Riemannian Non-commutative Geometry

Steven Lord

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Department of Pure Mathematics
University of Adelaide

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School of Mathematical Sciences
University of Adelaide

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Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

Steven Lord

Abstract

The elements of non-commutative geometry are presented from an operator algebraic viewpoint. Threaded through the presentation is the example of a spectral triple associated to a second countable metrisable locally compact oriented manifold without boundary and without the assumption of spin structure.

Generalisation of the spectral triple associated to such a manifold admits the new notion of a Riemannian representation of a C*-algebra which directly links to the standard theory of von Neumann algebras. The involvement of the standard theory and the reformulation of the axioms of non-commutative geometry in the absence of spin structure are investigated and presented.

The construction of Riemannian representations of C*-algebras is also considered. A new generalisation of a symmetric derivation on a von Neumann algebra R provides the means of constructing Riemannian representations of a C*-subalgebra $A \subset R$ associated to a faithful finite trace on R. The interaction between the standard theory and the generalised symmetric derivation provides new analysis into the structure of K-cycles.

Contents

Summary

There exist many notions of geometry in mathematics inherited from the elements of Euclid. Non-commutative geometry is a broad term used in many fields for the generalisation of algebraic geometric notions to non-Abelian base rings and non-commutative algebraic structures.

Non-commutative geometry, in this thesis, refers to the field drawn and established from the parent fields of operator algebras and differential geometry by the work of numerous mathematicians.

From the inception of operator algebras in the papers of Murray and von Neumann [MN], the inception and identification of C*-algebras by the works of Gelfand, Naimark and Segal [GN] [Se], the modular theory of Tomita [To] [Tk], the development of standard forms, crossed products of and derivations on von Neumann algebras by Araki, Haagerup, Takesaki, Sakai [Ar] [Ha] [Tk2] [Sak1] and others, and the classification of hyperfinite factors by Connes [C1] [C2], C*-algebras and their automorphisms have been inherently viewed as topological and ergodic dynamics. The topological view broadened with the development of K-theory, homology and cohomology theories for C*-algebras. The measure theoretic view broadened with the development of non-commutative integration [Se2] and generalised Radon-Nikodym theorems. The introduction of Dirac operators by Dirac [Dir] and the culminating index theory of Atiyah and Singer that linked Fredholm operators associated to Dirac operators and characteristic classes [AS] drew algebraic differential geometry into the field of operator algebras. From these pieces Connes drew out a generalisation of differential geometry and established it by a series of foundation papers [C3]-[C5] that began a new paradigm in geometry.

This thesis, essentially, views the field of non-commutative geometry inside the parent field of operator algebras. An overview of differential geometry in Section 1.1 presents a differential calculus as the elements of a topological space, a space of functions on the topological space, derivation and integration of the functions, and a metric. The remainder of chapter 1, following for the most part the approach of Connes, describes the manner in which the field of operator algebras provides the elements of a (generalised) differential calculus. This is done, where possible, through collation of results in the theory of C*-algebras and von Neumann algebras, for example Theorems 1.2.1, 1.2.8, 1.2.9, 1.2.11, 1.5.2, 1.5.6. Where necessary direct citation of results are used as background to the theory, such as Theorems 1.2.6 (Gel'fand-Naimark-Segal), 1.2.12 (Gel'fand), 1.5.5 (Reisz-Markov), and Section 1.6.2 (Radon-Nikodym).

There are three original facets to the presentation of the background of non-commutative geometry contained in chapter 1.

Firstly, the presentation itself is the collation of an extensive field.

Secondly, we introduce the notion of a base representation of a C*-algebra in Definition 1.4.3. Let A be a C*-algebra. A base representation (H, π, D) of the C*-algebra A is a separable representation (H, π) of A and a selfadjoint operator $D : DomD \to H$. A base representation of a C*-algebra leads to the notion of a C^1_c-representation, a C^∞_c-representation and an integrable representation of a C*-algebra, Definitions 1.4.4, 1.4.8 and 1.7.7 respectively. In the literature the same information is contained in the notion of a smooth spectral triple. Retaining an explicit definition in terms of the representation theory of the C*-algebra has notational and conceptual advantages. As an example, Section 1.5.3 introduces the notion of disintegration of base representations in terms of the established theory of disintegration of representations of C*-algebras and spectral representations of the selfadjoint operator D.

Thirdly, there is a sequential presentation, through Proposition 1.3.6, Section 1.3.6, Example 1.4.13, Example 1.5.4, Example 1.5.10, Example 1.6.10, Example 1.7.17 and Example 1.8.3, of the base representation $(L^2(X, \Lambda^*X), \pi_l, d + d^*)$ associated to the C*-algebra $C_0(X)$. Here X is a second countable metrisable locally compact oriented manifold without boundary, $C_0(X)$ is the C*-algebra of complex valued functions on X that vanish at infinity, Λ^*X is the exterior bundle of differential forms on X, $d + d^*$ is the selfadjoint extension of the signature operator and π_l is the action of $C_0(X)$ on the Hilbert space $L^2(X, \Lambda^*X)$ by left multiplication. This presentation lays the framework for the original sections of Chapter 2.

The initial sections of Chapter 2, following still for the most part the approach of Connes, describes the deeper aspects of generalising algebraic differential geometry in the field of operator algebras. Section 2.1 introduces $\mathbb{Z}_2$-graded Hilbert modules over C*-algebras. This admits the discussion of finite projective modules over C*-algebras, Definition 2.1.4, the Serre-Swan Theorem, Example 2.1.5, and Kasparov's bivariant KK-theory for C*-algebras, Section 2.4. Section 2.5 introduces the Hochschild and Cyclic homology of a C*-algebra. This admits the discussion of non-commutative De-Rham differential forms and cohomology, Section 2.5.1 and Section 2.5.2, and non-commutative volume forms, Section 2.5.3. The background results of Sections 2.1, 2.2, 2.4 and 2.5, as in chapter 1, contain collations or direct citation such as Theorem 2.1.9 (Serre-Swan), Theorems 2.4.2, 2.4.5, 2.4.6 (Kasparov), and Theorems 2.4.11, 2.5.4, 2.5.5, 2.5.7, 2.5.8. The initial sections of Chapter 2 contain the following original facets.

Section 2.3 contains an original presentation of the concept of Riemannian algebraic structure in non-commutative geometry. Theorems 2.3.1, 2.3.2 and 2.3.3 detail the $\mathbb{Z}_2$-graded Hilbert modules associated to a second countable metrisable locally compact oriented manifold X with no boundary. Theorem 2.3.4 identifies the construction of the Hilbert modules associated to X with the standard form associated to the base representation $(L^2(X, \Lambda^*X), \pi_l, d + d^*)$ of the C*-algebra $C_0(X)$ as discussed in Remark 1.6.12 and Theorem 1.7.21. Theorem 2.3.4 in effect links Remark 1.6.12 and Theorem 1.7.21 with Theorems 2.3.2 and Theorem 2.3.3. This identification, as discussed in Sections 2.3.3 and 2.3.4, elucidates a Riemannian structure that generalises to base representations of arbitrary C*-algebras. The notion of a Riemannian representation of a C*-algebra, Definition 2.3.5, constitutes the first major contribution of the thesis.

We remark that no spin or complexified spin structure is assumed in Section 2.3. Spin representation are recovered through the notion of a Morita equivalence, when it exists, see Definitions 2.3.7 and 2.3.8. The absence of spin structure necessitates the reformulation of Poincaré Duality, Example 2.4.4 and Section 2.6.1, and a fundamental class, Section 2.4.4, distinct from the presentation of Connes [C3, C]. Section 2.4.4 contains minor original results and notions of a real grading and fundamental class for a Riemannian representation (H, π, D) of a C*-algebra A. This involves the introduction of an index algebra, Definition 2.4.15, intended to be the Poincaré dual of the C*-algebra A.

In section 2.5.3 the absence of spin structure necessitates a distinct relationship between a volume form and a real grading. We note that, independent of the existence of volume forms, parity gradings always exist for a Riemannian representation by virtue of the standard theory of von Neumann algebras, see Proposition 2.5.12. The relationship between volume forms and gradings is summarised in Theorem 2.5.13, which neatly generalises Theorem 2.3.1. The notion of a real Riemannian representation, necessarily more general than the notion of reality presented in [C4], appears in Theorem 2.5.14 and Definition 2.5.15. Section 2.5.4 includes a criteria for uniqueness of a volume form for a Riemannian representation, Proposition 2.5.20. This result is gained, as with the majority of the results of Section 2.3, by the direct link between Riemannian representations and the standard theory of von Neumann algebras.

The exposition of Sections 1.2 through to 2.5, extensive as they may be, are the required background to present the axioms of compact Riemannian geometry in Section 2.6. The axioms, see Section 2.6.2, are closely based upon the axioms presented in [C3] with modifications necessitated by the absence of spin structure. The purpose of the axioms is this: a commutative unital *-algebra A should satisfy the axioms of compact Riemannian geometry if and only if $A = C(X)$ where X is a metrisable compact manifold without boundary. The axioms entail the contribution of an original axiom, the axiom of symmetry, and the necessity of this axiom is demonstrated by Proposition 2.6.6.

We remark that Section 2.6 has been, hopefully, designed as a starting point for the reader familiar with the extensive background of non-commutative geometry. Section 2.6.1 summarises the contribution of a Riemannian representation, a notion of this thesis, and back references definitions, concepts and notations to the revelant preceding sections.

Section 2.6.3 contains the details of the statement: a commutative unital *-algebra A should satisfy the axioms of compact Riemannian geometry if and only if $A = C(X)$ where X is a metrisable compact manifold without boundary. Theorem 2.6.9 proves the 'if' direction. This culminates the exposition of the base representation associated to a manifold that is threaded through the thesis in the results Proposition 1.3.6, Section 1.3.6, Example 1.4.13, Example 1.5.4, Example 1.5.10, Example 1.6.10, Example 1.7.17, Example 1.8.3, Theorem 2.3.1, Theorem 2.3.4, Theorem 2.4.21 and Example 2.5.6. The reconstruction theorem of Connes [C3] is cited, modified as necessary in the absence of spin structure, as Theorem 2.6.10 and provides the 'only if' direction.

Sections 2.1 to 2.6 in addition to Sections 1.1 to 1.9 complete the presentation of Riemannian Non-commutative Geometry as set in the field of operator algebras. We note the key definition of a Riemannian representation (H_ρ, π_ρ, D) of a C*-algebra A, Definition 2.3.5, involves the GNS representation (H_ρ, π_ρ) associated to a faithful state ρ of a von Neumann algebra R such that $A \subset R$. The GNS representation (H_ρ, π_ρ) can be constructed from the abstract information (R, ρ). However, the selfadjoint operator $D : DomD \to H_\rho$ is concrete. The natural question to ask is whether Riemannian representations can be constructed from abstract information? Section 2.7 culminates in answering the question in the affirmative when ρ is a trace. In Section 2.7 the approach of a Riemannian representation finds it full application and validation.

Section 2.7.1 discusses the established notion of a symmetric derivation δ on von Neumann algebra R, see Definition 2.7.1. Established results, collated in Theorem 2.7.3, allows the construction of triples (H_ρ, π_ρ, D) from the abstract information of an inner K-cycle (R, ρ, δ) where D is the spatial implementer of the symmetric derivation δ, see Definition 2.7.7. The definition of an inner Riemannian cycle (R, ρ, δ) over a C*-algebra, Definition 2.7.9, follows with the result that (H_ρ, π_ρ, D) is a Riemannian representation of A, Theorem 2.7.10.

The kinds of Riemannian representations constructed from inner Riemannian cycles are limited however. They do not include the base representation $(L^2(X, \Lambda^* X), \pi_l, d+d^*)$ of the C*-algebra $C_0(X)$ discussed previously. Here X is a second countable metrisable locally compact oriented manifold with no boundary. Section 2.7.2 contains the second major contribution of the thesis. It introduces the concept of a symmetric A-derivation on a von Neumann algebra R such that $A \subset R$, see Definition 2.7.25. Section 2.7.3 defines an abstract K-cycle (R, ρ, δ) over a *-algebra A where R is a von Neumann algebra $A \subset R$, ρ is a faithful state on R and δ is a symmetric A-derivation, see Definition 2.7.29. The GNS representation associated to an abstract K-cycle, see Definition 2.7.31, is the field of interaction between symmetric A-derivations, K-cycles and the standard theory of von Neumann algebras. Intrinsic structural and geometric results follow in Remarks 2.7.35, 2.7.39, 2.7.44 and 2.7.48. The highlight of the remarks is probably the notion of the Laplacian $\mathcal{L}_\delta$ associated to a symmetric A-derivation δ and the relation between positivity of the Laplacian and the modular flow on R, see Theorem 2.7.41, Corollary 2.7.42, Remark 2.7.44 and Theorem 2.7.47. Section 2.7.3 concludes with the result that a Riemannian representation of a C*-algebra A arising from a trace is equivalent to the GNS representation of a tracial abstract K-cycle, see Theorem 2.7.52. This result reduces the study of tracial Riemannian representations of a C*-algebra A to symmetric A-derivations and traces on a von Neumann algebra containing A.

Section 2.7.4 attempts the converse to Theorem 2.7.52. It defines a Riemannian cycle over a C*-algebra A as an abstract K-cycle whose associated GNS representation is a Riemannian representation of A, see Definition 2.7.54. While the means of abstractly identifying a Riemannian cycle is beyond the present treatment, a partial converse to Theorem 2.7.52 is derived. Theorem 2.7.56 motivates a subclass of Riemannian cycles called uniform positive Riemannian cycles, see Definition 2.7.57. Theorem 2.7.59 is a generalised GNS result that constructs a Riemannian representation $(H_\rho, \pi_\rho, D_\delta)$ of the C*-algebra A associated to a uniform positive Riemannian cycle (R, ρ, δ) over A when ρ is a trace. The results and notions of Sections 2.7.2, 2.7.3 and 2.7.4 are completely new as far as we know.

We remark that the notion of a symmetric A-derivation produces a natural bilinear map $S_\delta : \mathcal{A} \times \mathcal{A} \to R$ where $\mathcal{A}$ is a Frechet pre-C*-algebra of the C*-algebra A contained in the von Neumann algebra R, see Remark 2.7.35. The map S_δ is called the metric sheer of the A-derivation δ, Definition 2.7.37. The metric sheer, a completely general notion applicable to any abstract K-cycle (R, ρ, δ) over A, is shown to correspond to the Riemannian metric when $A = C(X)$ and X is a compact metrisable oriented manifold with no boundary, see Example 2.7.61.

The thesis concludes with an example of the concepts in Section 2.7. Section 2.8 details the Riemannian geometry associated to an irrational rotation algebra A_θ. The original nature of this presentation lies in constructing a Riemannian cycle $(M_2(A_\theta), \delta, \rho)$ associated to an irrational rotation algebra A_θ, see Theorem 2.8.8. The general construction theorems of Section 2.7 provide a Riemannian representation associated to the cycle, see Corollary 2.8.9, which is proved to admit a Riemannian geometry, Theorem 2.8.12. The well-known spin geometry of an irrational rotation algebra, abundant in the literature, can then be derived if necessary from the Morita equivalence of A_θ and $M_2(A_\theta)$, Corollary 2.8.13. Section 2.8 concludes with the proof that the metric sheer of the A_θ-derivation δ, see Definition 2.7.37, provides an orthogonal bilinear map $S_\delta : \mathcal{A}_\theta \times \mathcal{A}_\theta \to \pi_\rho(\Omega^2_\delta(\mathcal{A}_\theta))$ in the sense $S_\delta(u,u) = 2\pi = S_\delta(v,v)$ and $S_\delta(u,v) = 0$ where $uv = e^{2\pi i\theta}vu$, see Theorem 2.8.14. This result, coupled with the result in Example 2.7.61 that $S_\delta = -g$ for the usual commutative torus where g is the Riemannian metric of the torus, makes the metric sheer S_δ a strong candidate for the role of metric on the non-commutative torus.

Contribution to the field

We summarise the contribution of this thesis to the field of non-commutative geometry in the following manner.

The results of the thesis are divided into six possible categories.

The results in category one provide context or background for the topic of the thesis. Obviously they entail no original contribution to the field and are unlisted in the table below.

The results in category two are original collations of results in category one. The collation and interweaving of existing results in the field are viewed as contributions, albeit contributions of review and exposition, that may be of interest to readers.

The results in category three include new presentations or minor extensions of existing material in the field. The results in category three include preparatory lemmas and propositions for results and notions in category four or five, and sufficiently original presentations in an original manner that extend existing material and/or precedes and/or elucidates results in category four or five.

The results in category four are new results and/or notions that contain a contribution to the field. These results include original extensions of existing material in the field.

The results in category five are new results and/or notions that contain a major original contribution to the field. These results include major extensions or generalisations of existing material in the field and solutions to conjectures in the field. There are four notions and two results of category five in this thesis.

The results in category six are contributions to mathematics that are field generative, field establishing or field unifying. These results include novel directions in mathematics, foundation theorems in a mathematical field, cross-field or para-field results, and solutions to major conjectures in a mathematical field. There are no results of category six in this thesis.

Table: Sequential Categorisation of Contributions

	Category 2	Category 3	Category 4	Category 5
1.1				
1.2	Theorem 1.2.1			
	Theorem 1.2.8			
	Theorem 1.2.9			
	Theorem 1.2.10			
	Theorem 1.2.11			
1.3		Section 1.3.6		
1.4		Definition 1.4.1		
			Theorem 1.4.2	
		Definition 1.4.3		
		Definition 1.4.4		
	Proposition 1.4.7			
		Definition 1.4.8		
	Proposition 1.4.9			
		Lemma 1.4.11		
			Theorem 1.4.12	
	Proposition 1.4.14			
1.5	Theorem 1.5.1			
	Theorem 1.5.2			
	Theorem 1.5.6			
		Proposition 1.5.7		
		Corollary 1.5.8		
	Proposition 1.5.11			
			Section 1.5.3	
	Theorem 1.5.14			
	Theorem 1.5.15			
		Definition 1.5.16		
			Proposition 1.5.17	
1.6	Theorem 1.6.1			
	Theorem 1.6.2			
	Section 1.6.2			
	Theorem 1.6.8			
		Proposition 1.6.11		

	Category 2	Category 3	Category 4	Category 5
			Remark 1.6.12	
1.7	Theorem 1.7.1			
		Definition 1.7.3		
		Definition 1.7.5		
		Definition 1.7.6		
	Definition 1.7.7			
	Theorem 1.7.10			
		Corollary 1.7.11		
		Corollary 1.7.12		
	Example 1.7.17			
		Lemma 1.7.20		
		Theorem 1.7.21		
1.8	Theorem 1.8.2			
1.9		Definition 1.9.1		
2.1	Section 2.1	Theorem 2.1.19		
2.2				
2.3		Theorem 2.3.1	Section 2.3	
		Theorem 2.3.2		
		Theorem 2.3.3		
		Theorem 2.3.4		
				Definition 2.3.5 (Riemannian representation)
		Definition 2.3.8		
2.4		Example 2.4.8		
		Example 2.4.9		
		Section 2.4.4	Definition 2.4.13	
			Theorem 2.4.14	
			Definition 2.4.15	
			Proposition 2.4.16	
			Theorem 2.4.17	
			Definition 2.4.18	
		Theorem 2.4.21		
2.5			Theorem 2.5.4	
	Example 2.5.6			
		Definition 2.5.11		
			Proposition 2.5.12	

	Category 2	Category 3	Category 4	Category 5
			Theorem 2.5.13	
			Theorem 2.5.14	
			Definition 2.5.15	
			Proposition 2.5.20	
2.6		Section 2.6	Section 2.6.2	
			Definition 2.6.3	
			Proposition 2.6.6	
			Theorem 2.6.9	
		Theorem 2.6.10		
		Corollary 2.6.11		
		Corollary 2.6.12		
2.7	Theorem 2.7.3			
				Definition 2.7.7
				Definition 2.7.9 (Inner K-cycle)
			Theorem 2.7.10	
			Definition 2.7.13	
			Definition 2.7.14	
			Definition 2.7.20	
			Proposition 2.7.15	
			Proposition 2.7.16	
				Definition 2.7.24
				Definition 2.7.25 (Symmetric A-derivation)
			Remark 2.7.28	
				Definition 2.7.29 (Abstract K-cycle)
			Lemma 2.7.30	
			Definition 2.7.31	
			Corollary 2.7.32	
			Proposition 2.7.36	
				Definition 2.7.37 (Metric Sheer)
			Theorem 2.7.38	
			Remark 2.7.39	
			Theorem 2.7.41	
			Remark 2.7.44	
			Theorem 2.7.47	
			Remark 2.7.48	
			Theorem 2.7.51	

	Category 2	Category 3	Category 4	Category 5
				Theorem 2.7.52
			Definition 2.7.54	
			Theorem 2.7.56	
			Definition 2.7.58	
				Theorem 2.7.59
			Theorem 2.7.62	
2.8			Section 2.8	
	Corollary 2.8.3			
	Theorem 2.8.4			
		Lemma 2.8.6		
			Lemma 2.8.7	
			Theorem 2.8.8	
			Corollary 2.8.9	
		Lemma 2.8.10		
		Lemma 2.8.11		
			Theorem 2.8.12	
			Theorem 2.8.14	
A		Theorem A.1.1		
			Theorem A.2.3	

Riemannian Non-commutative Geometry

Chapter 1

Elements of Non-commutative Geometry

The 'quantum' or non-commutative differential calculus is at the core of Connes' non-commutative geometry [C, IV,VI.1]. To start with the most basic form of this calculus we require

(i) a separable C*-algebra A,

(ii) a concrete representation of the C*-algebra A

$$\pi : A \to B(H)$$

onto a separable Hilbert space H,

(iii) and a selfadjoint linear operator

$$D : DomD \to H.$$

Demonstrating in what sense the triple (A, H, D) is a non-commutative generalisation of differential calculus is the purpose of this chapter. We review the essential elements of commutative calculus, from our point of view, and then explain the non-commutative emulations in the subsequent sections.

1.1 Review of Calculus and Differential Geometry

1.1.1 Basic Calculus

Consider the metric space $(\mathbb{R}, d)$ where the metric d is defined by $d(x, y) := |x - y|$ for $x, y \in \mathbb{R}$. The metric topology is defined by the base sets $I(x, h) := \{y \in \mathbb{R} | \, d(x, y) < h\} = (x - h, x + h)$ for $x \in \mathbb{R}, h > 0$. The directed set $\{I(x, h)\}_{h>0}$ for fixed $x \in \mathbb{R}$ defines a net in this topology that converges to $\{x\}$. We consider the point sets $\{x\}$ the irreducible components of the topology. Irreducible in the sense $E \subset \{x\} \Rightarrow E = \{x\}$. Components in the sense $F = \cup_{x\in F}\{x\}$ for all $F \subset \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\sim$ the equivalence relation defined by $x \sim y$ if $f(x) = f(y)$. Then f defines a new topological space $\mathbb{R}/\sim$ with base sets $f(I(x, h))$. By this method we gain new topological spaces and new (inequivalent) presentations of the standard interval topology. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. First of all, a continuous function has the property of preserving convergence of nets in the standard topology. Moreover, the condition

$$\lim_{h \rightharpoonup 0} f(I(x,h)) = f(\{x\}) = \{f(x)\}$$

quantifies a relationship between the topological spaces $\mathbb{R}$ and $\mathbb{R}/\sim$ at their most irreducible structural level. Specification of this relationship is exactly what a continuous function is.

Calculus carries the same ideas to the metric structure. Let f be a continuous function. The map $(d \circ f)(x,y) = d(f(x), f(y)) : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ defines a continuous semi-metric on $\mathbb{R}$, a metric on the quotient space $\mathbb{R}/\sim$, and from it the topology with base sets $f(I(x,h))$. We define

$$f'(x) := \lim_{h \rightharpoonup 0} \frac{f(x+h) - f(x-h)}{(x+h)-(x-h)} = \lim_{h \rightharpoonup 0} \operatorname{sgn}(f(x+h)-f(x-h)) \frac{(d \circ f)(x+h, x-h))}{d(x+h, x-h)}$$

if it exists as the derivative of f at the limit $\{x\}$. To say a function is differentiable for all $x \in \mathbb{R}$ is exactly to say a proportion or relation exists between topological structures defined by (semi)-metrics $d \circ f$ and d. For any continuous function f we have the set $R_f(x) := \{\operatorname{sgn}(f(x+h)-f(x-h)) \frac{(d \circ f)(x+h,x-h))}{d(x+h,x-h)}\}_{h>0}$ that encodes the metric relation between the base sets of the topologies. The distinction of differentiable functions amongst continuous functions is this relationship exists between the metric structures at the most irreducible level of the topology. In the case of the metric topology of $\mathbb{R}$ this means points $\{x\} \subset \mathbb{R}$. The function $f'(x) = \lim_{h \rightharpoonup 0} R_f(x)$ quantifies that relationship at each point. This is highlighted in the Leibniz notation,

$$\frac{df}{dx}(x) = f'(x)\ , \ df(x) = f'(x)dx.$$

In common terminology, the symbol $df(x)$ is an 'infinitesimal' in the metric structure $d \circ f$, defined as above in proportion to the 'infinitesimal' dx of the metric structure d at that point. We define the 'length' of the infinitesimal $df(x)$ as

$$|df|(x) := |f'(x)|dx.$$

In differentiation we started with a continuous function f, and for a particular few we obtained a function f' that quantified a pointwise relationship between the metric spaces $(\mathbb{R}, d)$ and $(\mathbb{R}/\sim, d \circ f)$. Newton's Fundamental Theorem of Calculus says that any continuous function g is itself quantifying a pointwise relationship between such spaces. In particular, there exists an anti-derivative G such that $g(x) = G'(x)$ is quantifying the pointwise relationship between $(\mathbb{R}, d)$ and $(\mathbb{R}/\sim, d \circ G)$. An anti-derivative G is defined by integration,

$$G(x) = \int_a^x g(x')dx',$$

for any fixed $a \in \mathbb{R}$.

The depth of this result in relation to the geometry of the real line cannot be understated. The metric relationship existing at the irreducible level of the topology allows us to define a new semi-metric on $\mathbb{R}$ for every differentiable function f from the integral and the derivative by,

$$d_f(x,y) := \int_{x,y} |df|(t) = \int_{x,y} |f'(t)|dt.$$

for all $x, y \in \mathbb{R}$. Here $\int_{x,y}$ denotes $\int_x^y$ if $x < y$ and $\int_y^x$ if $x > y$ in line with the standard orientation on $\mathbb{R}$. The re-presentation of the metric structure has allowed definitions of 'distance' between points with greater generality. Simultaneously, as far as pointwise proportional alteration of the metric is concerned, 'distance' has been classified.

1.1.2 Multivariable Calculus

Let $X = \mathbb{R}^n$ with metric $d(x,y) := \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$ for $x, y \in \mathbb{R}^n$. The standard topology is given by base sets $B(x,h) := \{y \in \mathbb{R}^n \,|\, d(x,y) < h\}$ for $x \in \mathbb{R}^n, h > 0$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function, which can be denoted using the standard basis of $\mathbb{R}^m$ as $f = (f_1, ..., f_m)$ where $f_j : \mathbb{R}^n \to \mathbb{R}$ are continuous functions for $j = 1, .., m$. Let $L(\mathbb{R}^n, \mathbb{R}^m)$ denote the continuous linear functions. As in basic calculus we are interested in relations between $(\mathbb{R}^n, d)$ and $(\mathbb{R}^m, d \circ f)$. We can begin to analyse the geometric consequences of the mapping f by using basic calculus on each of the independent variables. Define the partial derivatives,

$$(\partial_i f_j)(x_1, ..., x_n) := \lim_{h \to 0} \frac{f_j(x_1, ..., x_i + h, ..., x_n) - f_j(x_1, ..., x_i - h, ..., x_n)}{2h}.$$

for all $i = 1, ..., n, j = 1, ..., m$. We say f is continuously differentiable if the partial derivatives exist and are continuous. We denote this by $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ or the terminology f is C^1. The $m \times n$ matrix associated to a C^1-function $f = (f_1, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$,

$$J_f(x_1, ..., x_n) := [\partial_i f_j(x_1, ..., x_n)]_{i=1,...,n\,,\,j=1,...,m}$$

is called the Jacobian of f at $(x_1, ..., x_n) \in \mathbb{R}^n$ [Cr, 6.4]. The Jacobian of f is the pointwise standard matrix representation of the function

$$df : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$$

called the derivative of f, [Cr, Thm6.7,Thm6.8]. The chain rule in the multivariable setting for $g \circ f : \mathbb{R}^n \to \mathbb{R}^m$, where $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^p \to \mathbb{R}^m$ are differentiable, is [Cr, 6.12]

$$J_{f \circ g}(x_1, ..., x_n) = J_g(f(x_1, ..., x_n))J_f(x_1, ..., x_n).$$

Recall the bijection $L(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{nm}$, which introduces a metric topology on $L(\mathbb{R}^n, \mathbb{R}^m)$. Derivation is then a map

$$d : C^1(\mathbb{R}^n, \mathbb{R}^m) \to C(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$$

and the function $df \in C(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$, provided its partial derivatives exist, has the derivative,

$$d(df) : \mathbb{R}^n \to L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m)).$$

The partial derivatives of df exist and are continuous if $\partial_i\partial_j f$ exist and are continuous for $i,j = 1,...,n$. We denote those functions such that $\partial_i\partial_j f$ exist and are continuous for $i,j = 1,...,n$ by $C^2(\mathbb{R}^n,\mathbb{R}^m)$. From the isomorphism $L(\mathbb{R}^n, L(\mathbb{R}^n,\mathbb{R}^m)) \to L(\mathbb{R}^n \times \mathbb{R}^n,\mathbb{R}^m)$, where the right hand side denotes multilinear maps, we have the second derivative

$$d^2 : C^2(\mathbb{R}^n,\mathbb{R}^m) \to C(\mathbb{R}^n, L(\mathbb{R}^n \times \mathbb{R}^n,\mathbb{R}^m)).$$

We can continue this process indefinitely to define $C^p(\mathbb{R}^n,\mathbb{R}^m)$ for $p \in \mathbb{N}$ as functions such that $\partial_{i_1}...\partial_{i_p} f$ exist and are continuous and

$$d^p : C^p(\mathbb{R}^n,\mathbb{R}^m) \to C(\mathbb{R}^n, L((\mathbb{R}^n)^{\times p},\mathbb{R}^m)).$$

The class of smooth or infinitely differentiable functions $C^\infty(\mathbb{R}^n,\mathbb{R}^m)$ are those such that $f \in C^p(\mathbb{R}^n,\mathbb{R}^m)$ for all $p \in \mathbb{N}$.

Like the derivative in basic calculus, the Jacobian of f provides a measure of what the mapping f does to the metric relationships between the points of $\mathbb{R}^n$ in the context of the metric space $\mathbb{R}^m$. For example, the Jacobian of f contains information on the tangent spaces, which are the multidimensional versions of the ratio $f' = \frac{df}{dx}$ in basic calculus above. Specifically, let $n < m$, $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable and $x \in \mathbb{R}^n$. Then $J_f(x) \cdot (k_1,...,k_n)$ for $(k_1,...,k_n) \in \mathbb{R}^n$ defines vectors in the n-dimensional tangent space to the surface $f(\mathbb{R}^n) \subset \mathbb{R}^m$ at the point $f(x) \in \mathbb{R}^m$ [Cr, 6.5].

We also use the Jacobian of f, the generalisation of the derivative of f, to define 'infinitesimal volumes' on $f(\mathbb{R}^n)$. Recall from basic calculus the pointwise relationship

$$|df|(x) = |f'(x)|dx$$

of infinitesimal lengths in $(\mathbb{R}, d)$ and $(\mathbb{R}/\sim,\ d \circ f)$. Similarly we want to measure the pointwise variation in volumes under the mapping $f : \mathbb{R}^n \to f(\mathbb{R}^n)$. Let $V := \{v_1,...,v_n\}$ be n vectors in $\mathbb{R}^m$ and $M(V)$ be the $m \times n$ matrix formed by taking v_i as the i^{th}-column. Let $E(V) := \{x \in \mathbb{R}^m \,|\, x = \sum_{i=1}^n t_i v_i\ ,\ t_i \in [0,1], \sum_i t_i = 1\}$ be the closed convex hull of $\{v_1,...,v_n\}$. This can be intuitively thought of as a n-dimensional parallelogram in $\mathbb{R}^m$. Then the volume of this region is given by $\mathrm{Vol}(E(V)) = \sqrt{det(M(V)^*M(V))}$ where $*$ denotes the transpose of $M(V)$ [Sr, XI Cor2.2]. Let $e_i \in \mathbb{R}^n$ be the i^{th} standard basis vector. We recall that the vectors $u_i := J_f(x) \cdot e_i = i^{\text{th}}$-column of $J_f(x)$ span the tangent space of $f(\mathbb{R})$ at $f(x)$. Hence we define the infinitesimal 'volume' element by

$$|df|(x_1,...x_n) \equiv \sqrt{\det(J_f(x_1,...,x_n)^* J_f(x_1,...,x_n))}dx_1...dx_n.$$

Let $x \in \mathbb{R}^n$ and f be differentiable. We call the function

$$g_f : \mathbb{R}^n \to L(\mathbb{R}^n,\mathbb{R}^n)\ ,\ g_f(x) := J_f(x)^* J_f(x)$$

the Riemannian metric of $f(\mathbb{R})$. We commonly shorten the notation of infinitesimals to $|df|(x) \equiv \sqrt{\det(g_f(x))}dx$. The metric allows us, in particular, to redefine the volume of an open set $A \in \mathbb{R}^n$ with standard orientation via [Sr, XI.3,XI Prop2.4]

$$\int_{f(A)} |df|(x) := \int_A \sqrt{\det(g_f(x))}dx$$

which is also referred to as the 'surface area' of $f(A) \subset \mathbb{R}^m$.

Particularly, and importantly, we can now measure distance and direction along oriented 'paths' in $\mathbb{R}^m$. That is, orientation preserving embeddings $\gamma : \mathbb{R} \to \mathbb{R}^m$ such that γ_i' is piecewise continuous. The arclength distance between points on the path, $a, b \in \gamma(\mathbb{R})$, is given by

$$d_\gamma(a,b) := \int_{a,b} |d\gamma| = \int_{\gamma^{-1}(a),\gamma^{-1}(b)} \sqrt{\sum_i \gamma_i'(x)^2 dx}$$

using the fact $J_\gamma(x) = (\gamma_1'(x), ..., \gamma_m'(x))$.

There are, of course, many paths γ between two points $x, y \in \mathbb{R}^n$. If we were to measure $d_\gamma(x,y)$ for all such oriented paths, we would find that the straight line ℓ from x to y provides the 'shortest' path. That meaning

$$d_\ell(x,y) \leq d_\gamma(x,y)$$

for all oriented paths γ such that $x, y \in \gamma(\mathbb{R})$. In fact

$$d(x,y) = d_\ell(x,y) = \inf_{x,y\in\gamma(\mathbb{R})} d_\gamma(x,y).$$

This insight extended to surface embeddings $f : \mathbb{R}^n \to \mathbb{R}^m$ allows us to define metrics for subspaces of $\mathbb{R}^m$ not so geometrically uniform as the restriction of the standard distance. Due to the embedding property of f, all oriented paths in the subspace $f(\mathbb{R}^n)$ can be defined by the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}^m$ where $\gamma : \mathbb{R} \to \mathbb{R}^n$ is an oriented path in $\mathbb{R}^n$. We can then define a metric for $x, y \in f(\mathbb{R}^n)$ by

$$d_f(x,y) := \inf_\gamma \int_{x,y} |d(f \circ \gamma)|$$

where γ is a oriented path connecting $f^{-1}(x)$ and $f^{-1}(y)$. The Jacobian of the composition function can be calculated from the chain rule, $J_{f\circ\gamma} = J_f J_\gamma$. The introduction of J_f entails that 'shortest distance' on a path from $f(x)$ to $f(y)$ 'along' the surface $f(\mathbb{R})$ is not necessarily the straight line distance in $\mathbb{R}^m$.

Example A simple example is $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x,y) = (x, y, x^2 + y^2)$. The image $C^+ = f(\mathbb{R}^2)$ is the positive circular paraboloid in $\mathbb{R}^3$. The paths of shortest distance are straight lines in $f^{-1}(C^+) = \mathbb{R}^2$ which become curved paths on the surface $C^+ = f(\mathbb{R}^2)$ when mapped into the space $\mathbb{R}^3$ under f.

We have seen that multivariable calculus involves the principles of basic calculus of $\mathbb{R}$ in conjunction with linear algebra. This combination leads to a richer theory. We can define length along one-dimensional embeddings or in the multi-dimensional space, and surfaces have other metric related relationships between points other than just distance such as torsion and curvature. These concepts are defined and quantified in the theory of sub-manifolds of $\mathbb{R}^m$, which leads to the general abstract theory of differentiable manifolds.

1.1.3 Differentiable Manifolds

Let X be a second countable metrisable locally compact n-dimensional topological manifold with a chosen locally finite atlas

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

for $\alpha \in \Lambda$ a countable indexing set [Sr, II.1] [St, 5][1]. We recall U_α are open sets, sometimes called co-ordinate patches or just patches, such that $X = \cup_\alpha U_\alpha$ and $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are open and continuous injections. That X is a topological manifold means on any overlap of patches $W_{\beta,\alpha} := U_\alpha \cap U_\beta$ the overlap maps

$$\omega_{\beta,\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(W_{\beta,\alpha}) \to \phi_\beta(W_{\beta,\alpha})$$

are homeomorphisms. The overlap homeomorphisms $\omega_{\beta,\alpha}$ are maps $\mathbb{R}^n \to \mathbb{R}^n$, hence fall under the multivariable calculus. If the overlap maps are p-differentiable (resp. smooth), X is called a p-differentiable (resp. smooth) manifold. If the overlap maps preserve an orientation of $\mathbb{R}^n$ [Sr, XI.3] then X is called an oriented manifold. Note these designations are all with respect to the chosen locally finite atlas.

Let X be a topological manifold. Let $f : X \to \mathbb{R}^m$ be a continuous function. Then $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Let $f : X \to \mathbb{R}^m$ be a continuous function such that $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function in the multivariable calculus sense for each $\alpha \in \Lambda$. Define the Jacobian and metric of the function f at a point x in a chart U_α by

$$J_\alpha(f, x) := J_{f \circ \phi_\alpha^{-1}}(\phi_\alpha(x))$$

and

$$g_\alpha(f, x) := J_{f \circ \phi_\alpha^{-1}}(\phi_\alpha(x))^* J_{f \circ \phi_\alpha^{-1}}(\phi_\alpha(x)).$$

This is valid in each chart. However, in an overlap $x \in U_\alpha \cap U_\beta$, we could define a derivative and metric at x using the Jacobian for the function $f \circ \phi_\alpha^{-1}$ or $f \circ \phi_\beta^{-1}$. In general the two Jacobians will not agree, of course. They are matrix representations of a linear mapping called the derivative, hence basis dependent. However, they may not agree even up to a change in basis. Hence the Jacobian of f at a point $x \in X$ is potentially ambiguous. This ambiguity does not exist on a differentiable manifold X as differentiability of the overlap function $\omega_{\beta,\alpha}$ ensures $J_\beta(f, x) = J_\alpha(\phi_\beta, x) J_\alpha(f, x)$ [Cr, 6.12]. Hence, on a differentiable manifold X, we say a function $f : X \to \mathbb{R}^m$ is differentiable if $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function for each $\alpha \in \Lambda$. A differentiable function $f : X \to \mathbb{R}^m$ has a well-defined Jacobian $J_f(x)$ at each point $x \in X$.

