}\right\} = -\alpha^{3/2}\sqrt{x} \quad \text{.....(5.13)}$$

The zero order equation is therefore

$$\frac{\partial^2 g_0}{\partial x^2} - \frac{\partial g_0}{\partial x} + \frac{1}{2x}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial g_0}{\partial \mu}\right] = -\alpha^{3/2}\sqrt{x} .$$

This is a separable equation with a solution of the form $g(\mu,x) = U(\mu)X(x)$ where U and X satisfy

$$(1-\mu^2)\frac{\partial^2 U}{\partial \mu^2} - 2\mu\frac{\partial U}{\partial \mu} + CU = 0 \qquad \text{.....(5.14)}$$

$$x\frac{\partial^2 X}{\partial x^2} - x\frac{\partial X}{\partial x} - \tfrac{1}{2}CX = -\alpha^{3/2}x^{3/2} \qquad \text{.....(5.15)}$$

The U equation is Legendre's equation which indicates that the solution is an expansion in spherical harmonics. The solution to (14) must be finite for all μ and so must contain only Legendre polynomials. This restricts the arbitrary constant, C, to the form m(m+1) where m is a positive integer. Equation (15) thus becomes

$$x\frac{\partial^2 X}{\partial x^2} - x\frac{\partial X}{\partial x} - \tfrac{1}{2}m(m+1)X = -\alpha^{3/2}x^{3/2} . \qquad \text{.....(5.16)}$$

The solution to the homogeneous equation corresponding to equation (16) is a linear combination of confluent hypergeometric functions,

$$X = C_1 M(a,0,x) + C_2 U^*(a,0,x) .$$

Here $a = \frac{1}{2}m(m+1)$ is a non-negative integer, C_1 and C_2 are arbitrary constants and M and U^* are Kummer functions. The small-x form of U^* is $\sim x^{-1} + \ln x$ which will not give a Maxwellian distribution for small x so that $C_2 = 0$.

Knowing the solutions to the homogeneous equation the general solution to equation (16) can be obtained by the standard method as

$$X = C_1 M - \alpha^{3/2} M \int_0^x x^{1/2} e^{-x} U^* dx + \alpha^{3/2} U^* \int_0^x x^{1/2} e^{-x} M\, dx .$$

Using the asymptotic forms of M and U^* this can be reduced to

$$X = C_1 M + \frac{\alpha^{3/2} x^{3/2}}{\alpha + 3/2}$$

so that the general solution for $f(\mu,x)$ to this order is

$$f(\mu,x) = e^{-x} \frac{S}{2\lambda} \left[\alpha^{-3/2} \sum_{m=0}^{\infty} C_m M\left(\tfrac{m(m+1)}{2}, 0, x\right) P_m(\mu) + x^{3/2} \sum_{m=0}^{\infty} \frac{2 P_m(\mu)}{m(m+1)+3} \right]$$

for large x.

It is considered that continuing this expansion to higher orders will generate a form of the solution which would be particularly suitable to numerical computation methods since it can be seen from the form of equation (13) that higher order terms will involve similar Legendre polynomial expansions. It is hoped to do this in the future.

CONCLUDING REMARKS

In this part of the thesis the intuitively correct use of a Debye potential with a Boltzmann equation collision term as a Kinetic equation for a plasma has been justified for near equilibrium situations. The collision term has been evaluated explicitly and written in an operator form. It has been shown that the Kinetic equation can be solved for the unknown operator by a method similar to the Chapman-Enskog method, and that the usual transport coefficients can be obtained in a straightforward manner. An indication that the operator form of the Kinetic equation is a very convenient and powerful form has been given by obtaining an exact solution under certain restrictive conditions which correspond closely to those associated with runaway electrons. Finally the problem of runaway electrons where a steady-state is maintained by a source has been discussed. A number of avenues for future work remain open.

APPENDIX A

In chapter two the scattering angle for scattering in a Debye field is evaluated approximately. An accurate estimate requires integration of the integral (equation (2.7))

$$\frac{\pi}{2} - \frac{\theta}{2} = \int_0^1 dx \left[\frac{1 - Qxye^{-1/xy}}{1 - Qye^{-1/y}} - x^2 \right]^{-1/2} \qquad \text{.....(1)}$$

in which the integrand diverges at the upper limit of integration. Whenever an integrand diverges, provided that the integral is finite, the following technique can be used to evaluate the integral rapidly. Consider the behaviour of the integrand close to the point where it diverges (in this instance at x = 1). Suppose that h(x,y,Q) is defined as

$$h(x,y,Q) = \left[\frac{1 - Qxye^{-1/xy}}{1 - Qye^{-1/y}} - x^2 \right]^{-1/2}$$

Then, for small δ

$$h(1-\delta, y, Q) = H(y,Q)\, \delta^{-1/2}$$

which defines H(y,Q) as

$$H(y,Q) = \left[\frac{Qy(y+1)e^{-1/y}}{1 - Qye^{-1/y}} + 2 \right]^{-1/2}.$$

Using this expression, if Δ is small and $D<1$ (so that $\Delta D^n \to 0$ as $n \to \infty$) we have

$$\int_{1-\Delta}^{1-\Delta D} h(x,y,Q)\,dx = -2H(y,Q)\,\Delta^{1/2}(D^{1/2}-1).$$

Hence

$$\int_{1-\Delta}^{1} h(x,y,Q)\,dx = \left\{\int_{1-\Delta}^{1-\Delta D} + \int_{1-\Delta D}^{1-\Delta D^2} + \int_{1-\Delta D^2}^{1-\Delta D^3} + \cdots\cdots\right\} h\,dx$$

$$= \sum_{n=0}^{\infty} \Delta^{n/2} \int_{1-\Delta}^{1-\Delta D} h\,dx$$

$$= (1-\Delta^{1/2})^{-1} \int_{1-\Delta}^{1-\Delta D} h\,dx \,. \qquad \text{.....(2)}$$

The range of integration of (1) can now be split into two parts,

$$\frac{\pi}{2} - \frac{\theta}{2} = \int_{0}^{1-\Delta} h\,dx + \int_{1-\Delta}^{1} h\,dx \,.$$

Of these the first integral presents no computational difficulties and the second can be approximated using (2) to any required degree of accuracy by the rapidly converging sequence

$$\frac{1}{1-\sqrt{\Delta}}\int_{1-\Delta}^{1-\Delta D} h\,dx \,,\quad \left\{\int_{1-\Delta}^{1-\Delta D} + \frac{1}{1-\sqrt{\Delta}}\int_{1-\Delta}^{1-\Delta D^2}\right\} h\,dx \,,\quad \left\{\int_{1-\Delta}^{1-\Delta D^2} + \frac{1}{1-\sqrt{\Delta}}\int_{1-\Delta D^2}^{1-\Delta D^3}\right\} h\,dx \,, \ldots\ldots\ldots .$$

In the following program this method has been used with $D = {}^{1}/10$ and $\Delta = {}^{1}/10$. Values of the energy parameter Q (written as AA), the impact parameter B (actually B^2 written as BSQ), the required order of accuracy (DEL) and the value of $(1 - \sqrt{10})^{-1}$ (AAA) are fed in as initial data, From these values, y (Y(J)) is computed for various B using the relation (equation (2.8))

$$B^2 y^2 + Q y e^{-1/y} - 1 = 0.$$

The upper and lower limits of $\left|\sin\frac{\theta}{2}\right|$ (AHI $= \left|\frac{Qy(y+1)e^{-1/y}}{(2-Qye^{-1/y})y + Qye^{-1/y}}\right|$ and ALO $= \left|\frac{Qye^{-1/y}}{2-Qye^{-1/y}}\right|$, see equation (2.10)) are calculated and the integral of the function

$$h(x,y,Q) - \left[\frac{1-Qxye^{-1/y}}{1-Qye^{-1/y}} - x^2\right]^{-1/2}$$

is computed using Simpson's rule and the technique outlined above to the specified degree of accuracy. This integral gives the deviation of $\theta/2$ (half the scattering angle) from the approximate value which corresponds to ALO. $\left|\frac{\theta}{2}\right|$ (ANG) and $\cos\theta$ (THCOS) are then calculated from the value of the integral for each of the specified values of B^2.

The program is written in FORTRAN for use on a CDC 6400 model computor.

```
      PROGRAM TEGR (INPUT,OUTPUT)
      DIMENSION Y(80),DINT(900),BINT(900),CINT(80),BSQ(80),THETA(80)
      DIMENSION THCOS(80)
      DIMENSION B(80)
      DIMENSION THY(80),ALO(80),ANG(80),AHI(80)
      DIMENSION ANGS(900)
      DIMENSION BSQA(80),DIF(80),COR(80)
      COMMON/AA/AA
      AAA=1.46247529557426
      AA=-1.*10.**(-6)
      DEL=0.0000001
      BSQ(1)=1.009999E-8
      BSQ(2)=1.0009990005E-6
      BSQ(3)=1.000099005E-4
      BSQ(4)=1.0000090484E-2
      BSQ(5)=1.0000003679
      BSQ(6)=100.
      DO4J=1,6
      Y(J)=(-AA+SQRT(AA*AA+4.*BSQ(J)*(1.-AA)))/(2.*BSQ(J))
    1 BSQA(J)=(1.-AA*Y(J)*EXP(-1./Y(J)))/(Y(J)*Y(J))
      DIF(J)=BSQA(J)-BSQ(J)
      IF(ABS(DIF(J))/BSQ(J)-DEL)8,7,7
    7 COR(J)=DIF(J)*Y(J)*Y(J)*Y(J)/(2.-AA*EXP(-1./Y(J))*(Y(J)-1.))
      Y(J)=Y(J)+COR(J)
      GO TO 1
    8 B(J)=AA*Y(J)*EXP(-1./Y(J))
      ALO(J)=B(J)/(2.-B(J))
      ALO(J)=-ALO(J)
      AHI(J)=(B(J)*(Y(J)+1.))/((2.-B(J))*Y(J)+B(J))
      AHI(J)=-AHI(J)
      ANGS(1)=10.
      DO2M=1,2
    2 CALL SIMP(1.-10.**(1-M),1.-10.**(-M),DEL,DINT(M),Y(J))
      M=1
      BINT(M)=DINT(M)+AAA*DINT(M+1)
    3 M=M+1
      CALL SIMP(1.-10.**(-M),1.-10.**(-M-1),DEL,DINT(M+1),Y(J))
      BINT(M)=BINT(M-1)+(1.-AAA)*DINT(M)+AAA*DINT(M+1)
      THETA(M)=SIN(BINT(M))
      ANGS(M)=ALO(J)+THETA(M)
      IF(ABS((ANGS(M)-ANGS(M-1))/ANGS(M))-DEL)6,3,3
    6 ANG(J)=ANGS(M)
      THCOS(J)=2.*(ANG(J))*(ANG(J))
      BSQ(J)=(1.-AA*Y(J)*EXP(-1./Y(J)))/(Y(J)*Y(J))
      PRINT57,DEL,AA,Y(J),M
      PRINT5,Y(J),ALO(J),ANG(J),AHI(J)
    4 PRINT5,BINT(M),ANG(J),THCOS(J),BSQ(J)
    5 FORMAT(1H0,4E18.10)
   57 FORMAT(1H0,3F15.8,1I5)
      STOP
      END
```

```
      SUBROUTINE SIMP(A,B,DELTA,AREA,G)
      N=1
      V=(B-A)*10.E25
      H=(B-A)/2.
      FNX=FUNCX(A,G)
      RJ=FUNCX(B,G)
      RJ=(RJ+FNX)*H
   11 C=0.
      DO12K=1,N
      Z=(2*K-1)*H+A
      FNX=FUNCX(Z,G)
   12 C=C+FNX
      RI=4.*H*C+RJ
      IF(ABS(RI-V)-DELTA)14,13,13
   13 V=RI
      RJ=(RI+RJ)/4.
      N=2*N
      H=H/2.
      GO TO 11
   14 AREA=RI/3.
      RETURN
      END


      FUNCTION FUNCX(X,R)
      COMMON/AA/AA
      IF(X)24,24,25
   24 FUNCX=0.
      GO TO 23
   25 FUNCX=SQRT(1./((1.-AA*R*X*EXP(-1./(X*R)))/
     1(1.-AA*R*EXP(-1./R))-X*X))
      FUN=SQRT(1./((1.-AA*R*X*EXP(-1./R))/(1.-AA*R*EXP(-1./R))-X*X))
      FUNCX=FUNCX-FUN
   23 RETURN
      END
```

APPENDIX B

B1: PROOF OF THE RELATION (4.5)

This relation gives an evaluation of the integral

$$I_{ab} = \frac{\partial}{\partial \underline{\xi}_a} \cdot \int R(\rho) \exp\left\{-\tfrac{1}{2}\beta m_a \xi_a^2 - \tfrac{1}{2}\beta m_b \xi_b^2\right\} \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \, d\underline{\xi}_b \, .$$

For convenience we will write $E_a = \exp\left\{-\tfrac{1}{2}\beta m_a \xi_a^2\right\}$ and E_b similarly. The differentiation $\frac{\partial}{\partial \underline{\xi}_a} \cdot$ can be taken inside the integral to give

$$I_{ab} = \int R(\rho) \frac{\partial}{\partial \underline{\xi}_a}(E_a E_b) \cdot \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \, d\underline{\xi}_b + \int E_a E_b \frac{\partial}{\partial \underline{\xi}_a} \cdot \left(R(\rho) \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \right) \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \, d\underline{\xi}_b \, .$$

Now, since

$$\frac{\partial}{\partial \underline{\xi}_a}(E_a E_b) = -\beta m_a \underline{\xi}_a E_a E_b = \frac{m_a}{m_b}(-\beta m_b(\underline{\xi}_b - \underline{\rho})) E_b E_a = \frac{m_a}{m_b} \frac{\partial}{\partial \underline{\xi}_a}(E_a E_b) + \beta m_a \underline{\rho} E_a E_b$$

the above integral becomes

$$I_{ab} = \int \frac{m_a}{m_b} R(\rho) \frac{\partial}{\partial \underline{\xi}_b}(E_a E_b) \cdot \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \, d\underline{\xi}_b + \int \rho^3 R(\rho) \beta m_a E_a E_b \underline{\rho} \cdot (\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}) \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \, d\underline{\xi}_b$$
$$+ \int E_a E_b \frac{\partial}{\partial \underline{\xi}_a} \cdot \left(R(\rho) \frac{\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}}{\rho^3} \right) \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \, d\underline{\xi}_b \, .$$

The first term on the right-hand side can be integrated by parts to give an integral over the infinite b-velocity surface which is zero, and a non-zero integral over velocity space. In the second term the integrand is identically zero. Hence

$$I_{ab} = \int E_a E_b \left[\frac{m_a}{m_b} \frac{\partial}{\partial \underline{\xi}_b} + \frac{\partial}{\partial \underline{\xi}_a} \right] \cdot \left[\rho^{-3} R(\rho) \, (\rho^2 \underline{\underline{\delta}} - \underline{\rho}\underline{\rho}) \cdot \frac{\partial \phi_b}{\partial \underline{\xi}_b} \right] d\underline{\xi}_b$$