Let X be a second countable metrisable locally compact differentiable manifold. Let $x \in X$ and $v, w \in \mathbb{R}^n$. We say $(U_\alpha, \phi_\alpha, w)$ and (U_β, ϕ_β, v) are equivalent if

[1] We recall for topological spaces: (1) second countable, locally compact and Hausdorff $\Rightarrow$ Lindelöf, locally compact and Hausdorff $\Rightarrow$ σ-locally compact $\Rightarrow$ paracompact and every cover has a countable locally finite subcover [FI, 10] [St, 5], and (2) second countable, locally compact and Hausdorff $\Rightarrow$ regular with σ-locally finite base $\Leftrightarrow$ metrisable [FI, 10] [St, 5]. As a result second countable metrisable locally compact topological spaces are equivalent to second countable locally compact Hausdorff topological spaces. We also note: (3) second countable, metrisable $\Rightarrow$ paracompact with countable base [St, 5], however (4) second countable, metrisable $\not\Rightarrow$ locally compact.

$w = J_\alpha(\phi_\beta, x)v$. This is an equivalence relation on such triples and the equivalence class $[v]$ is called a tangent vector at $x \in X$. The tangent space T_xX is defined to be the set of all tangent vectors at x and proved to be a n-dimensional vector space. We denote by $L(W,V)$ the continuous linear functions from a topological vector space W to a topological vector space V, and by $L(W \times W, V)$ the continuous multilinear functions from $W \times W \to V$. We have the well defined derivative of $f : X \to \mathbb{R}^m$,

$$df : x \mapsto L(T_xX, \mathbb{R}^m) \ , \ df(x)[v] := J_\alpha(f,x)v$$

which is independent of the choice of $(U_\alpha, \phi_\alpha, v) \in [v]$. Similarly metrics are well defined independent of the chart

$$g(f) : x \mapsto L(T_xX, T_xX) \ , \ g(f,x)[v] := [g_\alpha(f,x)v].$$

Metrics are equivalently viewed as multilinear symmetric functionals,

$$g(f) : x \mapsto L(T_xX \times T_xX, \mathbb{R}) \ , \ g(f,x)([v],[w]) := v^* g_\alpha(f,x)w.$$

Let X be a second countable metrisable locally compact p-differentiable (resp. smooth) manifold. For each $x \in X$, let $u(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a p-differentiable (resp. smooth) bijective function. Define the p-differentiable (resp. smooth) function $h : X \to \mathbb{R}^n$ by $h(x) := u(x)(\phi_\alpha(x))$ when $x \in U_\alpha$ for some α. We call the metric $g(h)$ a Riemannian metric on the p-differentiable (resp. smooth) manifold X. Henceforth we shall refer only to Riemannian metrics of the manifold X.

Let X be a second countable metrisable locally compact differentiable manifold and $g(h)$ a Riemannian metric. Let $f : X \to \mathbb{R}$ such that $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable for each $\alpha \in \Lambda$. We can define the integral of f over an oriented patch $U_\alpha \subset X$ using multivariable calculus,

$$\int_{U_\alpha} f(x)dU_\alpha(x) := \int_{\phi_\alpha(U_\alpha)} f \circ \phi_\alpha^{-1}(x)\sqrt{\det(g_\alpha(h,x))}d\phi_\alpha(x).$$

To use this formula to obtain a linear positive definite functional on the set of all such functions we have to (1) sum the contributions from each chart without 'overcounting' the contributions on the overlaps, and (2) the orientation of the patches must be consistent so that cancellation does not occur in the summation. Let $f : X \to \mathbb{R}$ be such that $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable for each $\alpha \in \Lambda$. Define the support of the function f as $\mathrm{supp}(f) := \{x \in X \mid f(x) \neq 0\}$. As the atlas is locally finite each $x \in X$ is contained in the intersection of a finite number of patches and there exists a continuous partition of unity [Sr, II Cor3.4]. A partition of unity is a set of continuous functions $\{p_\alpha : X \to [0,1]\}$ such that $\mathrm{supp}(p_\alpha) = U_\alpha$ and

$$\sum_{\{\alpha \mid x \in U_\alpha\}} p_\alpha(x) = 1$$

for all $x \in X$. The functions p_α can be chosen to be smooth if X is a smooth manifold. The integral of f, if it exists, is defined by

$$I_g(f) := \sum_\alpha \int_{U_\alpha} p_\alpha(x) f(x) dU_\alpha$$

and is linear and positive definite when X is an oriented manifold. An alternative notation for this formula is

$$I_g(f) = \int_X f(x)\sqrt{\det(g(h,x))}dx.$$

Let X be a second countable metrisable locally compact differentiable manifold with Riemannian metric $g(h)$. Let E be a Borel subset of X and denote by $\chi_E(x)$ the characteristic function of E[2]. We refer to the measure ξ_g defined by $\xi_g(E) := I_g(\chi_E)$ as the Lebesgue measure of the pair $(X, g(h))$.

It is standard in differential geometry to denote the pair of a differentiable second countable metrisable locally compact manifold and a chosen metric by $(X, g(h))$. We shall not be considering changes of metric however, so after this section we shall usually denote such a pair by X and consider the metric to be present, fixed and denoted by g without reference to the function h. We denote the measure ξ_g associated to (X, g) as just ξ, and call ξ the Lebesgue measure of X.

With integration and derivation now defined on a differentiable manifold, distances and directions can be defined for oriented paths $\gamma : \mathbb{R} \to X$ in complete analogy with the multivariable case. This leads us to the metric distance on a differentiable manifold. Let $x, y \in X$, then

$$d(x,y) := \inf_\gamma \int_{x,y} |d\gamma|$$

for all oriented paths γ such that $x, y \in \gamma(\mathbb{R})$. The metric $d : X \times X \to [0, \infty)$ is called the geodesic metric on X.

We have overviewed the basic application of multivariable calculus in defining calculus on a differentiable manifold. We return later to algebraic structures on a differentiable manifold based on the calculus introduced above, see section 1.3. We note we have taken the derivative and the integral of real-valued functions thus far. We shall henceforth consider all functions to be complex valued unless explicitly stated. As $f(x) = f_1(x) + if_2(x)$ for f_1, f_2 real-valued, differentiation and integration are defined by linear extension.

Non-Commutative Calculus (Part 1)

In the review above of multivariable calculus we encountered the following basic elements that, together with linear algebra, provided the theory of multivariable calculus:

(i) the second countable metrisable locally compact topological space X,

(ii) the algebra of continuous complex valued functions $C(X)$ on the space X,

[2]We recall the Borel sets are the elements of the algebra $\mathcal{B}(X)$ of subsets of X generated by complements, countable unions and countable intersections of open subsets of X. The characteristic function of a Borel set E is the function that is 1 on E and 0 on $X \setminus E$.

(iii) the derivative operation on differentiable functions $C^1(X) \subset C(X)$,

$$d : C^1(X) \to C(X, L(TX, \mathbb{C})),$$

where $TX := \cup_{x \in X} T_x X$ is the disjoint union of tangent spaces and $C(X, L(TX, \mathbb{C}))$ are the continuous functions $X \to L(TX, \mathbb{C})$,

(iv) the integral of a continuous function over the space X, which can be viewed as a linear positive definite functional

$$I : C(X) \to \mathbb{C} \cup \{\infty\}.$$

(v) a geodesic metric derived from the integral and derivative.

In this chapter we are concerned with the emulation of the elements (i)-(v) in the non-commutative environment of the triple (A, H, D) defined in the introduction. In part 1 we deal with the elements (i)-(iii). Section 1.2 identifies the separable C*-algebra A as a counterpart of the continuous vanishing at infinity complex valued functions on a second countable metrisable locally compact topological space, and the structure spaces of the C*-algebra $PS(A) \stackrel{[\cdot]_u}{\to} \hat{A} \stackrel{\ker}{\to} \mathrm{Prim}(A)$ as the counterpart of a second countable metrisable locally compact topological space. Section 1.3 introduces the exterior derivative on a differentiable manifold X, which is a generalisation of the derivative operation. Section 1.4 identifies the derivation $\pi(a) \mapsto [D, \pi(a)]$ for a subset $A^1 \subset A$ as a counterpart to the derivative operation and introduces the counterpart to exterior derivation.

Basic Definitions

Let V and W be topological vector spaces over $\mathbb{C}$. All vector spaces we consider shall be vector spaces over the field of complex numbers $\mathbb{C}$. We denote by $L(V, W)$ the continuous linear functions $V \to W$. Let $V^{\times p}$ denote the p^{th} Cartesian product of V. We denote by $L(V^{\times p}, W)$ the continuous p-multilinear functions $V^{\times p} \to W$, and $L_a(V^{\times p}, W)$ the antisymmetric elements of $L(V^{\times p}, W)$. An involution $^* : V \to V$ of a topological vector space is a conjugate linear map such that $v^{**} = (v^*)^* = v \ \forall v \in V$.

Let A be a vector space with product $m : (a, b) \mapsto ab$ such that

(i) (associative) $m(ab, c) = m(a, bc) \ \forall a, b, c \in A$,

(ii) (distributive) $m : A \times A \to A$ is a multilinear function.

Then A is called an (associative) algebra over $\mathbb{C}$. We define the commutator $[\cdot, \cdot] : A \times A \to A$ by

$$[a, b] := m(a, b) - m(b, a) = ab - ba.$$

The algebra A is called commutative (or Abelian) if the commutator map is trivial, that is, m is symmetric or $ab = ba \ \forall a, b \in A$. We define the centre of an algebra A by $Z(A) := \{b \in A \,|\, [a, b] = 0 \ \forall a \in A\}$. The algebra A is called unital if there exists an element, the unit or identity, $e \in A$ such that $m(e, a) = a = m(a, e) \ \forall a \in A$. A topological algebra A is an algebra A such that A is a topological vector space and $m \in L(A \times A, A)$. A topological algebra A is called separable if it admits a countable basis. An involution * of a topological algebra A is an continuous involution $^* : A \to A$

such that $(ab)^* = b^*a^* \ \forall\, ab \in A$. A topological algebra admitting an involution * is called a topological *-algebra. A normed (*-)algebra is a topological (*-)algebra with norm $\|\cdot\|$ (and isometric involution). A Banach (*-)algebra is a normed (*-)algebra that is closed in the norm topology. A C*-algebra A is a Banach *-algebra A such that $\|a^*a\| = \|a\|^2 \ \forall\, a \in A$.

A homomorphism between topological (*)-algebras A and B is map $\phi \in L(A,B)$ such that $\phi(ab) = \phi(a)\phi(b)$ (and $\phi(a^*) = \phi(a)^*$). An isomorphism $\phi : A \to A$ of a topological (*)-algebra A is called a (*)-automorphism of A. The set of (*)-automorphisms of A with the product of composition of maps and the weakest topology making each automorphism continuous is a topological group denoted $\mathrm{Aut}(A)$.

A module (W,π) of a topological algebra A is a topological vector space W with a homomorphism $\pi : A \to L(W,W)$. In this context π is referred to as a representation of A. Injective representations are referred to as faithful representations. Representations such that $\pi(A)W$ is dense in W are called non-degenerate. A representation $\pi : A \to L(W,W)$ of a topological *-algebra A shall be taken to include the condition $L(W,W)$ admits an algebraic involution $\dagger : L(W,W) \to L(W,W)$ such that $\pi(a^*) = \pi(a)^\dagger \ \forall\, a \in A$. Let A be a topological *-algebra and (W,π) a module. The topological *-subalgebra $\pi(A)' := \{T \in L(W,W) \,|\, [T,\pi(a)] = 0 \ \forall\, a \in A\}$ is called the commutant of $\pi(A)$.

Let H be Hilbert space. We say H is separable if it admits a countable orthonormal basis. We denote by $C(H)$ the closed linear operators on H, $B(H) := L(H,H)$ the C*-algebra of bounded linear operators on H, $K(H)$ the compact operators on H, and $FR(H)$ the finite rank operators on H. We recall the norm closure of $FR(H)$ is $K(H)$ and the compact operators form a norm closed *-ideal of $B(H)$ [S, Thm 1.3]. Let $\mathrm{sp}(S)$ denote the spectrum of a bounded or selfadjoint linear operator on H [RS, VI.3,VIII.1].

Let $K : DomK \to H$ be a selfadjoint linear operator. We recall if $S \in B(H)$ has the properties $SDomK \subset DomK$ and $\sup_{\eta \in DomK, \|\eta\| \leq 1} \|[K,S]\eta\| < \infty$ then the closure of the linear operator $[K,S]$ with domain $DomK$ is a bounded operator. We will abusively refer to $S \in B(H)$ satisfying the conditions of the last sentence for selfadjoint $K \in C(H)$ by the term $[K,S]$ is bounded.

We introduce the uniform, strong, σ-weak and weak operator topologies of $B(H)$ where H is separable. Let R be a *-subalgebra of $B(H)$. We recall a sequence a_n of elements of R converge to $a \in B(H)$

(i) uniformly if $\|a_n - a\| \to 0$. The uniform closure of R is a C*-subalgebra of $B(H)$ that we typically denote by $\overline{R}$.

(ii) strongly if $\|(a_n - a)\eta\| \to 0$ for all $\eta \in H$. The strong closure of R will be denoted $St(R) \subset B(H)$.

(iii) σ-weakly if $\sum_{j,k}\langle \eta_j, (a_n - a)\xi_k\rangle \to 0$ for all $(\eta_j,\xi_k) \in H \times H$ such that $\sum_j \|\eta_j\|^2, \sum_k \|\xi_k\|^2 < \infty$. The σ-weak closure of R will be denoted $W_\sigma(R)$.

(iv) weakly if $\langle \eta, (a_n - a)\xi\rangle \to 0$ for all $(\eta,\xi) \in H \times H$. The weak closure of R is a unital C*-algebra denoted by $W(R)$. The commutant R' of R is an example of a weakly closed C*-algebra. The terminology R acts non-degenerately on H

means RH is dense in H, or equivalently, the unit of $W(R)$ is the unit of $B(H)$. Let R act non-degenerately on H. Then we denote the weak closure by the double commutant $R'' = (R')' = W(R)$. Further, the strong, σ-weak and weak closure of R are identical (though the weak topology is weaker than the σ-weak topology which is weaker than the strong topology). These results follow from von Neumann's bi-commutant theorem, [Ped, 2.2.2,2.2.5] [vN].

Let A be a normed *-algebra and H a Hilbert space that admits a non-degenerate representation $\pi : A \to B(H)$. We refer to the module (H, π) as a (concrete) representation of the normed *-algebra A. When H is separable the representation (H, π) is called separable. In this case the above topologies can be applied to $\pi(A)$ as a *-subalgebra of $B(H)$.

Let A be a C*-algebra. Let $A^+ := \{a^*a \,|\, a \in A\}$ denote the positive elements of A [Mu, 2.2]. We recall a weight on A is an additive mapping

$$\tau : A^+ \to [0, \infty].$$

Since the positive elements A^+ complex linearly span A [BR, 2.2.11], from any weight we may uniquely form the linear mapping

$$\tau : A \to \mathbb{C} \cup \{\infty\}$$

by linear extension.

We recall any C*-algebra A possesses an approximate unit $\{u_\lambda\}_\Lambda$, [Mu, 3.1.1]. That is, a directed set Λ and $u_\lambda \in A^+$ such that $\lim_{\lambda \in \Lambda} \|u_\lambda a - a\| = 0$ for all $a \in A$. A positive linear form σ on A is a weight such that $\lim_\lambda \sigma(u_\lambda) < M$ for all approximate units $\{u_\lambda\}_\Lambda$ and some fixed $M > 0$.

1.2 Non-Commutative Topological Spaces

The theory of C*-algebras is often called non-commutative topology. We shall take a C*-algebra A to be the 'non-commutative functions' on the 'non-commutative space' $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ where $PS(A)$ is the pure state space, $\hat{A}$ the spectrum, and $\mathrm{Prim}(A)$ the primitive ideals of A respectively. To understand this association we review the structure of C*-algebras.

1.2.1 The GNS Construction

Let A be a C*-algebra. Let τ be a positive linear form on A and define $N_\tau := \{a \in A \,|\, \tau(a^*a) = 0\}$. We have the following 'GNS construction' from Gelfand and Naimark [GN] and Segal [Se]:

(i) The space N_τ is proved to be a closed left ideal of A.

(ii) We define the factor space
$$A_\tau := A/N_\tau.$$

(iii) We let $a_\tau := a + n$ for $a \in A, n \in N_\tau$ denote an element $a_\tau \in A_\tau$. Then

$$\langle a_\tau, b_\tau \rangle_\tau := \tau(a^*b)$$

defines an inner product on A_τ.

(iv) We can define the Hilbert space

$$H_\tau := \overline{A_\tau}$$

as the closure of the pre-Hilbert space A_τ in the inner product $\langle\cdot,\cdot\rangle_\tau$.

(v) The canonical inclusion map

$$\iota_\tau : A_\tau \to H_\tau$$

is a linear injection with dense range, and

$$\pi_\tau(a)\iota_\tau(b_\tau) = \iota_\tau(a_\tau b_\tau)$$

defines a non-degenerate representation $\pi_\tau : A \to B(H_\tau)$.

In summary, to each positive linear form τ of a (separable) C*-algebra A we construct the associated (separable) 'GNS representation' (H_τ, π_τ) of A. The positive linear form τ and the GNS representation π_τ are faithful if $N_\tau = \{0\}$.

1.2.2 Topological Spaces associated to a C*-algebra

There are three fundamental spaces associated to the structure of a C*-algebra A.

(1) Pure State Space, $PS(A)$

Let A be a C*-algebra. We recall the dual $A^* := L(A, \mathbb{C})$ of A consists of all continuous linear functionals

$$\tau : A \to \mathbb{C}.$$

The dual has two topologies we consider,

(i) the norm topology from the norm

$$\|\tau\| := \sup_{\|a\|\leq 1} |\tau(a)|$$

(ii) the weak*-topology, which is the locally convex topology generated by the family of semi-norms

$$p_a(\tau) := |\tau(a)| \ , \ a \in A.$$

The linear extension σ of a positive linear form

$$\sigma : A^+ \to [0, \infty)$$

has the defining property [Mu, 3.3.4]

$$\lim_\lambda \sigma(u_\lambda) = \|\sigma\|$$

for all approximate units $\{u_\lambda\}_\Lambda$ of A. Hence σ belongs to the dual A^* of continuous linear functionals. The extensions of positive linear forms on A are called positive linear functionals on A.

Let $S_+(A, \mathbb{C})$ denote the positive linear functionals on A of norm less than or equal to one. Then $S_+(A, \mathbb{C})$ is a convex weak*-compact subset of A^* [Dix, 2.5.5]. We denote by $\mathrm{Extr}(S_+(A, \mathbb{C}))$ the extremal points of $S_+(A, \mathbb{C})$. We define the pure states as the non-trivial extremal points,

$$PS(A) := \mathrm{Extr}(S_+(A, \mathbb{C})) \setminus \{0\}.$$

We give the space $PS(A)$ of pure states the topology of the restricted weak*-topology from A^*. We recall that $PS(A)$ separates A [Mu, 5.1.11]. This means for any non-zero $a \in A$ there exists $\rho \in PS(A)$ such that $\rho(a) \neq 0$. This implies the weak*-topology is Hausdorff. We recall our initial formulation of the triple (A, H, D) involved a *separable* C*-algebra.

Theorem 1.2.1 *(i) Let A be a C*-algebra. Then $PS(A)$ is a Hausdorff topological space (given the weak*-topology).*

(ii) Let A be a separable C-algebra. Then $PS(A)$ is a complete second countable metrisable topological space (given the weak*-topology)* [Ped, 4.3.2].

The realisation of $PS(A)$ as a complete second countable metrisable topological space determines the exclusive role of separable C*-algebras in non-commutative geometry.

(2) Spectrum, $\hat{A}$

A representation (H, π) of a C*-algebra A is called irreducible if $\pi(A)' = \{S \in B(H) \,|\, [\pi(a), S] = 0 \;\forall a \in A\} = \mathbb{C}1$ where $1 = \mathrm{id}_H$ [Mu, 5.1.5]. We denote the irreducible representations of A by $\mathrm{Irr}(A)$. A consequence of irreducibility is every non-zero vector $\xi \in H$ is cyclic for A, that means $\overline{\pi(A)\xi} = H$ [Mu, 5.1.5].

Let $\rho \in PS(A)$. Then the GNS representation (H_ρ, π_ρ) associated to ρ is irreducible [Se] [Mu, 5.1.6]. A cyclic vector is given by $\xi = \lim_\lambda \iota_\rho(u_\lambda)$ for any approximate unit $\{u_\lambda\}$ of A_ρ. Conversely given an irreducible representation (H, π) with unit cyclic vector $\xi \in H$, one can define the pure state $\rho(a^*a) := \langle \xi, \pi(a^*a)\xi \rangle_H : A^+ \to [0, \infty)$ such that (H, π) is the GNS representation associated to ρ [Se] [Mu, 5.1.7]. This leads to the following result at the core of the structure of C*-algebras,

$$\begin{array}{ccc} \mathrm{PS(A)} & \Longleftrightarrow & \mathrm{Irr}(A) \\ \rho & \longleftrightarrow & \pi_\rho \end{array}$$

We say that representations (H_1, π_1) and (H_2, π_2) of A are unitary equivalent if there exists a unitary $U : H_1 \to H_2$ such that $U\pi_1(a)U^* = \pi_2(a)$ for all $a \in A$. Unitary equivalence is an equivalence relation on representations of A which restricts to the irreducible representations of A. We denote unitary equivalence of two representations of A by $(H_1, \pi_1) \sim_u (H_2, \pi_2)$. We define the spectrum of a C*-algebra A as

$$\hat{A} := \mathrm{Irr}(A)/\!\sim_u$$

From the isomorphism $PS(A) \to \mathrm{Irr}(A)$ we will denote an element of the spectrum by the class $[(H_\rho, \pi_\rho)]_u \in \hat{A}$ or equivalently $[\rho]_u \in \hat{A}$.

With the identification of $\hat{A}$ as a quotient

$$PS(A) \to PS(A)/\sim_u \leftrightarrow \hat{A}$$

we can induce the quotient of the weak*-topology on $\hat{A}$. In general $\hat{A}$ with this topology is quasi-locally compact but not Hausdorff[3] [Dix, 3.3.7].

Example 1.2.2

a. Let $A = C_0(\mathbb{R})$ be the C*-algebra of continuous vanishing at infinity continuous complex valued functions $f : \mathbb{R} \to \mathbb{C}$. As $C_0(\mathbb{R})$ is commutative it commutes with itself. Hence for any representation (H, π), $\pi(C_0(\mathbb{R})) \subset \pi(C_0(\mathbb{R}))'$. In particular, all irreducible representations are one-dimensional as $\pi(C_0(\mathbb{R})) \subset \pi(C_0(\mathbb{R}))' = \mathbb{C}1$. From this one easily shows that

$$\operatorname{Irr}(C_0(\mathbb{R})) = \{(\mathbb{C}, \pi_x) \mid x \in \mathbb{R}\}$$

where

$$\pi_x : C_0(\mathbb{R}) \to \mathbb{C}\ ,\ \pi_x(f)z = f(x)z\ \ \forall z \in \mathbb{C}.$$

Since unitary equivalence is equality on one-dimensional representations,

$$PS(A) \cong \hat{A} \cong \mathbb{R}.$$

The pure states ρ_x are given by pointwise evaluation

$$\rho_x(f) := f(x).$$

Hence a base for the weak*-topology on $PS(A)$ is given by

$$B_f(\rho_x, \epsilon) := \{\rho_y \mid |f(x) - f(y)| < \epsilon, f \in C_0(\mathbb{R})\} = B(x, \delta_f)$$

where $\delta_f > 0$ is defined by $y \in B(x, \delta_f) \Rightarrow f(y) \in B(f(x), \epsilon)$ in the definition of continuity of f. Hence the weak*-topology is just the standard open interval topology on $\mathbb{R}$ and $\hat{A} \leftrightarrow PS(A)$ is homeomorphic to $\mathbb{R}$. These spaces are both locally compact and Hausdorff.

b. Let $A = M_n(\mathbb{C})$ be the C*-algebra of $n \times n$ complex matrices.

Proposition 1.2.3 *Let $A = M_n(\mathbb{C})$ be the C^*-algebra of $n \times n$ complex matrices. Then*

(i) $PS(A) \cong \mathrm{PU}_n(\mathbb{C})$

(ii) $\hat{A} \cong \{1\}$.

where $\mathrm{PU}_n(\mathbb{C})$ *is the projective unitary group.*

Proof Lectures Notes [ReL]. □

[3] We recall that a topological space X is quasi-locally compact if every point $x \in X$ has a quasi-compact open neighbourhood U. An open set $U \subset X$ is quasi-compact if every open cover of U has a finite subcover. A quasi-compact open set U is called compact if it is satisfies the T_0 separation axiom [FI].

(3) Primitive Spectrum, Prim(A)

Define a set of closed two sided ideals of A called the primitive ideals,

$$\mathrm{Prim}(A) := \{\ker \pi_\rho \mid \rho \in PS(A)\}.$$

The relation $\rho \sim_p \sigma$ if $\ker \pi_\rho = \ker \pi_\sigma$ is an equivalence relation on $PS(A)$ that commutes with $\sim_u$. We distinguish the equivalence classes of $\rho \in PS(A)$ where necessary by $[\rho]_u$ and $[\rho]_p$. A C*-algebra A has the sequence of surjective maps and quotients,

$$\begin{array}{ccccc} PS(A) & \xrightarrow{[\cdot]_u} & \hat{A} & \xrightarrow{\ker} & \mathrm{Prim(A)} \\ \\ \rho & \mapsto & [(H_\rho, \pi_\rho)]_u & \mapsto & \ker \pi_\rho. \end{array}$$

We will define a topology on Prim(A). Let I be a closed ideal of A and define

$$\mathrm{hull}(I) := \{y \in \mathrm{Prim}(A) \mid I \subset y\}.$$

For a subset $Y \subset \mathrm{Prim}(A)$ define

$$\mathrm{kernel}(Y) := \cap_{y \in Y} y$$

which is a closed ideal of A. Then there exists a unique topology such that hull(kernel(Y)) defines the closure of $Y \subset \mathrm{Prim}(A)$ [Mu, 5.4.6]. The topology is called the Jacobson topology. The induced Jacobson topology on $\hat{A}$ is the weakest topology such that $\hat{A} \xrightarrow{\ker} \mathrm{Prim(A)}$ is continuous.

Remark 1.2.4 The map $I \mapsto \mathrm{hull}(I)$ is a bijection between closed ideals of A and the closed subsets of Prim(A), and inverts the partial order, $I_1 \subset I_2$ iff $\mathrm{hull}(I_2) \subset \mathrm{hull}(I_1)$ [Mu, 5.4.7].

Example 1.2.5 As an example, consider the C*-algebra $C_0(\mathbb{R})$. We recall $\mathbb{R}$ was homeomorphic to $PS(C_0(\mathbb{R})) = \{\rho_x \mid \rho_x(f) = f(x)\}$ given the weak*-topology. Now

$$\mathrm{Prim}(C_0(\mathbb{R})) = \{\ker \pi_{\rho_x}\} = \{I_x = \{f \in C_0(\mathbb{R}) \mid f(x) = 0\}\}$$

Clearly $f \in I_x \not\Rightarrow f \in I_y$ and $f \in I_y \not\Rightarrow f \in I_x$ for $x \neq y$. Hence the the map

$$\mathbb{R} \to \mathrm{Prim}(C_0(\mathbb{R})) \ , \ x \to I_x$$

is a bijection. Let $Y \subset \mathbb{R}$ and $I_Y = \{I_x \mid x \in Y\}$. Then

$$\mathrm{kernel}(I_Y) = \cap_{x \in Y} I_x = \cap_{x \in Y}\{f \in C_0(\mathbb{R}) \mid f(x) = 0\} = \{f \in C_0(\mathbb{R}) \mid f(x) = 0 \ \forall x \in \overline{Y}\}$$

where $\overline{Y}$ denotes the closure of Y in the usual topology on $\mathbb{R}$. Hence

$$\overline{I_Y} = \mathrm{hull}(\mathrm{kernel}(I_Y)) = \{I_s \in \mathrm{Prim}(A) \mid I_s \supset \{f \in C_0(\mathbb{R}) \mid f(x) = 0 \ \forall x \in \overline{Y}\}\} = I_{\overline{Y}}.$$

The bijection $\mathbb{R} \to \mathrm{Prim}(C_0(\mathbb{R}))$ is a homeomorphism. We also see

$$PS(C_0(\mathbb{R})) \cong \hat{C_0(\mathbb{R})} \cong \mathrm{Prim}(C_0(\mathbb{R}))$$

as topological spaces. This is not true for a general C*-algebra A. We shall see that $PS(A) \cong \mathrm{Prim}(A)$ if and only if A is a commutative C*-algebra.

Let A be a C*-algebra. Then $PS(A)$ separates A. In particular for $a \in A$ there exists $\rho \in PS(A)$ such that $\rho(a) = \|a\|$ [Mu, 5.1.11]. Hence $\pi_\rho(a) \neq 0$. We note if $\pi_\rho(a) \neq 0$, which occurs iff $\pi_\rho(a) \notin \ker \pi_\rho$, then $\pi_\sigma(a) \neq 0$ for any $\sigma \in [\rho]_p$ by construction. Hence $PS(A)$ separates A implies Prim(A) (and so $\hat{A}$) separates A in this sense.

We can now realise the structure of C*-algebras with the following decomposition theorem [Mu, 3.4.1] [GN],

Theorem 1.2.6 (Gelfand-Naimark Theorem) *Let A be a C*-algebra. Let $H := \bigoplus_{[\rho]_u \in \hat{A}} H_\rho$ and $\pi := \oplus_{[\rho]_u \in \hat{A}} \pi_\rho$. Then (H, π) is a faithful representation of A.*

Remark 1.2.7 (H, π) is called the universal representation of A. It is unique up to unitary equivalence. There is some degeneracy in the universal representation, in the sense we use a representative of the unitary equivalence class $[\rho]_u$ to obtain a faithful representation, where all that was needed was a representative of $[\rho]_p$ since Prim(A) separates A. Let $\sigma_\rho \in [\rho]_p$ for each $\rho \in PS(A)$. Then $H' := \oplus_{\sigma_\rho} H_{\sigma_\rho}$ and $\pi' := \oplus_{\sigma_\rho} \pi_{\sigma_\rho}$ defines a faithful representation (H', π') of A. However, this representation is not uniquely determined up to unitary equivalence. Two representations $(H_{\sigma_\rho}, \pi_{\sigma_\rho})$ and $(H_{\sigma'_\rho}, \pi_{\sigma'_\rho})$ where $\sigma_\rho, \sigma'_\rho \in [\rho]_p$ may have the same kernel but not be unitary equivalent. This occurs as $\hat{A} \to$ Prim(A) is only a surjection in general.

There is significant structural difference when $\hat{A} \to$ Prim(A) is a bijection. A C*-algebra such that $\hat{A} \to$ Prim(A) is a bijection is called postliminal.

Theorem 1.2.8 *Let A be a C*-algebra. Then*

(i) *the surjection $PS(A) \to \hat{A}$ is continuous and open,*

(ii) *the surjection $PS(A) \to$ Prim(A) , $\rho \to \ker \pi_\rho$ is continuous and open,*

(iii) *the quotient topology on $\hat{A}$ as a quotient $\hat{A} = PS(A)/\!\sim_u$ and the induced Jacobson topology on $\hat{A}$ from $\hat{A} \overset{\ker}{\to}$ Prim(A) agree.*

Proof (i) [Dix, 3.4.11] (ii) [Ped, 4.3.3] (iii) [Dix, 3.4.11] □

Theorem 1.2.9 *Let A be a C*-algebra. Then*

(i) *Prim(A) is a locally compact T_0-space,*

(ii) *$\hat{A}$ is a locally quasi-compact space,*

(iii) *$\hat{A}$ is a locally compact T_0-space, iff $\hat{A} \xrightarrow{\ker}$ Prim(A) is an isomorphism,*

(iv) *A is unital $\Rightarrow$ $\hat{A}$ and Prim(A) are compact.*

Proof [Dix, 3.1.6] (iv) [Mu, 5.4.8] □

Note the converse of (iv) is false [Mu, 5.4.8]. For non-unital simple C*-algebras Prim$(A) = \{0\}$ is a one-point space, and a particular example is the compact operators on a Hilbert space H where $[(H, \mathrm{Id})]_u = \hat{K(H)} \cong$ Prim$(K(H)) = \{0\}$.

Theorem 1.2.10 *Let A be a C*-algebra and $Z(A) = \{b \in A \,|\, [a, b] = 0 \;\forall a \in A\}$ be the centre of A. Then the following are equivalent*

(i) $\mathrm{Prim}(A)$ *is a locally compact Hausdorff space,*

(ii) $Z(A) \cong C_0(\mathrm{Prim}(A))$.

Proof (i) $\Rightarrow$ (ii) that $Z(A) \subset C_0(\mathrm{Prim}(A))$ is a closed *-subalgebra is established by [Ped, 4.4.4]. Let $f \in C_0(\mathrm{Prim}(A))$ and $\lambda \in \mathrm{Prim}(A)$. Let $\Lambda \subset \hat{A}$ denote the set such that $\ker[\pi](\Lambda) = \lambda$. Then $\kappa(f) = \oplus_{\lambda \in \mathrm{Prim}(A)} \oplus_\Lambda f(\lambda)$ is a central element of $\pi_U(A)$ where (H_U, π_U) is the faithful universal representation of A (Gelfand-Naimark Theorem). Clearly κ is a a faithful representation such that $\pi_U(Z(A)) \supset \kappa(C_0(\mathrm{Prim}(A)))$. (ii) $\Rightarrow$ (i) By the Gelfand Theorem in the next section $Z(A) = C_0(\Sigma)$ for a locally compact Hausdorff space Σ. Hence $C_0(\Sigma) \cong C_0(\mathrm{Prim}(A))$ which implies Σ and $\mathrm{Prim}(A)$ are homeomorphic. □

To summarise, the structure of a C*-algebra is encoded in the triple of topological spaces

$$PS(A) \xrightarrow{[\cdot]_u} \hat{A} \xrightarrow{\ker} \mathrm{Prim(A)}$$

where $PS(A)$ is Hausdorff but not locally compact in general, and $\mathrm{Prim}(A)$ is locally compact but not Hausdorff in general. The triple $PS(A) \xrightarrow{[\cdot]_u} \hat{A} \xrightarrow{\ker} \mathrm{Prim}(A)$ is considered the 'non-commutative space' associated to the C*-algebra A. This interpretation comes from the form of the sequence $PS(A) \xrightarrow{[\cdot]_u} \hat{A} \xrightarrow{\ker} \mathrm{Prim}(A)$ in the sub-theory of commutative C*-algebras.

1.2.3 Commutative C*-algebras

Let A be a commutative algebra C*-algebra and $PS(A) \xrightarrow{[\cdot]_u} \hat{A} \xrightarrow{\ker} \mathrm{Prim}(A)$ the pure state space of A, the spectrum of A and primitive ideals of A respectively. The fundamental property of a commutative 'non-commutative space' is $PS(A) \xrightarrow{[\cdot]_u} \hat{A} \xrightarrow{\ker} \mathrm{Prim}(A)$ resolves to the single locally compact Hausdorff space $\Sigma(A)$.

Theorem 1.2.11 *Let A be a C*-algebra. Then A is commutative iff the continuous and open surjections*

$$PS(A) \xrightarrow{[\cdot]_u} \hat{A} \xrightarrow{\ker} \mathrm{Prim(A)}$$

are homeomorphisms.

Proof ($\Rightarrow$) Since A is commutative, then for any irreducible representation $\pi : A \to B(H)$ we have $\pi(A) \subset \pi(A)' \subset \mathbb{C}1$. Hence any irreducible representation is one dimensional, and unitary equivalence $\sim_u$ restricted to $\mathrm{Irr}(A)$ just becomes equality. Further, $\ker \pi$ has co-dimension one, hence $\ker \pi_1 = \ker \pi_2$ iff $\pi_1 = \pi_2$.

($\Leftarrow$) The isomorphism implies $[\pi_\rho] = \pi_\rho$ for $\rho \in PS(A)$. Then $U\pi_\rho(a)U^* = \pi_\rho(a) \Rightarrow [\pi_\rho(a), U] = 0$ for all $a \in A$ and $U \in B(H_\rho)$. Since $B(H_\rho)$ is a C*-algebra, then every element in $B(H_\rho)$ decomposes as a linear combination of four unitaries, [BR, 2.2.14]. Hence $\pi_\rho(A) \subset B(H_\rho)' \cong \mathbb{C}1$, and π_ρ is one-dimensional (so commutative) for every $\rho \in PS(A)$. Using the Gelfand-Naimark Theorem we have a faithful commutative representation $\pi = \oplus_{\rho \in PS(A)} \pi_\rho$ of A. Hence A is commutative. □

When A is a commutative C*-algebra we speak of 'the' spectrum $\Sigma(A) := \hat{A} \cong PS(A) \cong \mathrm{Prim}(A)$. Since the locally compact Hausdorff space $\Sigma(A)$ is central to the structure of the C*-algebra A, we can determine the form of all commutative C*-algebras [Mu, 1.3.5,2.1.10].

Theorem 1.2.12 (Gelfand Theorem) *Let A be a commutative C^*-algebra with spectrum $\Sigma(A)$. Define the Gelfand transform*

$$a \mapsto f_a$$

where the function $f_a : \Sigma(A) \to \mathbb{C}$ is defined by $f_a([\rho]_u) := \rho(a)$ for $\rho \in PS(A)$. Then the Gelfand transform provides an isomorphism of C^*-algebras,

$$A \to C_0(\Sigma(A))$$

Conversely, for any locally compact Hausdorff space X, $C_0(X)$ is a C*-algebra and $X = \Sigma(C_0(X))$, [Mu, 1.1.3,2.1.2]. This provides the theorem,

Theorem 1.2.13 *There is a bijective correspondence between locally compact Hausdorff spaces and continuous vanishing at infinity functions on them $(X, C_0(X))$, and commutative C^*-algebras and their spectrums $(\Sigma(A), A)$,*

$$(X, C_0(X)) \longleftrightarrow (\Sigma(A), A)$$

Example 1.2.14 We have already seen the correspondence $(\mathbb{R}, C_0(\mathbb{R}))$ as an example last section. We apply the result of the Gelfand-Naimark Theorem in this case. We recall the pure states of $C_0(\mathbb{R})$ are given by $\rho_x(f) = f(x)$, and the irreducible representations $(\mathbb{C}, c_x)$ by

$$c_x : C_0(\mathbb{R}) \to \mathbb{C} \ , \ (c_x(f)z) := f(x)z \ \forall z \in \mathbb{C}.$$

The spectrum $\mathbb{R}$ is defined by the isomorphisms

$$\{x\} \to \rho_x \to c_x \to \ker c_x = \{f \in C_0(\mathbb{R}) \,|\, f(x) = 0\}.$$

The universal representation Hilbert space is given by

$$L^2(\mathbb{R}) \cong \bigoplus_{\mathbb{R}} \mathbb{C}$$

and the universal representation

$$c = \oplus_{\mathbb{R}} c_x \ , \ (c(f)g)(x) = f(x)g(x) \ \ \forall g \in L^2(\mathbb{R}).$$

1.2.4 Non-Commutative Topological Spaces

It is the bijective correspondence

$$(X, C_0(X)) \longleftrightarrow (\Sigma(A), A)$$

between locally compact Hausdorff spaces X and commutative C*-algebras A which leads to the view general non-commutative C*-algebras A are the 'non-commutative continuous functions' on the 'non-commutative topological space' $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$. We shall take this view. Hence a C*-algebra A and the structure spaces $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ provide the basic elements of (i) a 'topological space', (ii) 'continuous functions' on that space, in non-commutative geometry.