On expanding the integrand by performing the differentiation, $\frac{m_a}{m_b}\frac{\partial}{\partial \underline{\xi}_b} + \frac{\partial}{\partial \underline{\xi}_a}$, this becomes

$$I_{ab} = \left(1+\frac{m_a}{m_b}\right)\int E_a E_b \rho^{-3} R(\rho)\left[2\underline{\rho}\cdot\frac{\partial \phi_b}{\partial \underline{\xi}_b} + \frac{1}{M}\underline{\rho}\underline{\rho}:\frac{\partial}{\partial \underline{\xi}_b}\frac{\partial \phi_b}{\partial \underline{\xi}_b} - \frac{\rho^2}{M}\frac{\partial}{\partial \underline{\xi}_b}\cdot\frac{\partial \phi_b}{\partial \underline{\xi}_b}\right] d\underline{\xi}_b$$

where $M = \dfrac{m_a m_b}{m_a + m_b}$. This can easily be expressed in the required form, as given in equation (4.5), as

$$I_{ab} = -\frac{2m_a m_b}{M}\int \frac{R(\rho)}{\rho^3}\exp\left\{-\frac{\beta m_a}{2}\xi_a^2 - \frac{\beta m_b}{2}\xi_b^2\right\}\left[-\frac{M}{m_b}\underline{\rho}\cdot\frac{\partial \phi_b}{\partial \underline{\xi}_b} - \frac{1}{2}\underline{\rho}\underline{\rho}:\frac{M^2}{m_b^2}\frac{\partial}{\partial \underline{\xi}_b}\frac{\partial \phi_b}{\partial \underline{\xi}_b} + \frac{1}{2}\rho^2\frac{M^2}{m_b^2}\frac{\partial}{\partial \underline{\xi}_b}\cdot\frac{\partial \phi_b}{\partial \underline{\xi}_b}\right] d\underline{\xi}_b \, .$$

B2: THE INTEGRALS IN CHAPTER 4.3

The integrals are I_1, I_2, I_3, I_4, $\mathcal{G}_1$ and $\mathcal{G}_2$ and will be evaluated in that order.

$$I_1 = \frac{2\pi}{y b^2} \int_0^\infty \int_{-y}^{y} z\, e^{-z^2 - y^2 - 2zx}\, dx\, dz \,. \qquad \text{.....(1)}$$

On integrating over x and defining $F_1(z, y)$ by

$$F_1(z, y) = e^{-(z+y)^2} - e^{-(z-y)^2} = -2 e^{-y^2} e^{-z^2} \sinh(2zy) \qquad \text{.....(2)}$$

this becomes

$$I_1 = -\frac{\pi}{b^2 y} \int_0^\infty F_1(z, y)\, dz$$

$$\therefore\ I_1 = \frac{\pi}{b^2 y} \sqrt{\pi}\, \mathrm{Erf}\, y \qquad \text{.....(3)}$$

where $\mathrm{Erf}\, y = 2(\pi)^{-1/2} \int_0^y e^{-x^2} dx$. The second integral is

$$I_2 = \frac{2\pi}{y^3 b^2} \int_0^\infty \int_{-y}^{y} z x^2 e^{-z^2 - y^2 - 2zx}\, dx\, dz \,.$$

The integration over x is straightforward and gives

$$I_2 = -\frac{\pi}{y^3 b^2} \int_0^\infty \left[\left(y^2 + \tfrac{1}{2} z^{-2}\right) F_1(z, y) + y z^{-1} F_2(z, y) \right] dz \qquad \text{.....(4)}$$

where $F_2(z, y)$ is defined by

$$F_2(z, y) = e^{-(z+y)^2} + e^{-(z-y)^2} = 2 e^{-y^2} e^{-z^2} \cosh(2zy) \qquad \text{.....(5)}$$

Using the relation $\frac{\partial F_1}{\partial z} = -2z F_1 - 2y F_2$, the expression (4) can be simplified to

$$I_2 = -\frac{\pi}{b^2 y^3}\left[(y^2-1)\int_0^\infty F_1\, dz - \int_0 \frac{\partial}{\partial z}\left(\tfrac{1}{2}z^{-1}F_1\right) dz\right]$$

$$\therefore I_2 = -\frac{\pi}{b^2 y^3}\left[\sqrt{\pi}\,(y^2-1)\,\mathrm{Erf}\, y + 2y\, e^{-y^2}\right] \qquad .....(6)$$

The third integral is

$$I_3 = \frac{2\pi}{y b^2}\int_0^\infty \int_{-y}^{y} z \ln\left(\frac{N}{z}\right) e^{-z^2-y^2-2zx}\, dx\, dz \qquad .....(7)$$

which, in the same way as I_1, reduces to

$$I_3 = -\frac{\pi}{y b^2}\int_0^\infty (\ln N - \ln z)\, F_1\, dz$$

$$\therefore I_3 = \frac{\pi}{y b^2}\left[\sqrt{\pi}\,\mathrm{Erf}\, y \ln N + g_1\right]. \qquad .....(8)$$

The fourth integral is

$$I_4 = \frac{2\pi}{y^3 b^2}\int_0^\infty \int_{-y}^{y} z \ln\left(\frac{N}{z}\right) x^2\, e^{-z^2-y^2-2zx}\, dx\, dz\,. \qquad .....(9)$$

Following the integration over x in I_2 this becomes

$$I_4 = -\frac{\pi}{b^2 y^3}\left[(y^2-1)\int_0^\infty (\ln N - \ln z)\, F_1\, dz - \int_0^\infty (\ln N - \ln z)\, \frac{\partial}{\partial z}\left(\tfrac{1}{2}z^{-1}F_1\right) dz\right]$$

$$\therefore I_4 = -\frac{\pi}{b^2 y^3}\left[(1-y^2)\left(\mathrm{Erf}\, y \ln N\,(\sqrt{\pi}) + g_1\right) - 2y e^{-y^2}\ln N + \int_0^\infty \ln z\, \frac{\partial}{\partial z}\left(\tfrac{1}{2}z^{-1}F_1\right) dz\right] .....(10)$$

To evaluate (10) we must find $I = \int_0^\infty \ln z\, \frac{\partial}{\partial z}\left(\tfrac{1}{2}z^{-1}F_1\right). dz$.

Integrating by parts twice and using the relation for

$\frac{\partial F_1}{\partial z}$ given earlier gives

$$I = \left[\tfrac{1}{2} z^{-1} F_1 \ln z\right]_0^\infty + 2y\, e^{-y^2} + \int_0^\infty F_1\, dz + y \int_0^\infty z^{-1} F_2\, dz .$$

Writing $F_2 = 2e^{-y^2} e^{-z^2}(\cosh 2zy - 1) + 2e^{-y^2} e^{-z^2}$, integrating by parts and combining terms this becomes

$$I = \left[2y e^{-y^2}\left(1 - \frac{\sinh 2zy}{2zy}\right) e^{-z^2} \ln z\right]_0^\infty + 2y e^{-y^2} - \sqrt{\pi}\, \mathrm{Erf}\, y + 2y e^{-y^2} \int_0^\infty 2z \ln z\, e^{-z^2} dz$$
$$+ 2y e^{-y^2} \int_0^\infty z^{-1} e^{-z^2} (\cosh 2zy - 1)\, dz .$$

Using this relation and $\int_0^\infty z \ln z\, e^{-z^2} dz = -\frac{1}{4}\gamma$ where γ is Euler's constant, enables (10) to be written

$$I_4 = \frac{\pi}{b^2 y^3}\left[\left\{\sqrt{\pi}(y^2-1)\mathrm{Erf}\, y + 2y e^{-y^2}\right\}\ln N + \sqrt{\pi}\,\mathrm{Erf}\, y + 2y e^{-y^2}\left(\frac{\gamma}{2} - 1 - g_2\right) + (y^2-1) g_1\right] \quad \text{.....(11)}$$

I_3 and I_4 depend on the integrals

$$g_1 = \int_0^\infty \ln z\ F_1\, dz$$
$$g_2 = \int_0^\infty z^{-1} e^{-z^2} (\cosh 2yz - 1)\, dz .$$

Both of these can be approximated for small or large y. Consider the case where y is small first. We use the first few terms in the expansions of cosh and sinh (from F_1) only to obtain

$$g_1 \approx -2y e^{-y^2} \int_0^\infty e^{-z^2} \ln z\, \left(2z + \tfrac{4}{3} y^2 z^3\right) dz$$
$$\therefore\ g_1 \approx -2y\, e^{-y^2}\left[-\frac{\gamma}{2} + \frac{2}{3} y^2 \left(\frac{1}{2} - \frac{\gamma}{2}\right)\right]$$
$$\therefore\ g_1 \approx \gamma\, y\, e^{-y^2} \quad \text{.....(12)}$$

and

$$\mathcal{G}_2 \approx \int_0^\infty z^{-1} e^{-z^2}\, 2z^2 y^2 \, dz$$

$$\therefore \mathcal{G}_2 \approx y^2 \qquad \text{..... (13)}$$

For large y we use the fact that $e^{-(y-z)^2} \gg e^{-(z+y)^2}$ over most of the range of integration and also the approximation

$$\int_0^\infty H(z)\, e^{-(z-Z)^2} dz \approx H(Z) \frac{\sqrt{\pi}}{2}(1 + \mathrm{Erf}\, Z) \approx \sqrt{\pi}\, H(Z)$$

valid for large Z where H(z) is an arbitrary, but not too badly behaved, function. $\mathcal{G}_1$ and $\mathcal{G}_2$ become

$$\mathcal{G}_1 \approx -\int_0^\infty \ln z \; e^{-(y-z)^2} dz$$

$$\therefore \mathcal{G}_1 \approx -\sqrt{\pi} \ln y$$

and, since $\cosh 2zy \gg 1$ over most of the range of integration,

$$\mathcal{G}_2 \approx \int_0^\infty z^{-1} e^{-z^2} \cosh 2yz \, dz$$

$$\therefore \mathcal{G}_2 \approx \tfrac{1}{2} e^{y^2} \int_0^\infty z^{-1} \, e^{-(y-z)^2} dz$$

$$\therefore \mathcal{G}_2 \approx \frac{\sqrt{\pi}}{2} y^{-1} e^{y^2}.$$

It is difficult to see the ranges of validity of the expressions (12) to (15). However the integrals for $\mathcal{G}_1$ and $\mathcal{G}_2$ can easily be evaluated numerically and the results compared with the approximations given here. This numerical integrating was carried out on a C.D.C. 3200 computer and gave the following results (where P.E. denotes percentage error).

$\lvert P.E.\rvert < 35\%$	:	$\Upsilon < .69;\ \Upsilon > 1.02$	for g_1
		$\Upsilon < .91;\ \Upsilon > 1.04$	for g_2
$\lvert P.E.\rvert < 10\%$	:	$\Upsilon < .42;\ \Upsilon > 2.04$	for g_1
		$\Upsilon < .53;\quad 1.32 < \Upsilon < 1.71,\ \Upsilon > 2.47$	for g_2.

Using the small-Υ and large-Υ approximations for g_1 and g_2, the corresponding limiting forms for I_3 and I_4 become, using (8) and (11),

$$I_3 = \frac{\pi}{\Upsilon b^2}\left[\sqrt{\pi}\,\mathrm{Erf}\,\Upsilon\,\ln N + \gamma\Upsilon e^{-\Upsilon^2}\right] \quad \text{, small } \Upsilon \qquad \text{.....(16)}$$

$$I_3 = \frac{\pi}{\Upsilon b^2}\left[\sqrt{\pi}\,(\ln N - \ln \Upsilon)\right] \quad \text{, large } \Upsilon \qquad \text{.....(17)}$$

$$I_4 = \frac{\pi}{\Upsilon^3 b^2}\left[\left\{\sqrt{\pi}(\Upsilon^2-1)\,\mathrm{Erf}\,\Upsilon + 2\Upsilon e^{-\Upsilon^2}\right\}\ln N + \sqrt{\pi}\,\mathrm{Erf}\,\Upsilon - 2\Upsilon e^{-\Upsilon^2}\left\{1+\Upsilon^2 - \frac{\gamma\Upsilon^2}{2}\right\}\right] \text{, small } \Upsilon \text{.....(18)}$$

$$I_4 = \frac{\pi}{\Upsilon^3 b^2}\left[\sqrt{\pi}\,(\Upsilon^2-1)(\ln N - \ln\Upsilon) + 2\Upsilon e^{-\Upsilon^2}\left(\tfrac{1}{2}\gamma - 1\right)\right] \quad \text{, large } \Upsilon \text{.....(19)}$$

An alternative method for deriving the limiting forms for g_2 is to notice that $\frac{\partial g_2}{\partial \Upsilon} = e^{\Upsilon^2}\int_0^\infty F_1\,dz = \sqrt{\pi}\,e^{\Upsilon^2}\,\mathrm{Erf}\,\Upsilon$. For small Υ, $\sqrt{\pi}\,\mathrm{Erf}\,\Upsilon \approx \Upsilon e^{-\Upsilon^2}$ so that $\frac{\partial g_2}{\partial \Upsilon} = 2\Upsilon$ and $g_2 = \Upsilon^2$ as before. For large Υ, $\mathrm{Erf}\,\Upsilon \approx 1 \approx 1 - \frac{1}{2}\Upsilon^{-2}$ so that $\frac{\partial g_2}{\partial \Upsilon} = \sqrt{\pi}\,e^{\Upsilon^2}(1 - 2\Upsilon^{-2})$. On integration this gives $g_2 = \sqrt{\pi}\,e^{\Upsilon^2}/2\Upsilon$ as before.

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PART II

CHAPTER ONE: INTRODUCTION

The Meteorology and Climatology of caves is a field which is of more interest to biological scientists, concerned with environmental influences on plant and animal life, than it is to physical scientists. One important physical aspect is, however, the study of the motion of air inside, and into and out of caves. This is an aspect of microclimatology which has been mentioned by a number of authors, notably Geiger[1]. Geiger classifies caves as either 'static' (subject to little circulation of air) or 'dynamic' (in which air movements of considerable magnitude can occur) according to whether they have one or more entrances. His classification of single-entrance caves as static indicates his unawareness (possibly because it is rarely encountered in European countries) of a most remarkable natural phenomenon; namely large-magnitude, reversing air currents which are observed in some single-entrance caves. This phenomenon, and its explanation, is the main concern of the present section of this thesis.

The term 'breathe' will be used specifically to describe such air movements. This word has often been used with a more general meaning to describe all forms of air movement encountered in caves. It is also used (in the United States) to describe the short-period oscillations which occur when the air in a cave resonates in response to external air movements (the 'Helmholtz resonator effect' explained by Faust[2], Schmidt[3] and Plummer[4]). No confusion should arise because these effects will not be discussed further.

Cave-breathing is a paradox since the question can be asked, 'where does the air come from or go to?' The answer may be associated with the variation of atmospheric pressure. However, simple considerations show that the magnitude of this effect would, in most cases, be almost immeasurably small and therefore re-establish the paradox. The possibility of air movement due to atmospheric pressure changes has been noted by Lawrence[5], who states it to be an effect of small magnitude. Measurements made by Polli[6] confirm that (at least in the cave he studied) this is so. Plummer[4], in reviewing the major factors which can cause air movement in caves, also mentions this possibility, but discards it as a factor of little importance except in extremely large caves. He does, however, refer to the unexplained large magnitude of air movement in Wind Cave, South Dakota. Moore and Nicholas[7] discuss the effect of air pressure changes on air in caves, but do not mention that inordinately large magnitudes have been observed.

It has been known for many years (although rarely reported in any literature) that a form of air movement exists in caves of certain regions where the variation of atmospheric pressure is the only possible cause, but where, should this be so, the magnitude of the phenomenon has been many times greater than would be expected. It is this which is referred to here as cave-breathing. Some examples of breathing caves (in this sense) in the United States are cited by Halliday[8].

Popular belief has been that strong breathing indicates, and can be explained by, the existence of a large volume of undiscovered, yet penetrable, cave. However, the number of breathing caves which are known, and the extensive exploration of them, makes this a most unlikely explanation applicable to all cases. A more realistic suggestion by Ward[9], (in relation to breathing wells rather than caves), that the large magnitude is due to the extreme porosity (in the non-technical sense of the word) of the rock, is apparently unique in the literature.

Probably the most remarkable instances of cave-breathing occur in the 'blowholes' of the Nullarbor Plain region of southern Australia. These are apertures in the ground, from a few inches to a few feet in diameter, 'through which there are draughts and which may make moaning or whistling noises because of the passage of air' (Jennings[10]). They frequently exude air at a sufficient rate to completely evacuate their penetrable volume in a matter of minutes. Although noticed by the earliest explorers, no attempt was made to study their breathing until 1957 (Bishop and Hunt[11]). This work was hampered by a lack of suitable equipment and the fairly commonly held belief that the phenomenon was due primarily to temperature effects. In spite of Ward's statement[9] and the knowledge of the high porosity (King[12]) and the remarkable extent of anastomosing and intense perforation of the whole mass of bedrock[10,14], this belief was still adhered to in 1964 (Anderson[13]).

The Nullarbor Plain is one of the largest limestone karst regions in the world covering tens of thousands of square miles[10,14]. It is, unfortunately, a rather remote region almost 1000 miles from the nearest population centers. The climate of the region ranges in aspect from semi-arid to desert. As a consequence it is rarely visited either for scientific or other reasons and further work on the blowholes lapsed until 1964. In the meantime an attempt at explaining the phenomenon of breathing in the Wupatki National Monument region of the United States met with little success (Sartor and Lamar[15]). The results of these authors did, however, show that the relation between air movement and pressure changes could not be a simple one since changes in direction of breathing lagged behind the changes in rate of change of pressure.

In 1964 a most significant cave discovery was made (see Anderson[13]) by means of aerial survey photographs of the Nullarbor Plain. This cave, 'Mullamullang Cave', which has only one entrance and is at present the largest cavern in Australia, exhibits the breathing common to the blowholes of the Plain to an unprecedented degree. At one 'constriction' in the cave (some 200 square feet cross-sectional area) 'winds' estimated at over ten miles per hour were observed. This remarkable example provided a unique opportunity to study cave breathing in detail. Preliminary measurements were made in 1966 during a large-scale privately organized expedition to the cave. The results confirmed a theoretical prediction of the present author (see Wigley, Wood and Smith in Hill[16]). The detailed explanation of the breathing and the presentation of further confirmatory evidence is the main subject of this part of the thesis.

Chapter two gives an account of the phenomenon as a problem in the non-steady flow of gas through a porous medium. The hypothesis that the accessible cave is a large cavity in a much larger mass of extremely porous material, and that external (to the cave) atmospheric porous surrounding material to move in response to them, is put forward. This hypothesis is examined initially from a meteorological point of view and it is shown that the conditions are such that the governing equations reduce to a diffusion equation. Two models, the 'cylinder' and 'long-slit' models, of idealized caves are proposed and the diffusion equation is solved under the boundary conditions appropriate to these models. Results are given from Mullamullang Cave which agree well with the predictions of the long-slit model, and the results of Schley[17], Sartor and Lamar[15], and Conn[18] are found also to be in accord with the theory. Further experimental work where readings were taken simultaneously in Mullamullang Cave and in a nearby blowhole (this latter though to be an example of the cylinder cave model) is discussed. The Mullamullang Cave study indicates that the 'fracture'* permeability dominates the 'matrix'* permeability of the rock in regions where cave breathing is observed. Some of the consequences of this discovery are examined.

In Chapter three the theory presented in the second chapter is extended to cover a more general hydrological problem, that of the non-steady flow of a fluid into a well in a confined aquifer. One of the boundary conditions previously written as a Fourier series is rewritten as a Fourier integral and a solution is obtained which is a generalization of that which is widely quoted in hydrological literature (see, for example,

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* Often called 'secondary' and 'primary' permeability.

Todd[19] and de Wiest[20]). This solution is, however, in accord with, and formally equivalent to, that of Ritchie and Sakakura[21] and is a well-known result of the theory of heat conduction. A possible hydrological consequence of the cave breathing theory is noted. This is the partial confinement of an aquifer in the vicinity of a well (or cave) due to the decrease in amplitude of air pressure fluctuations as one moves away from the well.

CHAPTER TWO: CAVE BREATHING

2.1 POROUS FLOW THEORY

Consider the problem of flow of air in a porous medium of thickness h which lies in or on the earth's surface. The medium is bounded by parallel impervious layers at the top and bottom and the air flow in it is induced by pressure variations in a cavity inside the medium. Two particular cavity shapes will be considered (see Figure 1). Later these shapes will be taken to represent two types of ideal cave; the 'long-slit' and 'cylinder' cave models.

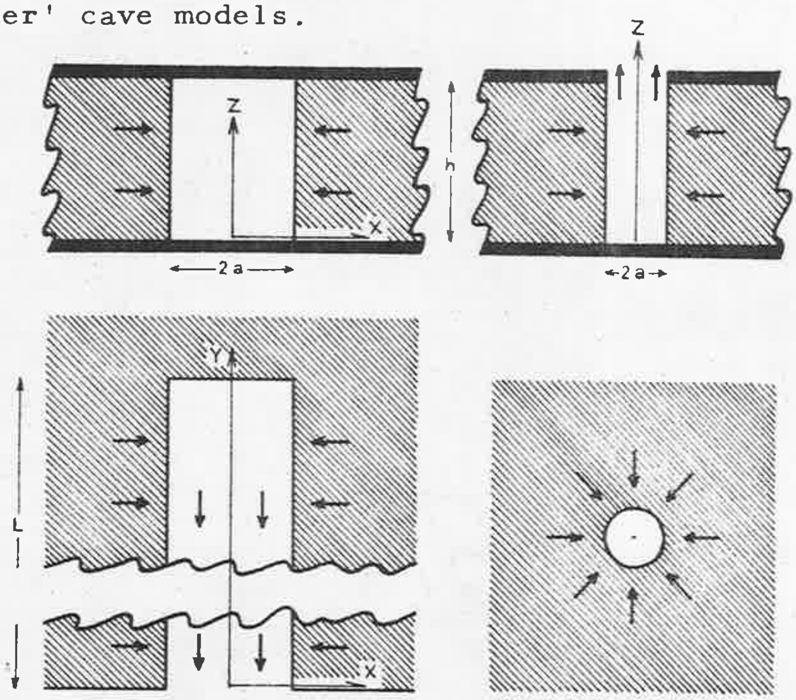

Fig. 1. Elevation and plan of the long slit and cylinder cave models. The arrows show direction of air movement during an out-flow (falling pressure) cycle. Long slit (*left*); cylinder (*right*).

The first cavity chosen is a long rectangular slit of length L, height h and half-width a. The slit is open at one end, and is assumed long enough for the effects of flow into the other end to be ignored. (The word 'slit' is perhaps a little misleading since it usually implies that one of the dimensions h or a is much larger than the other. This is not necessarily the case here). The geometry of this model suggests the use of a cartesian coordinate system with origin at the center of the lower side of the open end of the slit. The cavity is thus confined to the region $-a \leqslant x \leqslant a$, $o \leqslant y \leqslant L$, $o \leqslant z \leqslant h$.

The second cavity chosen is a circular cylinder of height h and radius a with axis normal to the two impervious layers. The upper end of the cylindrical cavity penetrates the upper impervious layer. Cylindrical polar coordinates are suggested here, with origin at the bottom of the cylinder, where the axis intersects the lower impervious layer. In this system the cavity is restricted to the region $r \leqslant a$, $0 \leqslant z \leqslant h$.

The equations governing motion in the porous medium are

$$p = \rho R T \qquad \ldots\ldots (2.1)$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \rho \underline{v} = 0 \qquad \ldots\ldots (2.2)$$

$$\frac{d\underline{v}}{dt} + \underline{\nabla}\phi + \frac{1}{\rho}\underline{\nabla}p + \underline{v} \times 2\underline{\Omega} + \frac{S\mu}{\rho k}\underline{v} = 0 \qquad \ldots\ldots (2.3)$$

where k is the permeability of the medium

S is the porosity of the medium

p, ρ, T are the pressure, density and temperature of the air

ϕ is the earth's geopotential

μ is the coefficient of viscosity of the air

$\underline{\Omega}$ is the angular velocity of the earth's rotation

and $\underline{v}$ is the (particle) velocity of the air, not to be confused with seepage velocity ($\underline{q} = S\underline{v}$) often used in porous flow studies. The flow is assumed to be laminar and, to make the system of equations complete, isothermal.

These equations are the equations of dynamic meteorology with an additional (linear) resistive term in the equation of motion which represents the drag due to motion through the porous medium. As given they are, of course, too general for this particular application and the Coriolis term and the effect of variation of the dependent variables in the vertical direction can be neglected (see below).

This approach to porous flow problems is the 'Drag Theory' proposed by Brinkman[22]. An alternative, though less general starting point which is frequently used is the semi-empirical law of Darcy. The two approaches are equivalent and Darcy's Law can easily be 'deduced' from the Drag Theory provided certain restrictive assumptions are made (see, for example, de Wiest[20]). The introduction of a drag term is equivalent to assuming that the effect of the small-scale tortuous paths through which the air travels in the porous medium is manifest as a resistance when the air motion is considered over a sufficiently large scale. The resistance is proportional to a 'coarse-ground' velocity; i.e. the velocity of the air averaged over a length of small-scale path sufficiently long to include a large number of small changes in direction. There are, therefore, two separate scales of motion through the medium; the 'microscopic' scale on which the shape of the small-scale path along which the air moves is important, and the 'macroscopic' scale which is described by equation

(3). It might be expected that this would be a valid working hypothesis only for a certain range of a 'pore-size' parameter which would be a characteristic of the microscopic motion. For instance, it is certainly valid for common porous materials where the pore-size is less than the order of 1/10 mm., provided that flow velocities are not too small or too large (the experimental validity of Darcy's Law indicates this), but is it valid when the pore-size is of the order of cms. or greater? The answer to this question is apparently 'yes', provided one considers phenomena on a sufficiently large scale compared with the pore-size, although it would be expected that the regime of flow velocities for which the hypothesis was valid would be restricted in some way. The so-called 'non-linear' effects which are known to occur at very low flow rates are ignored in this treatment. They would only be important for small time intervals and so will not affect any of the broad conclusions which will be drawn.

The boundary conditions which must be imposed are: (i) at an infinite distance from the cavity p = constant (= P_o, say), and (ii) inside the cavity $p(z = 0) = f(t)$. At this stage the function of time $f(t)$ need not be specified any more precisely than

$f(t) \approx$ constant (as is the case with atmospheric pressure variations).

Flow into a long slit.

The symmetry of the problem suggests that, if y is the coordinate axis measured along the slit, the vertical and y-components of the (coarse-grained) velocity may be neglected. Since the remaining component, u, is small, Coriolis terms may be ignored and the equations reduce to

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad , \quad (|x| \geqslant a) , \qquad \text{..... (2.4)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{S\mu}{\rho k} u = 0 \ , \ (|x| \geqslant a) , \qquad \text{..... (2.5)}$$

$$\frac{\partial p}{\partial y} = 0 , \qquad \text{..... (2.6)}$$

$$\frac{\partial p}{\partial z} = -\rho g . \qquad \text{..... (2.7)}$$

Using the isothermal condition, equations (6) and (7) can be integrated. The solution is

$$p = P(x,t) \exp\left(-\frac{gz}{RT}\right) \approx P(x,t) , \qquad \text{..... (2.8)}$$

since $o \leqslant z \leqslant h$ and $h \ll {}^{g}/RT$, so that variations in the vertical may be ignored and equations (4) and (5) become

$$\frac{\partial P}{\partial t} + \frac{\partial (Pu)}{\partial x} = 0 , \quad (|x| \geqslant a), \qquad \text{..... (2.9)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{RT}{P}\frac{\partial P}{\partial x} + \frac{S\mu RT}{pk} u = 0 , \quad (|x| \geqslant a). \qquad \text{..... (2.10)}$$

Differentiating equation (9) with respect to x and (10) with respect to t and subtracting gives, for $|x| \geqslant a$

$$S\mu \frac{\partial P}{\partial t} = P_o k \frac{\partial^2 P}{\partial x^2} \qquad \text{..... (2.11)}$$

where small terms have been neglected since, with

$f(t) \approx$ constant,

$\delta(\ln p) \ll \delta(\ln u)$. It is convenient now the expand $f(t)$ as a Fourier series,

$$f(t) = P_o + \sum_{n=1}^{\infty} P_n \sin(n\omega t - \epsilon_n).$$

Under these conditions, the solution of (11) is well-known (see, for example, Carslaw[23]), being

$$P = P_o + \sum_{n=1}^{\infty} P_n \, e^{-\alpha_n(|x|-a)} \sin(n\omega t - \epsilon_n - \alpha_n(|x|-a)),$$

where $\alpha_n^2 = (n\omega\mu S/(2P_o k))$, and where, to comply with the assumed close constancy of $f(t)$, $P_n \ll P_o$. Thus the complete solution is given by

$$p \approx \left[P_0 + \sum_{n=1}^{\infty} P_n\, e^{-\alpha_n(|x|+a)} \sin\left(n\omega t - \epsilon_n - \alpha_n(|x|-a)\right)\right], \quad |x| \geqslant a,$$

$$p \approx f(t) \quad , \quad |x| \leqslant a. \qquad \ldots\ldots (2.12)$$

The amount of air flowing through the slit at any point along its length can now be calculated. Integrating the continuity equation over the volume $-\infty \leqslant x \leqslant \infty$, $y \geqslant l$, $0 \leqslant z \leqslant h$, and using Gauss's Theorem yields the result

$$p_l UA = \iiint \frac{\partial p}{\partial t}\, d\tau, \qquad \ldots\ldots (2.13)$$

where p_l is the average pressure and U the average air speed (measured positive inward) over the cross-section of area A normal to the y-axis at y = l, and $d\tau$ is a typical volume element. Hence

$$p_l UA = 2h(L-l)\int_0^{\infty} \frac{\partial p}{\partial t}\, dx .$$

The length l should be small compared with L so that end effects can be ignored. Completing the integration either directly or using the fact that $\frac{\partial P}{\partial t}$ is proportional to $\frac{\partial^2 P}{\partial x^2}$ we find that

$$p_l UA = 2h(L-l)\sum_{n=1}^{\infty} P_n n\omega\left[a\cos(n\omega t - \epsilon_n) + \frac{\sqrt{2}}{2\alpha_n}\cos\left(n\omega t - \epsilon_n - \frac{\pi}{4}\right)\right] \qquad \ldots\ldots (2.14)$$

which becomes, for a very permeable medium, (say $k \gtrsim 10^{-11}$ square metres)

$$U = \left[\frac{2h(L-1)}{A\,p_1}\right]\left[\frac{P_o k}{\mu S}\right]^{1/2} \sum_{n=1}^{\infty} P_n \sqrt{n\omega}\,\cos\left(n\omega t - \epsilon_n - \frac{\pi}{4}\right). \qquad \ldots\ldots (2.15)$$