Remark 1.2.15 (i) Does the terminology 'non-commutative continuous functions' have more than a conceptual meaning? In the sense explained below, every non-commutative C*-algebra with $\mathrm{Prim}(A)$ Hausdorff is an algebra of continuous operator-valued functions. Recall Theorem 1.2.10, $Z(A) \cong C_0(C)$ for any C*-algebra A such that $C := \mathrm{Prim}(A)$ is Hausdorff. The elements of the C*-algebra can then be viewed as 'operator valued continuous functions on' or 'continuous sections of a bundle of simple C*-algebras over' the locally compact Hausdorff space C,

$$a : c \to \pi(a)$$

where $a \in A$ and $(H, \pi) \in \ker^{-1}(c)$ is irreducible[4] [Ped] [Dix]. Note that this view is not unique up to unitary equivalence. If further,

(a) (Postliminal) $\hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ is an isomorphism, then A is viewed as 'continuous sections of a bundle of simple C*-algebras' over the locally compact Hausdorff space C,

$$a : c \to \pi(a)$$

where $(H, \pi) \in \ker^{-1}(c) = [(H, \pi)]_u$. This presentation is unique up to unitary equivalence [Dix],

(b) (Liminal) $\pi(A) = K(H)$ for each $(H, \pi) \in \mathrm{Irr}(A)$, then A can be considered as 'compact operator valued functions' over the locally compact Hausdorff space $\mathrm{Prim}(A)$ [Dix].

Commutative 'non-commutative continuous functions', which are continuous functions on C in the ordinary sense, are the trivial case where each fibre is the one-dimensional simple C*-algebra $\mathbb{C}$. The simplest non-commutative example of the above situations is the C*-algebra $A = C_0(X) \otimes M_n(\mathbb{C}) = C_0(X, M_n(\mathbb{C}))$ where X is a locally compact Hausdorff space.

(ii) As our last point on 'non-commutative continuous spaces', we remark on C*-algebras such that $\mathrm{Prim}(A)$ is not Hausdorff. An 'operator valued function' view can be determined as in (i) above, but one needs to consider bounded Borel sections over

[4] If the reader has not encountered the terminology section and bundle, see [Sr, I.1,III.1], or 1.3.2 Vector Bundles.

the space of unitary equivalence classes of factor representations of the von Neumann closure $\pi(A)''$. These are deeper results in the theory of C*-algebras, [Ped, Dix].

However, that Prim(A) is locally compact but non-Hausdorff in general is one of the deepest generalising points of non-commutative topology. A. Connes has used this viewpoint to perform 'geometry' on non-Hausdorff spaces. These spaces are typically pathological and outside the reach of classical methods. Examples are in the work of Connes, the space of Penrose Tilings [C, II.3], the dual of non-type I discrete groups [C, II.4] and foliations [C7].

1.3 Exterior Derivation on Differentiable Manifolds

Let A be a separable C*-algebra. We have the 'non-commutative second countable metrisable topological space' $PS(A) \overset{[.]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ and its algebra of 'non-commutative continuous functions' A from the previous section. What should constitute the derivative of a 'non-commutative function'?

As an initial guide we reduce the derivative in basic calculus on $\mathbb{R}$ to algebraic terms. Denote the polynomial functions by $P[x]$. Define a linear map $\delta : P[x] \to P[x]$ by

$$\delta(a_n x^n + ... + a_1 x + a_0) := n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1.$$

We note that δ is completely determined by the relation

$$\delta(pq) = p\delta(q) + \delta(p)q \ \ \forall p, q \in P[x]$$

and $\delta(x) = 1$. The map δ is an example of a derivation (definition below). Define norms on $P[x]$,

$$\|p\|_k^m := \sup_{x\in[-m,m]} |p(x)| + \sup_{x\in[-m,m]} |\delta(p)(x)| + ... + \sup_{x\in[-m,m]} |\delta^k(p)(x)|$$

We make the convention $\delta^0(p) = p$. We say a sequence of polynomials is k-Cauchy if it is Cauchy in the norm $\|.\|_k^m$ for all m. We denote by $\mathcal{C}_k(P[x])$ the set of k-Cauchy sequences of $P[x]$ and by $p_i \overset{\mathcal{C}_k}{\to} f$ the limit f of the k-Cauchy sequence $\{p_i\}$. The result $\mathcal{C}_0(P[x]) = C(\mathbb{R})$ is a consequence of the Stone-Weierstrass Theorem. It can be shown $\mathcal{C}_k(P[x]) = C^k(\mathbb{R})$. Explicitly, if $p_i \overset{\mathcal{C}_k}{\to} f$ then $\delta^k(p_i) \overset{\mathcal{C}_0}{\to} f^{(k)}$. Differentiation is hence defined as the continuous closure

$$\overline{\delta} : \mathcal{C}_1(P[x]) \to \mathcal{C}_0(P[x])$$

of $\delta : P[x] \to P[x]$, and $\overline{\delta}$ is a derivation $C^1(\mathbb{R}) \to C(\mathbb{R})$.

Hence an immediate candidate for the role of differentiation in non-commutative calculus are derivations on C*-algebras. Derivations on C*-algebras is a well established theory [BR, 3.2].

Definition 1.3.1 [BR, 3.2.21, 3.2.54]

(i) *A symmetric norm-dense derivation δ of a C*-algebra A is a linear operator $\delta : Dom\delta \to A$ with norm dense domain $Dom\delta$ such that $\delta(a)^* = \delta(a^*)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in Dom\delta$.*

(ii) *A symmetric derivation δ of a C^*-algebra A is said to be spatially implemented by a symmetric operator D on a Hilbert space H if there exists a representation (H, π) of A such that $\pi(\delta(a)) = -i[D, \pi(a)]$.*

Example 1.3.2 Let $C_c^1(\mathbb{R})$ denote the continuous compactly supported functions $\mathbb{R} \to \mathbb{C}$ with continuous derivative. Then the derivative operation

$$d : C_c^1(\mathbb{R}) \to C_c(\mathbb{R}) \ , \ f \mapsto f'$$

is a derivation of the C*-algebra $C_0(\mathbb{R})$ with norm-dense domain $C_c^1(\mathbb{R})$. Let us examine a spatial implementer of d. Define the linear operator

$$i\frac{d}{dx} : C_c^1(\mathbb{R}) \to C_c(\mathbb{R}) \ , \ f(x) \mapsto if'(x).$$

As $C_c^1(\mathbb{R})$ is L^2-dense in $L^2(\mathbb{R}, \xi)$, this operator has a unique selfadjoint extension as an unbounded linear operator [RS, VIII.1]

$$i\frac{d}{dx} : Domi\frac{d}{dx} \to L^2(\mathbb{R}, \xi).$$

We recall the representation $(L^2(\mathbb{R}, \xi), \pi_l)$ of the commutative C*-algebra $C_0(\mathbb{R})$

$$\pi_l : C_0(\mathbb{R}) \to B(L^2(\mathbb{R}, \xi)) \ , \ (\pi_l(f)g)(x) = f(x)g(x) \ \xi\text{-a.e.} \ \forall g \in L^2(\mathbb{R}, \xi)$$

where ξ is the Lebesgue measure on $\mathbb{R}$.

Proposition 1.3.3 *Let $f \in C_c^1(\mathbb{R})$. Then*

$$-i[i\frac{d}{dx}, \pi_l(f)] = \pi_l(f').$$

Proof Let $f, g \in C_c^1(\mathbb{R})$ and $D = i\frac{d}{dx}$. Then $D\pi_l(f)g = D(fg) = if'g + fDg$ (*). Now, for $g \in DomD$ we have $g_n \in C_c^1(\mathbb{R})$ such that $g_n \to g$ in the graph norm. Clearly $\pi_l(f)g_n \to \pi_l(f)g$ in the graph norm from (*). Hence $\pi_l(f)DomD \subset DomD$. Furthermore from (*),

$$D\pi_l(f)g - \pi_l(f)Dg = if'g = i\pi_l(f')g$$

for $g \in DomD$. Thus $[i\frac{d}{dx}, \pi_l(f)] = i\pi_l(f')$ on $DomD$ and, as f' is compactly supported and continuous, $i\pi_l(f')$ is norm bounded on $DomD$. Hence it extends as a bounded operator on all of $L^2(\mathbb{R}, \xi)$. □

All the elements of basic calculus have been encoded in the triple

$$(C_0(\mathbb{R}), L^2(\mathbb{R}, \xi), i\frac{d}{dx}),$$

which is of the form (A, H, D) as described in the introduction to this chapter. We have the separable C*-algebra $A = C_0(\mathbb{R})$, a faithful representation $(L^2(\mathbb{R}, \xi), \pi_l)$ of $C_0(\mathbb{R})$, and a selfadjoint linear operator $D = i\frac{d}{dx}$ on $L^2(\mathbb{R}, \xi)$ that spatially implements the symmetric derivation

$$d(\cdot) = -i[i\frac{d}{dx}, \cdot] : C_c^1(\mathbb{R}) \to C_c(\mathbb{R}) \ , \ d(\pi_l(f)) = \pi_l(f').$$

The example above was the one-dimensional case. In the multivariable case,

$$d : C_c^1(\mathbb{R}^n) \to C_c(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{C}))$$

and on the differentiable manifold X

$$d : C_c^1(X) \to C_c(X, L(TX, \mathbb{C})).$$

The map d still satisfies the Leibniz relation

$$d(fg) = df\, g \;+\; f\, dg \qquad \forall f, g \in C_c^1(X)$$

but is clearly not a derivation of $C_0(X)$ in general. To develop a counterpart in non-commutative geometry to the derivative of a differentiable function f on a differentiable manifold X we review exterior differentiation on X. A summary of the algebraic structures involved in exterior differentiation appears in section 1.3.6. Section 1.4 develops a counterpart of exterior differentiation in the non-commutative calculus.

1.3.1 Exterior and Clifford Algebras

Let V be a vector space over $\mathbb{C}$. Let $T^i(V) = V^{\otimes i}$ be the i^{th} tensor product. Let $T(V) = \oplus_{i=0}^{\infty} T^i(V)$ be the tensor algebra of V and I the ideal generated by the elements $\{v \otimes w + w \otimes v \,|\, v, w \in V\}$. The exterior algebra of V is defined as the quotient [BGV, 3.1]

$$\Lambda(V) := T(V)/I,$$

and has the natural $\mathbb{N}$-grading $\Lambda^i(V) = T^i(V)/(I \cap T^i(V))$. The quotient product, denoted $\wedge$, is the called the exterior product and has the property

$$v \wedge w + w \wedge v = 0.$$

Let V be finite dimensional with $\dim V = n$ and $L_a(V^{\times p}, \mathbb{C})$ denote anti-symmetric multilinear functionals on V^p. We recall the isomorphism [Sr, V.3]

$$\Lambda(V) \to L_a(V^{\times n}, \mathbb{C}).$$

Suppose V admits an inner product $q : V \times V \to \mathbb{C}$. Let I_q be the ideal generated by the set of elements of the form $\{v \otimes w + w \otimes v + 2q(v, w) \,|\, v, w \in V\}$. The Clifford algebra of V (generated by q) is defined as the quotient algebra [BGV, Prop 3.2]

$$\mathrm{Cliff}(V, q) := T(V)/I_q$$

The quotient product, denoted $\cdot$, is called the Clifford product or Clifford multiplication and has the property

$$v \cdot w + w \cdot v = -2q(v, w).$$

With a quadratic form one can introduce the interior product on the exterior algebra,

$$v \intercal (w_1 \wedge \ldots \wedge w_n) := \sum_{i=1}^{n} (-1)^i q(v, w_i) w_1 \wedge \ldots \wedge w_{i-1} \wedge w_{i+1} \wedge \ldots \wedge w_n \;\; \forall v, w_i \in V.$$

The homomorphism
$$\iota : \mathrm{Cliff}(V,q) \to \Lambda(V)$$
defined by
$$\iota(v \cdot w) := \iota(v) \wedge \iota(w) + \iota(v) \top \iota(w)$$
is a linear isomorphism [BGV, 3.1]. We call the action
$$w_l(u) := \iota(w \cdot \iota^{-1}(u))$$
the left action of $w \in \mathrm{Cliff}(V,q)$ on $u \in \Lambda(V)$, and
$$w_r(u) := \iota(\iota^{-1}(u) \cdot w)$$
the right action of $w \in \mathrm{Cliff}(V,q)$ on $u \in \Lambda(V)$.

The Clifford and Exterior algebras are linearly isomorphic as graded vector spaces by giving the Clifford algebra the grading [BGV, Prop 3.6]
$$\mathrm{Cliff}^k(V,q) := \bigoplus_{i=0}^{k} \iota^{-1}(\Lambda^i(V)).$$
We point out the surjection $\iota_k : \mathrm{Cliff}^k(V,q) \to \Lambda^k(V)$ defined by
$$\iota_k : v_1 \cdot ... \cdot v_k \to v_1 \wedge ... \wedge v_k \;\; \forall v_1, .., v_k \in V$$
is not an isomorphism. The kernel of this map is $\mathrm{Cliff}^{k-2}(V,q)$. Finally we define an inner product on $\Lambda^i(V)$ by
$$q_i(w_1 \wedge ... \wedge w_i, v_1 \wedge ... \wedge v_i) := \det([q(w_m, v_n)]_{m,n=1,...,i}).$$
We extend this to an inner product of $\Lambda(V)$ by $q(v,w) = q_i(v,w)$ if $v, w \in \Lambda^i(V)$ and $q(v,w) = 0$ otherwise.

1.3.2 Vector Bundles

Let X be a locally compact Hausdorff space. Let $e : E \to X$ be a vector bundle. We recall this means there exists a topological space E, a topological vector space V and open covering $\{U_\alpha\}$ of X such that [Sr, III.1]

(i) (local triviality) there is a homeomorphism ($U_\alpha \times V$ is given the product topology)
$$\sigma_\alpha : e^{-1}(U_\alpha) \to U_\alpha \times V$$
that commutes with the canonical projection $p : U_\alpha \times V \to U_\alpha$. In particular there is the isomorphism
$$\sigma_\alpha(x) : e^{-1}(x) \to V.$$

(ii) (transition functions) the map $U_\alpha \cap U_\beta \to L(V,V)$ given by $x \mapsto \sigma_\beta(x) \circ \sigma_\alpha(x)^{-1}$ is a morphism. In particular, for $x \in U_\alpha \cap U_\beta$ we have

$$\sigma_\beta(x) \circ \sigma_\alpha(x)^{-1} : V \to V$$

is an isomorphism.

The vector space $E_x = e^{-1}(x)$ is called the fibre over x. A section s of a vector bundle $e : E \to X$ is a function $s : X \to E$ such that $e \circ s(x) = x$. We denote continuous sections of $e : E \to X$ by $C(X, E)$ and the continuous sections with compact support by $C_c(X, E)$. Let us suppose each fibre E_x admits a norm $\|.\|_x$, then we may define a norm on sections,

$$\|s\| := \sup_{x \in X} \|s(x)\|_x.$$

The closure of $C_c(X, E)$ in this norm is the Banach space $C_0(X, E)$ of continuous sections that vanish at infinity. Let us suppose further that each fibre E_x is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_x$. Then

$$\langle s_1, s_2 \rangle := \int_X \langle s_1(x), s_2(x) \rangle_x \sqrt{\det(g)} dx$$

defines an inner product on $C_c(X, E)$. The closure of $C_c(X, E)$ in the associated norm is a Hilbert space $L^2(X, E)$ called the square integrable sections of the vector bundle $e : E \to X$. Finally, suppose the fibre E_x is an algebra for each x. Then $C_c(X, E)$ is an algebra when given the product

$$(s_1 s_2) : X \to E \ , \ (s_1 s_2)(x) = s_1(x) s_2(x) \ \ \forall s_1, s_2 \in C_c(X, E).$$

Consequently one shows $C_0(X, E)$ is a Banach algebra.

We shall mean by the term 'Hermitian' vector bundle a vector bundle whose fibres are separable Hilbert spaces.

From a Hermitian vector bundle $e : E \to X$ we can define another vector bundle $e_L : L(E, E) \to X$ with fibres

$$e_L : L(E_x, E_x) \to x.$$

We note the C*-algebra $C_0(X, L(E, E))$ has a natural concrete faithful representation $(L^2(X, E), \lambda)$, where

$$\lambda : C_0(X, L(E, E)) \to B(L^2(X, E)) \ , \ (\lambda(s)g)(x) = s(x)(g(x)) \ \ \forall g \in L^2(X, E).$$

As $C_0(X) \cong C_0(X)\mathrm{Id}_E \subset C_0(X, L(E, E))$, the representation λ restricts to a faithful representation $(L^2(X, E), \pi_l)$ where

$$\pi_l : C_0(X) \to B(L^2(X, E)) \ , \ (\pi_l(f)g)(x) = f(x)\mathrm{Id}_{E_x}(g(x)) \ \ \forall g \in L^2(X, E).$$

We say the vector bundle $e : E \to X$ is finite dimensional if V is finite dimensional. Then $E_x \cong \mathbb{R}^N$ for some N. Hence we can apply the multivariable calculus, If the transition functions $\sigma_\beta(x) \circ \sigma_\alpha(x)^{-1} : \mathbb{R}^N \to \mathbb{R}^N$ are p-differentiable (resp. smooth), we say $e : E \to X$ is a p-differentiable (resp. smooth) bundle.

Let X be a p-differentiable (resp. smooth) second countable metrisable locally compact manifold and $e : E \to X$ a p-differentiable (resp. smooth) Hermitian vector bundle. A section $s : X \to E$ is called p-differentiable (resp. smooth) if the functions $\sigma_\alpha \circ \phi_\beta^{-1}(\cdot)(s \circ \phi_\beta^{-1}(\cdot)) : \mathbb{R}^n \to \mathbb{R}^N$ are p-differentiable (resp. smooth) for all α, β where σ_α are the trivialising maps for the vector bundle $e : E \to X$ and ϕ_β are the co-ordinate maps for X. We denote the p-differentiable sections of a Hermitian vector bundle $e : E \to X$ by $C^p(X, E)$ and smooth sections by $C^\infty(X, E)$.

We note, for notation purposes, that the sections of the trivial Hermitian bundle $X \times \mathbb{C} \to X$ that are continuous and vanishing at infinity, square integrable, or smooth, define the functions on the manifold X that are continuous and vanishing at infinity, square integrable, or smooth respectively.

1.3.3 Exterior and Clifford Bundles

Let X be a second countable metrisable locally compact n-dimensional differentiable manifold with metric g. Let T_xX be the n-dimensional tangent space at $x \in X$. The dual space of continuous linear functionals $T_xX^* = L(T_xX, \mathbb{C})$ is also an n-dimensional vector space, called the cotangent space at $x \in X$ [5]. Let $f : X \to \mathbb{C}$ be differentiable, then $df(x) \in L(T_xX, \mathbb{C}) = T_x^*X$. By construction, if $x \in U_\alpha$ for a chart $(U_\alpha, \phi_\alpha = (\phi_\alpha^1, ..., \phi_\alpha^n))$ then

$$T_x^*X \cong \mathrm{span}_{\mathbb{C}}\{d\phi_\alpha^1(x), ..., d\phi_\alpha^n(x)\}.$$

There exists a isomorphism between the tangent and cotangent spaces at $x \in X$ provided by the metric,

$$\kappa_g : [v] \mapsto g(x)([v], \cdot),$$

and we define an inner product on T_x^*X by

$$\begin{aligned} q_g(x) : \quad & T_x^*X \times T_x^*X \to \mathbb{C} \\ & (v, w) \mapsto q_g(x)(v, w) := g(x)(\kappa_g^{-1}(v), \kappa_g^{-1}(w)). \end{aligned}$$

We form the exterior algebra $\Lambda(T_x^*X)$ and Clifford algebra $\mathrm{Cliff}(T_x^*X, q_g(x))$ of the n-dimensional vector space T_x^*X with inner products $q_g(x)$ as in Section 1.3.1.

Take the chosen atlas $\{U_\alpha\}$ over X as an open covering, and define the disjoint unions,

$$\begin{aligned} TX &:= \cup_{x\in X} T_xX \\ T^*X &:= \cup_{x\in X} T_x^*X \\ \Lambda^*X &:= \cup_{x\in X} \Lambda(T_x^*X) \\ \mathrm{Cl}(X, q_g) &:= \cup_{x\in X} \mathrm{Cliff}(T_x^*X, q_g(x)) \end{aligned}$$

[5] The standard notation of the cotangent space at x is T_x^*X.

and maps

$$\begin{aligned} e_T : &\quad TX \to X \ , \ T_xX \mapsto x \\ e_{T^*} : &\quad T^*X \to X \ , \ T_x^*X \mapsto x \\ e_\Lambda : &\quad \Lambda^*X \to X \ , \ \Lambda(T_x^*X) \mapsto x \\ e_C : &\quad \mathrm{Cl}(X, q_g) \to X \ , \ \mathrm{Cliff}(T_x^*X, q_g(x)) \mapsto x \end{aligned}$$

Then $e_T : TX \to X, e_{T^*} : T^*X \to X, e_\Lambda : \Lambda^*X \to X$ and $e_C : \mathrm{Cl}(X, q_g) \to X$ are Hermitian vector bundles over X [Se, III.1], called the tangent bundle, cotangent bundle, exterior bundle and the Clifford bundle respectively. When X is a smooth manifold then the the tangent bundle, cotangent bundle, exterior bundle and the Clifford bundle are smooth. We can extend the natural linear identification of the exterior and Clifford algebras fibrewise to the bundles,

$$\iota : \mathrm{Cl}(X, q_g) \to \Lambda^*X \ , \ \mathrm{Cliff}(T_x^*, q_g(x)) \to \Lambda(T_x^*).$$

This provides the canonical inclusions

$$C_c(X, \mathrm{Cl}(X, q_g)) \to C_c(X, L(\Lambda^*X, \Lambda^*X))$$

called the left action,

$$w_l(x)(u) := \iota(w(x) \cdot \iota^{-1}(u)) \qquad \forall w \in C_c(X, \mathrm{Cl}(X, q_g)), u \in \Lambda(T_x^*X)$$

and right action

$$w_r(x)(u) := \iota(\iota^{-1}(u) \cdot w(x)) \qquad \forall w \in C_c(X, \mathrm{Cl}(X, q_g)), u \in \Lambda(T_x^*X)$$

respectively.

Warning: henceforth we consider only connected, oriented, geodesically complete manifolds with no boundary. That a manifold is oriented is equivalent to the statement that there exists a non-vanishing continuous section in $C(X, \Lambda^{\dim X} X)$. That the manifold is connected implies $C_0(X)$ contains no proper projections and the dimension of the fibres in any Hermitian vector bundle over X is constant.

1.3.4 Covariant Derivatives and Exterior Differentiation

Let X be a second countable metrisable locally compact n-dimensional p-differentiable manifold. We recall the operation of k^{th}-differentiation,

$$d^k : C_c^k(X) \to C_c(X, L(TX^{\times k}, \mathbb{C})),$$

for $k \leq p$. The k^{th}-derivatives of a function in $C_c^k(X)$ is seldom used in differential geometry. Instead we consider a co-ordinate independent form of partial or directional differentiation, called a covariant derivative. A covariant derivative then defines a fundamental graded derivation

$$d : C_c^1(X, L_a(TX^{\times(k-1)}, \mathbb{C})) \to C_c(X, L_a(TX^{\times k}, \mathbb{C})),$$

called an exterior derivative.

Let $E \to X$ be a Hermitian vector bundle. A covariant derivative on $E \to X$

$$\nabla : C_c^1(X, TX \otimes E) \to C_c(X, E)$$

is a map $v \otimes \sigma \to \nabla_v \sigma$ with the properties

(i) (linearity) $\nabla_{fv+gw} = f\nabla_v + g\nabla_w \quad \forall f, g \in C_c^1(X)$,

(ii) (Leibniz) $\nabla_v f\sigma = f\nabla_v\sigma + df(v)\sigma \quad \forall f \in C_c^1(X)$.

The Leibniz rule is equivalently stated $[\nabla_v, f] = df(v)\ \forall f \in C_c^1(X)$.

Let $(U_\alpha, \phi_\alpha = (\phi_\alpha^1, ..., \phi_\alpha^n))$ be a local trivialising chart of X for the vector bundle E. Denote the 'local coordinates' of U_α by $x_i = \phi_\alpha^i(U_\alpha)$ $i = 1, .., n$. Then the local frame $\partial_i = \frac{\partial}{\partial x_i}$ spans TU_α. Let ∇ be a covariant derivative on $E \to X$. We consider $\nabla_i := \nabla_{\partial_i}$ a generalised i^{th}-partial derivative. Defining grad $:= (\nabla_1, ..., \nabla_n)$ then $\nabla_v = \sum v_i \nabla_i = (v_1, ..., v_n) \cdot \text{grad}$ is a generalised directional derivative where $v|_{U_\alpha} = \sum_i v_i \partial_i$.

We dualise a covariant derivative on $E \to X$ as the map

$$\hat{\nabla} : C_c^1(X, E) \to C_c(X, T^*X \otimes E)$$

by defining

$$\hat{\nabla}(\sigma)(v) = \nabla(v \otimes \sigma)\ \ \forall v \in C_c^1(X, TX), \sigma \in C_c^1(X, E).$$

Then $\hat{\nabla}$ has the form $\nabla\sigma = \sum_i \nabla^i \sigma$ where locally

$$\nabla^i \sigma = dx_i \otimes \nabla_i \sigma\ \ \forall \sigma \in C_c^1(X, E).$$

In this form $\hat{\nabla}$ is called a connection on $E \to X$. We shall henceforth drop the dual notation and denote a connection or a covariant derivative by ∇. A connection ∇ on $E \to X$ is a linear map $C_c^1(X, E) \to C_c(X, T^*X \otimes E)$ and satisfies the Leibniz rule

$$[\nabla, f]\sigma = df \otimes \sigma\ \ \forall f \in C_c^1(X), \sigma \in C_c^1(X, E).$$

We can extend a connection on $E \to X$ to a linear map

$$\tilde{\nabla} : C_c^1(X, \Lambda^k X \otimes E) \to C_c(X, \Lambda^{k+1} X \otimes E)$$

as follows. We define

$$d : C_c^1(X, \Lambda^k X) \to C_c(X, \Lambda^{k+1} X)$$

by the rule in a local frame $\{dx_i\}_{i=1,...,n}$ of T^*U_α

$$d(f dx_{i_1} \wedge ... \wedge dx_{i_k}) := df \wedge dx_{i_1} \wedge ... \wedge dx_{i_k},$$

for a multi-index $i_j \in \{1, ..., n\}, j = 1, ..., k$. Then there exists a unique extension $\tilde{\nabla}$ such that $\tilde{\nabla}|_{\Lambda^0 X \otimes E} = \nabla$ and $\tilde{\nabla}$ satisfies a Leibniz rule

$$[\tilde{\nabla}, w]_k \sigma = dw \otimes \sigma\ \ \forall w \otimes \sigma \in C_c^1(X, \Lambda^k X \otimes E).$$

Here $[S,T]_k := ST + (-1)^k TS$ is the graded commutator. Henceforth we denote the extended connection $\tilde{\nabla}$ and the extended dual covariant derivative by just ∇.

Let $v \in C_c(X, \Lambda^k X)$ and $w \in C_c(X, \Lambda^j X)$. Let m denote the antisymmetrisation map $m(v \otimes w) := v \wedge w$. Let ∇ be a covariant derivative on the tangent bundle $TX \to X$. We define the exterior derivative associated to ∇ by

$$d_\nabla := m \circ \nabla : C_c^1(X, \Lambda^k X) \to C_c(X, \Lambda^{k+1} X).$$

This is illustrated by the action of d_∇ in a local frame $\{dx_i\}_{i=1,\dots,n}$ of T^*U_α,

$$d_\nabla w = \sum_i dx_i \wedge \nabla_i w \quad \forall w \in C_c^1(X, \Lambda^k X).$$

Let X be a smooth manifold. Then

$$d_\nabla : C_c^\infty(X, \Lambda^k X) \to C_c^\infty(X, \Lambda^{k+1} X).$$

Applying this to map simultaneously to each exterior power yields the linear operator

$$d_\nabla : C_c^\infty(X, \Lambda^* X) \to C_c^\infty(X, \Lambda^* X)$$

with the properties [Sr, V Prop 3.3]

(i) (graded derivation) $d_\nabla(v \wedge w) = d_\nabla v \wedge w + (-1)^{\deg v} v \wedge d_\nabla w$,

(ii) (nilpotency) $d_\nabla^2 = 0$.

Let $v \in C_c(X, \Lambda^k X)$ and $w \in C_c(X, \Lambda^j X)$. Let $m_\top$ denote the interior contraction map $m_\top(v \otimes w) := v \top w$. We define the interior derivative associated to ∇

$$d_\nabla^* := m_\top \circ \nabla : C_c^1(X, \Lambda^k X) \to C_c(X, \Lambda^{k-1} X).$$

This is illustrated by the action of d_∇^* in a local frame $\{dx_i\}_{i=1,\dots,n}$ of T^*U_α,

$$d_\nabla^* w = \sum_i dx_i \top \nabla_i w \quad \forall w \in C_c^1(X, \Lambda^k X).$$

The interior derivative d_∇^* is also a linear operator

$$d_\nabla^* : C_c^\infty(X, \Lambda^* X) \to C_c^\infty(X, \Lambda^* X)$$

such that $d_\nabla^{*2} = 0$ and d_∇^* is a graded derivation. In fact d_∇^* is the adjoint of d_∇ for the inner product [R, 4.2.3]

$$q_g(v, w) = \int_X q_g(x)(v(x), w(x)) \sqrt{\det g}\, dx \quad \forall v, w \in C_c^\infty(X, \Lambda^* X).$$

Let X be a smooth second countable metrisable locally manifold. Then there exists a unique covariant derivative ∇' on the tangent bundle $TX \to X$ such that $d_{\nabla'} = d$ [BGV, Prop 1.22]. The connection associated to ∇' is called the Levi-Cevita connection and the operators d and d^* are usually called 'the' exterior and interior derivative. Elements of $C_c^\infty(X, \Lambda^* X)$ are called the compactly supported smooth exterior differential forms on X.

Remark 1.3.4 The exterior derivative and the exterior differential forms on a second countable locally compact smooth manifold X are fundamental objects in the study of differential geometry. We leave the theory of differential geometry at this point however. We are concerned with emulating the structure of the exterior derivative and the exterior differential forms primarily before we consider emulating their geometric consequences.

The exterior derivative $d : C_c^\infty(X, \Lambda^* X) \to C_c^\infty(X, \Lambda^* X)$ on a second countable metrisable locally compact smooth manifold X and its metric adjoint d^* define two fundamental operators.

1.3.5 The Laplacian and the Signature operator

The Laplacian

Let X be a second countable metrisable locally compact smooth manifold with exterior derivative d and interior derivative d^*. We define the Laplacian operator

$$\Delta := (d + d^*)^2 : C_c^\infty(X, \Lambda^k X) \to C_c^\infty(X, \Lambda^k X).$$

The Laplacian operator derives the name Laplacian from the following identification. Let $(U_\alpha, \phi_\alpha = (\phi_\alpha^1, ..., \phi_\alpha^n))$ be a chart of X with $x_i := \phi_\alpha^i$ the local co-ordinates and $\{dx_i\}_{i=1,...,n}$ a frame of T^*U_α. We define the 'components of the metric' on U_α as the functions $g_\alpha^{ij}(x) := q_g(x)(dx_i, dx_j)$ (which provide the matrix representation of the metric $g_\alpha(x) = [g_\alpha^{ij}(x)]_{i,j=1,...,n}^{-1}$). We can then identify from $\langle h, \Delta f\rangle = \langle dh, df\rangle \ \forall f, h \in C_c^\infty(X)$ [R, 1.2.3],

$$\begin{aligned} \Delta f &= -\textstyle\sum_{i,j} \sqrt{\det g_\alpha}^{-1} \partial_j (g_\alpha^{ij} \sqrt{\det g_\alpha} \partial_i f) \\ &= -\textstyle\sum_{i,j} g_\alpha^{ij} \partial_i \partial_j f + (\sum_{i,j} \sqrt{\det g_\alpha}^{-1} (\partial_j g_\alpha^{ij} \sqrt{\det g_\alpha})(\partial_i f)) \\ &= -\textstyle\sum_{i,j} g_\alpha^{ij} \partial_i \partial_j f + \text{first order derivatives.} \end{aligned}$$

When $g_\alpha(x) = \mathrm{id} \in L(T_xX, T_xX) \ \forall x \in U_\alpha$, for instance when $X = \mathbb{R}^n$ with standard metric, then Δ is the usual Laplacian on $\mathbb{R}^n$.

Remark 1.3.5 The form of the 'generalised Laplacian' above is central to the philosophy of non-commutative geometry. From the form of the Laplacian, it could be considered that the metrics $g_\alpha(x)$ determine Δ. However, the converse is equally valid. The metrics $g_\alpha(x)$ are determined by Δ and $C_c^\infty(X)$. Explicitely,

$$g_\alpha^{ij} = -\Delta x_i x_j = -\frac{1}{2}[[\Delta, c(x_i)], c(x_j)]$$

where $x_i = \phi_\alpha^i$ for the chart $(U_\alpha, \phi_\alpha = (\phi_\alpha^1, ..., \phi_\alpha^n))$ [BGV, Prop 2.3]. Non-commutative geometry is not formulated in terms of co-ordinate charts or Riemannian metrics, a non-commutative space may have none of these. Non-commutative geometry is formulated in operator algebra theory, hence it takes operators as its fundamental objects. Hence, conceptually, an unbounded linear operator we take as the 'Laplacian operator' Δ and a separable C*-algebra A determines a geometric structure on the non-commutative space $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$. We reiterate this is a conceptual view.

The signature operator

Let X be a second countable metrisable locally compact smooth manifold with exterior derivative d and interior derivative d^*. The signature operator is defined by the linear combination,

$$d + d^* : C_c^\infty(X, \Lambda^* X) \to C_c^\infty(X, \Lambda^* X).$$

This operator is essentially selfadjoint considered as a linear operator

$$d + d^* : C_c^\infty(X, \Lambda^* X) \to C_c^\infty(X, \Lambda^* X) \subset L^2(X, \Lambda^* X)$$

and has an unbounded selfadjoint closure [LM, II Thm5.7]

$$D : DomD \to L^2(X, \Lambda^* X)$$

such that $\Lambda = C_c^\infty(X, \Lambda^* X)$ is an invariant core for D, $D|_\Lambda = d + d^*$ and $D^2|_\Lambda = \Delta$. We shall often abuse notation and write $D = d+d^*$ and call D the signature operator.

For the purposes of Section 1.4, where a counterpart to the exterior derivative d and the compactly supported smooth exterior differential forms $C_c^\infty(X, \Lambda^* X)$ is derived in the non-commutative calculus, we discuss how the operation of exterior derivation d is transferred to the *-algebra $C_c^\infty(X, \mathrm{Cl}(X, q_g))$. We follow the treatment of Connes in [C, VI.1]. Define $d_\iota := \iota^{-1} d \iota$,

$$\begin{array}{ccc} C_c^\infty(X, \mathrm{Cliff}^k(X, q_g)) & \xrightarrow{\iota} & C_c^\infty(X, \Lambda^k X) \\ d_\iota \downarrow & & \downarrow d \\ C_c^\infty(X, \mathrm{Cliff}^{k+1}(X, q_g)) & \xrightarrow{\iota} & C_c^\infty(X, \Lambda^{k+1} X). \end{array}$$

We recall the left Clifford action on the exterior bundle in section 1.3.3,

$$w_l : C_c(X, \mathrm{Cl}(X, q_g)) \to C_c(X, L(\Lambda^* X, \Lambda^* X))$$

and the canonical representation $(L^2(X, \Lambda^* X), \lambda)$ in section 1.3.2 where

$$\lambda : C_c(X, L(\Lambda^* X, \Lambda^* X)) \to B(L^2(X, \Lambda^* X)).$$

The composition $\pi_l := \lambda \circ w_l$ forms a faithful concrete representation $(L^2(X, \Lambda^* X), \pi_l)$ of the C^*-closure $C_0(X, \mathrm{Cl}(X, q_g))$.

Proposition 1.3.6 [LM, II Lemma 5.5] *Let X be a second countable metrisable locally compact smooth manifold with signature operator $d + d^*$. Then*

$$[d + d^*, \pi_l(f)] = \pi_l(d_\iota f)$$

for all $f \in C_c(X)$.

Remark 1.3.7 We note that

$$d + d^* = (m + m_{\mathsf{T}}) \circ \nabla = c \circ \nabla$$

where c denotes Clifford multiplication and ∇ is the Levi-Civita connection. Hence

$$[d + d^*, \pi_l(f)] = \pi_l(d_\iota f)$$

is an equivalent statement of the Leibniz rule. The signature operator in the form $d + d^* = c \circ \nabla$ is a Dirac operator [LM].

Define a *-subalgebra of $B(L^2(X,\Lambda^* X))$,

$$\Omega_{d+d^*}(C_c^\infty(X)) := \langle\, \pi_l(C_c^\infty(X)) \;,\; [d+d^*, \pi_l(C_c^\infty(X))] \,\rangle$$

generated by $\pi_l(f), [d+d^*, \pi_l(g)]$ $\forall f,g \in C_c^\infty(X)$. From Proposition 1.3.6 and [C, VI.1 Lemma 6]

$$\Omega_{d+d^*}(C_c^\infty(X)) = \pi_l(C_c^\infty(X, \mathrm{Cl}(X,q_g))),$$

and there exists a canonical isomorphism,

$$\pi_l^{-1} : \Omega_{d+d^*}(C_c^\infty(X)) \to C_c^\infty(X, \mathrm{Cl}(X,q_g))$$

given by $\pi_l^{-1} : \pi_l(f_0)[d+d^*, \pi_l(f_1)]...[d+d^*, \pi_l(f_k)] \mapsto f_0 d_\iota f_1 \cdot ... \cdot d_\iota f_k$.

Hence we can reconstruct $C_c^\infty(X, \mathrm{Cl}(X,q_g))$ given just the signature operator $d+d^*$ and the C*-algebra $C_0(X)$. What about the exterior derivative?

Define

$$\Omega^k_{d+d^*}(C_c^\infty(X)) := \{\pi_l(f_0)[d+d^*, \pi_l(f_1)]...[d+d^*, \pi_l(f_k)] \,|\, f_0, ..., f_k \in C_c^\infty(X)\},$$

and a linear map

$$\delta_d : \Omega^k_{d+d^*}(C_c^\infty(X)) \to \Omega^{k+1}_{d+d^*}(C_c^\infty(X))$$

by

$$\delta_d\Big(\pi_l(f_0)[d+d^*, \pi_l(f_1)]...[d+d^*, \pi_l(f_k)]\Big) := [d+d^*, \pi_l(f_0)][d+d^*, \pi_l(f_1)]...[d+d^*, \pi_l(f_k)].$$

We note $\pi_l^{-1}(\Omega^k_{d+d^*}(C_c^\infty(X)))$ is not

$$C_c^\infty(X, \mathrm{Cliff}^k(X,q_g)) := \iota^{-1}(C_c^\infty(X, \oplus_{i=0}^k \Lambda^i X)).$$

Hence

$$\delta_d \circ \pi_l \neq \pi_l \circ d_\iota.$$

To rectify this, we recall the map from section 1.3.1

$$\iota_k : w_1 \cdot ... \cdot w_k \to \iota(w_1) \wedge ... \wedge \iota(w_k) \qquad \forall w_1, .., w_k \in C_c^\infty(X, T^*X)$$

is a surjection. Hence the surjective map

$$\iota_k \circ \pi_l^{-1} : \Omega^k_{d+d^*}(C_c^\infty(X)) \to C_c^\infty(X, \Lambda^k X),$$

defines a quotient algebra

$$\Lambda^k_{d+d^*}(C_c^\infty(X)) := \Omega^k_{d+d^*}(C_c^\infty(X)) / \ker \iota_k \circ \pi_l^{-1}.$$

Then the following diagram commutes, moreover the top and bottom surjections are isomorphisms [C, IV.1 Lemma 6],

$$\begin{array}{ccc} \Lambda^k_{d+d^*}(C_c^\infty(X)) & \xrightarrow{\cong} & \pi_l(C_c^\infty(X, \Lambda^k X)) \\ \delta_d \downarrow & & \downarrow d \\ \Lambda^{k+1}_{d+d^*}(C_c^\infty(X)) & \xrightarrow{\cong} & \pi_l(C_c^\infty(X, \Lambda^{k+1} X)). \end{array}$$

Hence this treatment of Connes has captured exterior differentation on a manifold. The map δ_d we can generalise in section 1.4.1 to the non-commutative situation, the map d_ι we cannot since it relies on anti-commutation relations specific to the Clifford algebra.