Flow into a circular cylinder

Following the argument presented above it can be shown that, for $r \geqslant a$,

$$p \approx P(r,t)\ ,$$

$$S\mu \frac{\partial P}{\partial t} = P_o k\left(\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r}\right),$$

with the boundary conditions

$$P = P_o + \sum_{n=1}^{\infty} P_n \sin(n\omega t - \epsilon_n)\ ,\ r \leqslant a\ ,$$

$$P \rightarrow P_o \quad \text{as} \quad r \rightarrow \infty\ .$$

The solution of the diffusion equation under these conditions is not so well known, but it can be obtained by considering the inverse Fourier transform of P(Q, say) as a new dependent variable. The general solution for Q can be expressed in terms of Kelvin functions. On taking the Fourier transform of the solution which satisfies the appropriate boundary conditions the required solution for P is obtained as

$$P = P_o$$

$$+ \sum_{n=1}^{\infty} P_n \sin(n\omega t - \epsilon_n) \left\{ \frac{\mathrm{ker}(\sqrt{2}\alpha_n r)\,\mathrm{ker}(\sqrt{2}\alpha_n a) + \mathrm{kei}(\sqrt{2}\alpha_n r)\,\mathrm{kei}(\sqrt{2}\alpha_n a)}{\mathrm{ker}^2(\sqrt{2}\alpha_n a) + \mathrm{kei}^2(\sqrt{2}\alpha_n a)} \right\}$$

$$- \sum_{n=1}^{\infty} P_n \cos(n\omega t - \epsilon_n) \left\{ \frac{\mathrm{ker}(\sqrt{2}\alpha_n r)\,\mathrm{kei}(\sqrt{2}\alpha_n a) - \mathrm{ker}(\sqrt{2}\alpha_n a)\,\mathrm{kei}(\sqrt{2}\alpha_n r)}{\mathrm{ker}^2(\sqrt{2}\alpha_n a) + \mathrm{kei}^2(\sqrt{2}\alpha_n a)} \right\}$$

where $\alpha_n^2 = (n\omega\mu S/(2P_o k))$, as before, and 'ker' and 'kei' are Kelvin functions of the second type and zero order ($\mathrm{ker}(q) + i\,\mathrm{kei}(q) = K_o(q\sqrt{i})$ where K_o is the zero order modified Bessel function of the second type). The functions ker and kei can be replaced by their phase and amplitude functions, which satisfy the following relationships

$$N^2(q) = \mathrm{ker}^2(q) + \mathrm{kei}^2(q)\,,$$

$$\Phi(q) = \arctan\left[\mathrm{kei}(q)/\mathrm{ker}(q)\right],$$

so that

$$p = P_o + \sum_{n=1}^{\infty} P_n \frac{N(\sqrt{2}\alpha_n r)}{N(\sqrt{2}\alpha_n a)} \sin(n\omega t - \epsilon_n + \Phi(\sqrt{2}\alpha_n r) - \Phi(\sqrt{2}\alpha_n a)) \quad , \quad r \geqslant a \qquad \text{.....} \quad (2.16)$$

and $p = f(t) \quad , \quad r \leqslant a.$

The rate of flow of air from the top of the cylinder can be found by integrating the continuity equation. Hence, as before,

$$p_e UA = 2\pi h \int_0^{\infty} r \frac{\partial p}{\partial t} dr \quad ,$$

where U is the average wind speed through the top of the cylinder (measured positive inward) and p_e is the average pressure over the top of the cylinder. Integrating $r \frac{\partial p}{\partial t}$, as obtained from equation (16), gives

$$p_e UA = 2\pi h \Big\{ \sum_{n=1}^{\infty} \tfrac{1}{2} P_n \omega n a^2 \cos(n\omega t - \epsilon_n)$$

$$- \sum_{n=1}^{\infty} \frac{a P_o P_n k \sqrt{2}\alpha_n}{\mu S} \; \frac{N_1(\sqrt{2}\alpha_n a)}{N(\sqrt{2}\alpha_n a)} \cos\left(n\omega t - \epsilon_n - \Phi(\sqrt{2}\alpha_n a) + \Phi_1(\sqrt{2}\alpha_n a) - \frac{3\pi}{4}\right) \Big\} \text{.....} (2.17)$$

where the first summation term can be neglected for very permeable media, and N_1 and Φ_1 are the phase and amplitude functions corresponding to the Kelvin functions of the second type and first order.

In the limit as $a \to \infty$ this result reduces to that obtained for flow into a long slit (equation (15)), thus providing a valuable consistency check.

Flow for a simple pressure variation

It is of interest now to consider a simple example in order to appreciate more readily the implications of the above results.

First, however, consider the solution for the case of zero permeability. This corresponds to a cavity which has walls which are completely impervious. From either equation (14) or (17) this is found to be

$$p_1 UA = V \frac{\partial p}{\partial t} , \qquad \text{..... (2.18)}$$

where V is the volume of the cavity beyond the point where U is measured. This solution holds for a cavity of arbitrary shape. The most important consequence of this result is that, in the impervious wall limit, the wind speed and rate of change of pressure are in phase with each other.

Returning now to the porous flow case, suppose that the boundary pressure variation f(t) is a simple sinusoidal function of time

$$f(t) = P_0 + P_1 \sin \omega t \quad , \quad (|x| \text{ or } r \leqslant a)$$

where as before, $P_1 \ll P_0$. Under these conditions, for a long slit, the pressure inside the porous medium is given by

$$p = P_o + P_1 \exp\left[-(|x|-a)\left(\frac{S\mu\omega}{2P_o k}\right)^{1/2}\right] \sin\left[\omega t - (|x|-a)\left(\frac{S\mu\omega}{2P_o k}\right)^{1/2}\right],$$

This solution shows, firstly, that the amplitude of the pressure oscillation decreases exponentially, and secondly, that the oscillation lags further and further behind the boundary variation, as one moves away from the cavity deeper into the porous medium. The average wind speed across an area A at distance l along the slit is

$$U = \frac{2(L-l) P_1 h\sqrt{\omega P_o k}}{p_1 A\sqrt{\mu S}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

$$= \frac{2(L-l)\, h\sqrt{P_o k}}{p_1 A\sqrt{\mu S}} \left[\frac{\partial p}{\partial t}\right]_{t-\frac{\pi}{4\omega}} . \qquad \text{..... (2.19)}$$

The wind speed thus lags behind the rate of change of pressure by an eighth of a period. The magnitude of the wind speed is the same as that for an impervious walled slit of half-width a_{eff} where

$$a_{eff} = \sqrt{\frac{P_o k}{\mu\omega S}} . \qquad \text{..... (2.20)}$$

a_{eff} can be called the 'effective half-width' for the porous medium. In general it will be much larger than the physical half-width of the slit in a very permeable medium so that, as a consequence, the magnitude of the wind speed will be much larger than it would be if the cavity had impervious walls.

As seen above, in this simple case the lag is one eighth of a period: for more complicated boundary conditions the lag can easily be shown to vary considerably either side of this value.

For the circular cylindrical hole the results are similar. The pressure oscillation amplitude decreases and lags further behind the boundary oscillation as one proceeds deeper into the porous medium. Substituting limiting forms for the phase and amplitude Kelvin functions the average wind speed at the top of the cylinder becomes

$$U = \frac{2\pi h P_o P_1 k}{S p_e A \mu \beta} \cos\left(\omega t - \frac{\pi}{2} - \frac{\pi}{4(\gamma - \beta)}\right) \qquad \text{..... (2.21)}$$

where γ is Euler's constant ($\approx .57721$) and

$$\beta = -\ln\left\{\frac{a\sqrt{S\mu\omega}}{2\sqrt{P_o k}}\right\} = -\ln\left\{\frac{a}{2a_{eff}}\right\}$$

Although equation (21) appears to give an infinite lag when $\gamma = \beta$ this possibility is never realised. This is because a is always much smaller than a_{eff} so that β is always larger than ln2 (and $\ln 2 > \gamma$). Generally, in a very permeable medium and for a of order 1 metre, $\beta \gtrsim 3$.

Hence, the lag of wind speed behind the rate of change of pressure is greater than for the case of a long slit and tends to one quarter of a period as a becomes vanishingly small.

The 'effective radius', R_{eff}, can be seen to be

$$R_{eff} = \left\{ \frac{\sqrt{2}\, P_0 k}{\mu \omega \beta S} \right\}^{1/2} \qquad \text{.....(2.22)}$$

which is generally of the same order of magnitude as the effective half-width a_{eff} for the long slit case. Since limiting (small a) forms have been used for the Kelvin functions in the original expression (21) it should be noted that R_{eff} will not tend to a_{eff} in the large-a limit.

2.2 CAVE BREATHING

The results of section (2.1) can be used to explain the inordinately large wind speeds encountered in some single entrance caves. If typical values of pressure and its rate of change, and of cave volume and cross-sectional area are substituted into the equation for a cave with impervious walls (equation 18) the value of the expected wind speed would be quite small (of order 0.1 metres per minute, i.e. 1/200 ft. per sec., through an entrance hole of area 0.2 square metres (2 sq. feet) leading to a cave of volume 1000 cubic metres). For caves of similar volume on the Nullarbor Plain wind speeds of the order of 3 metres per second are actually observed. The implication of such observations is that, either the potentially accessible volume of every cave which exhibits the breathing phenomenon is much greater (by a factor of order 2000) than has so far been detected, or, alternatively, most of the air comes from the porous walls of the cave. It is this latter possibility which appears to gain favour when considered in terms of the preceding theory.

Caves which breathe in response to changes in atmospheric pressure certainly occur in the United States and in Australia. Examples are cited in a number of references(7,11,13,14,15,16,17,18), and many other examples which probably fit this category exist in popular speleological literature. Reliable meteorlogical observations of the phenomenon of cave breathing are, however, scarce and limited to a few authors(15,16,17,18). On inspection of these observations it is found that they invariably indicate that, not only is the magnitude of the breathing much larger than would be expected, but also that there is a lag in changes of direction of breathing behind changes in sign of the derivative of pressure with respect to time. Only in (16) is the significance of this fact pointed out.

Some of the world's most remarkable examples of cave breathing occur in caves in the Nullarbor Plain region of southern Australia, where numerous 'blowholes' (small caves with volumes ranging from 10 to 10,000 cubic metres) are observed to breathe at rates of the order of 1 cu. m. per sec. (35 cu. ft. per sec.). Far surpassing this, however, at a constriction near the entrance of a much longer cave (Mullamullang Cave) wind speeds are regularly found averaging 2 m. per sec. over an area of 20 square metres. (This cave will sometimes be referred to by its alpha-numeric code symbol 'N37').

The morphology of the region is such that the caves fall into two classes, 'deep' and 'shallow' caves (Jennings[10]), which can be idealized to fit the long slit and cylindrical cavity models discussed above. Mullamullang, by far the largest of the deep caves (see Figure 2), has been the subject of intense speleological study since its recent discovery. This study included a short period of meteorological observations early in 1966[16].

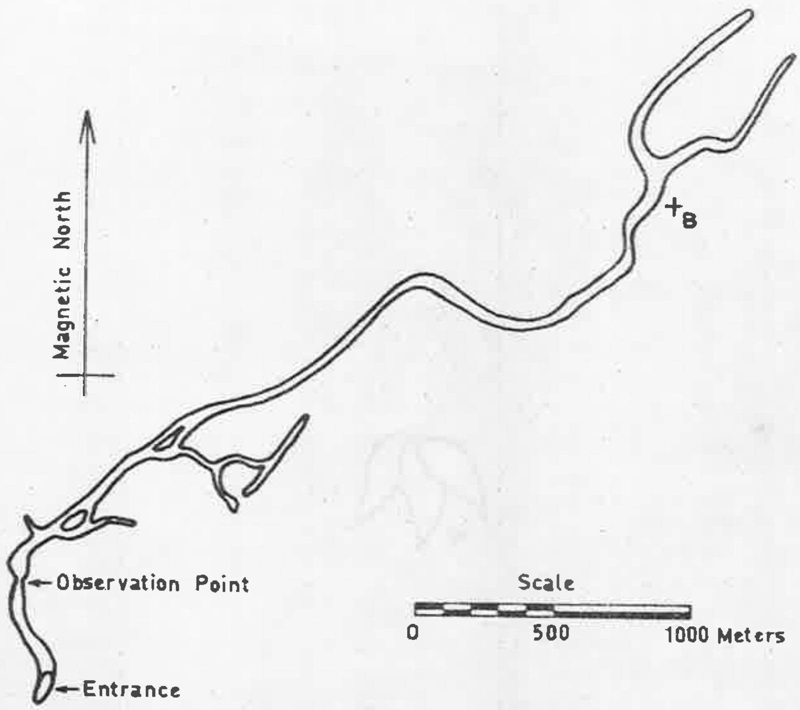

Fig. 2. Simplified map of Mullamullang Cave, Western Australia.

The cave itself shows a marked similarity to the long-slit model and the wind speed and pressure observations taken inside the cave are in excellent agreement with the theoretical predictions. Typical results are shown in Figure 3 which has been adapted from (16). Since pressure variations are reasonably complex, varying lags of wind speed fluctuations behind changes in the time derivative of pressure would be expected, although the magnitude of the lag should still be of the order of one eighth of a period. Also one would expect, on theoretical grounds, that higher frequency pressure oscillations superimposed on the general trend would not be so noticeable in the observed wind speed. These predictions are borne out by the results.

It is fortunate that the tidal semi-diurnal atmospheric pressure fluctuation in the Nullarbor Plain region is of reasonably large amplitude. The observations cited in (16) were made (intentionally) during a period when the fluctuations in pressure due to the movement of large-scale weather systems were small, so that the regular tidal effects would dominate any other trends. In this way an ideal situation existed for the comparison of the simpler theoretical prediction (equation (19)) with experiment.

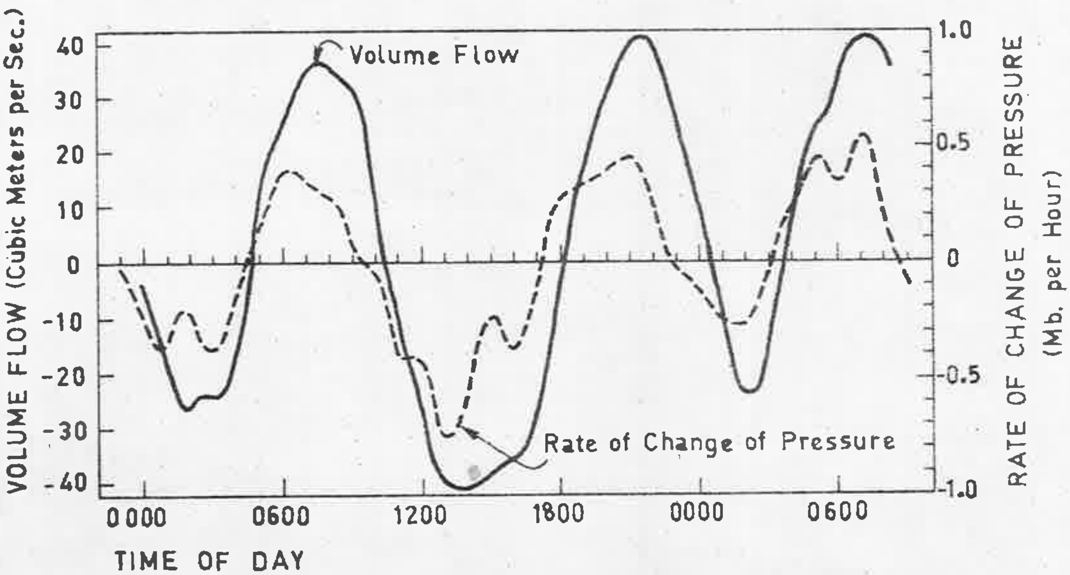

Fig. 3. Simultaneous measurements of wind speed and rate of change of pressure illustrating the lag of fluctuations in speed behind those in $\partial p/\partial t$ and the damping of higher-frequency fluctuations in $\partial p/\partial t$.

Following on from this work a more extensive verification using more accurate instruments was undertaken during January 1967. A blowhole (alpha-numeric designation N73) situated approximately at point B in Figure 2 and shown in relation to the larger cave, N37, in Figure 4, and which was discovered in the course of the 1966 experiment, was chosen as a second observation point. Over the major part of the Nullarbor Plain there is a marked stratigraphic separation between the blowholes and the large (deep) caves, of which Mullamullang Cave is one. The former are wholly situated in the upper layer of Nullarbor Limestone, while the latter, except in the vicinity of their entrances, are in the lower Wilson Bluff Limestone; these two layers being separated by impervious intermediate beds (Jennings[10]). It would be expected that a blowhole situated close to a large cave would thus be separated from it and that the two would breathe independently of each other. However, the stratigraphy of the Plain near N37 and the blowhole N73 is apparently much more complex. Indeed its geologic description is at present an active project of the Western Australian Geological Survey. Even so, it was hoped that, due to their great difference in shape (N37 being physically similar to the long-slit model and N73 being similar in appearance to the cylinder model) simultaneous observations of the two would show some degree of independence.

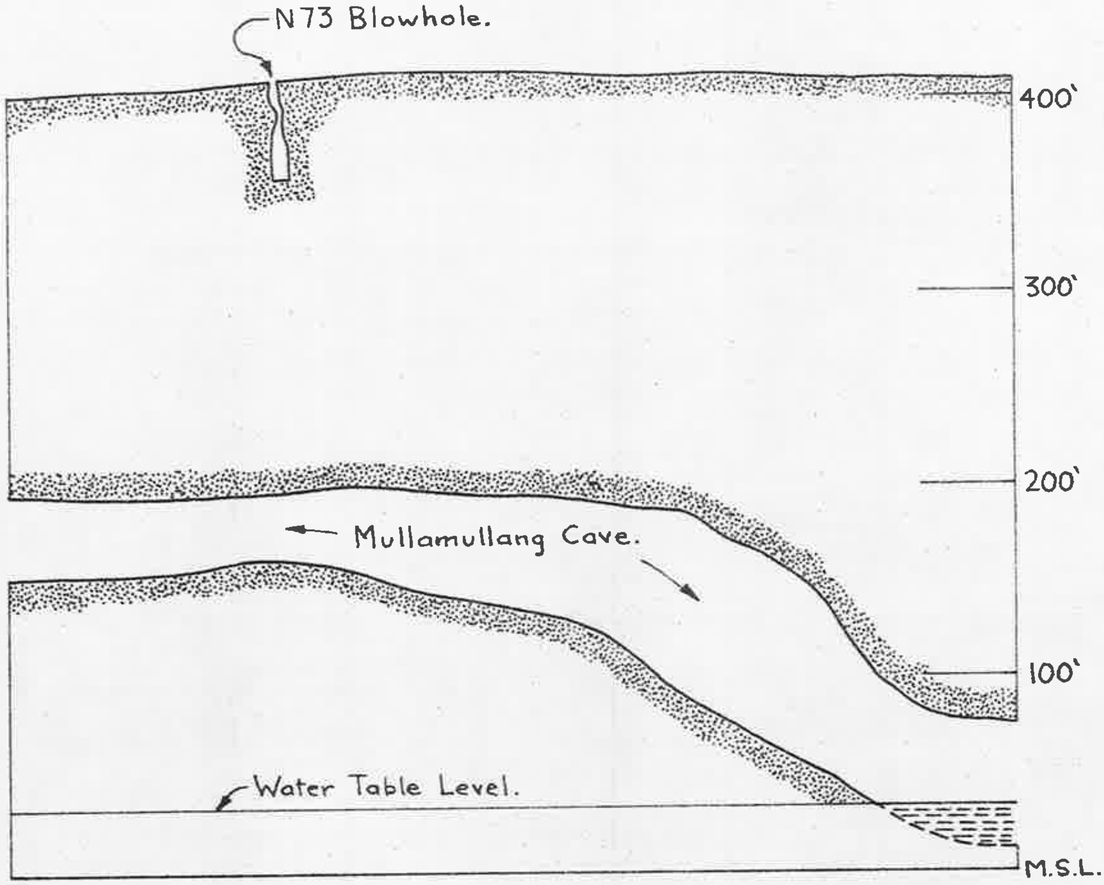

Figure 4. Cross-section showing the relative positions of Mullamullang Cave and the blowhole, N73. Vertical scale shows heights above Mean Sea Level in feet.

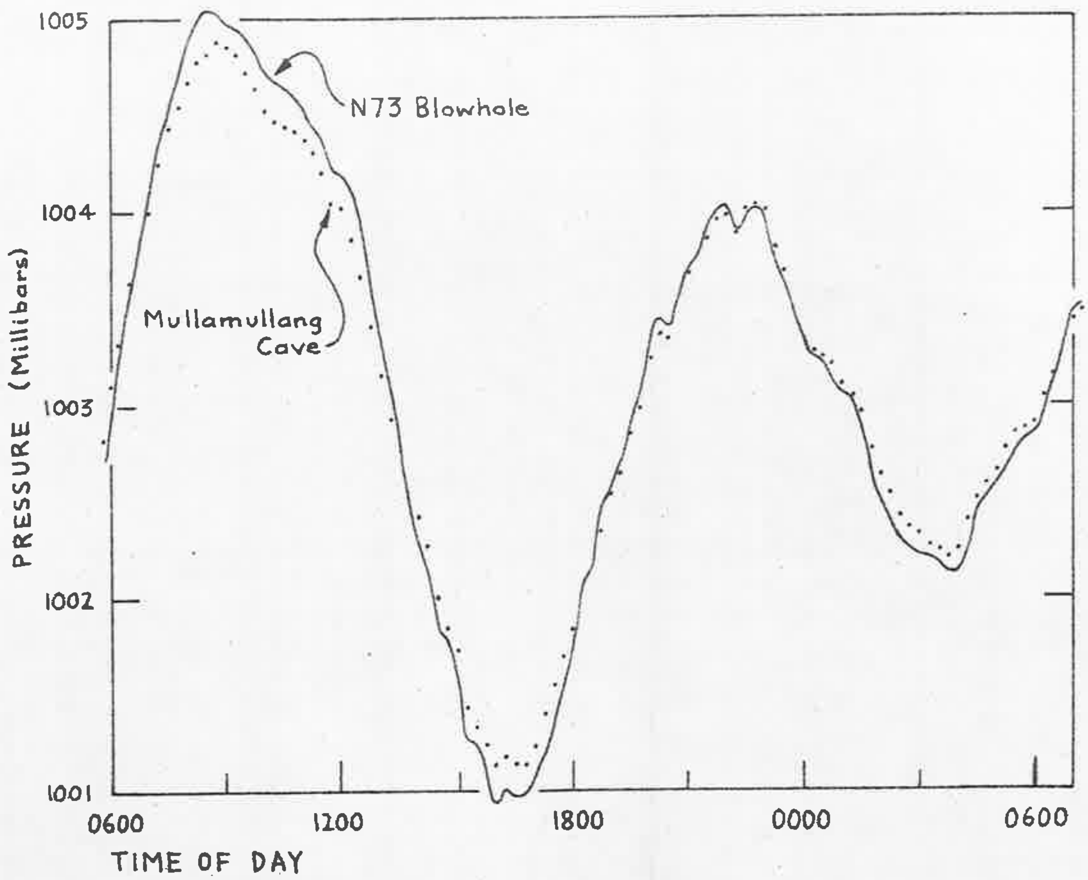

Figure 5. Pressure at the surface (bold line) and underground (dots) observation points showing reduced amplitude at the underground station. (12.40 mb. has been added to the underground readings to make the observations comparable).

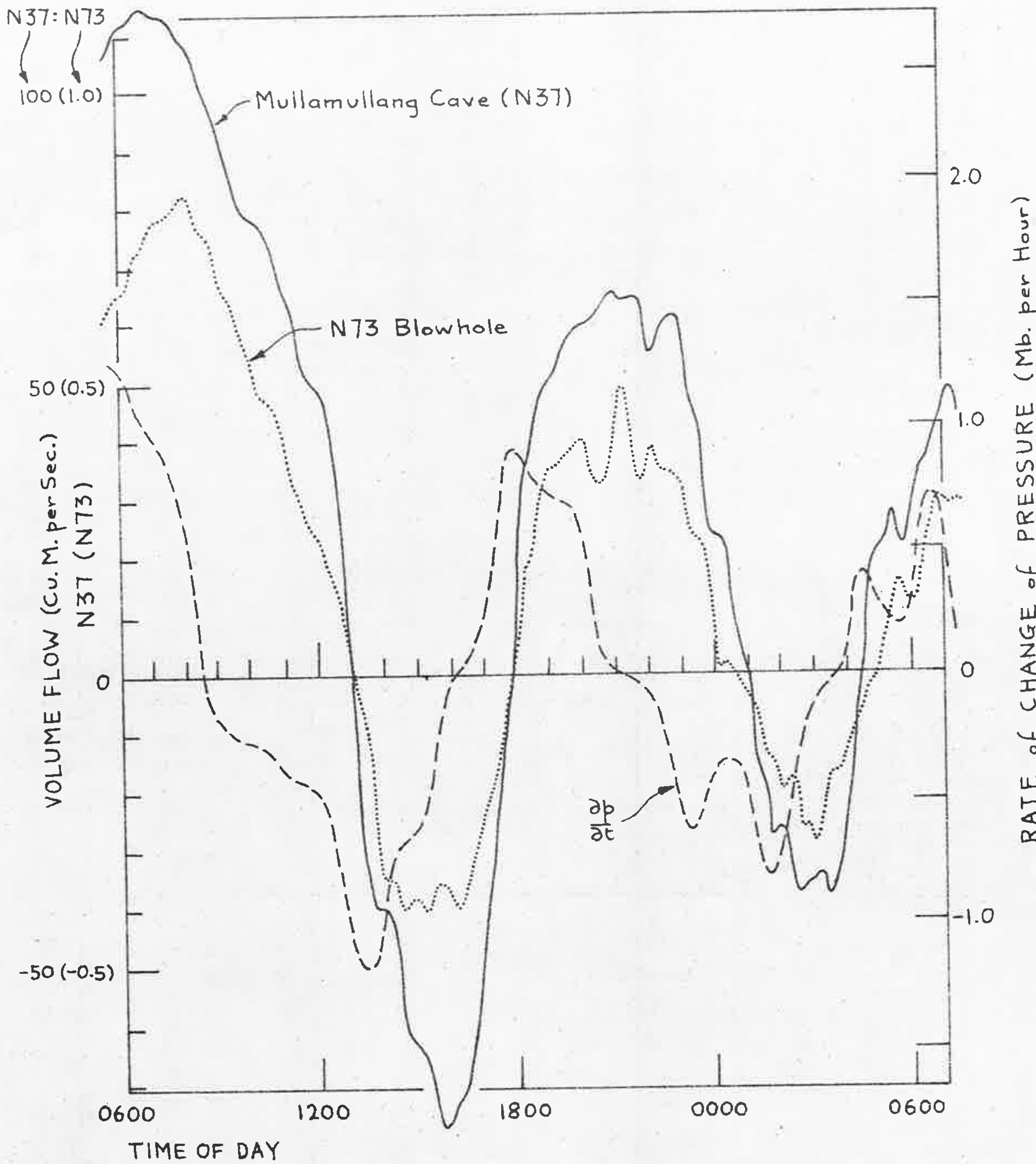

Figure 6. Rate of change of pressure and volume flow at Blowhole entrance and in Mullamullang Cave. Rapid fluctuations in pressure which do not contribute significantly to the rate of breathing have been smoothed out.

Two observation points were set up, one as shown in Figure 2 inside Mullamullang Cave and the other on the surface at point B, the blowhole entrance. The results are shown in Figures 5 and 6. Extremely sensitive Digital Aneroid barometers were used; properly calibrated these can measure pressure absolutely to 0.1 mb. and relatively to 0.01 mb. It was hoped that micro-fluctuations in pressure would be observed which could be correlated between the two observation stations by the use of these instruments.

The results obtained are generally in accord with those obtained during 1966[16]. Two new points are outstanding. Firstly, there is a decrease in the amplitude of the pressure fluctuation in going from the surface to the underground station and some of the small-scale fluctuations noticeable at the surface station are absent from the underground results. This is quite a remarkable result since it indicates that there is an observable damping of the pressure fluctuation along the length of cave (a macroscopic 'pore') between the entrance and the observation point (although this effect may be due in part to the difference in elevation of the two observation points and the fact that the surface observation point was situated some distance from the entrance to the cave N37).

It would be possible to estimate from this amplitude reduction a 'permeability' corresponding to flow through a macroscopic pore, the cave itself. (It should be noted that although Figure 2 shows a relatively free path for air travel between the observation point and the cave entrance this is a result of the degree of simplification used in the drawing). The assigning of a 'permeability' for flow through a pore or pores of the size of the cave is a rather doubtful procedure since in many parts of such a pore the rate of air movement is sufficiently large for the flow to be fully turbulent. Inside the porous medium the flow is certainly slow enough for turbulence effects to be ignored, but in the cave itself the interpretation of the results as a permeability is a somewhat arbitrary step. However, if such a permeability were estimated, its magnitude would be large enough not to invalidate the hypothesis of infinite permeability along the cave length inherent in the theoretical treatment of the long-slit model. This result does, however, indicate that the drag hypothesis on which the original equation of motion, (3), is based may be of more general validity in this application than would appear at first sight. Certainly the hypothesis is valid for discussing effects where the pore-size is of the order of centimetres.

This conclusion cannot necessarily be extrapolated to cover the liquid flow regime. Some preliminary observations of ground-water movement by the author do, however, indicate that this extrapolation may be a good one, at least in this region. Ollos[24] has shown, using laboratory experiments, that the permeability concept is not valid for the discussion of ground-water flow in certain types of karstic region. His conclusion, being not in accord with the above, illustrates the complexity of the problem.

The second point noticeable in the 1967 observations is that the difference in air-flow characteristics between the blowhole and the cave is only small. In particular the times of change of direction of air movement generally differ by only a few minutes. If the blowhole behaved like the cylinder cave model it resembles, then the air flow in it should lag considerably further behind the rate of change of pressure than does the air flow in Mullamullang Cave (see discussion after equation (21)). An explanation of this unexpected behaviour will be given near the end of this section.

Because the models discussed are such broad idealizations of the caves they represent, good quantitative agreement between model and experiment would not be expected. A detailed comparison would require the spectral analysis of a long period of observations. With the type of instruments used (non-recording) and the necessity to use a voluntary labour force to obtain data it was not possible to obtain the required length of observations. It is doubtful whether much more information could have been obtained from more extensive data.

It is possible to estimate the permeability of the limestone from the N37 results, and the good qualitative agreement between theory and experiment in this case suggests that this is not an overambitious task. Calculation of the permeability requires a knowledge of the porosity of the rock and the thickness of the material which participates in the breathing, although only to order of magnitude accuracy in view of the other approximations involved. The porosity is known to be high and is quoted as 26% by King[12]. A thickness estimate of 100 metres is based on stratigraphic work of Lowry (in Hill[16]), the surveys of Hill[16], and aneroid levels established in the area by Wigley and Hill[25]. The permeability, found by using equation (19) with the frequency estimated graphically, is

$$k \approx 10^{-9} \text{ square metres.}$$

It is known that, when the drag theory is valid, the permeability of a medium for liquid flow often differs from the value for gas flow (Klinkenberg[26], Scheidegger[27]). However, the difference is within the limits of accuracy of this experiment and there is evidence (Rigden[28]) that it becomes smaller as pore size increases.

A value of 10^{-9} for the permeability is rather a high one for well-consolidated limestones, but it is not inconsistent with the limestone of this region of the Nullarbor Plain. According to Jennings[10,14] the degree of anastomosing and joint- and bedding-plane enlargement is extremely high. The whole mass of rock is thought to be riddled with anastomosing small tubes and the calculated value of permeability strongly supports this view. The rock in this case can be thought of as the superposition of two porous media with pores of widely differing sizes, similar to the way in which Barenblatt and Zheltov[29] represent a fractured porous medium. The permeability measured is that of the combined media and the two parts are separately considered to contribute 'matrix' and 'fracture' effects, where 'fracture' in this instance denotes the large-scale pores. The value, $k \approx 10^{-9}$, thus represents the combined 'matrix and fracture' permeability, dominated by the effects of anastomoses and fracture, rather than the 'matrix' permeability. No other measurements of the combined permeability of limestones seem to be available; however they are expected to give a value much larger than the matrix permeability (Scheidegger[30]). It should be noted that the matrix permeability of most limestones is appreciably less than would be required for the rock to breathe to any great extent. It is thus probable that breathing caves will be found only in limestones of high fracture permeability.

In the light of this morphological discussion the results from the blowhole, N73, which were found not to fit the cylinder cave model, are not so unlikely. They can be considered, rather, to substantiate the opinion that solutional development occurs preferentially along certain directions in limestone regions of high porosity (Pinchemel[31]) and, in particular, in the Nullarbor Plain region (Jennings[10,14]). The obvious linearity of the deep caves and of many surface features of the Plain is the observational basis for this opinion.