1.3.6 Summary of Riemannian Structure

We recall the term Riemannian manifold refers to a second countable, metrisable, locally compact, connected, oriented, geodesically complete smooth manifold with no boundary and given Riemannian metric g. Let X be a Riemannian manifold. Then section 1.3.1 through to section 1.3.5 have discussed the following structures:

(i) the map
$$q_g : C_c(X, T^*X) \times C_c(X, T^*X) \to C_c(X)$$
defined by $q_g(v,w)(x) := q_g(x)(v(x), w(x)) \quad \forall\, v, w \in C_c(X, T^*X)$. The map q_g is often called the metric (since the matrix representation of the metric g at x is the inverse of the matrix of co-efficients of $q_g(x)$ as an inner product), see Section 1.3.3,

(ii) the Hilbert space $L^2(X, \Lambda^*X)$ defined as the closure of $C_c(X, \Lambda^*X)$ in the inner product, see Section 1.3.2,
$$\langle h_1, h_2 \rangle := \int_X q_g(h_1, h_2)(x)\sqrt{\det(g)}dx \;\; \forall\, h_1, h_2 \in C_c(X, \Lambda^*X),$$

(iii) the faithful representation $(L^2(X, \Lambda^*X), \pi_l)$ of the C*-algebra $C_0(X, \mathrm{Cl}(X, q_g))$ defined by $\pi_l := \lambda \circ w_l$, see Section 1.3.5,

(iv) from the inclusion $C_0(X) \hookrightarrow C_0(X, \mathrm{Cl}(X, q_g))$ we obtain the representation $(L^2(X, \Lambda^*X), \pi_l)$ of the C*-algebra $C_0(X)$ referred to as 'representation by multiplication operators',

(v) the signature operator $d+d^*$ which is an unbounded selfadjoint linear operator
$$d + d^* : Dom(d + d^*) \to L^2(X, \Lambda^*X)$$
that implements differentiation, see Section 1.3.5,
$$[d + d^*, \pi_l(f)] = \pi_l(d_\iota f) \quad \forall\, f \in C_c^\infty(X),$$

(vi) the Laplacian $\Delta = (d + d^*)^2$ determines the metric g, and hence the geometry of X, see Section 1.3.5.

Let X be a Riemannian manifold. Analagous to Example 1.3.2, the above information provides the triple $(C_0(X), L^2(X, \Lambda^*X), d + d^*)$ of a separable C*-algebra $C_0(X)$, a representation $(L^2(X, \Lambda^*X), \pi_l)$ of $C_0(X)$ and a selfadjoint linear operator $d+d^* : Dom(d+d^*) \to L^2(X, \Lambda^*X)$. Hence, as in Example 1.3.2, differential calculus on a Riemannian manifold is encoded in the triple
$$(C_0(X), L^2(X, \Lambda^*X), d + d^*)$$
which is of the form (A, H, D) described in the introduction to this chapter. In the literature the triple $(C_0(X), L^2(X, \Lambda^*X), d+d^*)$ is referred to as the spectral triple of the Riemannian manifold X. We prefer to consider the information $(L^2(X, \Lambda^*X), \pi_l, d + d^*)$ as a representation for the separable C*-algebra $C_0(X)$ which generates differential geometry on the spectrum $X = \Sigma(C_0(X))$. This non-standard view aids considerations to follow. Hence we shall retain it and reiterate we make distinct definitions from [C3].

1.4 Exterior Derivation on C*-algebras

1.4.1 Non-Commutative Differential Forms

The purpose of last section was to review the structure of the exterior derivative on a differentiable manifold as a guide to how we could conceive an exterior derivative in non-commutative geometry. We introduce in this section a variant of Connes' non-commutative counterpart to section 1.3.5 above.

Let A be a C*-algebra. Let $\mathcal{P}(X)$ denote the power set of a set X. Define a map

$$\mathrm{supp} : A \to \mathcal{P}(\mathrm{Prim}(A)) \ , \ a \mapsto \mathrm{supp}(a) := \{\ker \pi_\rho \,|\, \rho \in PS(A), a \notin \ker \pi_\rho\}$$

called the support map.

Definition 1.4.1 *We say $a \in A$ has compact support if* $\mathrm{supp}(a)$ *is contained in a compact subset of* $\mathrm{Prim}(A)$.

Let A_c denote the subset of A consisting of all elements of compact support. If $\mathrm{Prim}(A)$ is compact then $A = A_c$.

Theorem 1.4.2 *Let A be a C^*-algebra. Then A_c is a norm dense two-sided *-ideal of A.*

Proof Let $a \in A_c$ and $\rho \in PS(A)$ such that $\pi_\rho(a) = 0$. Then $\pi_\rho(a^*) = \pi_\rho(a)^* = 0$ and $\pi_\rho(ab) = \pi_\rho(a)\pi_\rho(b) = 0 = \pi_\rho(b)\pi_\rho(a) = \pi_\rho(ba)$ $\forall\, b \in A$. This proves A_c is a two-sided *-ideal. Let $C = \mathrm{Prim}(A)$ and $c \in C$. Let $[a]_c$ denote the class of $a \in A^+$ in the quotient A/c. Define a function $f_a : \mathrm{Prim}(A) \to [0, \infty)$ by $f_a(c) := \|[a]_c\|$. The map $a \to f_a$ extends to an isomorphism $Z(M(A)) \to C_b(C)$ where $M(A)$ is the multiplier algebra of A and C_b denotes continuous bounded functions (Dauns-Hofmann Theorem [Ped, Cor 4.4.8]). Let $\{f_\mu\}$ be a net of continuous bounded functions with compact support such that $\|(1 - f_\mu)f\| \to 0$ for all $f \in C_0(C)$. Let $\{u_\lambda\}$ be an approximate unit of A. Then $\{f_\mu u_\lambda\}$ is a compactly supported approximate unit for A. This is sufficient for norm density of A_c. □

Definition 1.4.3 *Let (H, π) be a non-degenerate separable concrete representation of a normed *-algebra A. Let D be a (bounded or unbounded) selfadjoint operator $D : DomD \to H$. Then we call (H, π, D) a base representation of the normed *-algebra A.*

Definition 1.4.4 [C3] *Let (H, π, D) be a base representation of a C^*-algebra A. Then we call (H, π, D) a C^1_c-representation if there exists a norm-dense *-subalgebra A^1_c of A_c such that*

(i) *$\pi(a)DomD \subset DomD$ for $a \in A^1_c$,*

(ii) *$[D, \pi(a)]$ is norm bounded on $DomD$ for $a \in A^1_c$.*

There exists a unique bounded operator extending $[D, \pi(a)]$ for $a \in A^1_c$. We abuse notation and denote the extension $[D, \pi(a)]$ as well. Let (H, π, D) be a C^1_c-representation of a C*-algebra A. We define

$$\Omega_D(A^1_c) := < \pi(A^1_c) \ , \ [D, \pi(A^1_c)] >$$

as the *-subalgebra of $B(H)$ generated by $\pi(a), [D, \pi(b)]$ for all $a, b \in A^1_c$. We can $\mathbb{Z}$-grade this algebra by $\Omega^0(A^1_c) := \pi_l(A^1_c)$ and

$$\begin{aligned}\Omega^k_D(A^1_c) :=& \ \{w \in \Omega_D(A^1_c) \,|\, w \text{ is degree } k \text{ in } [D, \pi(A^1_c)] \text{ terms}\}\\ =& \ \{\pi(a_0)[D, \pi(a_1)] \ldots [D, \pi(a_k)] \,|\, a_0, ..., a_k \in \hat{A}^1_c\}.\end{aligned}$$

for $k \geq 1$ where $\hat{A}^1_c$ is the unitisation of A^1_c [6]. We view the operation

$$[D, \cdot\,] : \pi(A^1_c) \to \Omega^1_D(A^1_c)$$

as 'differentiation'. We extend this to

$$\delta_D : \Omega^k_D(A^1_c) \to \Omega^{k+1}_D(A^1_c)$$

given by $\delta_D\Big(\pi(a_0)[D, \pi(a_1)] \ldots [D, \pi(a_k)]\Big) := [D, \pi(a_0)][D, \pi(a_1)] \ldots [D, \pi(a_k)]$.

The map δ_D is designed to be the generalised exterior derivative. It satisfies $\delta^2_D = 0$ by construction. However, it is not a graded derivation in general. Following [C, VI.1] we will quotient the algebra $\Omega^k_D(A^1_c)$ by the obstruction to δ_D being a graded derivation, and hence obtain the generalisation of differential forms. Let us calculate the obstruction.

The universal graded differential algebra $(\Omega(B), \delta)$ of a unital associative algebra B is given by

(i) $\Omega^0(B) := B$,

(ii) $\Omega^1(B) := \ker\{a \otimes b \mapsto ab \,|\, a, b \in B\} \subset B \odot B$,

(iii) the derivation $\delta : B \to \Omega^1(B)$ defined by $\delta(b) := 1 \otimes b - b \otimes 1$,

(iv) $\Omega^k(B) := \underbrace{\Omega^1(B) \odot_B \ldots \odot_B \Omega^1(B)}_{k}$ for $k \geq 2$,

(v) the unique graded derivation $\delta : \Omega^k(B) \to \Omega^{k+1}(B)$ that extends $\delta : B \to \Omega^1(B)$ [Sb, II Lemma 1.1.2]

(vi) $\Omega(B) := \oplus_{k \geq 0} \Omega^k(B)$ with multiplication by tensor product over B.

Let $\hat{B}$ be the unitisation of a non-unital *-algebra B. We then define the universal graded differential algebra of B by

$$\Omega(B) := B \oplus (\oplus_{k=1}^{\infty} \Omega^k(\hat{B})).$$

Let (H, π, D) be a C^1_c-representation of a C*-algebra A. The map

$$\pi_D : \Omega^k(A^1_c) \to \Omega^k_D(A^1_c) \ , \ \delta \to \delta_D$$

[6] The equality between the sets defining $\Omega^k_D(A^1_c)$ is not a triviality. For instance, consider $w \in \Omega_D(A^1_c)$ given by $w = \pi(a_0)[D, \pi(a_1)]\pi(a_2)$ for $a_0, a_1, a_2 \in A^1_c$. This is of degree 1 in $[D, \pi(A^1_c)]$ terms. The fundamental relation

$$[D, \pi(a)\pi(b)] = \pi(a)[D, \pi(b)] + [D, \pi(a)]\pi(b) \qquad \forall\, a, b \in A^1_c$$

provides the form $w = \pi(a_0)[D, \pi(a_1 a_2)] + \pi(a_0 a_1)[D, \pi(a_2)]$.

defined by

$$\pi_D(a_0 \otimes \delta(a_1) \otimes \ldots \otimes \delta(a_k)) := \pi(a_0)[D, \pi(a_1)] \ldots [D, \pi(a_k)]$$

is an algebraic homomorphism [C, Pg 186]. We use the terminology $(\Omega_D(A_c^1), \delta_D)$ is a representation of the universal graded differential algebra $(\Omega^k(A_c^1), \delta)$. However, it is not differential as $\pi_D \circ \delta \neq \delta_D \circ \pi_D$ in general.

Example 1.4.5 The universal graded differential algebra of an associative algebra B is universal in the following sense. Let $M = \oplus_k M^k$ be a graded B-bimodule with $M^0 = B$ and graded derivation $d : M^k \to M^{k+1}$ such that $d^2 = 0$ [7]. Then there exists a graded bimodule homomorphism $\theta : \Omega(B) \to M$ such that $\theta\delta = d\theta$ [JL] [P, 7.1,7.2].

For example, on a Riemannian manifold X there exist homomorphisms p_1, p_2

$$\begin{tikzcd}
 & \Omega(C_c^\infty(X)) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
C_c^\infty(X, \mathrm{Cl}(X, q_g)) \arrow[rr, "\iota"] & & C_c^\infty(X, \Lambda^* X)
\end{tikzcd}$$

such that $\iota p_1 = p_2$, $p_1\delta = d_\iota p_1$ and $p_2\delta = dp_2$. Let $D = d + d^*$ and define the surjection

$$p_3 : C_c^\infty(X, \mathrm{Cliff}^k(X, q_g)) \to \Omega_D^k(C_c^\infty(X)) \ , \ d_\iota \to \delta_D.$$

Then $p_3 d_\iota \neq \delta_D p_3$. This is due to the existence of elements $w \in \ker p_3$ such that $d_\iota w \notin \ker p_3$. As an example, an element with local representation in a chart U_α of X with the form, $w_\alpha(x) = f(x)\pi_l(dx_1)...\pi_l(dx_{k-1})$ such that $\partial_i f = 0$ for $i = 1, ..., k-1$ and $\partial_k f \neq 0$. As $\Omega^k(C_c^\infty(X))$ is universal then $p_3 p_1 = \pi_D$. Hence $\pi_D \delta \neq \delta_D \pi_D$ as $\pi_D \delta = p_3 p_1 \delta = p_3 d_\iota p_1 \neq \delta_D p_3 p_1 = \delta_D \pi_D$.

We can identify the obstruction to π_D being a differential representation. Define for $k \geq 0$,

$$J^k(A_c^1) := \{w_1 + \delta(w_2) \mid w_1 \in \Omega^k(A_c^1), w_2 \in \Omega^{k-1}(A_c^1), \pi_D(w_1) = \pi_D(w_2) = 0\}.$$

The algebra $J_D(A_c^1) := \oplus_k \pi_D(J^k(A_c^1))$ is a graded differential two sided ideal of $\Omega_D(A_c^1)$ [C, VI.1 Prop 4]. Define

$$\Lambda_D(A_c^1) := \Omega_D(A_c^1)/J_D(A_c^1)$$

then

$$\delta_D : \Lambda_D^k(A_c^1) \to \Lambda_D^{k+1}(A_c^1)$$

has the properties

(i) δ_D is a graded derivation on $\Lambda_D(A_c^1)$,

(ii) $\delta_D^2 = 0$,

that generalise the properties of the exterior derivative.

[7]Bimodules of associative algebras are discussed in Section 2.1.1.

1.4.2 Smooth Non-Commutative Differential Forms

Last section we defined the non-commutative exterior differential forms and non-commutative exterior differentiation associated to the C^1_c-representation (H,π,D) of a C*-algebra A as the differential representation $(\Lambda_D(A^1_c),\delta_D)$ of the universal graded differential algebra $(\Omega^k(A^1_c),\delta)$. The only detraction of this construction is the incapacity to define degrees of differentiability. The nilpotency of the 'exterior derivative' δ_D prevents repeated derivation of a function from determining degree of differentiability. We recall from Section 1.3.4 that covariant derivatives provided the analogue of partial or directional derivatives on a Riemannian manifold. A. Connes treatment of 'non-commutative covariant derivation' in [C5, CM] involves a non-commutative pseudo-differential calculus of a C^1_c-representation (H,π,D) of a C*-algebra A. We outline the definition of smooth non-commutative differential operators in this calculus.

Let (H,π,D) be a C^1_c-representation of a C*-algebra A. Define the subspaces of the Hilbert space H, [8]

$$H^s = Dom|D|^s$$

for all $s \geq 0$. We define $\mathrm{op}^{s,r}$ as the linear space of continuous operators

$$\mathrm{op}^{s,r} : H^s \to H^{s-r}$$

for all $s \geq r \geq 0$. Let $f_s(x) = (1+x^2)^{-\frac{s}{2}}, x \in \mathbb{R}, s \geq 0$. Introduce the norm

$$\|T\|_r := \sup\{\|f_s(D)Tf_t(D)\| \mid s+t=r\}.$$

Define the normed space

$$\mathrm{op}^{s,r}_o := \{T \in \mathrm{op}^{s,r} \mid \|T\|_r < \infty\}.$$

We will consider the operators in $\mathrm{op}^{s,p}_o$ $\forall\, s \geq p$ to be the p^{th}-differential operators in the calculus of (H,π,D).

Remark 1.4.6 We have $\mathrm{op}^{0,0} = \mathrm{op}^{0,0}_o = B(H)$. A zeroth order operator T is bounded and $T \in L(Dom|D|^s, Dom|D|^s)$ for all $s \geq 0$. The operators $D, |D| \in \mathrm{op}^{s,1}_o$ $\forall\, s \geq 1$ and are considered 'first order'. The Laplacian $D^2 \in \mathrm{op}^{s,2}_o$ $\forall\, s \geq 2$ is second order. A first order operator that is central in Connes local index formula is the 'covariant derivative' [C5, CM]

$$\nabla_{da} = \frac{1}{2}[D^2, \pi(a)]$$

for $a \in A^1_c$ that are zeroth order. Connes' definition of a covariant derivative provides generators of the 'non-commutative geodesic flow' and generalisations of the Levi-Civita connection, see [C5, Section 6].

[8] Let $h : \mathbb{R} \to \mathbb{C}$ be a bounded Borel function, then $h(D)$ will denote the operator defined by the spectral theorem for selfadjoint operators [RS, Thm VIII.5]. The spectral theorem extends to unbounded Borel functions in the following sense. Let h_n be bounded and Borel and $h_n \to h$ pointwise. Then there exists an unbounded closed operator $h(D)$ such that $h_n(D) \to h(D)$ in the strong resolvent sense (Spectral Theorem with Trotter-Kato and Trotter Theorems [RS, VIII.21-22]).

Define the covariant derivation,

$$\nabla(w) := \frac{1}{2}[D^2, w]$$

and the derivation

$$\delta_{|D|}(w) := [|D|, w]$$

for $w \in \pi(A_c^1)$ or $w \in \Omega_D^1(A_c^1)$ that are zeroth order.

Proposition 1.4.7 [C5, Lemma 1, Cor 1, Lemma 2]
Let $w \in \pi(A_c^1)$ or $w \in \Omega_D^1(A_c^1)$ be zeroth order and $p \in \mathbb{N}$. Then

$$w \in Dom\delta_{|D|}^m \ m = 1, ..., p \iff \nabla^m(w) \in \mathrm{op}_o^{m,m} \ m = 1, ..., p.$$

The result identifies p-differentiability of differential forms and the domain $\cap_{m=1}^p \delta_{|D|}^m$.

Definition 1.4.8 [C3] *Let (H, π, D) be a base representation of a C^*-algebra A. We say (H, π, D) is a C_c^∞-representation if there exists a norm-dense *-subalgebra A_c^∞ of A_c such that*

(i) $\pi(a)DomD \subset DomD$ *for* $a \in A_c^\infty$,

(ii) $[D, \pi(a)]$ *is norm bounded on* $DomD$ *for* $a \in A_c^\infty$,

(iii) $\pi(a), [D, \pi(a)] \in Dom\delta_{|D|}^k \ \forall k \in \mathbb{N}$ *for* $a \in A_c^\infty$.

Let (H, π, D) be a C_c^∞-representation of a C*-algebra A. As a consequence of the definition above $\Omega_D(A_c^\infty) \subset \mathrm{op}_o^{s,0}$ for each $s \geq 0$. In otherwords, defining

$$H^\infty := \cap_{s\geq 0} H^s = \cap_{s\geq 0} Dom|D|^s,$$

then

$$\Omega_D(A_c^\infty) \subset L(H^\infty, H^\infty).$$

We define a locally convex topology on $\Omega_D(A_c^\infty)$ by the family of semi-norms [BR, V.1]

$$p_m(w) := \|\delta_{|D|}^m(w)\| \ , \ m = 0, 1, 2, ...$$

with the convention $\delta_{|D|}^0(T) = T$ for $T \in B(H)$. We denote this locally convex topology $\mathcal{S}_D$ and the closure of a set $O \subset \cap_m Dom\delta^m$ by $\mathcal{S}_D(O)$. We define a stronger locally convex topology on A_c^∞ by the family of semi-norms

$$p_m^0(a) := \|\delta_{|D|}^m(\pi(a))\| \ , \ p_m^1(a) := \|\delta_{|D|}^m([D, \pi(a)])\| \ , \ m = 0, 1, 2, ...$$

with the convention $\delta_{|D|}^0(T) = T$ for $T \in B(H)$. We denote this locally convex topology $\mathcal{S}_D^1$ and the closure of a set $O \subset \cap_m Dom\delta^m$ by $\mathcal{S}_D^1(O)$. We have not specified that A_c^∞ be closed in the locally convex topology $\mathcal{S}_D^1$.

Let (H, π, D) be a C_c^∞-representation of a C*-algebra A. We introduce the notion of smoothness in the non-commutative calculus associated to the C_c^∞-representation (H, π, D). Define

$$\mathcal{A}_\pi := \{a \in A \,|\, p_m^n(\pi(a)) < \infty, n = 0, 1, m = 0, 1, 2, ...\}.$$

Proposition 1.4.9 *The *-algebra $\mathcal{A}_\pi$ as above has the properties,*

(i) *the topology $\mathcal{S}_D^1$ on $\mathcal{A}_\pi$ is metrisable,*

(ii) *$\mathcal{A}_\pi$ is closed in $\mathcal{S}_D^1$,*

(iii) *$\mathcal{A}_\pi$ is closed in the holomorphic functional calculus,*

(iv) *$\mathcal{A}_\pi$ is norm dense in A.*

Proof Checking that $\mathcal{A}_\pi$ is a *-algebra is straightforward. (i) $\mathcal{S}_D^1$ is generated by a countable family of semi-norms [RS, Thm V.5]. (ii) By construction.
(iii) This result is well known, and the same concept as [C, Lemma 6]. We outline the proof. Let $\pi(a) \in \mathcal{A}_\pi$. Let $f(\pi(a)) = \int_C f(\lambda)(\pi(a) - \lambda)^{-1} d\lambda$ where C is a closed contour with $\mathrm{sp}(a)$ interior to C and f is holomorphic on a region containing C and its interior. Let $M(C, \mathrm{sp}(a)) = \inf_{z \in C, w \in \mathrm{sp}(a)} \|z - w\|$ and $L(C)$ = arclength of C. From $[T, (\pi(a) - \lambda)^{-1}] = -(\pi(a) - \lambda)^{-1}[T, \pi(a)](\pi(a) - \lambda)^{-1}$. we obtain

$$\|[T, f(\pi(a))]\| \leq \|[T, \pi(a)]\| \max_C |f| M(C, \mathrm{sp}(a))^{-2} L(C).$$

This proves $p_1^0(f(\pi(a))), p_0^1(f(\pi(a))) < \infty$ letting $T = |D|, D$. Similar arguments provide the same result for the semi-norms p_m^n where $n = 0, 1$ and $m = 0, 1, 2, \ldots$. Hence $f(\pi(a)) \in \mathcal{A}_\pi$.
(iv) There exists a norm dense sub-algebra $A_c^\infty \subset \mathcal{A}_\pi$ by definition. □

A locally convex space that is closed and metrisable is called a Frechet space [Jn]. In the literature a smooth algebra is defined to be a Frechet *-algebra stable under the holomorphic functional calculus. A pre-C*-algebra is defined to be a norm dense *-subalgebra of a C*-algebra stable under the holomorphic functional calculus.

Corollary 1.4.10 *The pre-C*-algebra $\mathcal{A}_\pi$ of A is smooth.*

The *-algebra of multipliers,

$$\mathcal{M}(\mathcal{A}_\pi) := \{a \in M(A) \,|\, ab, ba \in \mathcal{A}_\pi \ \ \forall\, b \in \mathcal{A}_\pi\},$$

is a pre-C*-algebra of the multiplier algebra $M(A)$ that is closed in the non-metrisable locally convex topology given by the family of seminorms

$$p_{m,b}^n := p_m^n(ab) + p_m^n(ba) \ , \ b \in \mathcal{A}_\pi, n = 0, 1, m = 0, 1, 2, \ldots$$

We denote this locally convex topology by $\mathcal{MS}_D^1$ and the closure of $O \subset \mathcal{M}(\mathcal{A}_\pi)$ by $\mathcal{MS}_D^1(O)$.

The locally convex topology $\mathcal{S}_D^1$ can be placed on the *-algebra

$$\mathcal{M}(\mathcal{A})_\pi = \{a \in M(A) \,|\, p_m^n(\pi(a)) < \infty, n = 0, 1, m = 0, 1, 2, \ldots\},$$

making it a Frechet C*-subalgebra of the multiplier algebra $M(A)$.

Lemma 1.4.11 *Let $\mathcal{M}(\mathcal{A})_\pi$ and $\mathcal{M}(\mathcal{A}_\pi)$ be the closed locally convex *-algebras as above. Then*

$$\mathcal{MS}_D^1(\mathcal{A}_\pi) \subset \mathcal{M}(\mathcal{A})_\pi \subset \mathcal{M}(\mathcal{A}_\pi)$$

with $\mathcal{S}_D^1 << \mathcal{MS}_D^1$.

Proof Let $X = \mathcal{MS}_D^1(\mathcal{A}_\pi)$, $Y = \mathcal{M}(\mathcal{A})_\pi$ and $Z = \mathcal{M}(\mathcal{A}_\pi)$. Let $a \in Y$. Then (*)

$$p_{m,b}^n(a) = p_m^n(ab) + p_m^n(ba) \leq \sum_{i=0}^{m} \binom{m}{i} (p_i^n(a)p_{m-i}^n(b) + p_i^n(b)p_{m-i}^n(a)) < \infty.$$

Hence $a \in Z$. Moreover, let $\{a_i\} \subset Y$ such that $a_i \to a$ in the topology $\mathcal{S}_D^1$. Then $p_{m,b}^n(a_i - a) \to 0$ for each $b \in \mathcal{A}_\pi$ by (*). Hence $\mathcal{S}_D^1 << \mathcal{MS}_D^1$ and Y is closed in $\mathcal{MS}_D^1$. Then $X \subset Y$ since $\mathcal{A}_\pi \subset Y$. □

Theorem 1.4.12 *Let $\mathcal{M}(\mathcal{A})_\pi$ and $\mathcal{M}(\mathcal{A}_\pi)$ be the closed locally convex *-algebras as above. Then the following statements are equivalent*

(i)

$$\mathcal{MS}_D^1(\mathcal{A}_\pi) = \mathcal{M}(\mathcal{A})_\pi = \mathcal{M}(\mathcal{A}_\pi),$$

(ii) *the smooth *-algebra $\mathcal{A}_\pi$ admits an approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ for the C^*-algebra A such that $p_m^n(u_\lambda) \to 0$ for all $m, n \neq 0$.*

Proof Let $X = \mathcal{M}(\mathcal{A})_\pi$, $Y = \mathcal{M}(\mathcal{A}_\pi)$ and $Z = \mathcal{MS}_D^1(\mathcal{A}_\pi)$.

(i) $\Rightarrow$ (ii) We have $1 \in Z$ and by hypothesis there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_\pi$ such that $p_{m,b}^n(u_i - 1) \to 0$ for all $b \in \mathcal{A}_\pi, n = 0, 1, m = 0, 1, 2, \ldots$. Fix $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_2$. Let $u_i = b$ as above. Then there exists $j(i, m, n) \in \mathbb{N}$ such that $\forall j > j(i, m, n)$, $p_m^n(u_i u_j^2 u_i) \leq p_{m,u_i}^n(u_j) < i^{-1}$. Then $\{u_{i,j} := u_i u_j^2 u_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_\pi$ is an approximate unit such that $p_m^n(u_{i,j}) \to 0$ as $i \to \infty$ for any $j > j(i, m, n)$. Now define $u_i^p := u_{i,j}$ for any $j > \max\{j(i, k, n) \,|\, n = 0, 1, k = 0, 1, \ldots, p\}$. Then $\{u_i^p\}_{(i,p)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}_\pi$ is an approximate unit such that $p_m^n(u_i^p) \to 0$ as $i, p \to \infty$.

(ii) $\Rightarrow$ (i) The result follows from the previous lemma if we establish $Z \subset X$. Let $a \in Z$. Then $ab, ba \in \mathcal{A}_\pi$ for all $b \in \mathcal{A}_\pi$. By hypothesis there exists an approximate unit $\{u_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{A}_\pi$ such that $p_m^n(au_\lambda b - ab) \to 0$ and $p_m^n(bau_\lambda - ba) \to 0$ for all $n = 0, 1, m = 0, 1, 2, \ldots$. Hence there exists a sequence $\{au_\lambda\} \subset \mathcal{A}_\pi$ such that $p_{m,b}^n(au_\lambda - a) \to 0$ for all $b \in \mathcal{A}_\pi, n = 0, 1, m = 0, 1, 2, \ldots$. Then $a \in X$. □

An approximate unit $\{u_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{A}_\pi$ is called a smooth approximate unit for the C*-algebra A if $p_m^n(u_\lambda) \to 0$ for all $m, n \neq 0$. In the absence of $\mathcal{A}_\pi$ admitting a smooth approximate unit for A we consider the largest *-algebra $\mathcal{M}(\mathcal{A}_\pi)$ as the pre-C*-algebra of smooth multipliers. A corollary of Theorem 1.4.12 is that $\mathcal{M}(\mathcal{A}_\pi)$ is a smooth *-algebra admitting the topology $\mathcal{S}_D^1$ if and only if $\mathcal{A}_\pi$ admits a smooth approximate unit for A.

Let A be a C*-algebra and $\Sigma_\eta(A) := PS(A) \stackrel{[\cdot]_u}{\to} \hat{A} \stackrel{\ker}{\to} \mathrm{Prim}(A)$ the associated non-commutative space. In summary, we have introduced the algebras in the non-commutative smooth differential calculus associated to the C_c^∞-representation (H, π, D)

coincident with the following intuitive scheme

$$\begin{array}{rcl}
A_c^\infty & \overset{\text{dense}}{\subset} & \text{'non-commutative } C_c^\infty(\Sigma_\eta)\text{'} \\
A_c \cap \mathcal{A}_\pi & = & \text{'non-commutative } C_c^\infty(\Sigma_\eta)\text{'} \\
\mathcal{A}_\pi & = & \text{'non-commutative } C_0^\infty(\Sigma_\eta)\text{'} \\
\mathcal{M}(\mathcal{A}_\pi) & = & \text{'non-commutative } C_b^\infty(\Sigma_\eta)\text{'} \\
A_c & = & \text{'non-commutative } C_c(\Sigma_\eta)\text{'} \\
A & = & \text{'non-commutative } C_0(\Sigma_\eta)\text{'} \\
M(A) & = & \text{'non-commutative } C_b(\Sigma_\eta)\text{'}.
\end{array}$$

The next example demonstrates the algebras above are the counterparts to smooth, compactly supported and continuous functions on a Riemannian manifold.

Example 1.4.13 Let X be a Riemannian manifold.

Proposition 1.4.14 *Let $(L^2(X, \Lambda^* X), \pi_l, d + d^*)$ be the base representation of the commutative C^*-algebra $A = C_0(X)$ defined in section 1.3.6. Then the base representation $(L^2(X, \Lambda^* X), \pi_l, d + d^*)$ is a C_c^∞-representation of $C_0(X)$. In particular $A_c = C_c(X)$, $\mathcal{A}_{\pi_l} = C_0^\infty(X)$ and $\mathcal{M}(\mathcal{A}_{\pi_l}) = C_b^\infty(X)$.*

Proof Let $\pi_l := \pi$. It is trivial that $C_c(X) = C_0(X)_c$ by $\mathrm{Prim}(C_0(X)) = X$. Let $f \in C_0^\infty(X)$ which is norm-dense in $C_0(X)$. We refer to [CM, Theorem I.1] for the proof that $[|d + d^*|, [|d + d^*|, \ldots, [|d + d^*|, T]..]]$ for $T = \pi_l(f)$ or $[d + d^*, \pi_l(f)] = \pi_l(df)$ are zero order because the principal symbol of $|d + d^*|$ is scalar. The same proof holds for any bounded smooth function $C_b^\infty(X)$. Then $C_0^\infty(X) \subset \mathcal{A}_\pi \subset C_0(X)$ and $C_b^\infty(X) \subset \mathcal{M}(\mathcal{A})_\pi \subset \mathcal{M}(\mathcal{A}_\pi) \subset C_b(X)$.
Equally it is an exercise in differential geometry to determine $\nabla(df) := \frac{1}{2}[\Delta, \pi_l(f)]$ indeed defines a covariant derivative up to scalar terms. Then $\|\nabla^k(f)\|_k < \infty$ implies the partial derivatives of f in any chart all exist and are continuous. Hence $\mathcal{A}_\pi \subset C_0^\infty(X)$ and $M(A)_\pi \subset C_b^\infty(X)$. Otherwise one may use the symbol calculus to obtain the partial derivatives as co-efficients of $[|d + d^*|, [|d + d^*|, \ldots, [|d + d^*|, \pi_l(f)]..]]$, such as [Re, 4.3] or the proof of [CM, Theorem I.1]. □

Non-Commutative Calculus (Part 2)

We recall the following basic elements that, together with linear algebra, provided the theory of multivariable calculus:

(i) the second countable metrisable locally compact topological space X,

(ii) the algebra of continuous complex valued functions $C(X)$ on the space X,

(iii) the derivative operation on differentiable functions $C^1(X) \subset C(X)$,

$$d : C^1(X) \to C(X, L(TX, \mathbb{C})),$$

(iv) the integral of a continuous function over the space X, which can be viewed as a linear positive definite functional

$$I : C(X) \to \mathbb{C} \cup \{\infty\}.$$

(v) a geodesic metric derived from the integral and derivative.

In this chapter we are concerned with the emulation of the elements (i)-(v) in the non-commutative environment of the triple (A, H, D) defined in the introduction. In part 2 we deal with the elements (iv)-(v). Section 1.5 identifies the predual of a von Neumann envelope of a separable C*-algebra A as the counterpart of the space of linear positive definite functionals on integrable functions. Section 1.6 extends the theory of non-commutative integration by introducing generalised Radon-Nikodym derivatives. Section 1.6 also introduces a foundation structural theory of von Neumann algebras called the Tomita-Takesaki or Modular theory. The Tomita-Takesaki theory is shown to have a fundamental link to the spectral triple $(C_0(X), L^2(X, \Lambda^*X), d + d^*)$ of a Riemannian manifold X. Section 1.7 introduces A. Connes formulation of the integral calculus associated to a base representation (H, π, D) of a C*-algebra A. We remark Connes formulation is not coincident with the established non-commutative integration theory of normal semi-finite weights on von Neumann algebras discussed in Section 1.5 and Section 1.6. Section 1.8 develops the counterpart of geodesic metric. Section 1.9 summarises the non-commutative calculus associated to a C_c^∞- and integrable representation (H, π, D) of a C*-algebra A as developed in Section 1.2 through to Section 1.8.

Basic Definitions

Let X be a topological space and μ a Borel measure. We call the pair (X, μ) a (Borel) measure space. We recall a Borel subset B of X is defined by

$$B = \cup\{C \subset B \,|\, C \text{ is compact}\} = \cap\{B \subset O \,|\, O \text{ is open}\}.$$

A regular Borel measure on X is a Borel measure μ such that for all Borel sets B,

$$\mu(B) = \sup_{C \subset B} \mu(C) = \inf_{B \subset O} \mu(O)$$

where O are open and C are compact and Borel. A Borel measure μ is called σ-finite if there exists a countable collection of Borel sets $\{E_n\}_{n\in\mathbb{N}}$ such that $\mu(E_n) < \infty \;\forall\, n \in \mathbb{N}$ and $\cup_n E_n = X$. A Borel measure μ is called finite if $\mu(X) < \infty$. When the topology on X is metrisable every finite measure is regular.

A Borel set B is called a null set for μ if $\mu(B) = 0$. Let μ and ν be Borel measures on X. Then ν is absolutely continuous with respect to μ, denoted $\nu << \mu$, if $\nu(B) = 0 \Leftarrow \mu(B) = 0$ for a Borel set B. The Borel measures ν and μ are called equivalent, denoted $\nu \equiv \mu$, if $\nu << \mu$ and $\mu << \nu$.

1.5 Non-Commutative Measure Theory

Let X be a locally compact Hausdorff space and μ a regular Borel measure on X. In the following sections we shall show a bijective correspondence

$$(X, \mu) \longleftrightarrow (W(A), \tau)$$

between regular Borel measure spaces (X, μ) and commutative von Neumann algebras and normal semifinite weights $(W(A), \tau)$. The existence of this bijective correspondence leads to the generalised theory of von Neumann algebras and normal semifinite weights being called 'non-commutative measure theory'.

1.5.1 Measure Theory on $\mathbb{R}$

Let μ be a regular Borel measure on $\mathbb{R}$. Let $L^1(\mathbb{R}, \mu)$ be the Banach space of integrable functions and $L^\infty(\mathbb{R}, \mu)$ the Banach space of essentially bounded functions. Let $f \in L^\infty(\mathbb{R}, \mu)$. The correspondence

$$f \mapsto T^\mu(f, \cdot) \ , \ T^\mu(f, g) := \int_{\mathbb{R}} f g d\mu \ \forall g \in L^1(\mathbb{R}, \mu)$$

defines a linear isometry

$$L^\infty(\mathbb{R}, \mu) \longleftrightarrow L^1(\mathbb{R}, \mu)^*$$

between essentially bounded functions and the dual of $L^1(\mathbb{R}, \mu)$. However, letting $g \in L^1(\mathbb{R}, \mu)$, the correspondence

$$g \mapsto T^\mu(\cdot, g) \ , \ T^\mu(f, g) := \int_{\mathbb{R}} f g d\mu \ \forall f \in L^\infty(\mathbb{R}, \mu)$$

defines a linear isometry of $L^1(\mathbb{R}, \mu)$ onto a proper closed subspace $L^\infty(\mathbb{R}, \mu)_*$ of $L^\infty(\mathbb{R}, \mu)^*$,

$$L^1(\mathbb{R}, \mu) \longleftrightarrow L^\infty(\mathbb{R}, \mu)_*.$$

We call $L^\infty(\mathbb{R}, \mu)_*$ the pre-dual of $L^\infty(\mathbb{R}, \mu)$. The Radon-Nikodym Theorem identifies $L^\infty(\mathbb{R}, \mu)_*$ with the finite Borel measures ν absolutely continuous to μ,

$$\nu(E) := T^\mu(\chi_E, f),$$

where χ_E is the characteristic function of a Borel set E. The ultraweak topology on $L^\infty(\mathbb{R}, \mu)$ is the weak*-topology induced on $L^\infty(\mathbb{R}, \mu)$ as the dual of $L^1(\mathbb{R}, \mu)$,

$$f_n \to f \iff T^\mu(f_n, g) \to T^\mu(f, g) \ \forall g \in L^1(\mathbb{R}, \mu).$$

The equivalences

finite Borel measure $\nu << \mu$ on $\mathbb{R}$

$\iff$ $T^\mu(\cdot, f) \in L^\infty(\mathbb{R}, \mu)_*$,

Borel measurable sets E s.t. $\mu(E) > 0$

$\iff$ characteristic functions χ_E in $L^\infty(\mathbb{R}, \mu)$,

$\iff$ the non-zero projections in $L^\infty(\mathbb{R}, \mu)$,

regularity of a finite Borel measure $\nu << \mu$

$\iff$ ultraweak continuity of $T^\mu(\cdot, f) \in L^\infty(\mathbb{R}, \mu)_*$ on the unit ball of $L^\infty(\mathbb{R}, \mu)$,

Borel measurable sets constructed from the topology of $\mathbb{R}$

$\iff$ $L^\infty(\mathbb{R}, \mu)$ is the ultraweak closure of $C_0(\mathbb{R})$, transfer measure theory from Borel sets and measures to the Banach space $L^\infty(\mathbb{R},\mu)$ and the predual $L^\infty(\mathbb{R},\mu)_*$.