It is probable that the blowhole, N73, is situated in a region of preferential solutional development. This is indicated by the close proximity of N73 to the almost linear large cave, N37, and the fact that, in this part of the Plain, the stratigraphic separation between near-surface and deeper limestone layers is less marked. In this case the resistance to air flow into the blowhole would be reduced along certain preferred directions. The cylindrical symmetry of the situation is therefore lost and the (inverse) permeability must be considered to have tensor properties. If the off-diagonal terms of this tensor are supposed small the problem can be solved following similar lines to the long-slit model theory. If the y direction is chosen as that along which the permeability is greatest and preferential development is confined to a finite range, -L to L, the solution is found to be

$$p_e UA = \frac{2hP_1\omega}{\alpha_x \alpha_y}\left[\cos\left(\omega t - \frac{\pi}{2}\right) - e^{-\alpha_y L}\cos\left(\omega t - \alpha_y L - \frac{\pi}{2}\right)\right] \quad \text{..... (2.23)}$$

(for an external pressure variation $P = P_o + P_1 \sin\omega t$)

where

$$(\alpha_x)^2 = \frac{S\mu\omega}{2P_o}\,(k^{-1})_{xx} \quad , \ \alpha_y \text{ similarly},$$

$$\text{and} \qquad (k^{-1})_{xx} \gg (k^{-1})_{yy} .$$

This result can also be applied to the long-slit case where the 'permeability' effect of motion of air along the slit is represented by $(k^{-1})_{yy}$.

The estimate of permeability from the N37 results can be used for $k_{xx} (\equiv ((k^{-1})_{xx})^{-1})$. Since $k_{yy} \gg k_{xx}$, $\alpha_y L$ must be small, and the appropriate limiting form of equation (23) is

$$p_e UA \approx \frac{2\sqrt{2}\,L h P_1 \omega}{\alpha_x} \cos\left(\omega t - \frac{\pi}{4}\right)$$

so that

$$U \approx \frac{2(2L)\,h\,P_1\sqrt{\omega P_o k_{xx}}}{p_e A\sqrt{\mu S}} \cos\left(\omega t - \frac{\pi}{4}\right) \qquad \text{..... (2.24)}$$

a result which is, as expected, identical to that obtained for the long-slit model (equation (19)). The similarity between the observations at the blowhole and in the cave thus may be considered as providing direct evidence for the morphological supposition of preferential solutional development.

In conclusion, it must be pointed out that a complete and detailed discussion of the experimental results which have been presented does not properly belong in a thesis of this type. Some of the points which have been made are only speculative. In regions such as the Nullarbor Plain, and similar regions elsewhere, it is considered that further work along these lines by more properly qualified observers may provide valuable insight into the structure of limestones which would otherwise be unobtainable.

2.3 DISCUSSION OF OTHER RESULTS

Other qualitative measurements of cave-breathing have been presented by Sartor and Lamar[15], Schley[17] and Conn[18]. Only Conn attempts a detailed explanation of his results. He assumes that the phenomenon is somehow connected with the complex structure of the caves he studies. These are Wind Cave and Jewel Cave, South Dakota, both of which are large systems consisting of numerous complicated passageways. Conn's over-simplified argument is partly successful in that it does predict that a lag should be observed between air movement and rate of change of pressure (although Conn does not state this explicitly). However, Conn's theoretical treatment is semi-empirical and involves fitting parameters to match theory to experiment. The chosen values of these parameters have little physical justification and they vary considerably from cave to cave. Conn also concludes, erroneously, that the large magnitude of the breathing indicates the existence of a large volume of undiscovered penetrable cave. Both Sartor and Lamar, and Schley, draw the same conclusion (or, rather, propose it as an explanation). This 'conclusion' is inadequate since the existence of large volumes of undiscovered cave associated with all breathing caves is a highly improbable situation. It is this very possibility that any explanation of breathing would hope to avoid.

The results given by these authors can be discussed with reference to the theory presented here. Both Schley, and Sartor and Lamar, studied blowholes in the Wupatki area of north-central Arizona. From the geological discussion in Sartor and Lamar it appears that these blowholes should fit the cylinder cave model of the present theory. Schley gives pressure and wind speed readings covering a 25 hour period. Assuming that the term β in equation (22) of 2.2 above is approximately 5 (β can only vary between about 3 and 10 in general), and using values for the other unknowns from the geological data in Sartor and Lamar, Schley's results indicate that $k \sim 10^{-9}$ $(\text{metres})^2$. His results also show a lag of the order $\frac{1}{8}$ to $\frac{1}{4}$ of a period. Sartor and Lamar present results of pressure and wind speed over a 72 hour period. These indicate a value of the permeability of order 10^{-9} $(\text{metre})^2$, consistent with that using Schley's results. The results of Sartor and Lamar show lags of about $\frac{1}{4}$ of a period (where the pressure variation is approximately sinusoidal). This is in complete accord with the cylinder cave model predictions. In particular, these results indicate a permeability value consistent with that obtained from the Nullarbor Plain results given in this thesis, and they show the predicted lag of the order $\frac{1}{8}$ to $\frac{1}{4}$ of a period. This latter point tends to support the hypothesis of tensor permeability put forward to explain the discrepancy between the N73 blowhole results and the cylinder cave model.

Conn gives extensive results from Wind and Jewel caves; wind speed and pressure over periods of 17, 11 and 12 days. Unfortunately they are presented on such a small scale that they cannot be analysed in detail. The pressure changes during the observation periods are also rather complicated (semi-diurnal tidal fluctuations being scarcely noticeable) and the two caves studied are such complex systems that it is not possible to fit them to either the cylinder or long-slit models with any confidence. However, Conn's results do show lags of the correct order of magnitude. Also, if the long-slit model theory is applied to them, chosing estimates of the cave dimensions L and h based on the data presented by Conn and the map shown by Halliday[8], Conn's results indicate that $k \sim 10^{-8}$ to 10^{-9} $(\text{metres})^2$. This value is far from certain, particularly since the geological structure of the limestone in the Wind and Jewel Caves region is very different from that of the Nullarbor Plain.

Some of the results of this chapter have been published as 'Non-Steady Flow through a Porous Medium and Cave Breathing' in Journal of Geophysical Research, 72, 3199-3205, (1967).*

In closing, some rather interesting aspects of cave breathing are worthy of mention. Firstly, the possibility of harnessing the effect to produce power. At peak flow-rate Mullamullang Cave breathes at an incredible rate (equal to the rate of flow of water from the world's largest fresh-water springs). The production of power on the Nullarbor Plain by this means may one day be realised since the region is remote from other energy

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* Reviewed in New Scientist 35 (555), 208 (1967).

sources. Secondly, the use of the cool air (about 68°F) which breathes from the blowholes of the Plain for air-conditioning is a possibility which has already been exploited. Out-breathing cycles were a welcome respite from the above-century conditions which prevailed during the observation period at the blowhole N73.

CHAPTER THREE: EXTENSION TO HYDROLOGY

The cylinder cave model discussed in the previous chapter is similar to a fundamental problem in Hydrology, the flow of water from a confined aquifer into a well. The cylinder model corresponds exactly to the case where the piezometric head (the free level to which water would rise in a well which penetrates the aquifer) is a periodic function of time. This is a circumstance which is not often realised and it is a logical step to extend the theory to cover more realistic cases. The simplest non-steady well flow problem, that of determining the shape of the piezometric surface for an idealized well with constant discharge rate, was first solved by Theis(32). A solution to the image problem of determining the discharge due to constant drawdown (i.e. the reduction of piezometric head at the well) was later given by Jacob and Lohman(33) using a solution to the heat conduction equation due to Smith(34). In the following the more general problem of flow into a well where the discharge (or recharge) is at an arbitrary rate will be solved by a straight-forward generalization of the cylinder model cave-breathing theory.

3.1 FLOW INTO A WELL WITH ARBITRARY DISCHARGE

In a homogeneous isotropic medium the equation describing the time evolution of the piezometric head ($h(\underline{r},t)$) is

$$\nabla^2 h = \frac{S^*}{T}\frac{\partial h}{\partial t}$$

where S* is the storage coefficient of the (confined) aquifer, which corresponds to the porosity (denoted by S previously) in an unconfined aquifer, and T is the transmissivity of the medium (related to the permeability). For the case of cylindrically symmetrical flow into a well which completely penetrates a confined aquifer of uniform thickness, infinite extent and with no lateral inflow the equation becomes

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S^*}{T}\frac{\partial h}{\partial t} \quad . \qquad \text{..... (3.1)}$$

Consider the problem of solving equation (1) with the boundary conditions

(a) $h \rightarrow h_o$ as r ,

(b) $h(r,t)$ is an arbitrary function of time at

$r = a_1$. ..... (3.2)

The well will be assumed to have a finite radius 'a' (not necessarily small). In the condition 2(b) above it will be assumed that $a_1 = a$, a convenient although not a necessary simplification. The correspondence between this problem and the cylinder model cave breathing problem is now apparant. S* (written S from now on) replaces the porosity; T replaces $\frac{P_0 k}{\mu}$; the piezometric head, h, replaces the pressure, P; and the well radius, a, corresponds to the radius of the cylindrical cave. However, where P was expanded as a Fourier series, the boundary condition for h(a,t) must be non-periodic and so, to complete the analogy, must be written in the form of a Fourier integral,

$$h(a,t) = h_0 - \int_{-\infty}^{\infty} G(x) \exp(2\pi i x t)\, dx . \qquad \text{.....} (3.3)$$

Equation (1) can be solved by considering the inverse Fourier transform of h(r,t) (g(r,x), say) as a new dependent variable as was done in the previous chapter. The general solution for g(r,x) is obtained in terms of Kelvin functions. On taking the Fourier transform of the particular solution obeying the appropriate (transformed) boundary conditions (corresponding to (2(a)) and (3)) the solution for h is found to be

$$h(r,t) = h_0 - \int_{-\infty}^{\infty} G(x) \frac{N(\alpha r)}{N(\alpha a)} \exp\left\{i\left(2\pi xt + \Phi(\alpha r) - \Phi(\alpha a)\right)\right\} dx \quad \ldots\ldots (3.4)$$

where $\alpha^2 = 2\pi Sx/T$. Here N and Φ are the amplitude and phase functions corresponding to the zero-order Kelvin functions of the second type as in chapter 2. In this instance it is more convenient to retain the modified Bessel function form

$$h(r,t) = h_0 - \int_{-\infty}^{\infty} G(x) \frac{K_0(\alpha r\sqrt{i})}{K_0(\alpha a\sqrt{i})} e^{2\pi ixt} dx \quad \ldots\ldots (3.5)$$

Substitution of the solution (5) into the continuity equation which determines the discharge rate,

$$Q(t) = -2\pi S \int_a^{\infty} r \frac{\partial h}{\partial t} dr ,$$

and integrating over r gives

$$Q(t) = \int_{-\infty}^{\infty} G_1(x) e^{2\pi ixt} dx \quad \ldots\ldots (3.6)$$

where

$$G_1(x) = 2\pi T \alpha a\sqrt{i} \frac{K_1(\alpha a\sqrt{i})}{K_0(\alpha a\sqrt{i})} G(x). \quad \ldots\ldots (3.7)$$

It is possible to determine not only the rate of discharge, Q, for known drawdown conditions, but also the drawdown as a general function of r and t if Q is known as a function of time. The former possibility will be illustrated by generating the solution of Jacob and Lohman[33] and the latter by generating the Theis solution[32].

3.2 CONSTANT DISCHARGE CASE

The boundary conditions generally assumed are

(a) $h \rightarrow$ constant (h_o) as $r \rightarrow \infty$,

(b) $h(r,t) = h_o$ for $t \leqslant 0$,

(c) If the rate of flow from the well is $Q(t)$ then $Q(t) = 0$ for $t < 0$ and $Q(t) =$ constant (Q_o) for $t \geqslant 0$. ...(3.8)

The solution for $h(r,t)$ for $t > 0$ is then [32]

$$h = h_o - \frac{Q_o}{4\pi T} E_1(u)$$

where $E_1(u)$ is the exponential integral, $\int_u^\infty \exp(-x)\, dx/x$, and $u = Sr^2/4Tt$.

Although readily applied to field conditions this solution (the 'Theis solution') is subject to fairly restrictive boundary conditions and is, in some ways artificial since the well must be assumed to have an infinitesimal radius while the solution diverges logarithmically at the origin. To use the solution (6) above it is necessary to assume that the discharge is a known function of time; in this case a step function given by (8(c)). This determines $G_1(x)$ which is related to $G(x)$ through (7) which in turn determines the form of $h(r,t)$. With this choice for Q the transform function G_1 is

$$G_1(x) = \frac{Q_o}{2\pi i x}$$

Equation (7) then determines G so that the general solution for $h(r,t)$ is

$$h(r,t) = h_o - \frac{Q_o}{2\pi T} \int_{-\infty}^{\infty} \frac{1}{\alpha a \sqrt{i}\; 2\pi i x} \frac{K_o(\alpha r \sqrt{i})}{K_1(\alpha a \sqrt{i})} e^{2\pi i x t} dx \qquad \ldots\ldots (3.9)$$

The integration contour can be displaced by an amount c(real and positive) to below and parallel to the real axis. This is permissable since a c exists such that no poles occur inside the infinite rectangle formed by the displacement and since the contribution from the ends of the rectangle at infinity is zero.

As the well radius approaches zero this solution should approach the Theis solution. In the limit of small q

$$q K_1(q\sqrt{i}) \rightarrow (\sqrt{i})^{-1} + O(q^2 \ln \tfrac{1}{2}q).$$

Hence

$$h(r,t) = h_o - \frac{Q_o}{2\pi T} \int_{-\infty-ic}^{\infty-ic} \frac{K_o(\alpha r\sqrt{i})}{2\pi i x} \left\{1 - O\left(\alpha^2 a^2 \ln \frac{\alpha a}{2}\right)\right\} e^{2\pi i x t} \, dx. \quad \ldots\ldots (3.10)$$

From known Fourier integrals (see, for example, Campbell and Foster [35]) the following relation can be obtained

$$\int_{-\infty-ic}^{\infty-ic} \frac{K_o(\sqrt{2\pi i x z})}{2\pi i x} e^{2\pi i x t} \, dx \begin{array}{ll} = \frac{1}{2} E_1(z/4t) & , \ t > 0 \\ = 0 & , \ t \leqslant 0 \end{array}$$

where z can be any finite complex quantity with positive real part.

Using this with $z = Sr^2/T$ equation (10) becomes

$$h(r,t) \approx h_o - \frac{Q_o}{4\pi T} E_1\left(\frac{r^2 S}{4Tt}\right) \quad , \ t > 0$$

$$h(r,t) = h_o \quad , \ t \leqslant 0$$

where terms which are smaller than of order a have been neglected. This is exactly the form obtained by Theis.

By using the ascending series expansion for $K_1(q)$ the higher order correction terms to this result can be obtained. The first correction term can be obtained after some manipulation so that the solution for $t > 0$ becomes

$$h(r,t) = h_0 - \frac{Q_0}{4\pi T}\left\{ E_1\left(\frac{r^2 S}{4Tt}\right) - \frac{Sa^2}{2Tt}\exp\left(-\frac{Sr^2}{4Tt}\right)\ln\left(\frac{Sar}{4Tt}\right)\right\}$$

valid for small a (more strictly the limiting process used above is valid only for $\frac{Sa^2}{Tt}$ small so that, for any small but finite radius, δ say, this solution is incorrect for times $\lesssim \frac{S\delta^2}{T}$). This solution is the same as that given by Ritchie and Sakakura[21] (see also Carslaw and Jaeger[36]) obtained using Laplace transforms, and the methods are formally equivalent because of the relationship between Fourier and Laplace transforms

$$\underline{F}\left\{f(p)\right\} = \int_{-\infty-ic}^{\infty-ic} f(p)\, e^{2\pi ixt}\, dx = \frac{1}{2\pi i}\int_{2\pi c-i\infty}^{2\pi c+i\infty} f(x)\, e^{xt}\, dt = L^{-1}\left\{f(x)\right\}$$

where $p = 2\pi ix$, $\underline{F}$ is the Fourier transform operator with displaced integration contour and L^{-1} is the inverse

Laplace transform operator.

3.3 CONSTANT DRAWDOWN CASE

A solution to this special case was first given by Jacob and Lohman[33] in the form of an integral

$$Q(t) = 8HT \frac{Tt}{Sa^2} \int_0^\infty y \exp\left(-\frac{Tt}{Sa^2} y^2\right) \left\{\frac{\pi}{2} + \tan^{-1}\left(\frac{Y_0(y)}{J_0(y)}\right)\right\} dx \qquad \text{..... (3.11)}$$

which they evaluated numerically, although the equivalent integral (equation (13) below) had previously been tabulated by Jaeger and Clarke[37].

Using the boundary conditions:

(a) $h \rightarrow$ constant (h_o) as $r \rightarrow \infty$,

(b) $h = h_o$ for $t \leqslant 0$, and

(c) $h = h_o - H$ for $r \leqslant a$, $t > 0$,

the solution is

$$Q(t) = 2HT\pi a \sqrt{\frac{S}{T}} \int_{-\infty}^{\infty} \frac{K_1\left(\sqrt{\frac{S}{T}}\sqrt{p}\, r\right)}{\sqrt{p}\, K_0\left(\sqrt{\frac{S}{T}}\sqrt{p}\, a\right)} e^{2\pi i x t} \, dx \qquad \text{..... (3.12)}$$

which is equivalent to that obtained using Laplace transforms. The Laplace transform solution can be reduced to (see, for example, Carslaw and Jaeger[36])

$$Q(t) = \frac{8HT}{\pi} \int_0^\infty \exp\left(-\frac{Tt}{Sa^2} y^2\right) \frac{dy}{y(Y_0^2 + J_0^2)} \qquad \text{.....} (3.13)$$

to which (11) reduces on integration by parts. The direct reduction of (12) to (13) can be obtained using an interesting integral involving Bessel functions which is derived below.

LEMMA:

$$\int_0^\infty \frac{dy}{y(y^2+z^2)(J_0^2+Y_0^2)} = \frac{\pi^2}{4} \frac{K_1(z)}{z K_0(z)}, \quad \mathcal{R}(z) > 0.$$

The relationship

$$\frac{K_1(-iq)}{K_0(-iq)} + \frac{K_1(iq)}{K_0(iq)} = \frac{4}{\pi q [Y_0^2(q) + J_0^2(q)]}$$

for real positive q, which can easily be established by first writing the modified Bessel functions in terms of Hankel functions, can be used to rewrite the integral as

$$\frac{\pi}{4} \int_{-\infty}^\infty \frac{K_1(iy)}{K_0(iy)} \frac{dy}{y^2+z^2}$$

The integration contour can now be extended to include the infinite semicircle in the lower half of the y-plane, thus avoiding the zeroes of $K_0(iy)$, which are all distributed in the upper half of the plane, and the cut along the positive imaginary axis. The particular pole of $\frac{1}{y^2+z^2}$ which lies within the contour depends on the sign of the real part of z. The Lemma is proved by a direct application of the residue theorem.

To integrate (12) the integrand is replaced by the integral form derived in the Lemma and the result

$$Q = \frac{8TH}{\pi} \int_0^\infty \exp\left(-\frac{Tt}{Sa^2} y^2\right) \frac{dy}{y(Y_o^2 + J_o^2)}$$

follows directly from interchanging the order of integration and evaluating the simple Fourier transform which remains.

3.4 PARTIALLY CONFINED AQUIFERS

One of the consequences of cave breathing is the possible effect of the phenomenon on the level of an unconfined aquifer. It is recognised that fluctuations in piezometric head can be caused by rainfall, well discharge or recharge, ocean tides, earth tides and atmospheric pressure changes. The latter two can only cause piezometric head fluctuations in confined aquifers and most authors state that atmospheric pressure changes can have no effect on the level of an unconfined aquifer. This is generally true; however, if a breathing cave (or well) intersects the free water table a rather unique situation exists. Throughout the aquifer there is hydrostatic balance between the water and the air. Inside the cave or well the air pressure changes are the same as those in the outside atmosphere. As one moves into the rock away from the well the amplitude of the pressure fluctuations is reduced. Thus, at a large distance from the well the air pressure above the water is effectively constant and could have the effect of 'confining' the aquifer in these regions. Nearer to the cave the increasingly larger pressure fluctuations will cause the level of the groundwater to rise and fall in response creating a situation resembling a giant water barometer. The magnitude of these level changes will be quite small and the relationship between them and the outside pressure variation would in general be quite complicated.

Mullamullang Cave intersects the water-table in this way and small-scale level changes were observed in the course of the cave-breathing observations cited in chapter two. Since the nearest point that the water level could be studied was some distance from the cave breathing observation point, level readings were only taken at widely spaced and random intervals. As a consequence only an inadequate picture of the way the level varied with time could be obtained. No simple relationship between pressure and level was evident.

Ward[9] states that level fluctuations have been observed in 'breathing' wells drilled in the porous limestones of the Nullarbor Plain and the Murray River valley (South Australia). In one such well, measurements showed a definite correlation between atmospheric pressure and water level; the details of these measurements are no longer available. Ward gives no physical explanation for the phenomenon other than that it is associated with the breathing which, in turn, he states to be a consequence of the high porosity of the limestone. Ineson[38] describes 'unexplained' level fluctuations (attributed to atmospheric pressure changes) in wells penetrating 'water-table' aquifers in chalk. He concludes that the aquifers must in reality be confined. It appears likely that the observations of Ward and Ineson are examples of the partial confining of an aquifer proposed above.

It is possible to construct models to predict these level fluctuations theoretically, based on the theory of cave breathing, but there would be no physical grounds on which to base the choice of a model (i.e. in the case of a cave the model is an idealization of what the cave actually looks like; since there is no data available on the shape and extent of the water-table it is more difficult to justify a model water-table). However, it may be possible to justify a particular model pragmatically. In the case of Mullamullang Cave there is some indication from this approach that the water is restricted to a channel of finite width parallel to the cave trend, but of much smaller extent than is involved in the cave-breathing (in chapter two the limestone which breathes is assumed to be infinite in the direction normal to the cave).

Any model, as proposed above, must assume that the drag theory of porous medium flow is applicable to the flow of ground-water. While the cave-breathing results imply that this theory is valid for the flow of air, it may not be so for the flow of a liquid. In fact the character of the pore spaces in the limestone below the water level may be markedly different from that of the pores above the water level. In view of the observation of Ollos[24], that the permeability concept is not valid for liquid flow in at least some types of limestone, a detailed theoretical discussion of a partially confined aquifer does not seem warranted at this stage. However, level fluctuations which are a consequence of partial confinement do occur and more accurate and more extensive measurements made in conjunction with accurate pressure observations would be worthwhile.

Although the results of sections 1, 2 and 3 of this chapter are fairly well-known in the theory of heat conduction, the general treatment appears to have been neglected in hydrological texts. Integral transform techniques have, however, been used in this field for more complex problems. Part of the contents of this chapter have been published as a short paper in _Journal of Hydrology_ (in press).

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Non-Steady Flow through a Porous Medium and Cave Breathing

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The theory of flow through a porous medium into a cavity under the action of an arbitrary pressure variation inside the cavity is used to explain the volume and phase shift of air moving into and out of caves in response to changes in atmospheric pressure. Observations of this phenomenon ('breathing') can be used in conjunction with the theory to estimate the combined 'matrix and fracture' permeability of limestones.

Introduction. Many caves are found in limestone regions throughout the world, and, in some of these regions, there are caves which are said to 'breathe.' The term breathe has been used to describe air movements caused by changes in atmospheric pressure and also to describe the shorter-period oscillations occurring when the cave air resonates in response to external air movements. This paper is concerned with the first of these types of breathing.

In this sense, caves that breathe as a result of changes in atmospheric pressure present an enigma since the magnitude of the breathing is much greater than might be expected. A number of attempts have been made to explain this phenomenon, but they have generally lacked a firm scientific foundation and have mainly been confined to popular speleological literature. In this paper the theory of time-dependent flow of air through a porous medium is developed and successfully applied to resolve the cave-breathing dilemma.

Porous flow theory. Consider the problem of air flow in a porous medium of thickness h which lies in or on the earth's surface. The medium is bounded by parallel impervious layers at the top and bottom, and the air flow in it is induced by pressure variations in a cavity inside the medium. Two particular cavity shapes will be considered (Figure 1).

The first cavity chosen is a long rectangular slit of length L, height h, and half-width a. The slit is open at one end and is assumed to be long enough for the effects of flow into the other end to be ignored. The geometry here suggests the use of a Cartesian coordinate system with origin at the center of the lower side of the open end of the slit. The cavity is thus confined to the region $-a \leq x \leq a$, $0 \leq x \leq L$, $0 \leq z \leq h$.

The second cavity chosen is a circular cylinder of height h and radius a with axis normal to the two impervious layers. The upper end of the cylindrical cavity penetrates the upper impervious layer. Cylindrical polar coordinates are suggested here, with origin at the bottom of the cylinder, where the axis intersects the lower impervious layer. In this system the cavity is restricted to the region $r \leq a$, $0 \leq z \leq h$.

The equations governing motion in the porous medium are

$$p = \rho R T \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{2}$$

$$\frac{d\mathbf{v}}{dt} + \nabla \varphi + \frac{1}{\rho} \nabla p + \mathbf{v} \times 2\mathbf{\Omega} + \frac{S \mu \mathbf{v}}{(\rho k)} = 0 \tag{3}$$

where

k is the permeability of the medium.
S is the porosity of the medium.
p, ρ, T are the pressure, density, and temperature of the air.
φ is the earth's geopotential.
μ is the coefficient of viscosity of the air.
$\mathbf{\Omega}$ is the angular velocity of the earth's rotation.
$\mathbf{v}$ is the (particle) velocity of the air, not to be confused with the seepage velocity ($\mathbf{q} = S\mathbf{v}$) often used in porous flow studies.

The flow is assumed to be laminar and to make the system of equations complete, isothermal.

The boundary conditions imposed are (1) at

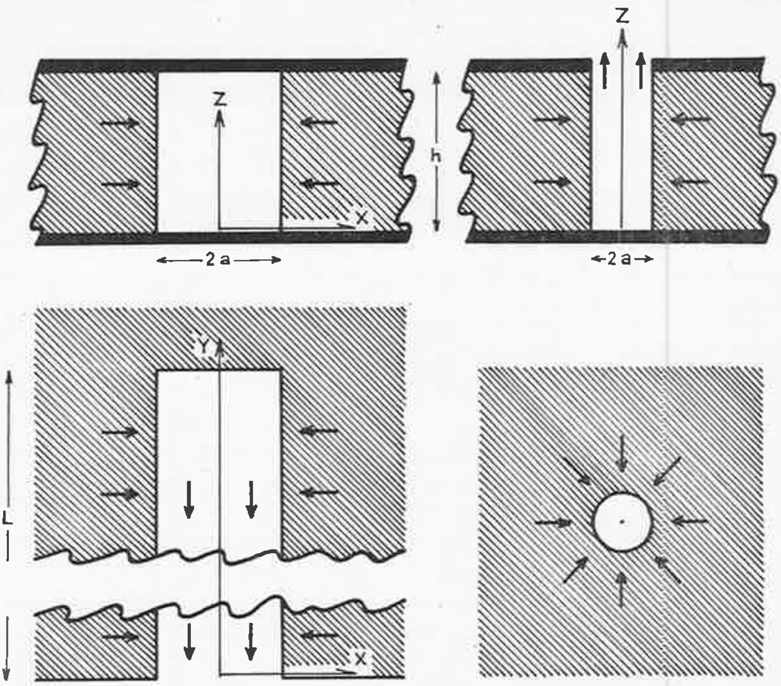

Fig. 1. Elevation and plan of the long slit and cylinder cave models. The arrows show direction of air movement during an out-flow (falling pressure) cycle. Long slit (*left*); cylinder (*right*).

an infinite distance from the cavity $p =$ constant ($=P_0$, say) and (2) inside the cavity $p(z = 0) = f(t)$. The function of time $f(t)$ need not be specified more precisely than $f(t) \approx$ constant (as is the case with atmospheric pressure variations).

Flow into a long slit. The symmetry of the problem suggests that, if y is the coordinate axis measured along the slit, the vertical and y components of velocity may be neglected. Since the remaining component, u, is small, Coriolis terms may be ignored and the equations reduce to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \qquad (|x| \geq a) \qquad (4)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{(\partial p/\partial x)}{\rho} - \frac{S\mu u}{(\rho k)} \quad (|x| \geq a) \quad (5)$$

$$\partial p/\partial y = 0 \qquad (6)$$

$$\partial p/\partial z = -\rho g \qquad (7)$$

Using the isothermal condition, equations 6 and 7 can be integrated. The solution is

$$p = P(x, t) \exp(-zg/(RT)) \approx P(x, t) \qquad (8)$$

since $0 \leq z \leq h$ and $h \ll g/(RT)$, so that equations 4 and 5 become

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}(Pu) = 0 \qquad (|x| \geq a) \qquad (9)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\left(RT\frac{\partial P}{\partial x}\right) \Big/ P - S\mu RTu/(pk)$$
$$(|x| \geq a) \qquad (10)$$

Differentiating (9) with respect to x and (10) with respect to t and subtracting gives, for $|x| \geq a$,

$$S\mu\frac{\partial P}{\partial t} = P_0 k\frac{\partial^2 P}{\partial x^2} \qquad (11)$$

where small terms have been neglected since, with $f(t) \approx$ constant, $\delta(\ln p) \ll \delta(\ln u)$. Now $f(t)$ can be expanded in a Fourier series,

$$f(t) = P_0 + \sum_{n=1}^{\infty} P_n \sin(n\omega t - \epsilon_n)$$

and under these conditions, the solution of (11) is well known [*Carslaw*, 1921], being

$$P = P_0 + \sum_{n=1}^{\infty} P_n e^{-\alpha_n(|x|-a)} \cdot \sin [n\omega t - \epsilon_n - \alpha_n(|x| - a)]$$

where $\alpha_n^2 = [n\omega\mu S/(2P_0 k)]$ and where, to comply with the assumed close constancy of $f(t)$, $P_n \ll P_0$. Thus the complete solution is given by

$$p = \left\{P_0 + \sum_{n=1}^{\infty} P_n e^{-\alpha_n(|x|-a)} \cdot \sin [n\omega t - \epsilon_n - \alpha_n(|x| - a)]\right\} \qquad |x| \geq a \tag{12}$$

$$p = f(t) \qquad |x| \leq a$$

The amount of air flowing through the slit at any point along its length can now be calculated. Integrating the continuity equation over the volume $-\infty \leq x \leq \infty$, $y \geq l$, $0 \leq z \leq h$, and using Gauss's theorem yields the result

$$p_l U A = \iiint \frac{\partial p}{\partial t}\, d\tau \tag{13}$$

where p_l is the average pressure and U the average air speed (measured positive inward) over the cross section of area A normal to the y axis at $y = l$, and $d\tau$ is a typical volume element. Hence,

$$p_l U A = 2h(L - l) \int_0^{\infty} \frac{\partial p}{\partial t}\, dx$$

The length l should be small compared with L, so that end effects can be ignored. Completing the integration we find that

$$p_l U A = 2h(L - l) \sum_{n=1}^{\infty} P_n n\omega \{a \cos (n\omega t - \epsilon_n) + (\sqrt{2}/(2\alpha_n)) \cos (n\omega t - \epsilon_n - \pi/4)\} \tag{14}$$

which becomes, for a very permeable medium, (say $k \gtrsim 10^{-11}$ meter^2)

$$U = \{2h(L - l)/(p_l A)\}\{P_0 k/(\mu S)\}^{1/2} \cdot \sum_{n=1}^{\infty} P_n \sqrt{n\omega} \cos (n\omega t - \epsilon_n - \pi/4) \tag{15}$$