We convert the Banach space concepts above into operator algebra theory. The Banach space $L^\infty(\mathbb{R},\mu)$ is a *-algebra given the involution of complex conjugation and the product $(fg)(x) = f(x)g(x)$ μ-a.e. Then $L^\infty(\mathbb{R},\mu)$ is a C*-algebra. It has a faithful representation $(L^2(\mathbb{R},\mu),\pi_l)$ where

$$\pi_l : L^\infty(\mathbb{R},\mu) \to B(L^2(\mathbb{R},\mu)) \ , \ \pi_l(f)g(x) = f(x)g(x) \ \mu\text{-a.e. } \forall g \in L^2(\mathbb{R},\mu).$$

The continuous linear functionals $L^\infty(\mathbb{R},\mu)^* = L(L^\infty(\mathbb{R},\mu),\mathbb{C})$ play the same role as section 1.2 in the structure of the C*-algebra $L^\infty(\mathbb{R},\mu)$. However $\sigma : L^\infty(\mathbb{R},\mu) \to \mathbb{C}$, while continuous in the $\|.\|_\infty$-norm sense

$$\sigma(f_\alpha) \to \sigma(f) \ \text{ when } \ \|f_\alpha - f\|_\infty \to 0,$$

is not necessarily continuous in an ultraweak sense

$$\sigma(f_\alpha) \to \sigma(f) \ \text{ when } \ f_\alpha \to f \text{ ultraweakly.}$$

Hence the weaker topology on $L^\infty(\mathbb{R},\mu)$ prescribes a restricted class of (norm-)continuous linear functionals and a new facet to the analysis:

Let $\{f_\alpha\}$ be any bounded monotonically increasing net of positive essentially bounded functions such that $f_\alpha \to f$ ultraweakly. Then a continuous linear functional $\sigma \in L(L^\infty(\mathbb{R},\mu),\mathbb{C})$ is called normal if $\lim_\alpha \sigma(f_\alpha) = \sigma(f)$. The normal linear functionals are denoted $L_*(L^\infty(\mathbb{R},\mu),\mathbb{C})$.

Theorem 1.5.1 *Let μ be a Borel measure on $\mathbb{R}$. Then*

(i) *$L^\infty(\mathbb{R},\mu)$ is ultraweakly closed,*

(ii) *the pre-dual consists of normal linear functionals,*

$$L^\infty(\mathbb{R},\mu)_* = L_*(L^\infty(\mathbb{R},\mu),\mathbb{C})$$

Proof (i) The dual of a Banach space is closed in the weak*-topology. (ii) [Ped, Cor 3.5.6, Thm 3.6.4]. □

The normal linear functionals on $L^\infty(\mathbb{R},\mu)$ are equivalent to the finite regular Radon measures in the measure theory on $L^\infty(\mathbb{R},\mu)$.

1.5.2 Von Neumann Algebras and Weights

Theorem 1.5.1 is the basis for the generalisation of measure theory to general operator algebras.

Let H be a Hilbert space. A von Neumann algebra R is a weakly closed C*-subalgebra of $B(H)$. For any C*-algebra A we have denoted by $L(A,\mathbb{C})$ the continuous linear functionals on A and $L^+(A,\mathbb{C})$ the positive linear functionals. Let $\{a_\alpha\}$ be any bounded monotonically increasing net of positive elements of R such that $a_\alpha \to a$ weakly. Then a continuous linear functional $\sigma \in L(R,\mathbb{C})$ is called normal if

$\lim_\alpha \sigma(a_\alpha) = \sigma(a)$. The normal linear functionals are denoted $L_*(R, \mathbb{C})$. The space of normal linear functionals on a von Neumann algebra R is also denoted R_* and called the pre-dual of R [9]. The pre-dual R_* is, conceptually, the space of non-commutative finite regular Radon measures on R.

Von Neumann algebras with separable pre-dual

Theorem 1.5.2 *Let R be a von Neumann algebra on a separable Hilbert space. Then there exists a separable C*-algebra A such that*

(i) *$\pi_\sigma(A)'' \cong R$, where (H_σ, π_σ) is the separable representation associated to a faithful positive linear functional $\sigma \in L^+(A, \mathbb{C})$,*

(ii) *R is isomorphic as a Banach space to the second dual A^{**},*

(iii) *the pre-dual $R_* \cong A^*$ is separable.*

Proof (i) The proof of [Ped, Prop 3.8.4] (ii) [Ped, Prop 3.7.8] (iii) [Ped, Thm 3.9.8] □

The statements (i)-(iii) of Theorem 1.5.2 are equivalent and provide the form of all von Neumann algebras with separable pre-dual.

Let A be a separable C*-algebra. We recall the structure spaces $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ of A, and that $PS(A)$ is a complete second countable metrisable Hausdorff topological space given the weak*-topology, see Theorem 1.2.1. We are not interested in generalised measure theory per se, but the 'non-commutative measure space(s)' arising from a 'second countable metrisable non-commutative topological space' $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$. Theorem 1.5.2 hence restricts our study to von Neumann algebras with separable pre-dual. Let (H, π) be a separable representation of A. We shall call $\pi(A)''$ the von Neumann envelope of A associated to (H, π) [10].

Remark 1.5.3 The fundamental relation $PS(A) \leftrightarrow \mathrm{Irr}(A)$ identifies $PS(A)$ as non-commutative 'point' measures for the 'points' given by the irreducible representations of the algebra. The pre-dual of the von Neumann envelope is generated by the closure of linear combinations of 'point' measures in the weak*-topology. This is precisely the case on $\mathbb{R}$, where the Dirac measures ρ_x for $x \in \mathbb{R}$ generate the finite Radon measures on $L^\infty(\mathbb{R}, \mu)$ [RS, variant problem 41, IV].

Relevant examples of Von Neumann algebras

Example 1.5.4

[9] Let B be a Banach space. The pre-dual B_* is the unique closed subspace of the dual B^* such that B is the dual of B_*. A Banach space is called reflexive if $B_* = B^*$. The spaces we deal with are not reflexive in general. For example, $L^\infty(\mathbb{R}, \mu)_* = L^1(\mathbb{R}, \mu) \neq L^\infty(\mathbb{R}, \mu)^*$. For a von Neumann algebra one identifies the normal linear functionals with the pre-dual [Ped, Cor 3.5.6, Thm 3.6.4].

[10] A von Neumann algebra with separable pre-dual will suffice as non-commutative measure theory for us. However, there are deeper reasons why hyperfinite von Neumann algebras are considered regular measure theory. Connes' early work completes the classification of hyperfinite von Neumann algebras [C1]. Amenable ($\equiv$ hyperfinite) von Neumann algebras arise by replacing separable C*-algebras in Theorem 1.5.2 with separable nuclear C*-algebras [EC].

a. Commutative von Neumann Envelopes

Let A be a commutative C*-algebra. Hence $A = C_0(\Sigma)$ for a locally compact Hausdorff space Σ [Gelfand Theorem, Thm 1.2.12]. The space of positive linear functionals on $C_0(\Sigma)$ are the finite regular Radon measures [RS, IV.18]

Theorem 1.5.5 (Riesz-Markov Theorem) *Let Σ be a locally compact Hausdorff space. Then every $\rho \in L^+(C_0(\Sigma), \mathbb{C})$ is of the form $\rho_\mu(f) = \int f d\mu$ for some finite regular Borel measure μ.*

Denote by $M(\Sigma)$ the space of complex finite regular Borel measures on Σ.

Theorem 1.5.6 *Let A be a commutative C*-algebra with spectrum Σ. Then*

(i) $M(\Sigma) \cong L(A, \mathbb{C}) = A^*$,

(ii) *the representation by multiplication operators*

$$\pi_\mu : A \to L^2(\Sigma, \mu),$$

provides the GNS representation associated to ρ_μ for $\mu \in M(\Sigma)$,

(iii) $\pi_\mu(A)'' = L^\infty(\Sigma, \mu)$ *for* $\mu \in M(\Sigma)$.

Proof (i) Riesz-Markov (ii),(iii) [Ped, 3.4.1, 3.4.4, 3.4.5] □

b. Essentially bounded sections

Let X be a Riemannian manifold and $p : E \to X$ be a Hermitian vector bundle. We recall from section 1.3.2 we have

(i) the Hilbert space $L^2(X, E)$ with inner product

$$\langle \sigma_1, \sigma_2 \rangle = \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_x \sqrt{\det g} dx \;\; \forall \sigma_1, \sigma_2 \in L^2(X, E)$$

(ii) the C*-algebra $C_0(X, L(E, E))$ with norm

$$\|T\| = \sup_{x \in X} \|T(x)\|_x$$

and representation $(L^2(X, E), \lambda)$ where

$$(\lambda(T)\sigma)(x) = T(x)(\sigma(x)) \;\; \forall T \in C_0(X, L(E, E)), \sigma \in B(L^2(X, E)).$$

Let ξ denote the Lebesgue measure of X, see Section 1.1.3. We define an essentially bounded sections of endomorphisms on E as a sections S taking values in $L(E, E)$ such that ess-sup $\|S(x)\|_x < \infty$. We denote the space of essentially bounded sections of endomorphisms on E by $L^\infty(X, L(E, E))$.

Proposition 1.5.7 $\lambda(C_0(X, L(E, E)))'' = L^\infty(X, L(E, E))$.

Proof For convenience let $R = L^\infty(X, L(E, E))$, $C = C_0(X, L(E, E))$ and $H = L^2(X, E)$. Both R and C act faithfully on the separable Hilbert space H

by left multiplication operators. Let us consider the commutant of R and C. As $L(E_x, E_x)' = Z(L(E_x, E_x)) = \mathbb{C}$ then $R' = C' \cong \pi_l(L^\infty(X,\xi))' \otimes \mathrm{id}$ where $\mathrm{id} : x \to \mathrm{id}_x$ and $c : L^\infty(X,\xi) \to B(L^2(X,\xi))$. Now $(\pi_l(L^\infty(X,\xi))' \otimes \mathrm{id})' = R$ as $\pi_l(L^\infty(X,\xi))$ is weakly closed. Hence $R = R'' = C''$. By von Neumann's bi-commutant theorem R is weakly closed [Ped, 2.2] [vN]. □

Let V be a choice of Hilbert space such that $V \cong E_x$. Let A be a C*-algebra that admits a representation (V, π). Then for each $x \in X$ there exists the representation (E_x, π_x) where $\pi_x : A \to L(E_x, E_x)$. We define a C*-subbundle $A(E)$ of the vector bundle $L(E,E)$ to be the vector subbundle of $X \to L(E,E)$ defined by the map $p_1 : \pi_x(A) \mapsto x$. Let $A(E)$ be a C*-subbundle of $L(E,E)$. Then $A''(E)$ denotes the associated C*-subbundle of $L(E,E)$ defined by the map $p_2 : \pi_x(A)'' \mapsto x$.

Corollary 1.5.8 *Let $A(E)$ be a C*-subbundle of $L(E,E)$. Then $\lambda(C_0(X, A(E)))'' = L^\infty(X, A''(E))$.*

Note $A''(E) = A(E)$ when V is finite dimensional. This follows as the weak and uniform topologies agree on the finite dimensional representation (V, π) of the C*-algebra A.

Weights on von Neumann algebras

Let A be a separable C*-algebra with separable representation (H, π). Let $\Sigma_\eta(A); = PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$. Define the 'volume' of $\Sigma_\eta(A)$ with respect to the 'measure' $\sigma \in \pi(A)''_*$ by

$$\mathrm{Vol}_\sigma(\Sigma_\eta(A)) := \sigma(1).$$

Elements of the pre-dual $\pi(A)''_*$ are considered 'finite measures' as $\sigma(1) < \infty$. This occurs for any linear functional. To generalise to 'non-finite measures' we introduce the notion of a weight on the von Neumann envelope $\pi(A)''$.

Definition-Lemma 1.5.9 [Ped, 5.1.1,5.1.2]

Let R be a C-algebra.*

(i) *A weight ρ on R is an additive form $\rho : R^+ \to [0, \infty]$.*

(ii) *A trace weight is a weight ρ such that $\rho(a^*a) = \rho(aa^*)$ for all $a \in R$.*

(iii) *The positive support of ρ, $R_\rho^+ = \{a \in R^+ \mid \rho(a) < \infty\}$, is a hereditary subspace of R^+.*

(iv) *The support of ρ, $R_\rho = \mathrm{Span}_{\mathbb{C}}(R_\rho^+)$, is a two-sided ideal of R.*

(v) *A weight ρ on a von Neumann algebra R is called*

  (a) *faithful if $\rho(a^*a) = 0 \Rightarrow a = 0$,*

  (b) *semifinite if R_ρ is σ-weakly dense in R,*

  (c) *normal if there is a set $\{\rho_\alpha\} \subset R_*$ such that $\rho(a) = \sup_\alpha \rho_\alpha(a)$ for all $a \in R_\rho$.*

(vi) *A weight ρ is a positive linear form if and only if $1 \in R_\rho$.*

The GNS construction can be performed for semifinite weights, exactly as in section 1.2.2, replacing the C*-algebra R by the σ-weakly dense two-sided ideal R_ρ [Ped, 5.1.6]. We will denote faithful normal semifinite by the acronym fns.

Example 1.5.10 Lebesgue Integration on $\mathbb{R}$ and Absolute Continuity

Let ξ denote the Lebesgue measure on $\mathbb{R}$. Let $C_0(\mathbb{R})$ be the C*-algebra of vanishing at infinity complex valued functions with representation $(L^2(\mathbb{R},\xi),\pi_l)$ given by multiplication operators $\pi_l : C_0(\mathbb{R}) \to B(L^2(\mathbb{R},\xi))$ as in Example 1.3.2. Let $C_0^+(\mathbb{R})$ denote the non-negative continuous functions. Define the Lebesgue integral of a non-negative continuous function f by

$$\lambda(f) := \int_{\mathbb{R}} f(x)d\xi(x).$$

Proposition 1.5.11

(i) *The integral λ is a faithful normal semifinite weight on the von Neumann algebra $L^\infty(\mathbb{R},\xi)$.*

(ii) *The representation $(L^2(\mathbb{R},\xi),\pi_l)$ of $L^\infty(\mathbb{R},\xi)$ is the GNS representation associated to the integral λ.*

Proof (i) By Theorem 1.5.6 $\pi_l(C_0(\mathbb{R}))'' = L^\infty(\mathbb{R},\xi)$. Hence $L^\infty(\mathbb{R},\xi)$ is a von Neumann algebra. Let $R = L^\infty(\mathbb{R},\xi)$ and $f : \mathbb{R} \to [0,\infty)$ be essentially bounded. Faithfulness of λ follows from $\lambda(f) = 0$ iff $f = 0$ ξ-a.e. The support $R_\lambda = \{f \in L^\infty(\mathbb{R},\xi) \mid \lambda(|f|) < \infty\} = L^1(\mathbb{R},\xi) \cap L^\infty(\mathbb{R},\xi)$ is strong dense in R, hence σ-weak dense. For normality, let $\chi_E(x)$ be the characteristic function for a Borel set $E \in \mathbb{R}$. Define

$$\lambda_n(f) = \int_{\mathbb{R}} \chi_{[-n,n]}(x)f(x)dx = \int_{-n}^{n} f(x)dx,$$

which is finite for any non-negative essentially bounded function. Hence λ_n is a positive linear functional. Let $f_\alpha \to f$ be an bounded increasing net of positive functions in $L^\infty(\mathbb{R},\xi)$ that converge weakly to f. Then $\chi_{[-n,n]}f_\alpha(x) \to \chi_{[-n,n]}f(x)$ ξ-a.e. for fixed n. Hence the Lebesgue Dominated Convergence Theorem implies λ_n is normal. As $\lambda(f) = \sup_n \lambda_n(f)$ then λ is a normal weight. (ii) The support of λ is $R_\lambda = L^1(\mathbb{R},\xi) \cap L^\infty(\mathbb{R},\xi)$. For $f,g \in R_\lambda$, $\lambda(f^*g) = \int_{\mathbb{R}} \overline{f(x)}g(x)d\xi$. Hence $\overline{R_\lambda} = L^2(\mathbb{R},\xi)$ and the associated representation is multiplication functions. Apply Theorem 1.5.6(ii). □

Let μ be a Borel measure on $\mathbb{R}$ and

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}} + \mu_{\mathrm{pp}}$$

be the Lebesgue decomposition of μ [11]. The Hilbert space $L^2(\mathbb{R},\mu)$ decomposes as the summand

$$L^2(\mathbb{R},\mu) = L^2(\mathbb{R},\mu_{\mathrm{ac}}) \oplus L^2(\mathbb{R},\mu_{\mathrm{s}}) \oplus L^2(\mathbb{R},\mu_{\mathrm{pp}}).$$

[11] Any Borel measure μ on $\mathbb{R}$ decomposes as $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}} + \mu_{\mathrm{pp}}$ where μ_{ac} is absolutely continuous with respect to ξ, μ_{s} is singular with respect to ξ but points are null sets, and μ_{pp} is a pure point measure [RS, Lebesgue Decomposition Theorem, I.14].

Hence the GNS representation π_μ of $C_0(\mathbb{R})$ given by Theorem 1.5.6 decomposes into

$$\pi_\mu = \pi_{\mu_{\mathrm{ac}}} \oplus \pi_{\mu_{\mathrm{s}}} \oplus \pi_{\mu_{\mathrm{pp}}}.$$

Definition 1.5.12 *Let A be a C^*-algebra with representation (H, π). The representation (H, π) is called an absolutely continuous representation if for all self-adjoint elements a of A the Borel set $\mathrm{sp}(\pi(a)) \subset \mathbb{R}$ is a null set for all singular and pure point Borel measures on $\mathbb{R}$.*

Theorem 1.5.6 implies for any Borel measure μ on $\mathbb{R}$ there exists the GNS representation $(L^2(\mathbb{R}, \mu), \pi_\mu)$ of $C_0(\mathbb{R})$. In Proposition 1.5.11 the Lebesgue measure ξ distinguished the representation $(L^2(\mathbb{R}, \xi), \pi_l)$ as the GNS representation arising from the Lebesgue integral. In the reverse direction, what identifies the Lebesgue measure ξ among GNS representations of $C_0(\mathbb{R})$?

Theorem 1.5.13 (Absolute Continuity) *Let $C_0(\mathbb{R})$ be the commutative C^*-algebra of continuous vanishing at infinity functions on $\mathbb{R}$. Up to equivalence of measures there exists a unique Borel measure ξ on $\mathbb{R}$ such that the representation by multiplication operators*

$$\pi_l : C_0(\mathbb{R}) \to B(L^2(\mathbb{R}, \xi))$$

provides a faithful absolutely continuous GNS representation $(L^2(\mathbb{R}, \xi), \pi_l)$ of $C_0(\mathbb{R})$.

Proof By Theorem 1.5.6 every GNS representation of $C_0(\mathbb{R})$ is of the form $(L^2(\mathbb{R}, \mu), \pi_\mu)$ for some regular Borel measure μ. By the Lebesgue decomposition theorem there exists elements of the C*-algebra $\pi_\mu(C_0(\mathbb{R}))$ that have singular spectrums with respect to Lebesgue measure, unless $\pi_{\mu_{\mathrm{s}}} \oplus \pi_{\mu_{\mathrm{pp}}} = 0$. Hence μ is absolutely continuous. If μ is absolutely continuous but inequivalent to ξ, there exists some open set O such that $\mu(O) = 0$. Hence $\pi_\mu(f) = 0$ for any $f \in C_c^+(\mathbb{R})$ with support in O. Then π_μ is not faithful. Hence μ must be equivalent to ξ. □

The results of this example can be extended to the Lebesgue measure on $\mathbb{R}^n$, subsequently to Lebesgue measure on a Riemannian manifold X. In Example 1.6.10 we discuss a similar result to Proposition 1.5.11 for Hermitian vector bundles over X.

1.5.3 Remark - Structure of C_c^∞-representations

Disintegration of Representations

Let A be a C*-algebra. In section 1.2.2 we reviewed the decomposition theory of a C*-algebra A over its structure space $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$. In particular, we found the universal representation of the C*-algebra A disintegrated into a direct sum of irreducible GNS representations (H_ρ, π_ρ) associated to the pure states $\rho \in PS(A)$. We describe below, following [Ped] and [Dix], the more sophisticated situation of the disintegration of *any* non-degenerate separable representation (H, π) of a separable C*-algebra A.

Let A be a C*-algebra. Denote by A'' the von Neumann envelope associated to the universal representation (H_U, π_U) given by Theorem 1.2.6. We call A'' the universal von Neumann envelope of A. We recall the statement of the Dauns-Hofmann Theorem [Ped, 3.12]: Let A be a C*-algebra and $M(A)$ the multiplier algebra of A. Then there exists an isomorphism $f : a \mapsto f_a$ from $Z(M(A))$ to the bounded continuous functions on $\hat{A}$ [Ped, 4.4.7,4.4.8]. As $Z(A)'' = Z(M(A))'' = Z(M(A))^{**}$ there exists an isomorphism $f : Z(A)'' \to C_b(\hat{A})^{**}$, and $Z(A)''$ corresponds to some class of functions on $\hat{A}$. We define a D-Borel subset $F \subset \hat{A}$ as the support of a characteristic function f_a where $a \in Z(A)''$ is a projection. Hence we have a bijective correspondence between central projections of A'' and the D-Borel structure on $\hat{A}$. We will denote a central projection by $p_F \in Z(A)''$. Let $\rho \in L_*^+(A'', \mathbb{C})$ and define the central measure $\mu_\rho(F) := \rho(p_F)$ on $\hat{A}$.

Theorem 1.5.14 *Let A be a separable C*-algebra and A'' the universal von Neumann envelope. Then*

(i) *for each $\rho \in L_*^+(A'', \mathbb{C})$ there is an isomorphism between $L^\infty(\hat{A}, \mu_\rho)$ and $Z(\pi_\rho(A))''$.*

(ii) *two separable representations of A are unitarily equivalent if and only if they have the same D-Borel null sets in $\hat{A}$,*

(iii) *there exists a bijective correspondence between classes of separable representations and central projections. In particular $(H, \pi) \sim_u (p_F H_U, p_F \pi_U p_F)$ for some support $F \in \hat{A}$.*

Proof (i) [Ped, Prop 4.7.6], (ii) [Ped, Thm 4.7.10] (iii) [Ped, Thm 3.8.2] □

Let A be a separable C*-algebra. We define the support of $\rho \in L_*^+(A'', \mathbb{C})$, or indeed any semifinite weight ρ on A'', as the central projection associated to the support of the equivalence class of the GNS representation (H_ρ, π_ρ). The previous theorem classifies the equivalence classes of separable representations in term of the D-Borel structure on $\hat{A}$. However, unless A is postliminal [Dix], then the D-Borel structure on $\hat{A}$ is insufficient to disintegrate the class of a representation (H, π) in terms of the support $F \subset \hat{A}$, see [Ped, 4.8.1]. The larger space required is called the factor or quasi-spectrum.

Let A be a separable C*-algebra. A factor representation (H, π) of A is a representation such that $Z(\pi(A)) = \mathbb{C}$. Define the quasi-spectrum $\check{A}$ to be the space of unitary equivalence classes of factor representation of A. Each irreducible representation is a factor representation, hence $\hat{A} \subset \check{A}$. Using an argument similar to above there exists an isomorphism between $Z(A)''$ and a class of functions on $\check{A}$. The D-Borel structure on $\check{A}$ is defined such that the characteristic function of a D-Borel set corresponds to a projection in $Z(A)''$. As above associate a central measure $\mu_\rho(F) := \rho(p_F)$ for a D-Borel set $F \subset \check{A}$ to each $\rho \in L_*^+(A'', \mathbb{C})$.

We recall from [Ped, 4.11] and [Dix, 8.1] the concept of a Borel field of Hilbert spaces $\{H_t\}_{t \in T}$ over a Borel space T and the direct integral Hilbert space $\int_T^\oplus H_t d\mu(t)$ for a bounded Borel measure μ on T. Let $\mathbb{C}_t$ denote the scalars of the Hilbert space H_t. A diagonalisable operator Y is an operator of the form $\int_T^\oplus \lambda(t) d\mu(t)$ where $\lambda(t) \in \mathbb{C}_t$ for μ-almost all t. Let $\mathcal{Y}$ denote the set of diagonalisable operators. An operator

$C \in B(\int_T^{\oplus} H_t d\mu(t))$ is called decomposable if $C = \int_T^{\oplus} C(t)d\mu(t)$ where $C(t) \in B(H_t)$ for μ-almost all t. Equivalently $C \in \mathcal{Y}'$ [Dix, (A80)]. An unbounded selfadjoint operator $S : DomS \to B(\int_T^{\oplus} H_t d\mu(t))$ is called decomposable if S is affiliated to $\mathcal{Y}'$.

Theorem 1.5.15 *Let (H,π) be a non-degenerate separable representation of a separable C^*-algebra A. Then there exists a central measure μ on $\check{A}$, a D-Borel subset $F_\pi \subset \check{A}$ and a D-Borel field of factor representations $\{(H_t,\pi_t)\}_{t\in\check{A}}$ such that*

$$H \sim_u \int_{F_\pi}^{\oplus} H_t d\mu(t) \ , \ \pi \sim_u \int_{F_\pi}^{\oplus} \pi_t d\mu(t).$$

Proof [Ped, Thm 4.12.4]. □

Disintegration of Base Representations

Let (H,π,D) be a C_c^∞-representation of a separable C*-algebra A as in Definition 1.4.8. The kind of disintegration in Theorem 1.5.15 cannot be performed in general for (H,π,D). The simple obstruction is the decomposition of D with respect to the D-Borel space (F_π,μ). Here (F_π,μ) is the pair associated to the representation (H,π) as in Theorem 1.5.15. There exist two extreme cases of the possible decomposition of D with respect to (F_π,μ), (1) it is completely indecomposable, or (2) it is decomposable.

Definition 1.5.16 *Let (H,π,D) be a base representation of a C^*-algebra A as in Definition 1.4.3. Then (H,π,D) is called base-irreducible if $[D,\pi(p)] \neq 0$ for any proper central projection $p \in Z(A)''$.*

Let (H,π,D) be a base representation of a separable C*-algebra A. Let (F_π,μ) be the Borel space corresponding to the representation (H,π) of A. As a consequence of base-irreducibility D is not decomposable for any pair $(F,\nu) \subset (F_\pi,\mu)$ where $\nu << \mu$. Hence base-irreducibility corresponds to the case (1).

Proposition 1.5.17 *Let (H,π,D) be a C_c^∞-representation of a separable C^*-algebra A and (F_π,μ) be the corresponding D-Borel space given by Theorem 1.5.15. The following statements are equivalent*

(i) $[D,\pi(p)] = 0$ *for all central projections $p \in Z(A)''$,*

(ii) *there exists a D-Borel field of factor C_c^∞-representations $\{(H_t,\pi_t,D_t)\}_{t\in\check{A}}$ such that*

$$H \sim_u \int_{F_\pi}^{\oplus} H_t d\mu(t) \ , \ \pi \sim_u \int_{F_\pi}^{\oplus} \pi_t d\mu(t) \ , \ D \sim_u \int_{F_\pi}^{\oplus} D_t d\mu(t).$$

Proof (i) $\Leftrightarrow$ (ii) Both statements are equivalent to the statement D is decomposable with respect to (F_π,μ). □

Spectral Representations

Let (H, π, D) be a C_c^∞-representation of a C*-algebra A.

Lemma 1.5.18 *Let (H, π, D) be a C_c^∞-representation of a C*-algebra A. Then any representative (H', π') of the class $[(H, \pi)]_u$ provides a C_c^∞-representation (H', π', D') of the C*-algebra A. In particular,*

$$(H', \pi', D') = (UH, U\pi U^*, UDU^*)$$

where $U : H \to H'$ is unitary.

Proof Define $DomD' := UDomD$, then $D' : DomD' \to H'$ is an unbounded selfadjoint operator and $|D'| = U|D|U^*$. The completion of the proof is straightforward. □

Lemma 1.5.18 implies that C_c^∞-representation is a property of the unitary equivalence class $[(H, \pi)]_u$. This point of view leads to several points to consider:

(i) Let (H, π) be a separable representation of a separable C*-algebra A. Define the outer C_c^∞-basespace of the unitary equivalence class $[(H, \pi)]_u$,

$$\mathcal{D}([(H, \pi)]_u) := \{D \in C(H) \setminus B(H) \mid (H, \pi, D) \text{ is a } C_c^\infty\text{-representation of } A.\}$$

Note $\mathcal{D}([(H, \pi)]_u)$ is defined using a fixed representative of $[(H, \pi)]_u$, but is independent of which representative is chosen. The study of $\mathcal{D}([(H, \pi)]_u)$ is of central interest in non-commutative geometry. Considerations include $\mathcal{D}([(H, \pi)]_u) \neq \emptyset$ (existence of C_c^∞-representations where D is unbounded), unitary equivalence classes in $\mathcal{D}([(H, \pi)]_u)$ (gauge transformations), and topologies and extremal points of $\mathcal{D}([(H, \pi)]_u)$.

(ii) We have observed C_c^∞-representation is a property of the class $[(H, \pi)]_u$ parametrised by the space $\mathcal{D}([(H, \pi)]_u)$. The natural question of a canonical representative in $[(H, \pi)]_u$ for $D \in \mathcal{D}([(H, \pi)]_u)$ arises. The spectral representation is an immediate candidate. We recall the statement of the Spectral Theorem for selfadjoint operators. There exists a measure space (M, μ) and a unitary $U_D : H \to L^2(M, \mu)$ such that $U_D D U_D^* = \pi_l(p)$ where $p : M \to \mathbb{R}$ is a measurable real-valued function on the measure space (M, μ) and $(\pi_l(p)g)(m) = p(m)g(m) \;\; \forall\, g \in L^2(M, \mu)$ is the usual representation by left multiplication. Let $H_D = U_D H$ and $\pi_D = U_D \pi U_D^*$. Then by Lemma 1.5.18,

$$(H, \pi, D) \sim_u (H_D, \pi_D, \pi_l(p)).$$

Hence $(H_D, \pi_D) \in [(H, \pi)]_u$ is a canonical representative for $D \in \mathcal{D}([(H, \pi)]_u)$.

1.6 Modular Theory and the Radon-Nikodym Theorem

Von Neumann algebras and normal semifinite weights have incredible structure theorems associated to them. We shall review the Tomita-Takesaki Modular Theory and the non-commutative Radon-Nikodym Theorem.

Basic Definitions

Let H be a Hilbert space and R be a *-subalgebra of $B(H)$. We recall a cyclic vector $\xi \in H$ for R is a vector such that $\overline{R\xi} = H$. A separating vector $\xi \in H$ for R is a vector $a\xi = 0 \Rightarrow a = 0$ for $a \in R$. A cyclic and separating vector $\xi \in H$ for R is a vector that is both cyclic and separating.

1.6.1 Modular Theory

Theorem 1.6.1 (Tomita-Takesaki) [Ha, Thm 1.6] [BR, Thm 2.7.14]

Let R be a von Neumann algebra. Then R is isomorphic to a von Neumann algebra in standard form $(\pi(R), H, J, \Delta, \mathcal{P})$. Here $\pi(R)$ is a von Neumann algebra on the Hilbert space H which admits the following structures

(i) (modular conjugation) *a conjugate linear isometric involution $J : H \to H$,*

(ii) (modular operator) *a positive operator $\Delta : Dom\Delta \to H$,*

(iii) (positive cone) *a self-dual cone $\mathcal{P}$ in H,*

with the properties

(iv) (symmetry) $J\pi(R)J = \pi(R)'$ *and* $J\pi(a)J = \pi(a)^*$ *iff* $a \in Z(R)$,

(v) (modular automorphism) $\Delta^{it}\pi(R)\Delta^{-it} = \pi(R)$ *for* $t \in \mathbb{R}$,

(vi) (reality) $J\eta = \eta$ *for all* $\eta \in \mathcal{P}$.

The information $(\pi(R), H, J, \Delta, \mathcal{P})$ is called a standard form of R. There is an associated standard form to every fns weight on a von Neumann algebra R.

Theorem 1.6.2 *Let R be a von Neumann algebra with fns weight ρ. Then there exists an associated standard form $(\pi_\rho(R), H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ where (H_ρ, π_ρ) is the GNS representation associated to ρ. Let ι_ρ be the dense injection $\iota_\rho : R_\rho \to H_\rho$ given by the GNS construction. Then we have the further properties,*

(i) $\mathcal{P}_\rho = \overline{\{\pi_\rho(a)J_\rho\iota_\rho(a) \mid a \in R_\rho\}} = \overline{\Delta_\rho^{1/4}\iota_\rho(R_\rho^+)}$

(ii) $\iota_\rho(R_\rho) \subset Dom\Delta_\rho$ *and* $J_\rho\Delta_\rho^{1/2} : \iota_\rho(a) \to \iota_\rho(a^*)$ *for* $a \in R_\rho$

(iii) *if ρ is a trace then* $\Delta_\rho = 1$

(iv) *if $1 \in R_\rho$ then $\iota_\rho(1) \in H_\rho$ is a separating and cyclic vector.*

Proof The closure of R_ρ in the inner product $\langle\cdot,\cdot\rangle_\rho$ provides an achieved left Hilbert algebra H_ρ. Hence, by the original Tomita-Takesaki theory, H_ρ admits a standard form. For the properties (i)-(iv), see [BR] or [Ha]. □

The one-parameter family of automorphisms of R

$$\sigma_t^\rho(a) := \Delta_\rho^{it}\pi(a)\Delta_\rho^{-it} \quad \forall a \in R \;\; t \in \mathbb{R}$$

is called the modular automorphism group associated to ρ. The modular automorphism group is fundamental in Connes' classification of hyperfinite factors [C1]. It has deep physical consequences, such as links to KMS states [Tk, Wn] and time flow in thermodynamic systems [C11].

1.6.2 Generalised Radon-Nikodym Theorems

Let R be a von Neumann algebra with fns weight ρ. Then there exists an associated standard form $(\pi_\rho(R), H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ where (H_ρ, π_ρ) is the GNS representation associated to ρ. Elements of the positive cone $\mathcal{P}_\rho$ are considered the generalised 'positive' L^2-functions associated to the fns weight ρ [Ha, Lemma 2.2],

Theorem 1.6.3 (Radon-Nikodym Theorem, positive cone version) *Let $(\pi(R), H, J, \Delta, \mathcal{P})$ be a standard form. The map $\eta \to \langle \eta, \cdot \eta \rangle$ is a homeomorphism of $\mathcal{P}$ onto R_*^+, the normal positive linear functionals.*

The Radon-Nikodym theorem induces (unique) unitary equivalence of standard forms [Ha, Thm 2.3],

Theorem 1.6.4 (Radon-Nikodym Theorem, standard form version) *Let $(R, H, J, \Delta, \mathcal{P})$ and $(\phi(R), H', J', \Delta', \mathcal{P}')$ be two standard forms where ϕ is an isomorphism. Then there exists a unique unitary $U : H \to H'$ such that $\pi'(\phi(a)) = U\pi(a)U^*$ for all $a \in R$, $J' = UJU^*$, $\Delta = U\Delta' U^*$ and $\mathcal{P}' = U\mathcal{P}$.*

Connes' Radon-Nikodym Theorem associates the modular automorphism groups arising from fns weights (stated as appears [C1]),

Theorem 1.6.5 (Connes' Radon-Nikodym Theorem, cocycle version) *Let R be a von Neumann algebra and $U(R)$ the unitary group of R equipped with the σ-weak topology. Let ρ be a fns weights on R. Then*

(i) *for each fns weight τ there exists a unique continuous map $u : \mathbb{R} \to U(R)$ such that*

$$u_{s+t} = (\sigma_t^\rho(u_s))u_t \qquad \forall\, s, t \in \mathbb{R},$$

$$\sigma_t^\tau(a) = u_t \sigma_t^\rho(a) u_t^* \ \ \forall\, t \in \mathbb{R}, a \in R$$

and

$$\tau(a) = \rho(u_{i/2} a u_{i/2}^*) \qquad \forall\, a \in R.$$

(ii) *for each continuous map $u : \mathbb{R} \to U(R)$ such that*

$$u_{s+t} = (\sigma_t^\rho(u_s))u_t \ \ \forall\, s, t \in \mathbb{R}$$

there exists a unique fns weight τ with the properties of (i).

Remark 1.6.6 The operator $u_{i/2}$ of Theorem 1.6.5(i) is ambiguous as stated. Let f_n be a compactly supported approximate unit of $C_b(\mathbb{C})$. Then $f_n(u_1)^{i/2}$ and $f_n(u_1^*)^{i/2} = (f_n(u_1)^{i/2})^*$ are normal. We then realise the operator $u_{i/2} = \lim_n f_n(u_1)^{i/2}$ as a strong resolvent limit in any faithful representation (H, π) of R [12]. The equality stated in the theorem is shorthand for

$$\tau(a) = \lim_n \rho(f_n(u_{i/2}) a f_n(u_{i/2})^*).$$

[12]Every normal (bounded) operator on $B(H)$ where H is separable can be written $a = b + ic$ where b, c are selfadjoint and commute. Hence the spectral representations for b and c coincide and there exists a unique operator $g(a)$, a complex measure space $L^2(M, \mu)$ and unitary $U : H \to L^2(M, \mu)$

The Radon-Nikodym derivatives in the standard form version and cocycle version of the Radon-Nikodym Theorem correspond to equivalences of measures. Let ρ and τ be normal semifinite weights on a von Neumann algebra R. We denote $\tau << \rho$ if $\tau(a) = 0 \Rightarrow \rho(a) = 0$ for $a \geq 0$. We say $\tau \equiv \rho$ if $\tau << \rho$ and $\rho << \tau$.

Corollary 1.6.7 (Radon-Nikodym Theorem, derivative version)

Let ρ, τ be normal semifinite weights of a von Neumann algebra R.

(i) *Let $\tau << \rho$. Then there exists an operator μ affiliated to R such that $\tau(a) = \rho(\mu a \mu^*)$.*

(ii) *Let ρ be a normal semifinite trace-weight and $\tau << \rho$. Then there exists a positive operator $\mu^*\mu = (\tau : \rho)$ affiliated with R such that $\tau(a) = \rho((\tau : \rho)\, a)$.*

Proof (i) Let p_τ and p_ρ be the central support projections for τ and ρ [Ped, Thm 3.8.2], see Section 1.5.3. As $\tau << \rho$ then $p_\tau p_\rho = p_\tau$. Define $\rho_\tau(\cdot) := \rho(p_\tau \cdot)$. Then ρ_τ and τ are fns weight on $p_\tau R$. One applies Connes' RN-Theorem to get unitaries $u_t \in p_\tau R$ and hence partial isometries $u_t p_\tau \in R$ with the results of Connes' RN-Theorem(i) for the fns weights ρ_τ and τ. In particular, the corollary is proven by setting $\mu = u_{i/2} p_\tau$. (ii) obvious from the tracial property and Remark 1.6.6. □

We note Corollary 1.6.7 (ii) is the original non-commutative Radon-Nikodym theorem, developed well before Tomita-Takesaki Theory and Connes' cocycle generalisation. For instance, see [Se2] or various formulations in [Ped, 5.3].

1.6.3 Modular Theory for von Neumann algebras with separable pre-dual

We introduced the general modular theory last section. For von Neumann algebras with separable pre-dual the theory can be formulated in terms of cyclic and separating vectors. This was the original exposition in the papers [C8] and [Ar]. A von Neumann algebra R is called countably generated if each set of pairwise orthogonal projections in R is countable.

Theorem 1.6.8 *The following are equivalent*

(i) *R is a von Neumann algebra with separable pre-dual,*

(ii) *R is a countably generated von Neumann algebra with a faithful normal state ν,* [13]

---

such that $(Ug(a)U^*f)(m) = g \circ F(m) f(m)$ where $g : \mathbb{C} \to \mathbb{C}$ is Borel and bounded and F is complex valued. When a is unbounded and $a^{**} = a$, then we say a is normal if the selfadjoint operators $b = a + a^*$ and $c = -i(a - a^*)$ commute. That is, they have the same spectral representations. Then a can be written $a = b + ic$ and the functional calculus can be defined on a using the spectral theory of b and c. An equivalent definition of affiliation of a selfadjoint operator m to a von Neumann algebra R is $f(m) \in R$ for all bounded Borel functions f. When a is normal and $a = b + ic$, then b and c affiliated to R implies a is affiliated to R in the same sense. It can be shown $u_{i/2}$ is normal and affiliated to R in the senses above.