Flow into a circular cylinder. Following the argument presented in the previous section it is found that, for $r \geq a$,

$$p = P(r, t)$$

$$S\mu \frac{\partial P}{\partial t} = P_0 k\left(\frac{\partial^2 P}{\partial r^2} + r^{-1} \frac{\partial P}{\partial r}\right)$$

with the boundary conditions

$$P = P_0 + \sum_{n=1}^{\infty} P_n \sin (n\omega t - \epsilon_n) \qquad r \leq a$$

$$P \to P_0 \quad \text{as} \quad r \to \infty$$

The solution of the heat equation under these conditions is not so well known, but it can be obtained by considering the inverse Fourier transform of P(Q, say) as a new dependent variable. The general solution for Q can be expressed in terms of Kelvin functions. On taking the Fourier transform of the solution that satisfies the appropriate boundary conditions the required solution for P is obtained as

$$P = P_0 + \sum_{n=1}^{\infty} P_n \sin (n\omega t - \epsilon_n) \frac{[\operatorname{ker} (\sqrt{2}\alpha_n r) \operatorname{ker} (\sqrt{2}\alpha_n a) + \operatorname{kei} (\sqrt{2}\alpha_n r) \operatorname{kei} (\sqrt{2}\alpha_n a)]}{[\operatorname{ker}^2 (\sqrt{2}\alpha_n a) + \operatorname{kei}^2 (\sqrt{2}\alpha_n a)]}$$
$$- \sum_{n=1}^{\infty} P_n \cos (n\omega t - \epsilon_n) \frac{[\operatorname{ker} (\sqrt{2}\alpha_n r) \operatorname{kei} (\sqrt{2}\alpha_n a) - \operatorname{ker} (\sqrt{2}\alpha_n a) \operatorname{kei} (\sqrt{2}\alpha_n r)]}{[\operatorname{ker}^2 (\sqrt{2}\alpha_n a) + \operatorname{kei}^2 (\sqrt{2}\alpha_n a)]}$$

where $\alpha_n^2 = (n\omega\mu S/(2P_0 k))$, as before, and 'ker' and 'kei' are Kelvin functions of the second type and zero order ($\operatorname{ker}(q) + i \operatorname{kei}(q) = K_0 (q \sqrt{i})$ where K_0 is the zero-order modified Bessel function of the second type). The functions ker and kei can be replaced by their phase and amplitude functions, which satisfy the following relationships,

$$N^2(q) = \operatorname{ker}^2(q) + \operatorname{kei}^2(q)$$

$$\Phi(q) = \arctan (\operatorname{kei}(q)/\operatorname{ker}(q))$$

so that

$$p = P_0 + \sum_{n=1}^{\infty} [P_n\{N(\sqrt{2}\alpha_n r)/N(\sqrt{2}\alpha_n a)\} \cdot \sin (n\omega t - \epsilon_n + \Phi(\sqrt{2}\alpha_n r) - \Phi(\sqrt{2}\alpha_n a))] \qquad r \geq a \tag{16}$$

and $p = f(t)$, $r \leq a$.

The rate of air flow from the top of the cylinder can be found by integrating the continuity equation. Hence, as before,

$$p_e U A = 2\pi h \int_0^\infty r \frac{\partial p}{\partial t} \, dr$$

where U is the average wind speed through the top of the cylinder (measured positive inward) and p_e is the average pressure over the top of the cylinder. Using equation 16, by integration we obtain

$$p_e U A = 2\pi h \Big\{ \sum_{n=1}^{\infty} P_n \tfrac{1}{2} n\omega a^2 \cos(n\omega t - \epsilon_n)$$

$$- \sum_{n=1}^{\infty} P_n a P_0 k \alpha_n \sqrt{2} N_1(\sqrt{2}\alpha_n a)$$

$$\cdot \cos\Big[n\omega t - \epsilon_n - \Phi(\sqrt{2}\alpha_n a)$$

$$+ \Phi_1(\sqrt{2}\alpha_n a) - \frac{3\pi}{4} \Big] \Big/ [\mu s N(\sqrt{2}\alpha_n a)] \Big\} \quad (17)$$

where the first summation term can be neglected for very permeable media and N_1 and Φ_1 are the phase and amplitude functions corresponding to the Kelvin functions of the second type and first order.

In the limit as $a \to \infty$ this result reduces to the result obtained for flow into a long slit (equation 15), thus providing a valuable consistency check.

Flow for a simple pressure variation. It is of interest now to consider a simple example in order to appreciate more readily the implications of the above results.

First, however, consider the solution for the case of zero permeability, where the walls of the slit or cylinder are impervious. From either (14) or (17), the solution is found to be

$$p_l U A = V \, \partial p / \partial t \quad (18)$$

where V is the volume of the cavity beyond the point where U is measured. This solution holds for a cavity of arbitrary shape. The most important consequence of this result is that, in the impervious wall limit, the wind speed and rate of change of pressure are in phase with each other. Returning now to the porous flow case, suppose that the boundary pressure variation $f(t)$ is a simple sinusoidal function of time,

$$f(t) = P_0 + P_1 \sin \omega t \qquad (|x| \quad \text{or} \quad r \le a)$$

where, as before, $P_1 \ll P_0$. Under these conditions, for a long slit, the pressure inside the porous medium is given by

$$p = P_0 + P_1 \exp\left(-(|x| - a)\sqrt{S\mu\omega/(2P_0 k)}\right)$$

$$\cdot \sin\left\{\omega t - (|x| - a)\sqrt{S\mu\omega/(2P_0 k)}\right\}$$

showing that the amplitude of the pressure oscillation falls off exponentially and that the oscillation lags behind the boundary variation by an increasing amount as one moves into the porous medium. The average wind speed across an area A at distance l along the slit is

$$U = \{2(L - l) P_1 h \sqrt{\omega P_0 k}$$

$$\cdot \cos(\omega t - \pi/4)\} / (p_l A \sqrt{\mu S})$$

$$= \{2(L - l) h \sqrt{P_0 k} / (p_l A \sqrt{\mu\omega S})\}$$

$$\cdot \left[\frac{\partial p}{\partial t}\right]_{(t - \pi/(4\omega))} \quad (19)$$

The wind speed thus lags behind the rate of change of pressure by one-eighth of a period. The magnitude of the wind speed is the same as that for an impervious walled slit of half-width a_{eff} where

$$a_{\mathrm{eff}} = \sqrt{P_0 k / (\mu\omega S)} \quad (20)$$

a_{eff} can be called the 'effective half-width' for the porous medium. In general, it will be much larger than the physical half-width of the slit in a very permeable medium; consequently, the magnitude of the wind speed will be much larger than it would be if the cavity had impervious walls.

As seen above, in this simple case the lag is one-eighth of a period: for more complicated boundary conditions the lag can easily be shown to vary considerably either side of this value.

For the circular cylindrical hole the results are similar. The pressure oscillation amplitude decreases and lags further behind the boundary oscillation as one proceeds deeper into the porous medium. Substituting limiting forms for the phase and amplitude Kelvin functions, the average wind speed at the top of the cylinder becomes

$$U = \frac{2\pi h P_0 P_1 k \cos\left(\omega t - \dfrac{\pi}{2} - \dfrac{\pi}{4(\gamma - \beta)}\right)}{S p_e A \mu \beta} \quad (21)$$

where γ is Euler's constant ($\approx$0.57721) and

$$\beta = -\ln\{(a\sqrt{S\mu\omega})/(2\sqrt{P_0k})\}$$

The lag of wind speed behind the rate of change of pressure is thus greater than for the case of a long slit and tends to one-quarter of a period as a becomes vanishingly small.

The 'effective radius,' R_{eff}, can be seen to be

$$R_{eff} = \{\sqrt{2P_0k/(\mu\omega\beta S)}\}^{1/2} \qquad (22)$$

which is generally of the same order of magnitude as the effective half-width a_{eff} for the long slit case. Since limiting (small a) forms have been used for the Kelvin functions in the original expression (21) it should be noted that R_{eff} will not tend to a_{eff} in the large-a limit.

Cave breathing. The above results can be used to explain the inordinately large wind speeds encountered in single entrance caves. Substitution of typical values of pressure and its rate of change, and of cave volume and cross-sectional area into equation 18, shows that observed wind speeds should be quite small (of order 0.1 m/min through a 0.2-meter^2 entrance hole in a cavity of volume 1000 meters^3, compared with speeds of order 3 m/sec which are actually observed). The implication is that either the potentially accessible volume of every cave that exhibits this breathing phenomenon is much greater than has been observed or, alternatively, the walls of the caves are, in fact, porous. It is this latter possibility that appears to gain favor when considered in terms of the preceding theory.

Breathing caves have been mentioned in popular speleological literature of the United States [*Halliday*, 1966] and Australia [*Bishop*, 1957]. The small number of meteorological data that are available nevertheless invariably indicates not only that the magnitude of the breathing is much larger than would be expected, but also that there is a lag in changes of direction of breathing behind changes in sign of the derivative of pressure with respect to time [*Conn*, 1966].

Some of the world's most remarkable examples of cave breathing occur in caves in the Nullarbor Plains region of southern Australia, where numerous 'blowholes' (small vertical caves with volumes ranging from 10 to 10^4 meters^3) are observed to breathe at rates of the order of 1 m^3/sec. Far surpassing this, however, at a constriction near the entrance of a much longer cave (Mullamullang Cave) wind speeds are regularly found averaging 2 m/sec over an area of 20 meters^2.

The stratigraphy of the region is such that the caves fall into two classes 'deep' and 'shallow' caves [*Jennings*, 1963], which can be idealized to fit the long slit and cylindrical cavity models discussed above. Mullamullang, by far the largest of the deep caves (see Figure 2), has been the subject of intense speleological study since its recent discovery. This study included a short period of meteorological observations early in 1966 [*Wigley et al.*, 1966].

The cave itself shows a marked similarity to the long slit model, and the wind speed and pressure observations taken inside the cave show excellent agreement with the theoretical predictions. Typical results are shown in Figure 3. Since pressure variations are reasonably complex, varying lags of windspeed fluctuations behind changes in the time derivative of pressure would be expected, although the magnitude of the lag should still be of the order of one-eighth of a period. Also one would expect, on theoretical grounds, that small-amplitude pressure oscillations superimposed on the general trend would not be so noticeable in the observed wind speed. These predictions are borne out by the results.

No measurements have been made of the permeability of the Nullarbor limestones. The porosity is known to be about 26% [*King*, 1950]. Using this value, the breathing observations in Mullamullang Cave can be used to estimate the permeability. The results indicate that $k \approx 10^{-9}$ meter^2. This value is representative of the combined 'matrix and fracture' permeability rather than the 'matrix' permeability of the limestone and, thus, incorporates the effects of anastomoses and fracture. No measurements of the combined permeability of limestones seem to be available; however, they would be expected to give a value much larger than the matrix permeability [*Scheidegger*, 1961]. It should be noted that the matrix permeability of most limestones is appreciably less than would be required for the rock to breathe to any great extent. It is thus probable that breathing caves will be found only in limestones of high fracture permeability.

Conclusions. The theory of time-dependent porous flow into a cavity has been presented

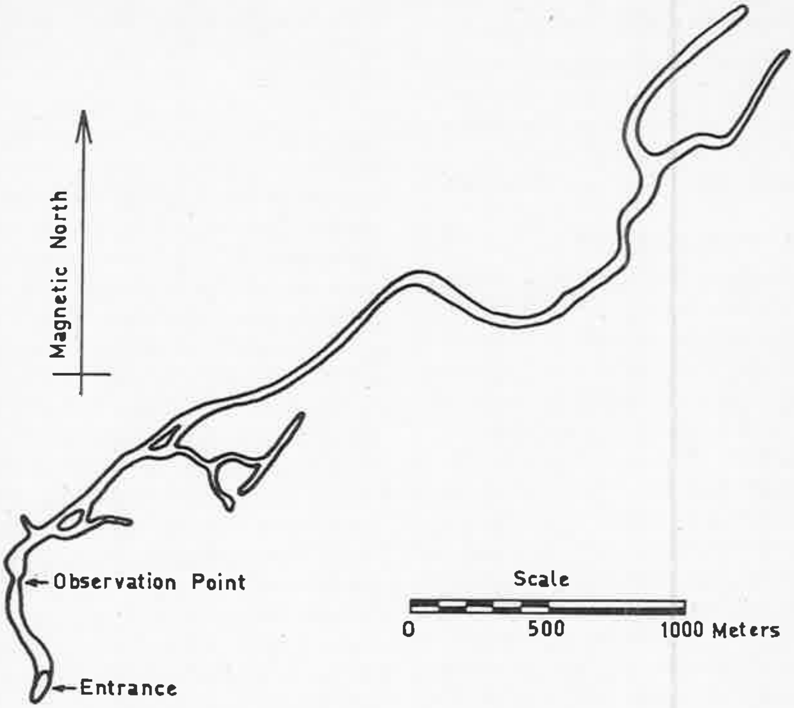

Fig. 2. Simplified map of Mullamullang Cave, Western Australia.

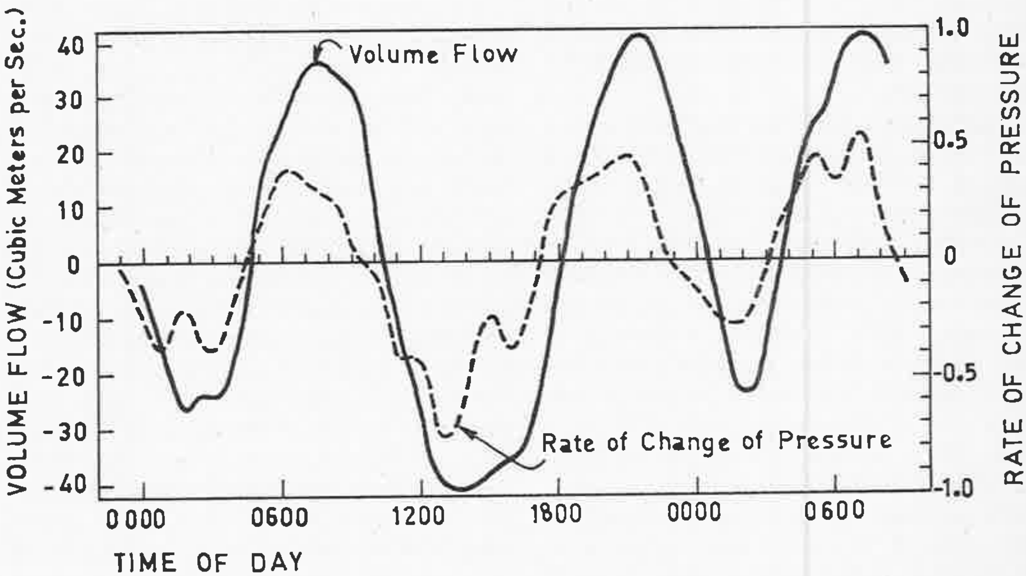

Fig. 3. Simultaneous measurements of wind speed and rate of change of pressure illustrating the lag of fluctuations in speed behind those in $\partial p/\partial t$ and the damping of higher-frequency fluctuations in $\partial p/\partial t$.

and has been used to explain the breathing of caves. Two types of cave occurring in the Nullarbor Plains region of Australia fall conveniently into the two classes that are theoretically most tractable. The development of this paper opens avenues for further research into the theoretical discussion of more complex cavity systems and into the further experimental verification of the two systems considered. The theory presented also gives a new method for estimating the combined permeability of limestones in areas where breathing caves are found.

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