[13] A state ν on a C*-algebra A is a positive linear functional with $||\nu|| = 1$. Alternatively, $\lim_\lambda \nu(u_\lambda) = 1$ for any approximate unit u_λ of A. States correspond to probability measures, since $1 = ||\nu|| =$ 'volume' of the non-commutative space $\Sigma_\eta(A)$.

(iii) *R is isomorphic to a von Neumann algebra $\pi(R)$ on a separable Hilbert space H that admits a cyclic and separating vector for R,*

Proof (i) $\Rightarrow$(ii) $\Rightarrow$(iii) $\Rightarrow$(i) is contained in [Ped, 3.8.4,3.9.9] □

Corollary 1.6.9 (Radon-Nikodym Theorem) *Let ν be a faithful normal state on a von Neumann algebra R with separable pre-dual. Let $(\pi(R), H, J, \Delta, \mathcal{P})$ be a standard form. Then $(\pi(R), H, J, \Delta, \mathcal{P})$ is unitary equivalent to a standard form $(\pi_\nu(R), H_\nu, J_\nu, \Delta_\nu, \mathcal{P}_\nu)$ with a cyclic and separating vector $\iota_\nu(1) \in \mathcal{P}_\nu$.*

Example 1.6.10 This example is a continuation of Example 1.5.4(b).

Proposition 1.6.11 *Let X be a Riemannian manifold with Lebesgue measure ξ. Let $p : E \to X$ be a Hermitian vector bundle over X. Let $A(E)$ be a C^*-subbundle of $L(E,E)$. Suppose there exists a section $\eta \in L^\infty(X,E)$ such that $\eta(x) \in E_x$ is a cyclic and separating vector for $\pi_x(A)''$. Then*

(i) *The weight*

$$I_{\eta,X}(\sigma) = \int_X \langle \eta(x), \sigma(x)\eta(x)\rangle_x d\xi$$

is a faithful semifinite normal weight on $L^\infty(X, A''(E))$.

(ii) *The faithful representation $(L^2(X,E), \pi_l)$ of $L^\infty(X, A''(E))$ as in Example 1.5.4(b)(ii) is the GNS representation associated to $I_{\eta,X}$.*

Proof (i) The form $I_{\eta,X}$ is linear. Let $\sigma \in L^\infty(X, A''(E))$. Since

$$\langle \eta(x), \sigma(x)^*\sigma(x)\eta(x)\rangle_x = \langle \sigma(x)\eta(x), \sigma(x)\eta(x)\rangle_x = \|\sigma(x)\eta(x)\|_{2,x}^2,$$

the form is positive. Moreover, the section η is separating, so $I_{\eta,X}(\sigma^*\sigma) = 0$ iff $\sigma = 0$ ξ-a.e. Hence $I_{\eta,X}$ is faithful. Let $\|\eta\| = K$. By the Cauchy-Schwartz inequality, $\langle \eta(x), \sigma(x)\eta(x)\rangle_x \leq K^2\|\sigma(x)\|_x$. Hence the support of $I_{\eta,X}$ contains $L^1(X, A''(E)) \cap L^\infty(X, A''(E))$, which is strong dense in $L^\infty(X, A''(E))$. Hence $I_{\eta,X}$ is semifinite. Let U_k be a countable set of compact subsets of X. Let χ_k be the characteristic function of the set U_k. An argument identical to the proof in Proposition 1.5.11 implies $I_{\eta,U_k}(\sigma) = I_{\eta,X}(\chi_k\sigma)$ is a positive normal linear functional for each k. Hence normality of $I_{\eta,X}$ follows from σ-compactness of the Riemannian manifold. Let $X = \cup_k U_k$ where $U_k \subset U_{k+1}$ are compact. Then $I_{\eta,X}(\sigma) = \sup I_{\eta,U_k}(\sigma)$ for non-negative sections σ. (ii) Let $R_I = L^1(X, A(E)) \cap L^\infty(X, A(E))$. Then the closure $\overline{R_I}$ in the inner product $I_{\eta,X}(\tau^*\sigma)$ is $L^2(X, \overline{A(E)\xi}) = L^2(X,E)$ by the cyclic property of ξ. By the GNS construction the representation is given by left multiplication operators. □

For any Hermitian bundle $E \to X$ and C*-subbundle $A(E)$ satisfying the hypothesis of Proposition 1.6.11, Theorem 1.6.2 provides a standard form associated to the fns weight $I_{\eta,X}$

$$(\pi_l(L^\infty(X, A''(E))), L^2(X,E), J_\eta, \Delta_\eta, \mathcal{P}_\eta).$$

Remark 1.6.12 (Riemannian Structure)

Let X be a n-dimensional Riemannian manifold. Let Cliff(V,q) be the Clifford algebra for a n-dimensional vector space V as in Section 1.3.1. The exterior bundle $\Lambda^*X \to X$ is the unique Hermitian bundle over X, up to isomorphism, that satisfies the conditions of Proposition 1.6.11 for the C*-algebra $A = \text{Cliff}(V,q)$. Consequently there exists a unique Hilbert space $L^2(X,\Lambda^*X)$, up to isomorphism, which admits a standard form for the von Neumann algebra $R = L^\infty(X, \text{Cl}(X,q_g))$,

$$\mathbf{Riem} = (L^\infty(X,\text{Cl}(X,q_g)), L^2(X,\Lambda^*X), J, 1, L^2(X,\Lambda^*X)^+).$$

Proposition 1.6.11 further implies the standard form **Riem** is constructed from R using the fns trace weight λ given by the Lebesgue integral and the metric q_g,

$$\lambda(w) = \int_X q_g(1,\pi_l(w)1)(x)\sqrt{\det(g)}dx \;\; \forall w \in L^\infty(X,\text{Cl}(X,q_g)).$$

The standard form **Riem**, and its construction from R using the fns weight λ, characterises the representation $(L^2(X,\Lambda^*X),\pi_l)$ amongst all other representations (H,π) of the C*-algebra $C_0(X)$.

1.7 Non-Commutative Integral Calculus

We have introduced C_c^∞-representations (H,π,D) of a C*-algebra A in the role of differential calculus, see Section 1.4.1 and Section 1.4.2. The representation (H,π) provides a generalised measure theory of semifinite weights on the von Neumann envelope $\pi(A)''$ as seen in Section 1.5 and Section 1.6. However, it is not apparent that a general semifinite weight constitutes an 'integral' in the calculus sense (a summation of 'infinitesimals distances' in the sense of Sections 1.1.1 and Section 1.1.2). A generalisation of infinitesimals and summation procedures on them is contained in the theory of what are termed symmetric functionals. Symmetric functionals are still an area of active research [DPSSS] [DPSSS2] [LSS].

Basic Definitions

Let $S \in B(H)$. The projection $p_S := 1 - P_{\ker S} = P_{\ker S^\perp}$ is called the support projection of S.

1.7.1 Symmetric Norm Ideals

Let $a \in K(H)$ and $\mu_n(a)_{n\in\mathbb{N}}$ denote the sequence of the singular values of a (the decreasing rearrangement of the eigenvalues of $|a|$). Let $\tilde{\phi}$ be a symmetric norm on ℓ^∞ [S]. Define $\phi(a) := \tilde{\phi}(\mu_1(a),\mu_2(a),\ldots)$, $I(\phi) := \{a \in K(H) \,|\, \phi(a) < \infty\}$, and $I_0(\phi) := \{a \in K(H) \,|\, \lim_m \phi(a - a_m) = 0 \text{ for some } \{a_m\}_{m\in\mathbb{N}} \in FR(H)\}$.

Theorem 1.7.1 [S, Theorem 2.7]

(i) *ϕ is a norm on $I(\phi)$ such that $\phi(ab\pi_l) \leq \|a\|\|c\|\phi(b)$ and $\phi(b) \geq \|b\|\tilde{\phi}(1,0,\ldots)$ for $b \in I(\phi)$ and $a,c \in B(H)$.*

(ii) *$I(\phi)$ and $I_0(\phi)$ are Banach spaces in the norm ϕ. For any $a \in I_0(\phi)$ its canonical decomposition as a compact operator converges in ϕ.*

(iii) **(Non-commutative Fatou Lemma)** *Let $a_m \in I(\phi), a_m \to a$ weakly and $\sup_m \phi(a_m) < \infty$. Then $a \in I(\phi)$ and $\phi(a) \leq \sup_m \phi(a_m)$*

For brevity we also call the norm ϕ on the two sided *-ideal $I(\phi)$ of $K(H)$ a symmetric norm. The symmetric norm ϕ is called regular if $I(\phi) = I_0(\phi)$.

Example 1.7.2 Let $a \in K(H)$ with singular values $\mu_n(a)_{n\in\mathbb{N}}$.

(i) The uniform operator norm

$$\phi(a) := \|a\| = \sup_i \mu_i(a)$$

is a regular symmetric norm [S]. The associated ideal is the compact operators themselves $K(H) \equiv I(\|.\|) = I_0(\|.\|)$.

(ii) The norm

$$\phi(a) := \|a\|_p = (\textstyle\sum_i \mu_i(a)^p)^{\frac{1}{p}}$$

is a regular symmetric norm [S]. The associated ideals, the Schatten ideals [S], are denoted $L_p := I(\|.\|_p) = I_0(\|.\|_p)$. If $a \in L_p$ then $|a|^p \in L_1$ and

$$\|a\|_p = Tr(|a|^p)^{1/p} \tag{1.1}$$

(iii) Define the sequences for $p \in [1, \infty)$

$$\alpha_p(a) := \left\{ f(p, N) \sum_{i=1}^{N} \mu_i(a) \right\}_{N\in\mathbb{N}} \tag{1.2}$$

where $f(1, N) := (\ln N)^{-1}$ and $f(p, N) := N^{-1+\frac{1}{p}}$ for $p > 1$. Then the norm

$$\phi(a) := \|a\|_{p,\infty} = \sup \alpha_p(a)$$

defines non-regular symmetric norms for all $p \in [1, \infty)$. The associated ideals, the weak ideals [S], are denoted by $L_{p,\infty} := I(\|.\|_{p,\infty})$ and $L^0_{p,\infty} := I_0(\|.\|_{p,\infty})$. The non-regularity of $\|.\|_{p,\infty}$ allows the definition of a distinct semi-norm on $L_{p,\infty}$

$$\rho_{p,\infty}(a) := \limsup \alpha_p(a).$$

The semi-norm $\rho_{p,\infty}$ vanishes exactly on $L^0_{p,\infty}$ and induces a norm on the factor space $L_{p,\infty}/L^0_{p,\infty}$.

Let ϕ be a symmetric norm. Neither weak operator convergence, strong operator convergence, nor uniform operator convergence of a sequence of compact operators $a_m \to a$ will guarantee $\phi(a_m - a) \to 0$ in general.

1.7.2 Symmetric Functionals

Let ϕ be a symmetric norm. A positive linear functional τ on a two-sided *-ideal $I(\phi)$ of $B(H)$ is called a hypertrace if $\tau(aT) = \tau(Ta)$ for all $a \in I(\phi)$ and $T \in B(H)$.

Definition 1.7.3 *We call a positive linear functional τ on $I(\phi)$ such that $|\tau(a)| \leq \phi(a)$ for all $a \in I(\phi)$ a symmetric functional. We call a symmetric functional on $I(\phi)$ that is a hypertrace a symmetric hypertrace.*

Example 1.7.4 **(a)** The canonical trace Tr is a symmetric hypertrace on L_1.
(b) Let ω be a dilation and translation invariant positive linear functional on ℓ^∞ such that $\omega(1) = 1$. Then

$$Tr_\omega(a) := \omega(\alpha_1(a))$$

defines a symmetric hypertrace on $L_{1,\infty}$ [C, LSS]. These symmetric hypertraces are called Dixmier traces after their discovery by J. Dixmier [Dix]. The symmetric hypertrace Tr_ω relates to the seminorm $\rho_{1,\infty}$ rather that the symmetric norm [LSS, Theorem 6.4]

$$\rho_{1,\infty}(a) = \sup_\omega Tr_\omega(|a|).$$

That $\rho_{1,\infty}(a)$ vanishes on $FR(H) \subset L_1 \subset L^0_{1,\infty}$ implies every Dixmier trace is a non-normal trace on the factor $B(H)$ and singular to the canonical trace Tr.

1.7.3 Symmetric Measures

Definition 1.7.5 *Let τ be a symmetric hypertrace associated to a symmetric norm ϕ. Let K be a fixed positive bounded operator with trivial kernel. Define the spaces*

$$I^{-1}(\tau, K) := \{S \in B(H) \mid SK, KS \in I(\phi)\}$$

and

$$I^{-c}(\tau, K) := \{S \in B(H) \mid p_{|S|} \in I^{-1}(\tau, K)\}$$

with norm

$$\phi_K(S) := \phi(SK).$$

We call the pair (τ, K) a symmetric measure. We say the symmetric measure (τ, K) is finite if $K \in I(\phi)$.

The linear functional on $I^{-1}(\tau, K)$

$$\tau_K(S) := \tau(SK)$$

is positive on $I^{-c}(\tau, K)$ and positive on $I^{-1}(\tau, K)$ when $I(\phi)$ a geometrically stable symmetric norm ideal [K1, K2] [14]. The Shatten ideals L_p and their weak variants $L_{p,\infty}$ are geometrically stable.

Definition 1.7.6 *Let (τ, K) be a symmetric measure. The linear functional τ_K on $I^{-1}(\tau, K)$ defined by $\tau_K(S) := \tau(SK)$ is called a weighted symmetric functional. A weighted symmetric functional τ_K that is a trace is called a weighted symmetric trace.*

[14]This result and results on the spaces $I^{-1}(\tau, K)$ and $I^{-c}(\tau, K)$ are discussed in the paper [LS]. They are not considered relevant here since the predominant situation in this thesis involves $K \in I(\phi)$, for which the results become greatly simplified.

Definition 1.7.7

(A) *Let A be a C^*-algebra. Let*

(i) (H, π, K) *be a base representation of A such that $K \geq 0$,*

(ii) (τ, K) *be a symmetric measure associated to a symmetric norm ϕ.*

(iii) $\pi(A_c) \subset I^{-1}(\tau, K)$ *for the ideal $A_c \subset A$ as in Definition 1.4.1.*

Then (H, π) is called a (τ, K)-integrable representation of the C^*-algebra A.

(B) *Let A be a C^*-algebra. Let*

(i) (H, π, D) *be a base representation of A,*

(ii) $f : \mathbb{R} \to (0, \infty)$ *be bounded and Borel measurable,*

(iii) $(\tau, f(D))$ *be a symmetric measure associated to a symmetric norm ϕ.*

(iv) $\pi(A_c) \subset I^{-1}(\tau, f(D))$ *for the ideal $A_c \subset A$ as in Definition 1.4.1.*

Then (H, π, D) is called a $(\tau, f(D))$-integrable base representation of the C^*-algebra A.

Corollary 1.7.8 *Let (H, π, D) be a $(\tau, f(D))$-integrable base representation of a separable C^*-algebra A. Then $\pi(A)''$ admits a semifinite weight ρ such that $\rho(\pi(a)) = \tau_{f(D)}(\pi(a))$ for all $a \geq 0$.*

Proof By definition τ is a positive linear functional in $I(\phi)$. Define $\rho(S) = \tau_{f(D)}(S)$ for $S \geq 0$. We prove $\rho : \pi(A)''_+ \to [0, \infty]$ and is additive. By Theorem 1.7.1(ii) $I(\phi)$ and $I_0(\phi)$ are Banach spaces. Hence they are geometrically stable by [K1, 3.2]. It follows from [K2, 2.6] that $\tau(S) = \tau(T)$ for any $S \in I(\phi)$ such that T has the same eigenvalues with multiplicity as S. As SK and $K^{1/2}SK^{1/2}$ have the same eigenvalues with multiplicity for S positive and K positive with trivial kernel, $\tau(Sf(D)) = \tau(f(D)^{1/2}Sf(D)^{1/2})$. Positivity and additivity of $\tau_{f(D)}$ now follows from positivity and additivity of τ.

Hence ρ is a positive linear functional on $\pi(A)'' \cap I^{-1}(\tau, K)$. Let $R_\rho = \pi(A)'' \cap I^{-1}(\tau, K)$. Norm density of A_c in A, from Theorem 1.4.2, implies $\pi(A_c)$ is σ-weak dense in $\pi(A)''$ by von Neumann's bi-commutant and density theorem. As $\pi(A_c) \subset \pi(A)'' \cap I^{-1}(\tau, K)$ by Definition 1.7.7(B)(iv), R_ρ is σ-weak dense in $\pi(A)''$. Hence ρ is semifinite. □

Definition 1.7.9 *Let (H, π, D) be a $(\tau, f(D))$-integrable representation of a separable C^*-algebra A. Then the support of the symmetric measure $(\tau, f(D))$ and the support of the weighted symmetric functional $\tau_{f(D)}$ shall identically mean the central support projection $p \in A''$ of the semifinite weight ρ on $\pi(A)''$ in Corollary 1.7.8.*

The next result displays the fundamental role of the canonical trace and the trace class operators in the theory of von Neumann algebras with separable pre-dual.

Theorem 1.7.10 (Characterisation of the pre-dual) [Ped, 3.6.4]

Let R be a von Neumann algebra on a separable Hilbert space H. Let $\rho \in R^*$. Then the following are equivalent:

(i) $\rho \in R_*$,

(ii) *ρ is weakly continuous on the unit ball of R,*

(iii) *ρ is σ-weakly continuous on R,*

(iv) *there is an operator $k_\rho \in L_1$ such that $\rho(a) = Tr(ak_\rho)$ for all $a \in R$.*

In particular, $k_\rho \geq 0$ if $\rho \in R_*^+$.

The result can be rephrased. Define

$$\mathrm{SM}_1 := \{(Tr, k) \mid k \geq 0, k \in L_1\}$$

$$\mathrm{SF}_1 := \{Tr_k \mid (Tr, k) \in \mathrm{SM}_1\}.$$

Corollary 1.7.11 *Let R be a von Neumann algebra on a separable Hilbert space H. Then $R_*^+ \subset \mathrm{SF}_1$.*

Corollary 1.7.12 *Let A be a separable C^*-algebra. Let $\sigma \in L^+(A, \mathbb{C})$. Then there exists a finite symmetric measure $(Tr, k) \in \mathrm{SM}_1$ such that*

(i) *the GNS representation (H, π) is (Tr, k)-integrable,*

(ii) $\sigma(a) = Tr_k(\pi(a)) \ \forall a \in A$.

The results indicate the non-commutative integration theory of von Neumann algebras with separable pre-dual discussed in section 1.5 and section 1.6 is equivalent to the symmetric measures SM_1. A. Connes' non-commutative calculus does not involve symmetric measures from SM_1.

1.7.4 Connes' Non-commutative Integral

Let D_s be the set of dilation and translation invariant positive linear functionals ω on ℓ^∞ such that $\omega(1) = 1$. Then

$$Tr_\omega(a) := \omega\Big(\Big\{\frac{1}{\ln N}\sum_{n=1}^{N} \mu_n(a)\Big\}_{N=2}^{\infty}\Big)$$

defines a Dixmier trace for $\omega \in D_s$ (see Example 1.7.4(b)). Define the set of finite symmetric measures

$$\mathrm{SM}_{1,\infty} := \{(Tr_\omega, K) \mid \omega \in D_s, K \geq 0, K \in L_{1,\infty}\}$$

Then Connes' integral is a weighted symmetric trace resulting from a measure in $\mathrm{SM}_{1,\infty}$ as follows.

Let $f_n(x) := (1 + x^2)^{-n/2}$. Let (H, π, D) be a $(Tr_\omega, f_n(D))$-integrable base representation of a unital separable C*-algebra A. Then the positive linear functional on $\pi(A)''$,

$$\tau_\omega(a) := Tr_\omega(\pi(a) f_n(D)),$$

is considered a non-commutative integral[15].

[15] We note that there are alternate definitions of integrability that lead to the same integral. The condition of $(Tr_\omega, f_n(D))$-integrable is generally the weakest.

Remark 1.7.13 **(a)** The separable C*-algebra A is taken to be *unital.* It is immediate that $A_c = A$ when A is unital. This implies $f_n(D) \in L_{1,\infty}$ as required for $(Tr_\omega, f_n(D)) \in \mathrm{SM}_{1,\infty}$. We note that results using Connes' integral exist only in final form for the case A unital[16].

(b) The functionals τ_ω are not necessarily normal linear functionals on $\pi(A)''$. This is a difficulty, but not a weakness of the theory. In some cases it provides a conceptual evolution, demonstrated by Connes in [C, IV.3]. We denote by (H_ω, π_ω) the GNS representation of A associated to the semifinite weight τ_ω for $\omega \in D_s$. By construction τ_ω is a faithful normal semifinite weight on $\pi_\omega(A)''$.

Results on Connes' Integral

Lemma 1.7.14 *Let (τ, K) be a finite symmetric measure associated to a symmetric norm ϕ. Then the weighted symmetric functional τ_K is a uniformly continuous linear functional on $B(H)$.*

Proof Let $S \in B(H)$. Then $SK \in I(\phi)$ as $K \in I(\phi)$. Hence $I^{-1}(\tau, K) = B(H)$. Let $S_n \to S$ uniformly where $S_n, S \in B(H)$. Then

$$|\tau_K(S_n - S)| \leq \phi((S_n - S)K) \overset{\text{Thm1.7.1(i)}}{\leq} \|S_n - S\|\phi(K).$$

Hence $\tau_K(S_n) \to \tau_K(S)$. □

Theorem 1.7.15 [C, IV.2.δ.15] *Let (H, π, D) be a C_c^1-representation of a unital separable C*-algebra A. Let $f_n(x) = (1 + x^2)^{-n/2}$ and $f_n(D) \in L_{1,\infty}$ for some $n \geq 1$. Let $\omega \in D_s$. Then*

(i) *the C*-algebra A admits a trace state,*

(ii) *(H, π, D) is $(Tr_\omega, f_n(D))$-integrable and*

$$\tau_\omega(a) := Tr_\omega(\pi(a) f_n(D))$$

is a positive trace on A,

(iii) *Let $p \in \mathbb{N} \setminus \{1\}$, $a_1, ..., a_p$ be commuting selfadjoint elements of A, and $E_{\mathrm{ac}} \subset \mathbb{R}^p$ the absolutely continuous support of their joint spectral measure μ. Let ρ_{ac} be the Radon measure given by μ_{ac},*

$$\rho_{\mathrm{ac}}(f) = \int_{E_{\mathrm{ac}}} f(x) d\mu_{\mathrm{ac}} \ \forall f \in C_0(\mathbb{R}^p).$$

Define the measure

$$\tau_\omega(f) := \tau_\omega(f(a_1, ..., a_n))$$

for $f \in C_0(\mathbb{R}^p)$. Then $\rho_{\mathrm{ac}} << \tau_\omega$.

[16] Equivalent results for non-unital C*-algebras are still being finalised. Conditions such as local approximate units [Re3] have allowed non-unital versions of the local index theorem and Theorem 1.7.15, see [Re, I,II] [Re4].

Proof Note [C, IV.2.δ.15] considers the pre-C*-algebra $A^1 = \{a \in A \mid \|[D,\pi(a)]\| < \infty\}$ and smooth functions $f \in C_c^\infty(\mathbb{R})$. However, the functional τ_ω is uniformly continuous by Lemma 1.7.14. So the result, once established for A^1 and $f \in C_c^\infty(\mathbb{R})$, can be extended by uniform continuity to A and $f \in C_0(\mathbb{R})$. The original statement also involves the operator $|D|^{-p}$, which requires $|D|$ have trivial kernel. Connes refined argument appears in [C6] using $f_n(D)$. (i) The trace state here is not necessarily τ_ω. The trace τ_ω is possibly trivial (for instance, $f_n(D) \in L^0_{1+}$ is not excluded). If it is zero, one uses the fact $f_{n+1}(D) \in L_1$ and [C9, Thm 8]. (ii) [CGS] (iii) section 2.δ of [C, IV]. □

Corollary 1.7.16 *Let (H,π,D) be a $(Tr_\omega, f_n(D))$-integrable C_c^1-representation of a unital separable C*-algebra A for $n \in \mathbb{N} \setminus \{1\}$ for any $\omega \in D_s$. Let π be absolutely continuous as in Definition 1.5.12. Let $a_1, ..., a_n$ be commuting selfadjoint elements of A. Then $\tau_\omega \equiv \lambda$ on the commutative C*-subalgebra $C^*(a_1,...,a_n) = C(E_{\text{ac}})$, where λ is the Lebesgue integral as in Example 1.5.10.*

Proof Let f be positive and continuous on E_{ac}. Suppose $\lambda(f) = 0$. Then $f = 0$ by absolutely continuity of the representation π and continuity of f. Hence $\tau_\omega(f) = 0$. This implies $\tau_\omega << \lambda$. Theorem 1.7.15(iii) provides $\lambda << \tau_\omega$. □

The relationship between the weighted symmetric function τ_ω associated to a $(Tr_\omega, f_n(D))$-integrable C_c^∞-representation (H,π,D) of a unital C*-algebra A and the Lebesgue measure on the commutative C*-subalgebra $C^*(a_1,...,a_n)$ is very deep. Connes views the integer n such that $f_n(D) \in L_{1+}$ as the (finite) dimension of the non-commutative space $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \text{Prim}(A)$ [17]. We complete the correlation suggested by Corollary 1.7.16.

Example 1.7.17 Lebesgue Integration and the Laplacian

Let X be a compact Riemannian manifold of dimension n. Let $C(X)$ denote the continuous complex-valued functions on X. Then $(L^2(X,\Lambda^*X), \pi_l, d+d^*)$ is a C_c^∞-representation of the unital separable C*-algebra $C(X)$, see Example 1.4.13.

Theorem 1.7.18 (Hodge Theorem and Decomposition)

Let X be a compact Riemannian manifold. Let $\Lambda^*X \to X$ be the exterior bundle. Then the eigenvectors of the Laplacian $\Delta : Dom\Delta \to L^2(X,\Lambda^*X)$ form an orthonormal basis of $L^2(X,\Lambda^*X)$. Each of the eigenvalues are positive, of finite multiplicity and they accumulate only at infinity.

Let $\Lambda := C^\infty(X,\Lambda^*X) \subset L^2(X,\Lambda^*X)$. Then the eigenvectors of Δ belong to Λ and we have the identification

$$\Lambda = \ker(d + d^*|_\Lambda) \oplus \text{im}\,(d + d^*|_\Lambda)$$

[17] Infinite dimensional spaces correspond to $(Tr, E_t(D))$-integrability for all t, where the function $E_t(x) = e^{-tx^2}$. This condition is called θ-summability. We shall not consider θ-summability in this thesis. The extension to θ-summable geometry is natural and necessary. Connes shows in [C9, Thm 16,Thm 19] there exist discrete groups G with no $(Tr_\omega, f_n(D))$-integrable C_c^1-representations for the reduced group C*-algebra. See [C, IV.7-9] for the properties of θ-summability and links to quantum field theory.

or in terms of the closure of $d + d^*|_\Lambda$

$$\Lambda = \cap_{s\geq 0} Dom(d+d^*)^s.$$

Proof The statement of the first paragraph and first identification are from the reference [R, Thm 1.30, Thm 1.37], and an instructive proof using the spectral theory of $d + d^*$ is contained in [R, Ex 34]. Let $k \geq 0$ and H_k be the k^{th} Sobolev space for sections of the bundle $\Lambda^* X$ [R, 1.3.3]. We know $d + d^* : H_{p+1} \to H_p \hookrightarrow H_0 = L^2(X, \Lambda^* X)$ is continuous for $p \in \mathbb{N}$. In fact, we have the Garding inequality [R, Thm 2.44]; there exists a constant C such that $\|w\|_{p+1} \leq C(\|w\|_p + \|(d+d^*)w\|_p)$ for all $w \in H_{p+1}, p \in \mathbb{N}$. We use this to prove the second identification. Let $w \in H_0$ and $w \in \cap_{p=1}^k Dom(d+d^*)^p$. Then by induction using the Garding inequality, $w \in H_k$. Hence $\cap_{p\in\mathbb{N}} Dom(d+d^*)^p \subset \cap_{p\in\mathbb{N}} H_p$. The Sobolev embedding theorem on a compact Riemannian manifold has the following corollary $w \in \cap_{p\in\mathbb{N}} H_p \iff w \in C^\infty(X, \Lambda^* X)$ [R, 1.3.3]. Hence $\cap_{p\in\mathbb{N}} Dom(d+d^*)^p \subset \Lambda$. From the definition of $d + d^*$ it is clear $\cap_{p\in\mathbb{N}} Dom(d+d^*)^p \supset \Lambda$. This provides the equality with intersection over the natural numbers. To extend to $s \geq 0$, we know from the spectral calculus that $Dom(d+d^*)^s \subset Dom(d+d^*)^n$ for $s \in (n, n+1], n \in \mathbb{N}$. □

Corollary 1.7.19 *Let X be a compact Riemannian manifold, Δ denote the Laplacian $\Delta : Dom\Delta \to L^2(X, \Lambda^* X)$ and $f \in C_0(\mathrm{sp}(\Delta))$. Then $f(\Delta)$ is a compact operator.*

Proof Let $f \in C_0(\mathrm{sp}(\Delta))$. The spectral theorem for unbounded selfadjoint operators provides a bounded operator $f(\Delta)$ on $H = L^2(X, \Lambda^* X)$ with spectrum $\mathrm{sp}(f(\Delta)) = f(\mathrm{sp}(\Delta))$. Hence the spectrum of $f(\Delta)$ is a set with zero the only limit point, identifying $f(\Delta)$ as a compact operator [S, Thm 1.1]. □

Lemma 1.7.20 *Let X be a compact Riemannian manifold of dimension n. Let $f_p(x) = (1+x^2)^{-p/2} \in C_0(\mathbb{R})$ and $(d+d^*)^2 = \Delta : Dom\Delta \to L^2(X, \Lambda^* X)$ be the Laplacian. Then*

(i) $f_p(d+d^*) = (1+\Delta)^{-\frac{p}{2}} \in L_1$ *for* $p > n$.

(ii) $f_n(d+d^*) = (1+\Delta)^{-\frac{n}{2}} \in L_{1,\infty} \setminus L^0_{1,\infty}$.

Moreover for any $\omega \in D_s$,

$$Tr_\omega((1+\Delta)^{-\frac{n}{2}}) = C(n)\mathrm{Vol}(X)$$

where $C(n) = \pi^{-\frac{n}{2}}\Gamma(\frac{n}{2}+1)^{-1}$.

Proof Let λ_N be the N^{th} eigenvalue of Δ listed in increasing order with multiplicity. We have the following statement of Weyl's theorem [BGV, Cor 2.43],

$$\lim_{N\to\infty} \lambda_N^{-n/2} N = \frac{\mathrm{Vol}(X)}{\pi^{n/2}\Gamma(\frac{n}{2}+1)}.$$

Let $c(n) = \mathrm{Vol}(X)(\pi^{n/2}\Gamma(\frac{n}{2}+1))^{-1}$. Since $\lim_{N\to\infty} \lambda_N^{-n/2}(1+\lambda_N)^{n/2} = 1$ we have $\lim_{N\to\infty}(1+\lambda_N)^{-n/2} N = c(n)$. Let $\alpha_N(n) = (1+\lambda_N)^{-n/2} N$. Then

$\alpha(n) = \{\alpha_N(n)\}_N$ is a bounded, convergent sequence with strictly positive terms, $\alpha_N(n) > 0 \ \forall N$, and strictly positive limit $c(n) > 0$. Hence $\alpha(n)$ has a positive supremum U and positive infinum L. Let $U_k = \sup_{N \geq k} \alpha_N(n) \leq U$ and $L_k = \inf_{N \geq k} \alpha_N(n) \geq L$.
Let $\delta \geq 0$ and define $g_\delta(x) = (1+x)^{-\frac{n}{2}(1+\delta)}$. Then $g_\delta \in C_0([0,\infty))$ for all $\delta \geq 0$. As g is monotonically decreasing the value $g_\delta(\lambda^N)$ is the N^{th}-singular value of the compact operator $g_\delta(\Delta)$. From continuity of the function $x^{1+\delta}$ on the interval $[L, U]$

$$\lim_{N\to\infty} g_\delta(\lambda^N) N^{1+\delta} = c(n)^{1+\delta}.$$

This sequence has supremum $U^{1+\delta}$ and infinum $L^{1+\delta}$. Using

$$g_\delta(\lambda^M) = (g_\delta(\lambda^M) M^{1+\delta}) M^{-(1+\delta)}$$

we derive (*)

$$L^{1+\delta} f(N) \sum_{M=1}^{N} M^{-(1+\delta)} \leq f(N) \sum_{M=1}^{N} g_\delta(\lambda^M) \leq U^{1+\delta} f(N) \sum_{M=1}^{N} M^{-(1+\delta)}$$

where $f(N) = 1$ or $f(N) = (\ln(1+N))^{-1}$.
When $\delta > 0$, then $\lim_N \sum_{M=1}^{N} M^{-(1+\delta)} = F_\delta < \infty$ is convergent. Let $f \equiv 1$ and take the limit as $N \to \infty$ in (*). Then

$$L^{1+\delta} F_\delta \leq Tr(g_\delta(\Delta)) \leq U^{1+\delta} F_\delta$$

using . The compact operator $g_\delta(\Delta)$ is trace class. This completes the proof of (i).
Let $\delta = 0$. The harmonic series $\sum_{M=1}^{N} M^{-1}$ is logarithmically divergent but not convergent. Let $F_0 = \|\{M^{-1}\}_{M=1}^{\infty}\|_{1^+} < \infty$ and recall $\lim_N \frac{1}{\ln(1+N)} \sum_{M=1}^{N} M^{-1} = 1$. Taking the supremum over N in (*),

$$\|g_0(\Delta)\|_{1^+} \leq U F_0$$

and $g_0(\Delta) \in L_{1^+}$. Considering

$$L \sum_{M=1}^{N} M^{-1} \leq \sum_{M=1}^{N} g_0(\lambda^M)$$

and taking the limit $N \to \infty$ implies $g_0(\Delta)$ is not trace class.
Let us show the Dixmier trace of $g_0(\Delta)$ does not vanish. We have

$$\lim_{N\to\infty} \frac{1}{\ln(1+N)} \sum_{M=1}^{N} a_M = \lim_{N\to\infty} \frac{1}{\ln(1+N)} \sum_{M=k}^{N} a_M$$

for any $k \in \mathbb{N}$ and any $\{a_M\} \in L_{1^+}$. We can re-derive (*)

$$L_k \frac{1}{\ln(1+N)} \sum_{M=k}^{N} M^{-1} \leq \frac{1}{\ln(1+N)} \sum_{M=k}^{N} g_0(\lambda^N) \leq U_k \frac{1}{\ln(1+N)} \sum_{M=k}^{N} M^{-1}.$$

Taking the limit as $N \to \infty$,

$$L_k \leq \lim_{N\to\infty} \frac{1}{\ln(1+N)} \sum_{M=1}^{N} g_0(\lambda^N) \leq U_k.$$

It follows the limit exists and is given by $c(n)$ as $\lim_k L_k = c(n) = \lim_k U_k$. $\square$

Let $g_n = (1+x^2)^{-n/2}$ for $x \in \mathbb{R}$. Define the weighted symmetric traces on $C(X)$

$$\tau_\omega(f) := Tr_\omega(\pi_l(f)g_n(d+d^*)) \ \forall f \in C(X)$$

for $\omega \in D_s$. Lemma 1.7.14 implies $\tau_\omega \in C(X)^*$. Theorem 1.5.2 provides the normal extension $\hat{\tau}_\omega$ of τ_ω to $L^\infty(X,\xi) = \pi_l(C(X))''$,

$$\hat{\tau}_\omega(f) := \lim_n \hat{\tau}_\omega(f_m) \ \text{ when } \ f_m \in C(X) \text{ s.t. } f_m \to f \in L^\infty(\mathbb{R}) \text{ ultraweakly.}$$

Hence, to clarify[18], we consider the weighted symmetric trace

$$\tau_\omega \in C(X)^*$$

and consider the normal extension on the von Neumann closure

$$\hat{\tau}_\omega \in L^\infty(X,\xi)_*.$$

These traces are distinguished by the following result.

Theorem 1.7.21 (Riemannian Structure)

Let (X,g) be a compact Riemannian manifold of dimension n with Lebesgue measure ξ. Let $\mathrm{Cl}(X) := \mathrm{Cl}(X,q_g)$. Let $\omega \in D_s$. Then

(i) *the C_c^∞-representation $(L^2(X,\Lambda^*X),\pi_l,d+d^*)$ of the unital separable C^*-algebra $C(X)$ is $(Tr_\omega, f_n(d+d^*))$-integrable,*

(ii) *$\hat{\tau}_\omega$ is a faithful normal trace on $L^\infty(X,\xi) = \pi_l(C(X))''$. Moreover*

$$\hat{\tau}_\omega(\pi_l(f)) = C(n)\int_X f(x)d\xi(x)$$

where

$$C(n) = \pi^{-\frac{n}{2}}\Gamma(\frac{n}{2}+1)^{-1}$$

for all $f \in L^\infty(X,\xi)$,

[18]Define the weighted symmetric trace on $L^\infty(X,\xi)$,

$$\tau_\omega(f) := Tr_\omega(\pi_l(f)g_n(d+d^*)) \ \forall f \in L^\infty(\mathbb{R},\xi).$$

Lemma 1.7.14 implies $\tau_\omega \in L^\infty(X,\xi)^*$. However, nowhere in the literature is it demonstrated $\tau_\omega \in L^\infty(X,\xi)_*$. The possibility,

$$(\tau_\omega - \hat{\tau}_\omega)(f) \neq 0\ , \text{ for some } f \in L^\infty(X,\xi)\setminus C(X),$$

remains nethier confirmed nor denied at present.

(iii) $\Omega_{d+d^*}(C^\infty(X)) = \pi_l(C^\infty(X, \mathrm{Cl}(X)))$,

(iv) *the von Neumann algebra* $\Omega_{d+d^*}(C^\infty(X))'' = L^\infty(X, \mathrm{Cl}(X))$ *has a cyclic and separating vector given by the one-section* $1 \in L^\infty(X, \Lambda^* X)$, *and admits the faithful normal trace*

$$\tilde{\tau}_\omega(\pi_l(w)) := \hat{\tau}_\omega(q_g(1, w \cdot 1)) = C(n) \int_X q_g(1(x), w(x)1(x))(x) d\xi(x),$$

(v) *the representation* $(L^2(X, \Lambda^* X), \pi_l)$ *of* $\Omega_{d+d^*}(C^\infty(X))''$ *is the GNS representation associated to* $\tilde{\tau}_\omega$,

(vi) *the trace* $\tilde{\tau}_\omega$ *is the unique faithful normal trace (up to Radon-Nikodym derivatives) on the finite hyperfinite von Neumann algebra* $L^\infty(X, \mathrm{Cl}(X))$.

Proof Let $D := d + d^*$. (i) Lemma 1.7.20

(ii) Let $n \neq 1$. Theorem 1.5.14 and Corollary 1.7.16 identifies $\hat{\tau}_\omega \equiv \xi$ as measures on $C(X)$. The identification $\pi_l(C(X))'' = L^\infty(X, \xi)$ is Theorem 1.5.6. Since the Lebesgue integral is a fns trace when extended to $\pi_l(C(X))''$ then $\hat{\tau}_\omega \equiv \xi$ as measures on $L^\infty(\mathbb{R}, \xi)$. The further identification of $\hat{\tau}_\omega$ as a scalar multiple of the Lebesgue integral can be proven as follows (including the case $n = 1$). In [C10, Thm 1] it is shown

$$Tr_\omega(\pi_l(f)(1 + \Delta)^{-n/2}) = \lim_{N \to \infty} \frac{1}{\ln N} \sum_{n=1}^{N} \mu_n(\pi_l(f)(1 + \Delta)^{-n/2})$$

for $f \in C^\infty(X)$ and any $\omega \in D_s$. Then (see the proof of Proposition I.2 [CM])

$$\int_X f(x) d\xi(x) \overset{(i)}{=} \lim_{t \to 0} (\pi t)^{\frac{n}{2}} Tr(\pi_l(f) e^{-t\Delta}) \overset{(ii)}{=} \pi^{\frac{n}{2}} \Gamma(1 + \frac{n}{2}) Tr_\omega(\pi_l(f)(1 + \Delta)^{-n/2})$$

for $f \geq 0$ and smooth. Equality (i) is the consequence of the asymptotics of the heat kernel of the Laplacian on the bundle $\Lambda^* X$ [R, 3.3], and equality (ii) is proven in [CPS, Thm 5.3]. This proves $\hat{\tau}_\omega$ is a scalar multiple of the Lebesgue integral on $C(X)$. The result follow from uniqueness of normal extensions of elements of $C(X)^*$ to $L^\infty(X, \xi)_*$.

(iii) [C, VI.1 Lemma 6]

(iv),(v) using norm density of $C^\infty(X)$ in $C(X)$, $\Omega_D(C^\infty(X))'' = \Omega_D(C(X))'' = \pi_l(C(X, \mathrm{Cl}(X)))''$. Then we have $\pi_l(C(X, \mathrm{Cl}(X)))'' = L^\infty(X, \mathrm{Cl}(X))$ from Corollary 1.5.8 where $E = \Lambda^* X$ is finite-dimensional and $C(E) = \mathrm{Cl}(X)$.

As X is oriented, $\Lambda^* X$ admits a one-section. Clearly $1(x) \in \Lambda(T_x^* X)$ is a separating and cyclic vector for the canonical left action w_l of $\mathrm{Cliff}(T_x^*, q_g(x))$ on $\Lambda(T_x^* X)$ by the isomorphism ι (section 1.3.3). We apply Proposition 1.6.11 to obtain $\tilde{\tau}_D$ is a faithful normal state. The tracial property follows as $q_g(x)(A) := \langle 1(x), A1(x) \rangle(x)$ equivalent to the matrix trace on $\mathrm{Cliff}(T_x^*, q_g(x))$.

(vi) Uniqueness of $\tilde{\tau}_D$ as a faithful trace on $L^\infty(X, \mathrm{Cl}(X))$ up to positive L^2-functions is a consequence of Corollary 1.6.7(ii). Since $Z(C(X, \mathrm{Cl}(X))) = C(X) = Z(C(X))$, then $C(X, \mathrm{Cl}(X))$ and $C(X)$ have the same factor representations. Indeed the space of unitary equivalence classes of factor representations of $C(X)$

is exactly its spectrum $\hat{C(X)} = X$. Hence the quasi-spectrum of $C(X, \mathrm{Cl}(X))$ is the spectrum X and the central factor decomposition of $L^\infty(X, \mathrm{Cl}(X))$ is given as a direct integral over X [Ped, Thm 4.12.4, Cor 4.12.5]. One need only prove the factor $L^\infty(X, \mathrm{Cl}(X))_x$ over $x \in X$ is hyperfinite and finite for $L^\infty(X, \mathrm{Cl}(X))$ to be hyperfinite and finite [C, V.7.α] [Ped, Thm 5.8.9]. As $L^\infty(X, \mathrm{Cl}(X))_x = \mathrm{Cliff}(T_x^* X, q_g)$ is finite dimensional, finiteness and hyperfiniteness of the factor is trivial. □

Remark 1.7.22 Remark 1.3.5 discussed the capacity of the Laplacian Δ and the C*-algebra $\pi_l(C(X))$ to determine the Riemannian metric, and so all the local geometric information. We have seen in Theorem 1.7.21 the equally deep capacity of the Laplacian operator to produce the measure class of the Lebesgue measure, and infact the Lebesgue integral, from its spectral properties.

1.8 The metric on pure states

Definition 1.8.1 *Let (H, π, D) be a C_c^1-representation of a separable C*-algebra A. We call (H, π, D) geometrically irreducible if the set*

$$B_D(A) = \{a \in A \setminus \mathbb{C} \,|\, \|[D, \pi(a)]\| \leq 1\}$$

is norm bounded in $\pi(A)$.

Theorem 1.8.2 *Let (H, π, D) be a C_c^1-representation of a separable C*-algebra A. Let (H, π, D) be geometrically irreducible. Then*

(i) *(H, π, D) is base irreducible in the sense of Definition 1.5.17,*

(ii)
$$d(\rho_1, \rho_2) := \sup\{|\rho_1(a) - \rho_2(a)| \,|\, a \in A \setminus \mathbb{C}, \|[D, \pi(a)]\| \leq 1\}$$
defines a metric on $PS(A)$, the pure state space of A,

(iii) *the metric topology induced by the metric d on $PS(A)$ is stronger than the weak*-topology on $PS(A)$.*

(iv) *$(PS(A), d)$ is a complete metric space.*

Proof (i) Let $[D, \pi(a)] = 0$ for any $a \in A \setminus \mathbb{C}$. Then $[D, \pi(\lambda a)] = 0$ for any $\lambda \in \mathbb{C}$. This contradicts the hypothesis on $B_D(A)$ as a norm-bounded subset of A. Consequently $[D, \pi(a)] \neq 0$ for all non-scalar $a \in Z(A)$. Since any proper central projection is a spectral projection of some $a \in Z(A)$, this imples $[D, \pi(p)] \neq 0$ for a proper central projection p.

(ii) [C9, Prop 3]

(iii) Let $\rho_\alpha \to \rho$ in the metric topology. Let $a \in A_c^1$. Then $a' = \|[D, \pi(a)]\|^{-1} a \in B_D(A)$ and $|\rho_\alpha(a') - \rho(a')| \to 0$. Hence $|\rho_\alpha(a) - \rho(a)| \to 0$. By hypothesis A_c^1 is norm dense in A. Let $b \in A$ and $\{a_m\}$ be a sequence of A_c^1 such that $a_m \to b$. Then $|\rho_\alpha(b) - \rho(b)| \leq |\rho_\alpha(a_m - b)| + |\rho_\alpha(a_m) - \rho(a_m)| + |\rho_\alpha(a_m - b)| \to 0$. Hence $\rho_\alpha \to \rho$ in the weak*-topology.

(iv) By Theorem 1.2.1(ii) $PS(A)$ is complete in the weak*-topology. Hence, by (iii), it is complete in the metric topology. □

Note all that is required for Theorem 1.8.2 is a C_c^1-representation. This theorem appeared as [C9, Prop 3] and the next example is [C9, Rem 2]. For a non-commutative example [C9, Lemma 5].

Example 1.8.3 Let X be a Riemannian manifold. Let $(L^2(X,\xi),\pi_l,d+d^*)$ be the geometrically irreducible C_c^1-representation of the C*-algebra $C_0(X)$ from Example 1.4.13. We recall from Remark 1.5.3 that the elements of $PS(C_0(X)) \cong X$ are the Dirac point measures on X. Then we have the equality of metrics

$$d(\rho_x,\rho_y) = d_\gamma(x,y) \ \forall \rho_x,\rho_y \in PS(C_0(\mathbb{R}))$$

where $d_\gamma(x,y)$ is the geodesic distance between $x,y \in X$.

1.9 Summary of Non-Commutative Calculus

We summarise our introduction to the non-commutative calculus.

Definition 1.9.1 *Let (H,π,D) be a $(Tr_\omega, f_n(D))$-integrable C_c^∞-representation of a C*-algebra A, see Definition 1.4.8 and Definition 1.4.9. Then we call (H,π,D) a $C_c^{n,\infty}$-representation of the C*-algebra A.*

Motivation and Philosophy

Through various examples we have seen the signature operator $d+d^*$ on a Riemannian manifold is a realisation of the philosophy of a $C_c^{n,\infty}$-representation of a C*-algebra A. That philosophy explicitly:

> the addition of the concrete selfadjoint linear operator $D : DomD \to H$ to the representation (H,π) of the C*-algebra A provides local differential geometry, through exterior derivations as in Section 1.4, and global integration, through weighted symmetric functionals as in Section 1.7.

Let (H,π,D) be a $C_c^{n,\infty}$-representation of a separable C*-algebra A.

The metric space

The structure space of the C*-algebra A

$$\Sigma_\eta(A) := PS(A) \stackrel{[\cdot]_u}{\to} \hat{A} \stackrel{\ker}{\to} \mathrm{Prim}(A)$$

is considered a non-commutative space. It involves a triple of topological spaces linked by continuous and open surjections. The pure state space $PS(A)$ is a complete second countable metrisable topological space in the weak*-topology and Prim(A) is locally compact in the Jacobson topology. With geometric irreducibility the operator D induces a metric on $PS(A)$,

$$d(\rho_1,\rho_2) = \sup\{|\rho_1(a)-\rho_2(a)| \,|\, a \in A\setminus\mathbb{C}, \|[D,\pi(a)]\| \le 1\}$$

and $(PS(A),d)$ is a complete metric space.

Differential Forms

Define the seminorms

$$p^0_m(T) := \|\delta^m_{|D|}(T)\| \ , \ p^1_m(T) := \|\delta^m_{|D|}([D,T])\| \ , \ m = 0,1,2,...$$

with the convention $\delta^0_{|D|}(T) = T$ for $T \in B(H)$ such that $T \in \cap_{k=1}^{\infty} Dom\delta^k_{|D|}$ and $[D,T]$ is bounded. Define the *-algebra

$$\mathcal{A}_\pi := \{a \in A \,|\, p^n_m(\pi(a)) < \infty, n = 0,1, m = 0,1,2,...\}.$$

Let S^1_D be the metrisable locally convex topology generated by the seminorms p^m_n, $n = 0,1, m = 0,1,2,....$ Then $\mathcal{A}_\pi$ is a Frechet pre-C*-subalgebra of A in the topology S^1_D.

We form the graded *-subalgebra of $B(H)$,

$$\Omega_D(\mathcal{A}_\pi) := \mathcal{A}_\pi \oplus \oplus_{k=1}^{\infty} \Omega^k_D(\hat{\mathcal{A}}_\pi)$$

where $\hat{\mathcal{A}}_\pi$ is the unitisation of $\mathcal{A}_\pi$ and for $k \geq 1$

$$\Omega^k_D(\hat{\mathcal{A}}_\pi) = \{\pi(a_0)[D,\pi(a_1)]...[D,\pi(a_k)] \,|\, a_0, a_1, ..., a_k \in \hat{\mathcal{A}}_\pi\}$$

We view the operation

$$[D, \cdot\,] : \pi(\mathcal{A}_\pi) \to \Omega^1_D(\mathcal{A}_\pi)$$

as 'differentiation'. We extend this to

$$\delta_D : \Omega^k_D(\mathcal{A}_\pi) \to \Omega^{k+1}_D(\mathcal{A}_\pi)$$

given by

$$\delta_D\Big(\pi(a_0)[D,\pi(a_1)]\dots[D,\pi(a_k)]\Big) := [D,\pi(a_0)][D,\pi(a_1)]\dots[D,\pi(a_k)].$$

The pair $(\Omega_D(\mathcal{A}_\pi), \delta_D)$ is a representation π_D of the universal graded differential algebra $(\Omega(\mathcal{A}_\pi), \delta)$ that is not differential in general

$$\pi_D \circ \delta \neq \delta_D \circ \pi_D.$$

Quotienting by the obstruction to a differential representation we obtain the *-algebra of exterior differential forms $\Lambda_D(\mathcal{A}_\pi)$ and a graded differential representation

$$\pi_\Lambda : (\Omega(\mathcal{A}_\pi), \delta) \to (\Lambda_D(\mathcal{A}_\pi), \delta_D).$$

Integration

The normal semifinite weights on the von Neumann algebra A'' provide the regular non-commutative Radon measure theory on A.

We consider the integral calculus as particular measures constructed from the spectral properties of D. The condition of $(Tr_\omega, f_n(D))$-integrability for some smallest $n \in [1,\infty)$ introduces the notion of finite dimensionality and provides the weighted symmetric traces τ_ω on the norm dense ideal A_c,

$$\tau_\omega(a) := Tr_\omega(\pi(a)f_n(D)) \qquad \forall\, a \in A_c$$

for any $\omega \in D_s$. There potentially exists multiple measure classes defined by an integral calculus.

Let (H_ω, π_ω) be the GNS representation of A associated to any $\omega \in D_s$.

Analytic Regime

A. Connes introduction of non-commutative differential calculus via $C_c^{n,\infty}$-representations of C*-algebras spans the existing analytic regime in operator algebra theory:

algebra	:	Frechet pre-C*-algebra		C*-algebra		von Neumann algebra
regime	:	smooth		topological		measure
notation	:	$\mathcal{A}_\pi$	$\subset$	A	$\subset$	$\pi_\omega(A)''$
topology	:	$\mathcal{S}_D^1$		$\|\cdot\|$		weak

Chapter 2

Riemannian Non-Commutative Geometry

We recall the theory of C*-algebras had the bijective correspondence

$$(X, C_0(X)) \longleftrightarrow (\Sigma(A), A)$$

between the pair of a locally compact Hausdorff spaces X and the vanishing at infinity continuous functions $C_0(X)$ on X, and a commutative C*-algebras A and its spectrum $\Sigma(A)$, see Theorem 1.2.12. This was the basis for considering general C*-algebras A and their structure spaces $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ to be the theory of non-commutative topology.

In the theory of von Neumann algebras we had the bijective correspondence

$$(L^\infty(X, \mu), M(\mu)) \longleftrightarrow (W(A), W(A)_*)$$

between the pair of essentially bounded functions on a Borel measure space (X, μ) and the finite regular Borel measures absolutely continuous to μ and the pair of a commutative von Neumann algebra $W(A)$ and its predual $W(A)_*$, see Theorem 1.5.6. This was the basis for considering general von Neumann algebras $W(A)$ and their preduals $W(A)_*$ to be non-commutative measure theory.

However, we do not have a bijective correspondence

$$(C_0(X), X, (L^2(X, \Lambda^* X), \pi_l, d + d^*)) \longleftrightarrow (A, \Sigma(A), (H, \pi, D))$$

between the triple associated to a Riemannian manifold X and the triple associated to a base-irreducible faithful $C_c^{n,\infty}$-representation (H, π, D) of a commutative separable C*-algebra A. Hence, though we have a non-commutative calculus, it is not considered that C*-algebras and $C_c^{n,\infty}$-representations constitute the theory of non-commutative differential manifolds.

Last chapter we introduced the non-commutative calculus of Connes. The initial sections of this chapter will be involved with more advanced aspects of generalising differentiable manifolds to operator algebra theory. We shall discuss Hilbert modules and finite projective modules ('non-commutative vector bundles'), Kasparov's KK-theory ('non-commutative algebraic topology'), and Hochschild and cyclic homology ('non-commutative Kähler-de Rham complex and cohomology of exterior differential forms'). In this framework Connes introduced a list of sufficient conditions on a $C_c^{n,\infty}$-representation of a unital C*-algebra that defined a non-commutative compact manifold. We shall follow the 'axiomatic' treatment presented in [C3, C4]. However, we shall not introduce the axioms that result in a correspondence with compact *spin* manifolds[1]. We define a Riemannian representation of a C*-algebra based upon the Riemannian structure detailed in Theorem 1.7.21, Theorem 2.3.2 and Theorem 2.3.3. We adjust the axiomatic approach to these representations to gain a correspondence with compact Riemannian manifolds, no spin structure assumed. One of the advantages of a Riemannian representation is the introduction of the modular theory of von Neumann algebras, which plays a central part in naturalising some of the 'axioms'. Indeed, we can construct Riemannian representations from abstract information called a Riemannian cycle using the modular theory and the theory of A-symmetric derivations of a C*-algebra A. The chapter is concluded by introducing a Riemannian cycle associated to the irrational rotation C*-algebra A_θ and deriving the Riemannian geometry of the non-commutative torus.

2.1 Hilbert Modules

Henceforth we shall be concerned with both C*-algebras and norm dense Frechet pre-C*-algebras. Let $\mathcal{A}$ be a Frechet *-algebra stable under the holomorphic functional calculus. We shall consider only those locally convex topologies such that one of the semi-norms is a C*-norm $\|.\|$. Let A be the C*-closure of $\mathcal{A}$ in $\|.\|$. Consequently $\mathcal{A}$ is a pre-C*-algebra of A. Hence there is an equivalence between the Frechet *-algebras we consider and Frechet pre-C*-algebras.

Let A be a (Frechet pre-)C*-algebra. In this section we introduce (pre-)Hilbert A-modules. A Hilbert A-module will provide a generalisation of the concrete representation theory of the C*-algebra A. Hilbert A-modules shall also provide 'non-commutative vector bundles' and 'non-commutative algebraic topology' in the form of finite projective A-modules and the KK-theory of A.

Tensor Products

The tensor product $A \otimes B$ of C*-algebras A and B shall always denote the spatial tensor product [Lc]. That is, the closure of the algebraic product $A \odot B$ in the spatial C*-norm $\|a \odot b\| = \|a\|_A \|b\|_B$ where $a \in A$, $b \in B$ and $\|.\|_A, \|.\|_B$ are the respective C*-norms on A and B. Let $A^{\otimes n}$ denote the n^{th} spatial tensor product of a C*-algebra A.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of Frechet pre-C*-algebras $\mathcal{A}$ and $\mathcal{B}$ shall always denote the projective tensor product. That is, the closure of the algebraic product $\mathcal{A} \odot \mathcal{B}$ in the locally convex topology generated by the family of semi-norms $p_{m,k}(a \otimes b) = \max\{p_m^{\mathcal{A}}(a), p_k^{\mathcal{B}}(b)\}$ $\forall a \in \mathcal{A}, b \in \mathcal{B}$ where $\{p_m^{\mathcal{A}}\}$ and $\{p_k^{\mathcal{B}}\}$ are the seminorms that generate the locally convex topologies on $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\mathcal{A}^{\otimes n}$ denote the n^{th} projective tensor product of a Frechet pre-C*-algebra $\mathcal{A}$.

[1]For the notion of a spin manifold see [LM].

2.1.1 Definition of Hilbert Modules

Let A be a topological algebra. Let $\iota : A \to A$ be the identity homeomorphism. Denote by A^{op} the topological vector space $\iota(A)$ given the product $\iota(a) \times \iota(b) \mapsto \iota(ba)$. Then A^{op} is a topological algebra homeomorphic to A such that $Z(A) = Z(A^{\mathrm{op}})$. Let A have an involution. Then A^{op} has an involution. In particular if A is a Frechet *-algebra, a C*-algebra or a von Neumann algebras, then A^{op} is a Frechet *-algebra, a C*-algebra or a von Neumann algebra respectively. We call A^{op} the opposite algebra of A.

We have already denoted the continuous linear functions between topological vector spaces V and W by $L(V,W)$. We recall a module W of a topological algebra A is a topological vector space W with a continuous representation $\pi : A \to L(W,W)$. We will denote A-modules by (W,π). An A-module (W,π) has a basis $\{w_\alpha\}$ if for any $w \in W$, $w = \sum_\alpha \pi(a_\alpha)w_\alpha$ for some 'co-ordinates' $\{a_\alpha\} \subset A$.

Let (V,π_1) and (W,π_2) be A-modules. We will denote the continuous module homomorphisms

$$E_A((V,\pi_1),(W,\pi_2)) = \{f \in L(V,W) \,|\, \pi_2(a)f = f\pi_1(a) \;\; \forall a \in A\}.$$

Let $E_A(W,\pi)$ denote $E_A((W,\pi),(W,\pi))$. We call an A^{op}-module (W,π) a right A-module and hence an A-module is called a left A-module by default. We will sometimes denote a right A-module (W,π^{op}), indicating an opposite (product reversing) representation of A is involved.

Let A and B be topological algebras. Let (W,π_B) be a B-module and (W,π_A^{op}) be a right A-module. We say $(W,\pi_B,\pi_A^{\mathrm{op}})$ is a B-A-bimodule if $\pi_B(B) \subset E_A(W,\pi^{\mathrm{op}})$. Alternatively, when A,B are (Frechet pre-)C*-algebras, a B-A-bimodule is an $B\otimes A^{\mathrm{op}}$-module $(W,\pi_B \otimes \pi_A^{\mathrm{op}})$. We refer to an A-A-bimodule as an A-bimodule.

Definition 2.1.1 *Let A be a topological *-algebra that admits a C*-norm. A pre-Hilbert A-module (W,π^{op}) is a right A-module with an 'A-valued inner product'. That is, a sesquilinear function $\langle\cdot,\cdot\rangle_A : W \times W \to A$ with the properties [Pa]*

(i) $\langle v,\pi^{\mathrm{op}}(a)w\rangle_A = \langle v,w\rangle_A a \quad \forall a \in A, v,w \in W,$

(ii) $\langle v,w\rangle_A = \langle w,v\rangle_A^* \quad \forall v,w \in W,$

(iii) $\langle w,w\rangle_A \geq 0$, $\langle w,w\rangle_A = 0$ *iff* $w = 0 \quad \forall w \in W.$

Define a norm on W by $\|w\| = \sqrt{\|\langle w,w\rangle_A\|}$. The completion of W is called a Hilbert A-module.

Define supp(W) as the closure of $\mathrm{Span}_{\mathbb{C}}\{\langle v,w\rangle_A \,|\, v,w \in W\}$ in the topology of A. The pre-Hilbert A-module is called (1) full if supp$(W) = A$ and (2) separable if it admits a countable basis as an A-module.

Let (V,π_1) and (W,π_2) be pre-Hilbert A-modules with A-valued inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ respectively. A map $f : V \to W$ is adjointable if there exists a map $f^* : W \to V$ such that $\langle v,f^*w\rangle_1 = \langle fv,w\rangle_2$ for all $v \in V, w \in W$. We will denote the adjointable continuous module homomorphisms by $B_A((V,\pi_1),(W,\pi_2)) \subset E_A((V,\pi_1),(W,\pi_2))$. We denote by $B_A(W,\pi^{\mathrm{op}})$, or sometimes $B_A(W)$, the elements of $E_A(W,\pi^{\mathrm{op}})$ that are adjointable. Given the operator norm $B_A(W)$ is a (pre-)C*-algebra [B1, Prop 13.2.2].

Let A and B be (Frechet pre-)C*-algebras. Let $(W, \pi_B, \pi_A^{\mathrm{op}})$ be a B-A-bimodule such that (W, π_A^{op}) is a (pre-)Hilbert A-module and $\pi_B : B \to B(W, \pi^{\mathrm{op}})$ is a *-representation. Then we call $(W, \pi_B, \pi_A^{\mathrm{op}})$ a B-A-(pre-)C*-bimodule.

Remark 2.1.2

(i) Hilbert A-modules and B-A-C*-bimodules generalise Hilbert spaces and concrete representations of C*-algebras on Hilbert spaces respectively. (Separable) Hilbert $\mathbb{C}$-modules are exactly (separable) Hilbert spaces and a B-$\mathbb{C}$-C*-bimodule (H, π, λ) is equivalent to a concrete representation (H, π) of the C*-algebra B.

(ii) Let $\mathcal{A}$ be a Frechet pre-C*-algebra, A the C*-closure of $\mathcal{A}$ and (W, π) a pre-Hilbert $\mathcal{A}$-module. Then the closure $(\tilde{W}, \tilde{\pi})$ is a Hilbert A-module [Lc2].

(iii) Let (W_i, π_i) be a countable family of (pre-)Hilbert A-modules with A-valued inner products $\langle \cdot, \cdot \rangle_i$. The direct sum $(\oplus_i W_i, \oplus \pi_i)$ is a pre-Hilbert A-module with A-valued inner product $\langle \oplus v_i, \oplus w_i \rangle = \sum_i \langle v_i, w_i \rangle_i$, [Lc2].

(iv) Many of the standard notions of Hilbert spaces carry through to Hilbert A-modules. Let (V, π_1) and (W, π_2) be Hilbert A-modules.

An operator $U \in B_A((V, \pi_1), (W, \pi_2))$ such that $U^*U = \mathrm{id}_W, UU^* = \mathrm{id}_V$ is called a unitary. We denote $(V, \pi_1) \sim_u (W, \pi_2)$ if there exists a unitary $U \in B_A((V, \pi_1), (W, \pi_2))$.

Let (W, π^{op}) be a Hilbert A-module. Let $v, w \in W$. We define an operator $F_{v,w}(z) = \pi^{\mathrm{op}}(\langle w, z \rangle_A) v$. Then $FR_A(W) = \mathrm{Span}_{\mathbb{C}}\{F_{v,w} \mid v, w \in W\}$ is a two-sided *-ideal of $B_A(W)$, considered finite rank operators. The operator norm closure of $FR_A(W)$ is denoted $K_A(W)$ and considered compact operators [Lc2].

Let $M(A)$ be the multiplier algebra of a C*-algebra A. Then we have the result $M(K_A(W)) = B_A(W)$ [B1, Thm 13.4.1].

Example 2.1.3

(i) A closed two-sided *-ideal I of a (Frechet pre-)C*-algebra A provides a canonical A-bimodule. The representation $\pi : A \otimes A^{\mathrm{op}} \to B_A(I)$ is given by $\pi(a \otimes b^{\mathrm{op}})c = acb \ \forall a, b \in A, c \in I$.

Hence a closed two-sided *-ideal I provides a (pre-)Hilbert A-module and an A-A-C*-bimodule with the A-valued inner product $\langle c, d \rangle_I = c^* d \ \forall c, d \in I$.

(ii) Let I be a closed two-sided *-ideal of a (Frechet pre-)C*-algebra A. We denote by I^k the direct sum pre-Hilbert A-module $\oplus_{i=1}^k I$ and $\tilde{I}^k$ the completion. We note that $K_A(\tilde{I}^k) = M_k(\tilde{I})$ and $B_A(\tilde{I}^k) = M_k(M(\tilde{I}))$ where $M(B)$ is the multiplier algebra of a C*-algebra B, [B1, Cor 13.4.2]. Here $\tilde{I}^k$ is considered to consist of $\tilde{I}$-valued column vectors and the matrices $M_k(M(\tilde{I}))$ act on the left by matrix multiplication.

Let A be a C*-algebra. The Hilbert A-module H_A defined by

$$H_A = \{\oplus_{i=1}^{\infty} v_i \mid \langle \oplus v_i, \oplus v_i \rangle = \sum_i \langle v_i, v_i \rangle_i \text{ converges in } A\}.$$

is called the Hilbert space of A [Lc2]. There exists a projection $p^k \in B(H_A)$ such that $p^k H_A = A^k$ for each $k \in \mathbb{N}$. The submodules A^k are fully complemented in the sense $(1-p^k)H_A \sim_u H_A$. Indeed, any countably generated Hilbert A-module (W,π) is unitarily equivalent to a fully complemented submodule of H_A [Lc2, Thm 6.2, Cor 6.3].

Let A be a C*-algebra and K the C*-algebra of compact operators on a Hilbert space. Then [Lc2, 6]

$$K_A(H_A) \cong A \otimes K,$$

and

$$B_A(H_A) \cong M(A \otimes K).$$

(iii) Let I be as in (ii). Let $p \in M_k(M(\tilde{I}))$ be a (self-adjoint) projection. Then $p\tilde{I}^k$ is a sub-module of $\tilde{I}^k$ such that $\tilde{I}^k = p\tilde{I}^k \oplus (\mathrm{id}_k - p)\tilde{I}^k$. Moreover $B_A(p\tilde{I}^k) = pB_A(\tilde{I}^k)p = pM_k(M(\tilde{I}))p$ and $K_A(p\tilde{I}^k) = pK_A(\tilde{I}^k)p = pM_k(\tilde{I})p$.

Definition 2.1.4 *Let A be a C*-algebra and $M(A)$ the multiplier algebra of A. We call a Hilbert A-module (W,π) a finite projective A-module if $(W,\pi) \sim_u pA^k$ for some projection $p \in M_k(M(A))$.*

Let $\mathcal{A}$ be a Frechet pre-C*-algebra with C*-closure A. We call a pre-Hilbert $\mathcal{A}$-module (W,π) a finite projective $\mathcal{A}$-module if $(\tilde{W},\tilde{\pi})$ is a finite projective A-module.

Example 2.1.5 The Serre-Swan Theorem

Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the C*-algebra of continuous vanishing at infinity functions on X. Let $E \to X$ be a Hermitian vector bundle with fibres E_x isometric to a Hilbert space H (see Section 1.3.2). Let $\phi_x : E_x \to H$ denote the isometric isomorphism between E_x and the Hilbert space H. Let σ, σ' be continuous sections of $E \to X$ with compact support. Then

$$\langle \sigma, \sigma' \rangle(x) := \langle \phi_x(\sigma(x)), \phi_x(\sigma'(x)) \rangle$$

defines a continuous function of compact support on X. It is immediate $\langle \sigma, \sigma' \rangle$ defines a sesquilinear function $\langle \cdot, \cdot \rangle : C_0(X,E) \times C_0(X,E) \to C_0(X)$. Let $C_0(X,E)$ be the C*-algebra of vanishing at infinity sections on E. Define the representation $\pi_r : C_0(X) \to L(C_0(X,E), C_0(X,E))$ given by

$$\pi_r(f)\sigma(x) = \sigma(x)f(x) \ \forall f \in C_0(X), \sigma \in C_0(X,E).$$

Lemma 2.1.6 *Let X be a locally compact Hausdorff space and $E \to X$ a Hermitian vector bundle. Then $(C_0(X,E), \pi_r)$ is a full Hilbert $C_0(X)$-module.*

Proof Denote $(C_0(X,E),\pi_r)$ by (C,π) for convenience. That (C,π) is a pre-Hilbert $C_0(X)$-module is discussed in [Sw]. We check that (C,π) is full and complete.

Suppose $f \in C_0(X) \setminus \mathrm{Span}(C)$ exists. Let Y be the closure of $\mathrm{supp}(f)$. Then $I_Y = \{f \in C_0(X) \,|\, f(x) = 0 \ \forall x \in Y\}$ is a proper closed ideal of $C_0(X)$. As $\mathrm{Span}(C) \subset I_Y$ this implies from the positive definite property of the $C_0(X)$-valued inner product that $C \subset \tilde{I}_Y = \{\sigma \in C \,|\, \sigma(x) = 0 \ \forall x \in Y\}$. This implies

C is contained in a proper closed ideal of itself, which is a contradication. Hence $f \in C_0(X) \setminus \mathrm{Span}(C)$ does not exist.
The norm $\|.\|_C$ induced on C as a pre-Hilbert $C_0(X)$-module is given by

$$\|\sigma\|_C = \sqrt{\sup_{x\in X} \langle\sigma,\sigma\rangle(x)} = \sup_{x\in X} \sqrt{\langle\sigma,\sigma\rangle(x)} = \|\sigma\|_\infty.$$

Hence $\|.\|_C$ and $\|.\|_\infty$ are identical. So C is complete. □

Lemma 2.1.7 *Let X be a locally compact Hausdorff space and $E \to X$ a Hermitian vector bundle. Let (W,π) be a Hilbert $C_0(X)$-module such that $(W,\pi) \sim_u (C_0(X,E),\pi_r)$. Then there exists a Hermitian vector bundle $F \to X$ such that $W = C_0(X,F)$.*

Proof Let $C_E := C_0(X,E)$. Let $x \in X$ and consider the ideal $I_x = \{f \in C_0(X) \mid f(x) = 0\}$. There exists the corresponding submodule

$$\tilde{I}_x = \pi_r(I_x)C_E = C_0(X \setminus \{x\}, E|_{X\setminus\{x\}}).$$

The quotient module

$$C_E/\tilde{I}_x \cong E_x \cong H,$$

where $f \in C_0(X)$ acts on $e \in E_x$ by $c_x^{op}(f)(e) = f(x)e$ [Dix].
Define the submodule $J_x = \pi(I_x)W$. Define the quotient module $W_x = W/J_x$ Let $U \in B_{C_0(X)}(C_E, W)$ be unitary. Then $U\pi(f)\sigma w = \pi(f)U^*\sigma = U^*\pi_r(f)\sigma$. Hence $U\pi_r(I_x)C_E = \pi(I_x)UC_E = \pi(I_x)W$. Explicitly $U\tilde{I}_x = J_x$. Hence the quotient map is well defined $U(C_E/\tilde{I}_x) = UC_E/U\tilde{I}_x = W/J_x = W_x$. Hence define the bundle by the disjoint union $F = \cup_{x\in X} W_x$ with fibres

$$W_x \cong C_E/\tilde{I}_x \cong E_x \cong H.$$

It is immediate that F has the same trivialising charts as E and that $W = C_0(X,F)$ as $C_0(X)$-modules by [Dix]. □

A Hermitian vector bundle with finite dimensional fibres means H is finite dimensional.

Lemma 2.1.8 *Let X be a locally compact Hausdorff space and $E \to X$ a Hermitian vector bundle with finite trivialising cover and finite dimensional fibres. Then $(C_0(X,E),\pi_r)$ is a finite projective $C_0(X)$-module.*

Proof Let $\{U^i\}_{i=0}^p$ be the finite trivialising cover of E. Let $T : E \to TE$ be the map defined by $T|_{U^i} : E_{U^i} \to U^i \times H$ where H is finite dimensional. From the definition of a vector bundle [Sr] we obtain a fibrewise isometric isomorphism $T|_x : E_x \to TE_x$. This provides a unitary equivalence as follows. Let $\{e_i\}$ be an orthonormal basis of H and define $e_i(x) := \phi_x^{-1}(e_i)$. By definition $T|_x e_i(x) = e_i$. Hence $T|_x\sigma(x) = \sum_i \lambda_i Te_i(x) = \sum_i \lambda_i e_i$. Then $T|_x(\sigma(x)f(x)) = (T|_x\sigma(x))f(x)$. Hence T defines a unitary element of $B_{C_0(X)}(C_E, C_{TE})$ and $(C_E,\pi_r) \sim_u (C_{TE},\pi_r)$ as $C_0(X)$-modules.

We now show $C_{TE} = pC_0(X)^k$ for a projection $p \in M_k(M(C_0(X))$. Note $M(C_0(X)) = C_b(X)$ by the Dauns-Hoffman Theorem [Ped, Cor 4.4.8]. Consider a continuous partition of unity $\{p_i\}$ of X subordinate to the finite cover $\{U^i\}$. Let N be the dimension of the Hilbert space H. Define the diagonal matrix $P_i \in M_N(C_b(X))$ by $P_i = p_i \mathrm{id}_N$. Define the selfadjoint matrix $P \in M_p(M_N(C_b(X)))$ by $P_{ij} = \sqrt{P_i P_j}$ for $i, j = 1, .., p$. Let $k = Np$. One checks the property $\sum P_i = \mathrm{id}_N$ implies $P^2 = P$. Hence there exists a projection $P \in M_k(C_b(X)) = M_k(M(C_0(X)))$ such that $C_{TE} = PC_0(X)^k$. □

The main result about (full) finite projective $C_0(X)$-modules is their one-to-one correspondence with Hermitian vector bundles $E \to X$ with a finite trivialising cover and finite dimensional fibres.

Theorem 2.1.9 (Serre-Swan) *Let X be a locally compact Hausdorff space. Let (W, π) be a full $C_0(X)$-Hilbert module. Then (W, π) is finite projective if and only if $W = C_0(X, E)$ for some Hermitian vector bundle $E \to X$ with finite trivialising cover and finite dimensional fibres.*

Remark 2.1.10 The above theorem is a generalisation of the original statement of the Serre-Swan theorem [Sw]. The original statement for compact Hausdorff spaces is recovered from the fact that every Hermitian vector bundle on a compact Hausdorff space has a finite trivialising cover. The finite trivialising cover condition for a locally compact Hausdorff space appeared in [Sw] and [HgR] as remarks and a detailed discussed appears in [Re3].

2.1.2 $\mathbb{Z}_2$-graded C*-algebras

Let A be a C*-algebra. Denote by $\mathrm{Aut}_2(A)$ the continuous *-automorphisms α of A such that $\alpha^2 = \mathrm{id}$.

Definition 2.1.11 *A $\mathbb{Z}_2$-graded C*-algebra is the pair (A, α) of a C*-algebra A and $\alpha \in \mathrm{Aut}_2(A)$.*

We introduce notions and notations,

1. Let A be a C*-algebra. Then the $\mathbb{Z}_2$-graded C*-algebra (A, id) is called a trivially graded C*-algebra, and A is referred to as trivially graded.
2. Let A be a C*-algebra. A $\mathbb{Z}_2$-grading $\alpha \in \mathrm{Aut}_2$ is unitary implemented if there exists a unitary $u \in M(A)$ such that $\alpha(a) = uau^*$.
3. An isomorphism of $\mathbb{Z}_2$-graded C*-algebras (A, α) and (B, β) is an isomorphism $\phi : A \to B$ such that $\phi \circ \alpha = \beta$.
4. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Define

   (i) the even elements, $A^{\mathrm{e}} = \{a \in A \,|\, \alpha(a) = a\}$,

   (ii) the odd elements, $A^{\mathrm{o}} = \{a \in A \,|\, \alpha(a) = -a\}$,

   We say $a \in A$ is even if $a \in A^{\mathrm{e}}$ and odd if $a \in A^{\mathrm{o}}$. Any $a \in A$ has a unique decomposition

$$a = a_{\mathrm{e}} + a_{\mathrm{o}},$$

where $a_e = \frac{1}{2}(a + \alpha(a)) \in A^e$ and $a_o = \frac{1}{2}(a - \alpha(a)) \in A^o$. We call $a \in A$ homogeneous if it is even or odd. From the decomposition above we can define properties on the homogeneous elements and extend by linearity to all of A. Let a be a homogeneous elements. Then $\deg(a) \in \{0,1\}$ is defined by $\alpha(a) = (-1)^{\deg(a)}a$. Define the graded commutator on homogeneous elements of A by

$$[a,b]_g = ab - (-1)^{\deg(a)\deg(b)}\, ba,$$

and extend it linearly to a bilinear map $A \times A \to A$.

5. Let $(W, \sigma^{\mathrm{op}})$ be a Hilbert B-module for any C*-algebra B. A grading element Γ is an adjointable operator $\Gamma \in L(W,W)$ such that $\Gamma^2 = 1$. We have a similar decomposition of W as above

   (i) the even elements, $W^e = \{w \in W \mid \Gamma w = w\}$,

   (ii) the odd elements, $W^o = \{w \in W \mid \Gamma w = -w\}$,

   (iii) $w = w_e + w_o\ \forall w \in W$ where $w_e = w + \Gamma w \in W^e$ and $w_o = w - \Gamma w \in W^o$.

   An element $w \in W$ is called homogeneous if it is even or odd. For a homogeneous element $w \in W$ define $\deg(w) \in \{0,1\}$ by $\Gamma w = (-1)^{\deg(w)}w$.

6. Let $(W, \sigma^{\mathrm{op}})$ be a Hilbert B-module for any C*-algebra B. Let (A,α) be a $\mathbb{Z}_2$-graded C*-algebra. Then $(\pi,\Gamma) : A \to B_B(W)$ is a graded representation of A if $\pi : A \to B_B(W)$ is a representation of A and Γ is a selfadjoint grading element such that $\mathrm{ad}_\Gamma \circ \pi(a) = \pi \circ \alpha(a)\ \ \forall a \in A$.

Definition 2.1.12 *Let (A,α) and (B,β) be $\mathbb{Z}_2$-graded C*-algebras. Then a graded Hilbert B-module $(W, \sigma^{\mathrm{op}}, \Gamma)$ is a Hilbert B-module $(W, \sigma^{\mathrm{op}})$ with selfadjoint grading element Γ such that $\Gamma\sigma^{\mathrm{op}}(b)w = \sigma^{\mathrm{op}}(\beta(b))\Gamma w\ \ \forall b \in B, w \in W$. A graded A-B-C*-bimodule $(W,\pi,\sigma^{\mathrm{op}},\Gamma,\Upsilon)$ is a graded Hilbert B-module $(W,\sigma^{\mathrm{op}},\Upsilon)$ and a graded representation $(\pi,\Gamma) : A \to B_B(W)$ such that $[\Gamma,\Upsilon] = 0$, $[\Gamma,\sigma^{\mathrm{op}}(b)] = 0\ \ \forall b \in B$, $[\pi(a),\Upsilon] = 0\ \ \forall a \in A$.*

2.1.3 Tensor Products of Hilbert modules

Let (A,α) and (B,β) be $\mathbb{Z}_2$-graded C*-algebras.

Graded Tensor Products of C*-algebras

The spatial tensor product $A \otimes B$ has the natural $\mathbb{Z}_2$-grading

$$\gamma(a \otimes b) = \alpha(a) \otimes \beta(b).$$

However, the graded C*-algebra typically used is the skew-commutative tensor product $A\hat{\otimes}B$, defined as follows. Let $(A \odot B, \gamma)$ be the graded algebraic tensor product. Define a product and involution on homogeneous elements that extends linearly to $A \odot B$,

$$\begin{aligned}(a \odot b)(a' \odot b') &= (-1)^{\deg(a')\deg(b)}\, aa' \odot bb' \\ (a \odot b)^* &= (-1)^{\deg(a)\deg(b)}\, a^* \odot b^*\end{aligned}$$

The closure in the graded spatial C*-norm [Ks1, 2.6] is the $\mathbb{Z}_2$-graded C*-algebra $(A\hat{\otimes}B, \gamma)$.

Let A or B be trivially graded. Then $A\hat{\otimes}B \cong A \otimes B$ [Ks1, 2.6].

Exterior Tensor Product of Hilbert modules

Let $(W, \pi^{\mathrm{op}}, \Gamma)$ and $(Y, \sigma^{\mathrm{op}}, \Upsilon)$ be graded Hilbert modules of A and B respectively.

Let $W \odot Y$ be the algebraic tensor product with grading element $\Phi = \Gamma \odot \Upsilon$. Then $(W \odot Y, \Phi)$ forms a graded pre-Hilbert $A \odot B$-module with the representation (for homogeneous element extended linearly)

$$(\pi^{\mathrm{op}} \odot \sigma^{\mathrm{op}})(a \odot b)(w \odot y) = (-1)^{\deg(y)\deg(a)}\, \pi^{\mathrm{op}}(a)w \odot \sigma^{\mathrm{op}}(b)y,$$

and $A \odot B$-valued inner product (for homogeneous element extended linearly)

$$\langle w \odot y, w' \odot y' \rangle = (-1)^{\deg(y)(\deg(w)+\deg(w'))}\, \langle w, w' \rangle \odot \langle y, y' \rangle.$$

The closure of $W \odot Y$ yields the Hilbert $A\hat{\otimes}B$-module $(W\hat{\otimes}Y, \pi^{\mathrm{op}}\hat{\otimes}\sigma^{\mathrm{op}})$ [Ks1, 9].

There is a continuous injection [Ks1, 9]

$$B_A(W)\hat{\otimes}B_B(Y) \to B_{A\hat{\otimes}B}(W\hat{\otimes}Y),$$

and a continuous isomorphism

$$K_A(W)\hat{\otimes}K_B(Y) \to K_{A\hat{\otimes}B}(W\hat{\otimes}Y).$$

Let W or Y be trivially graded (implying A or B are trivially graded). Then the skew-commutative tensor product $\hat{\otimes}$ reduces to the construction involving the tensor product $\otimes$ detailed in [Lc2, 4].

Interior Tensor Product of Hilbert modules

Let $(W, \pi^{\mathrm{op}}, \Gamma)$ be a graded Hilbert A-module and $(Y, \tau, \sigma^{\mathrm{op}}, \Gamma', \Upsilon)$ be a graded A-B-C*-bimodule. The interior product $W\hat{\otimes}_\tau Y$ is designed to yield a graded Hilbert B-module.

Define the subspace Z_τ of the algebraic tensor product $W \odot Y$ above

$$Z_\tau = \langle \pi^{\mathrm{op}}(a)w \odot y - w \odot \tau(a)y \mid w \in W, y \in Y, a \in A \rangle.$$

Define the grading element on $W \odot Y$ by $\Phi = \Gamma \odot \Gamma'\Upsilon$. As $\Phi Z_\tau = Z_\tau$, then

$$W \odot_\tau Y = W \odot Y / Z_\tau$$

is a graded right B-module

$$\sigma^{\mathrm{op}}_\tau(b)(w \odot_\tau y) = w \odot_\tau \sigma^{\mathrm{op}}(b)y.$$

with B-valued inner product [Lc2, 4.5]

$$\langle w \odot_\tau y, w' \odot_\tau y' \rangle = \langle y, \tau(\langle w, w' \rangle) y' \rangle.$$

The closure of $(W \odot_\tau Y, \sigma_\tau^{\mathrm{op}})$ defines the Hilbert B-module $(W \hat{\otimes}_\tau Y, \sigma_\tau^{\mathrm{op}})$.

The correspondence $T \mapsto T \otimes 1$ induces a graded representation [Lc2, 4] [Ks1, 2.8]

$$\tau_* : B_A(W) \to B_B(W \hat{\otimes}_\tau Y).$$

The statements of this section apply equally to Frechet pre-C*-algebras A and B with the substitution of the phrase (Frechet pre-C*) for (C*).

2.1.4 Morita Equivalence

Basic Definitions

Let A be a C*-algebra. Then A is called σ-unital if it admits a countable approximate unit. Indeed, A is σ-unital if and only if A is countably generated as an A-module [Lc2, 6]. It follows every separable C*-algebra is σ-unital.

Let K be the C*-algebra of compact operators on a Hilbert space. The stabilisation of a C*-algebra A is the C*-algebra $A_K = A \otimes K$. A C*-algebra A is called stable if $A \cong A_K$.

(Strong) Morita Equivalence

Definition 2.1.13 *Let A and B be C*-algebras. Then A is Morita equivalent to B, denoted $A \sim_M B$, if there exists a full Hilbert A-module (W, π^{op}) such that $B \cong K_A(W)$.*

Morita equivalence is an equivalence relation [Lc2, Prop 7.5], and sometimes called strong Morita equivalence.

Theorem 2.1.14 [BGR] *Let A and B be σ-unital C*-algebras. Then $A \sim_M B$ if and only if $A_K \cong B_K$.*

Example 2.1.15 Let A be a C*-algebra and $k \in \mathbb{N}$. Consider the full Hilbert A-modules A^k and H_A as in Example 2.1.3(i),(ii). Then

(i) $M_k(A) \sim_M A$ as $M_k(A) = K_A(A^k)$,

(ii) $A_K \sim_M A$ as $A_K = A \otimes K \cong K_A(H_A)$.

Graded Morita Equivalence

There exists an extended notion of Morita equivalence for $\mathbb{Z}_2$-graded C*-algebras.

Definition 2.1.16 *Let (A, α) and (B, β) be $\mathbb{Z}_2$-graded C*-algebras. Then A is graded Morita equivalent to B, denoted $A \sim_M B$, if there exists a full graded C*-A-B-bimodule $(W, \pi, \sigma^{\mathrm{op}}, \Gamma, \Upsilon)$ such that $(A, \alpha) \cong (K_B(W), \mathrm{ad}_\Gamma)$.*

Example 2.1.17 Graded Morita equivalence of A and $M_2(A)$

Let A be a $\mathbb{Z}_2$-graded C*-algebra and u a selfadjoint unitary in $M(A)$. Recall from Section 2.1.2 that $\mathrm{ad}_u \in \mathrm{Aut}_2(A)$ where $\mathrm{ad}_u(a) = uau^* \ \forall a \in A$. Define the operations $\Gamma_u, \Gamma'_u : A^2 \to A^2$ by

$$\begin{aligned}\Gamma_u(a,a') &= (ua, -ua') \ \ \forall a, a' \in A \\ \Gamma'_u(a,a') &= (au, -a'u) \ \ \forall a, a' \in A.\end{aligned}$$

Lemma 2.1.18 *Let (A, ad_u) and Γ'_u be as above. Then (A^2, Γ'_u) is a graded Hilbert A-module.*

Proof Let $u \in M(A)$ be unitary and selfadjoint. Then $u = u^*$ and $u^2 = 1$. Hence $\Gamma'^2_u(a,a') = (au^2, (-1)^2 a'u^2) = (a,a') \ \forall a, a' \in A$ and $\Gamma'^2 = 1$. Now $\Gamma'_u b^{\mathrm{op}}(a,a') = (abu, -a'bu) = (auubu, -a'uubu) = (au(ubu), -a'(ubu)) = (ubu)^{\mathrm{op}}(au, a'u) = \mathrm{ad}_u(b)^{\mathrm{op}}\Gamma'_u(a,a') \ \forall a, a' \in A$. □

Let $u \in M(A)$ be a selfadjoint unitary. Let γ_u be the selfadjoint unitary in $M_2(M(A)) = M(M_2(A))$ defined by

$$\gamma_u = \begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}.$$

Theorem 2.1.19 *Let A be a C*-algebra and $u \in M(A)$ be a selfadjoint unitary. Then $(A, \mathrm{ad}_u) \sim_M (M_2(A), \mathrm{ad}_{\gamma_u})$ as $\mathbb{Z}_2$-graded C*-algebras.*

Proof From Lemma 2.1.18 (A^2, Γ'_u) is a graded Hilbert A-module. From Example 2.1.15(i), $M_2(A) = K_A(A^2)$. The proof is complete if it is shown $\Gamma_u = \gamma_u$. Note $\Gamma_u \in B_B(A^2) = M_2(M(A))$ is immediate from its definition. Moreover $\Gamma_u^2 = 1$ and $\langle \Gamma_u(a,a'), (b,b')\rangle = (ua)^*b + (ua')^*b' = a^*ub + a'^*ub' = \langle (a,a'), \Gamma'_u(b,b')\rangle$. Hence Γ is a selfadjoint unitary of $M_2(M(A))$. It is immediate from using an approximate unit of A that the matrix representation of Γ_u is γ_u. □

2.2 Non-commutative Vector Bundles

Let X be a locally compact Hausdorff space. The Serre-Swan Theorem (Theorem 2.1.9) asserted the one-to-one correspondence between finite projective $C_0(X)$-modules and Hermitian vector bundles $E \to X$ with finite dimensional fibres and finite trivialising cover. When X is compact every Hermitian vector bundle has a finite trivialising cover. This leads to the definition in the literature,

Definition *A non-commutative vector bundle of a unital C*-algebra A is a finite projective A-module.*

2.3 Graded Hilbert Modules in Riemannian Geometry

Let X be a Riemannian manifold and $C_0(X)$ be the C*-algebra of continuous vanishing at infinity functions on X. Graded Hilbert $C_0(X)$-modules provide examples of the concepts in section 2.1 which are of central importance in the generalisation of Riemannian structure to non-commutative geometry.

Basic Definitions

The two-dimensional Clifford algebra C_1 is abstractly defined by

$$C_1 = \{\lambda_1 + h\lambda_2 \mid \lambda_1, \lambda_2 \in \mathbb{C}, h^2 = 1\}$$

with a $\mathbb{Z}_2$-grading $\beta : \lambda_1 + h\lambda_2 \to \lambda_1 - h\lambda_2$. A concrete graded representation $c_1 : C_1 \to B(\mathbb{C}^2) = M_2(\mathbb{C})$ is provided by

$$\lambda_i \mapsto \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}, \quad h \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \beta \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

2.3.1 Structure and Gradings on the Clifford Bundle

Let $\Lambda^* X$ and $\mathrm{Cl}(X)$ denote the exterior bundle and Clifford bundles of X respectively as in Section 1.3.3.

Parity and the Volume Form

Let V be a n-dimensional vector space with inner product q and $\{v_i\}_{i=1}^n$ be an orthonormal basis of V. We recall from Section 1.3.1 the Clifford algebra $\mathrm{Cliff}(V,q)$ with Clifford product $\cdot$ satisfying $w \cdot v + v \cdot w = -2q(w,v)$ $\forall v, w \in V$. Define the parity map ϵ_V on homogeneous elements of $\mathrm{Cliff}(V,q)$ by

$$\epsilon_V(v_{i_1} \cdot \ldots \cdot v_{i_j}) := (-1)^j v_{i_1} \cdot \ldots \cdot v_{i_j}$$

and extend it linearly to be an order two automorphism ϵ_V of $\mathrm{Cliff}(V,q)$. We define the volume form $\gamma_V \in \mathrm{Cliff}^n(V,q)$ by

$$\gamma_V := i^n v_1 \cdot \ldots \cdot v_n.$$

The relations $v \cdot \gamma_V = (-1)^{n-1} \gamma_V \cdot v$ for all $v \in V$ and $\gamma_V \cdot \gamma_V = 1$ follow from the Clifford product. This implies γ_V is a central element when n is odd. We note $\epsilon_V(\gamma_V) = (-1)^n \gamma_V$.

Extend the parity map to the C*-algebra $C_0(X, \mathrm{Cl}(X))$ of continuous vanishing at infinity sections of the Clifford bundle by

$$\epsilon(w)(x) := \epsilon_{T_x^* X}(w(x)) \ \forall w \in C_0(X, \mathrm{Cl}(X)).$$

Then ϵ is an order two *-automorphism of the C*-algebra $C_0(X, \mathrm{Cl}(X))$. Hence $(C_0(X, \mathrm{Cl}(X)), \epsilon)$ is a $\mathbb{Z}_2$-graded C*-algebra. The C*-algebra $C_0(X)$ inherits a trivial $\mathbb{Z}_2$-grading by the restriction of ϵ to $C_0(X) \subset C_0(X, \mathrm{Cl}(X))$.

The complex volume form of a n-dimensional Riemannian manifold X is the element $\gamma \in C_b(X, \mathrm{Cl}(X)) = M(C_0(X, \mathrm{Cl}(X)))$ defined by

$$\gamma(x) := \gamma_{T_x^* X}$$

The complex volume form has the properties $\epsilon(\gamma) = (-1)^n\gamma$, $\gamma = \gamma^*$, $\gamma \cdot \gamma = 1$, $\gamma \cdot f = f \cdot \gamma$ for all $f \in C_b(X)$ and $v \cdot \gamma = (-1)^{n-1}\gamma \cdot v$ for all $v \in C_b(X, T^*X)$. Note $\gamma \in Z(C_b(X, \mathrm{Cl}(X)))$ when $\dim X$ is odd.

Define a unitarily implemented $\mathbb{Z}_2$-grading α of the C*-algebra $C_0(X, \mathrm{Cl}(X))$ by

$$\alpha(w) := \gamma \cdot w \cdot \gamma.$$

The relationship between the $\mathbb{Z}_2$-gradings ϵ and α is summarised in the next theorem. We recall the two-dimensional Clifford algebra

$$C_1 = \{\lambda_1 + \lambda_2\beta \,|\, \lambda_1, \lambda_2 \in \mathbb{C}, \beta^2 = 1\}$$

is $\mathbb{Z}_2$-graded by the map $\lambda_1 + \lambda_2\beta \to \lambda_1 - \lambda_2\beta$.

Theorem 2.3.1 *Let X be an n-dimensional Riemannian manifold. Let ϵ be the parity map on $C_0(X, \mathrm{Cl}(X))$ and $\gamma \in C_b(X, \mathrm{Cl}(X))$ be the complex volume form.*
Let $\dim X$ be even. Then

(i) *ϵ is unitarily implemented by γ,*

$$\epsilon(w) = \gamma \cdot w \cdot \gamma$$

for all $w \in C_0(X, \mathrm{Cl}(X))$,

(ii) *$Z(C_0(X, \mathrm{Cl}(X))) = C_0(X)$,*

(iii) *ϵ is trivial on the centre $Z(C_0(X, \mathrm{Cl}(X)))$*

Let $\dim X$ be odd. Then

(iv) *γ is a central element of $C_b(X, \mathrm{Cl}(X))$, hence α is a trivial $\mathbb{Z}_2$-grading,*

(v) *$\epsilon(\gamma) = -\gamma$, hence ϵ is a non-trivial $\mathbb{Z}_2$-grading,*

(vi) *$Z(C_0(X, \mathrm{Cl}(X))) = \{f_1 + f_2\gamma \,|\, f_1, f_2 \in C_0(X)\} \cong C_0(X) \otimes C_1$ where C_1 is the graded two-dimensional Clifford algebra.*

(vii) *ϵ provides the $\mathbb{Z}_2$-grading on $Z(C_0(X, \mathrm{Cl}(X))) \cong C_0(X) \otimes C_1$.*

Proof Straightforward from the properties of the volume form and the Clifford algebra. □

2.3.2 Graded Representations

We consider representations of the C*-algebras $M(C_0(X))$ and $M(C_0(X, \mathrm{Cl}(X)))$ into the space $L(C_0(X, \Lambda^*X), C_0(X, \Lambda^*X))$. We recall from section 1.3.1 and section 1.3.3 the left and right actions of the Clifford bundle on the Exterior bundle. The left action provides a representation

$$\pi_l : M(C_0(X, \mathrm{Cl}(X))) \to L(C_0(X, \Lambda^*X), C_0(X, \Lambda^*X))$$

given by

$$(\pi_l(w)u)(x) = \iota(w(x) \cdot \iota^{-1}(u(x)))$$

for all $w \in M(C_0(X, \mathrm{Cl}(X))), u \in C_0(X, \Lambda^* X)$. The right action provides an opposite representation

$$\pi_r : M(C_0(X, \mathrm{Cl}(X))) \to L(C_0(X, \Lambda^* X), C_0(X, \Lambda^* X))$$

given by

$$(\pi_r(w)u)(x) = \iota(\iota^{-1}(u(x)) \cdot w(x))$$

for all $w \in M(C_0(X, \mathrm{Cl}(X))), u \in C_0(X, \Lambda^* X)$. We note $[\pi_r(w), \pi_l(v)] = 0$ for all $w, v \in M(C_0(X, \mathrm{Cl}(X)))$ from associativity of the Clifford product. We denote the restricted representation to the centre $Z(C_b(X, \mathrm{Cl}(X)))$ by π_l as well and

$$\pi_l : Z(C_0(X, \mathrm{Cl}(X))) \to L(C_0(X, \Lambda^* X), C_0(X, \Lambda^* X)).$$

Note $\pi_l(w) = \pi_r(w)$ for all $w \in Z(C_b(X, \mathrm{Cl}(X)))$. Then π_l is a representation of $C_0(X)$ as $C_0(X) \subset Z(C_b(X, \mathrm{Cl}(X)))$. As mentioned in section 1.3.6 the representation π_l extends to a concrete representation $(L^2(X, \Lambda^* X), \pi_l)$ of the C*-algebra $M(C_0(X, \mathrm{Cl}(X)))$ and $C_0(X)$ by restriction. Let $\pi_l(\gamma)$ and $\pi_r(\gamma)$ be the representatives of the volume form $\gamma \in M(C_0(X, \mathrm{Cl}(X)))$ of the Riemannian manifold X.

Theorem 2.3.2 (Riemannian Structure)

Let X be an even dimensional Riemannian manifold. Then

(i) $(C_0(X, \Lambda^* X), \pi_r)$ *is a Hilbert $C_0(X)$-module, that is finite projective if X is compact, such that the Riemannian metric q_g defines the $C_0(X)$-valued inner product,*

(ii) $(C_0(X, \Lambda^* X), \pi_r, \pi_r(\gamma))$ *is a graded finite projective $C_0(X, \mathrm{Cl}(X))$-Hilbert module,*

(iii) $(C_0(X, \Lambda^* X), \pi_l, \pi_r)$ *is a $C_0(X)$-$C_0(X)$-C*-bimodule,*

(iv) $(C_0(X, \Lambda^* X), \pi_l, \pi_r, \pi_r(\gamma))$ *is a graded $C_0(X)$-$C_0(X, \mathrm{Cl}(X))$-C*-bimodule,*

(v) $(C_0(X, \Lambda^* X), \pi_l, \pi_r, \pi_l(\gamma))$ *is a graded $C_0(X, \mathrm{Cl}(X))$-$C_0(X)$-C*-bimodule,*

(vi) $(C_0(X, \Lambda^* X), \pi_l, \pi_r, \pi_l(\gamma), \pi_r(\gamma))$ *is a graded*
$C_0(X, \mathrm{Cl}(X))$-$C_0(X, \mathrm{Cl}(X))$-C-bimodule,*

(vii) $(L^2(X, \Lambda^* X), \pi_l \otimes \pi_r, \pi_l(\gamma)\pi_r(\gamma))$ *is a graded*
$C_0(X, \mathrm{Cl}(X)) \otimes C_0(X, \mathrm{Cl}(X))^{\mathrm{op}}$-$\mathbb{C}$-C-bimodule.*

Proof (i) Lemma 2.1.6, Theorem 2.1.9 and Section 1.3.6. (ii) Define the $C_0(X, \mathrm{Cl}(X))$-valued inner product on $C_0(X, \Lambda^* X)$ by $\langle u, z \rangle = \iota(u)^* \iota(z)$ for all $u, z \in C_0(X, \Lambda^* X)$. The module $C_0(X, \Lambda^* X)$ is full in this inner product. Since $C_0(X, \Lambda^* X) \cong C_0(X, \mathrm{Cl}(X))$ linearly it is clearly a finite and projective $C_0(X, \mathrm{Cl}(X))$-module. That the grading element required is $\pi_r(\gamma)$ follows from $\pi_r(\gamma)^2 = 1$ and $\pi_r(\gamma)\pi_r(w) = \pi_r(\epsilon(w))\pi_r(\gamma)$ by Theorem 2.3.1. (vi) Follows from $\pi_l(\gamma)^2 = 1 = \pi_r(\gamma)^2$, $\pi_l(\gamma)\pi_l(w)\pi_l(\gamma) = \pi_l(\epsilon(w))$ and $[\pi_l(w), \pi_r(\gamma)] = 0 = [\pi_l(\gamma), \pi_r(w)]$ for all $w \in C_0(X, \mathrm{Cl}(X))$. (iii),(iv),(v) follow directly from (vi). (viii) Follows from (vi) by restricting to compactly supported sections and closing in the inner product of section 1.3.6(ii). □

Let X be a Riemannian manifold with odd dimension. We recall from Theorem 2.3.1 that $\pi_l(\gamma) = \pi_r(\gamma)$ is a central element and that

$$Z(\pi_l(C_0(X, \mathrm{Cl}(X))) = \mathrm{Span}_{C_0(X)}(1, \pi_l(\gamma)) \cong C_0(X) \otimes C_1$$

where C_1 is the graded two-dimensional Clifford algebra. Any grading induced by $\pi_l(\gamma)$ will be a trivial grading. We cannot obtain a graded representation of the $\mathbb{Z}_2$-graded C*-algebra $(C_0(X, \mathrm{Cl}(X)), \epsilon)$ on the module $(C_0(X, \Lambda^* X), \pi_r)$ using the complex volume form.

Let V be a n-dimensional vector space with inner product q and orthonormal basis $\{v_i\}_{i=1}^n$. We recall from Section 1.3.1 the Exterior algebra $\Lambda(V)$ with exterior product satisfying $w \wedge v + v \wedge w = 0 \ \forall v, w \in V$. Define the parity map ε_V on homogeneous elements of $\Lambda(V)$ by

$$\varepsilon_V(v_{i_1} \wedge ... \wedge v_{i_j}) := (-1)^j v_{i_1} \wedge ... \wedge v_{i_j}$$

and extend it linearly to be an order two automorphism ε_V of $\Lambda(V)$. Let $\Lambda^{\mathrm{even}}(V)$ denote the even elements of $\Lambda(V)$ with respect to the parity grading and $\Lambda^{\mathrm{odd}}(V)$ the odd elements. The grading ϵ_V splits the exterior algebra into a direct sum

$$\Lambda(V) \equiv \Lambda^{\mathrm{even}}(V) \oplus \Lambda^{\mathrm{odd}}(V)$$

and ε_V has the matrix representation

$$\varepsilon_V \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is immediate from the form of the left action

$$w_l(u) := \iota(w \cdot \iota^{-1}(u))$$

of $w \in \mathrm{Cliff}(V, q)$ on $u \in \Lambda(V)$ that

$$w_l^{\mathrm{even}}(\Lambda^{\mathrm{even}}(V)) = \Lambda^{\mathrm{even}}(V)\ ,\ w_l^{\mathrm{odd}}(\Lambda^{\mathrm{odd}}(V)) = \Lambda^{\mathrm{even}}(V)$$

$$w_l^{\mathrm{odd}}(\Lambda^{\mathrm{even}}(V)) = \Lambda^{\mathrm{odd}}(V)\ ,\ w_l^{\mathrm{even}}(\Lambda^{\mathrm{odd}}(V)) = \Lambda^{\mathrm{odd}}(V).$$

Identical relations hold for the right action

$$w_r(u) := \iota(\iota^{-1}(u) \cdot w)$$

of $w \in \mathrm{Cliff}(V, q)$ on $u \in \Lambda(V)$. Hence the matrix representation of the left action of $w \in \mathrm{Cliff}(V, q)$ on $\Lambda(V)$ with respect to the grading ε_V is

$$w_l \equiv \begin{bmatrix} w_l^{\mathrm{even}} & w_l^{\mathrm{odd}} \\ w_l^{\mathrm{odd}} & w_l^{\mathrm{even}} \end{bmatrix},$$

and

$$\mathrm{ad}_{\varepsilon_V}(w_l) = (\epsilon_V(w))_l.$$

Identical relations hold for the right action. Then γ_V has the matrix representation in the left action or right action of Cliff(V, q) on $\Lambda(V)$,

$$\gamma_V \equiv \begin{bmatrix} 0 & (\gamma_V)_l \\ (\gamma_V)_l & 0 \end{bmatrix} = \begin{bmatrix} 0 & (\gamma_V)_r \\ (\gamma_V)_r & 0 \end{bmatrix}$$

and

$$\mathrm{ad}_{\varepsilon_V}(\gamma_V) = -\gamma_V.$$

Extend the parity grading ε_V to the $C_0(X)$-Hilbert module $(C_0(X, \Lambda^* X), \pi_r)$ by

$$\varepsilon(u)(x) := \varepsilon_{T_x^* X}(u(x)) \ \ \forall\, u \in C_0(X, \Lambda^* X).$$

Then ε is a grading element of $(C_0(X, \Lambda^* X), \pi_r)$ that is $C_0(X)$-linear. The direct sum decomposition into sub $C_0(X)$-Hilbert modules is given by

$$C_0(X, \Lambda^* X) \equiv C_0(X, \Lambda^{\mathrm{even}}(X)) \oplus C_0(X, \Lambda^{\mathrm{odd}}(X)).$$

The matrix representation of ε is given by

$$\varepsilon \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\pi_l(\gamma) \equiv \begin{bmatrix} 0 & \pi_l(\gamma) \\ \pi_l(\gamma) & 0 \end{bmatrix}$$

with

$$\mathrm{ad}_\varepsilon(\pi_l(\gamma)) = -\pi_l(\gamma).$$

Let U denote the unitary operator on the Hilbert space $L^2(X, \Lambda^* X)$ such that $U \equiv \varepsilon$ on the dense subspace $C_0(X, \Lambda^* X)$ [2].

Theorem 2.3.3 (Riemannian Structure)

Let X be an odd dimensional Riemannian manifold and C_1 be the graded two dimensional Clifford algebra. Then

(i) $(C_0(X, \Lambda^* X), \pi_r)$ *is a Hilbert $C_0(X)$-module, that is finite projective if X is compact, such that the Riemannian metric q_g defines the $C_0(X)$-valued inner product,*

(ii) $(C_0(X, \Lambda^* X), \pi_r, \varepsilon)$ *is a graded Hilbert $C_0(X) \otimes C_1$-module*

(iii) $(C_0(X, \Lambda^* X), \pi_r, \varepsilon)$ *is a graded finite projective $C_0(X, \mathrm{Cl}(X))$-Hilbert module,*

(iv) $(C_0(X, \Lambda^* X), \pi_l, \pi_r, \varepsilon)$ *is a graded $C_0(X)$-$C_0(X) \otimes C_1$-C^*-bimodule,*

(v) $(C_0(X, \Lambda^* X), \pi_l, \pi_r, \varepsilon)$ *is a graded $C_0(X) \otimes C_1$-$C_0(X)$-C^*-bimodule,*

(vi) $(C_0(X, \Lambda^* X), \pi_l, \pi_r, \varepsilon)$ *is a graded $C_0(X, \mathrm{Cl}(X))$-$C_0(X)$-C^*-bimodule,*

[2] The existence of U is detailed in Theorem 2.3.4(iv) and Lemma 2.5.18.

(vii) $(L^2(X,\Lambda^*X),\pi_l\otimes\pi_r,U)$ *is a graded* $C_0(X,\mathrm{Cl}(X))\otimes C_0(X,\mathrm{Cl}(X))^{\mathrm{op}}$*-*$\mathbb{C}$*-*$C^*$*-bimodule.*

Proof (i) Lemma 2.1.6, Theorem 2.1.9 and Section 1.3.6. (iii) Same argument as the proof of Theorem 2.3.2(ii) except for the grading. That $\mathrm{ad}_\varepsilon(\pi_r(w))=\pi_r(\epsilon(w))$ follows from the statements preceding the theorem. (ii) Follows from (iii) since $C_0(X)\otimes C_1\cong Z(C_0(X,\mathrm{Cl}(X)))$. (iv),(v),(vi) Straightforward. (vii) Follows from (vi) and (iii) by restricting to compactly supported sections and closing in the inner product of section 1.3.6(ii). The existence of U we defer to Theorem 2.3.4(iv). Note that $U=U^*$ and $U\pi_l\otimes\pi_r(w)U=\pi_l\otimes\pi_r(\epsilon(w))$. □

2.3.3 Riemannian Structure

Let X be a Riemannian manifold. Remark 1.6.2, Theorem 1.7.21, Theorem 2.3.2 and Theorem 2.3.3 have been labeled 'Riemannian Structure'. We compare the results and emphasise their unification and generalisation in the Tomita-Takesaki modular theory.

The finite projective $C_0(X)$-module $(C_0(X,\Lambda^*X),\pi_r)$ can be determined, by Remark 1.6.12 with Theorem 2.3.2 and Theorem 2.3.3, amongst other Hilbert $C_0(X)$-modules by its properties as a graded $C_0(X,\mathrm{Cl}(X))$-bimodule. In particular, $(C_0(X,\Lambda^*X),\pi_r)$ is a free graded finite $C_0(X,\mathrm{Cl}(X))$-module of module dimension one and a graded $C_0(X,\mathrm{Cl}(X))$-$C_0(X)$-C*-bimodule. Equivalently:

Riem′: The representation $(L^2(X,\Lambda^*X),\pi_l)$ of the C*-algebra $C_0(X)$ is the unique graded $C_0(X,\mathrm{Cl}(X))\otimes C_0(X,\mathrm{Cl}(X))^{\mathrm{op}}$- $\mathbb{C}$-C*-bimodule that admits a dense subspace isomorphic to $C_c(X,\mathrm{Cl}(X))$.

Compare the statement **Riem′** to Remark 1.6.12. That there exists exactly a unique Hilbert space $L^2(X,\Lambda^*X)$ which admits a standard form for the von Neumann algebra $L^\infty(X,\mathrm{Cl}(X))$,

$$\mathbf{Riem}=(L^\infty(X,\mathrm{Cl}(X)),L^2(X,\Lambda^*X),J,1,L^2(X,\Lambda^*X)^+)$$

and the standard form **Riem** is constructed from the pair of the von Neumann algebra $L^\infty(X,\mathrm{Cl}(X))$ and the fns trace weight λ given by the Lebesgue integral and the metric q_g,

$$\lambda(w)=\int_X q_g(1,\pi_l(w)1)(x)\sqrt{\det(g)}dx\ \ \forall w\in L^\infty(X,\mathrm{Cl}(X)).$$

Define

$$\epsilon(w)(x):=\epsilon_{T^*_xX}(w(x))\ \ \forall w\in L^\infty(X,\mathrm{Cl}(X)).$$

Then ϵ is an order two *-automorphism of the von Neumann algebra $L^\infty(X,\mathrm{Cl}(X))$. We claim the standard form **Riem** contains the statement **Riem′**.

Theorem 2.3.4 *Let* **Riem** *be the standard form as above.*

(i) *The GNS representation associated to the Lebesgue integral*

$$\pi_\lambda : L^\infty(X, \mathrm{Cl}(X)) \to B(L^2(X, \Lambda^* X))$$

restricts identically to the representation

$$\pi_l : C_0(X, \mathrm{Cl}(X)) \to L(C_c(X, \Lambda^* X), C_c(X, \Lambda^* X))$$

on $C_0(X, \mathrm{Cl}(X)) \subset L^\infty(X, \mathrm{Cl}(X))$ *and* $C_c(X, \Lambda^* X) \subset L^2(X, \Lambda^* X)$.

(ii) *The opposite representation of* $L^\infty(X, \mathrm{Cl}(X))$ *provided by the Tomita conjugation operator*

$$\pi_\lambda^{\mathrm{op}}(w) := J\pi_\lambda(w)^* J$$

for all $w \in L^\infty(X, \mathrm{Cl}(X))$ *restricts identically to the right representation*

$$\pi_r : C_0(X, \mathrm{Cl}(X))^{\mathrm{op}} \to L(C_c(X, \Lambda^* X), C_c(X, \Lambda^* X))$$

on $C_0(X, \mathrm{Cl}(X)) \subset L^\infty(X, \mathrm{Cl}(X))$ *and* $C_c(X, \Lambda^* X) \subset L^2(X, \Lambda^* X)$.

(iii) *The representation*

$$\pi \otimes \pi^{\mathrm{op}} : L^\infty(X, \mathrm{Cl}(X)) \otimes L^\infty(X, \mathrm{Cl}(X))' \to B(L^2(X, \Lambda^* X))$$

provided by

$$\pi \otimes \pi_\lambda^{\mathrm{op}}(w \otimes u) := \pi_\lambda(w) J \pi_\lambda(u)^* J$$

restricts identically to the representation

$$\pi_l \otimes \pi_r : C_0(X, \mathrm{Cl}(X)) \otimes C_0(X, \mathrm{Cl}(X))^{\mathrm{op}} \to B_{C_0(X)}(C_c(X, \Lambda^* X))$$

on $C_0(X, \mathrm{Cl}(X)) \subset L^\infty(X, \mathrm{Cl}(X))$ *and* $C_c(X, \Lambda^* X) \subset L^2(X, \Lambda^* X)$.

(iv) *Let* $\alpha \in \mathrm{Aut}_2(L^\infty(X, \mathrm{Cl}(X)))$. *Then there exists a selfadjoint unitary* $U_\alpha \in B(L^2(X, \Lambda^* X))$ *such that*

$$\mathrm{ad}_{U_\alpha}(\pi_\lambda(w)) = \pi_\lambda(\alpha(w))$$

for all $w \in L^\infty(X, \mathrm{Cl}(X))$ *and*

$$[U_\alpha, J] = 0.$$

Let α *be the parity automorphism* ϵ. *Then* $U_\epsilon \equiv \varepsilon$ *when restricted to the subspace* $C_c(X, \Lambda^* X) \subset L^2(X, \Lambda^* X)$.

(v) *The GNS inclusion map*

$$\iota_\lambda : L^1(X, \mathrm{Cl}(X)) \cap L^\infty(X, \mathrm{Cl}(X)) \to L^2(X, \Lambda^* X)$$

induces a dense subspace $\iota_\lambda(C_c(X, \mathrm{Cl}(X)))$ *of* $L^2(X, \Lambda^* X)$ *isomorphic to* $C_c(X, \mathrm{Cl}(X))$.

Proof (i),(ii),(iii) Immediate from Theorem 1.6.1 and Proposition 1.6.11. (iv) Theorem 1.6.4 with **Riem** and **Riem** as the two standards forms and α the isomorphism. (v) Immediate from Section 1.2.1 and Definition-Lemma 1.5.9. $\square$

The moral of Remark 1.6.12 and Theorem 2.3.4 is that one obtains the statement **Riem′**, indeed one *constructs the graded finite* $C_0(X, \mathrm{Cl}(X))$*-bimodule* $(L^2(X, \Lambda^* X), \pi_r)$ *complete with grading element* U_ϵ, from the $\mathbb{Z}_2$-graded von Neumann algebra $(L^\infty(X, \mathrm{Cl}(X)), \epsilon)$ with semifinite faithful normal trace λ using the GNS construction and Tomita-Takesaki theory.

Let X be a compact Riemannian manifold. The moral of A. Connes viewpoint and Theorem 1.7.21 is one *constructs the von Neumann algebra* $L^\infty(X, \mathrm{Cl}(X))$ *and the trace provided by the Lebesgue integral* λ from the Frechet pre-C*-algebra $C^\infty(X)$ and the signature operator $d + d^*$ (statements (ii), (iii) and (iv) of Theorem 1.7.21). Hence the statement **Riem′** reaches its most general form for a compact Riemannian manifold X in statement (v) of Theorem 1.7.21:

> The representation $(L^2(X, \Lambda^* X), \pi_l)$ of $\Omega_{d+d^*}(C^\infty(X))''$ is the GNS representation associated to the trace $\tilde{\tau}_\omega$.

This statement characterises the representation $(L^2(X, \Lambda^* X), \pi_l)$ amongst all other representations (H, π) of the C*-algebra $C(X)$. It also a statement that we can generalise to an arbitrary C*-algebra.

2.3.4 Riemannian Representations

Let (H, π, D) be a C_c^∞-representation of a C*-algebra A. Let $\mathcal{A}_\pi$ be the 'smooth elements' of A for this representation (the Frechet pre-C*-subalgebra of A defined in Section 1.4.2). Let $(\Omega_D(\mathcal{A}_\pi), \delta_D)$ be the representation $\tilde{\pi}$ of the universal graded differential algebra $(\Omega(\mathcal{A}_\pi), \delta)$ as in Section 1.4.1. The universal differential algebra $(\Omega(\mathcal{A}_\pi), \delta)$ has the natural parity grading

$$\epsilon_\Omega(a_0 \delta a_1 ... \delta a_k) := (-1)^k a_0 \delta a_1 ... \delta a_k$$

and the direct sum decomposition

$$\Omega(\mathcal{A}_\pi) = \Omega^{\mathrm{even}}(\mathcal{A}_\pi) \oplus \Omega^{\mathrm{odd}}(\mathcal{A}_\pi).$$

This parity grading cannot be transferred to the representation $(\Omega_D(\mathcal{A}_\pi), \delta_D)$ in general [3]. We say the representation $\tilde{\pi}$ of $(\Omega(\mathcal{A}_\pi), \delta)$ is *parity preserving* if the map

$$\epsilon(\pi(a_0)[D, \pi(a_1)]...[D, \pi(a_k)]) := (-1)^k \pi(a_0)[D, \pi(a_1)]...[D, \pi(a_k)]$$

is well defined on $\Omega_D(\mathcal{A}_\pi)$.

Definition 2.3.5 *Let R be a von Neumann algebra with separable pre-dual and (H_ρ, π_ρ) be the GNS representation associated to a faithful normal semifinite weight ρ on R. Let A be a C*-subalgebra of R.*

If there exists a selfadjoint operator $D : DomD \to H_\rho$ such that

(i) (H_ρ, π_ρ, D) *is a* C_c^∞*-representation of* A,

(i′) (H_ρ, π_ρ, D) *is a* $C_c^{n,\infty}$*-representation of* A,

[3] For example, if D is bounded and $D \in M(\mathcal{A}_\pi)$ then $[D, \pi(a)] \in \mathcal{A}_\pi$ for all $a \in \mathcal{A}_\pi$. The parity grading with $\mathcal{A}_\pi$ even and $\Omega_D^1(\mathcal{A}_\pi)$ odd is then not well defined.

(ii) $\tilde{\pi}_\rho(\Omega(\mathcal{A}_{\pi_\rho}))$ *is a weak dense *-subalgebra of* $\pi_\rho(R)$*, and*

(iii) $\tilde{\pi}_\rho$ *is parity preserving,*

then we call (H_ρ, π_ρ, D) (1) *a Riemannian representation of the* C^**-algebra* A *when conditions* (i), (ii), *and* (iii) *are satisfied, and* (2) *an* n*-dimensional Riemannian representation of the* C^**-algebra* A *when conditions* (i′), (ii), *and* (iii) *are satisfied.*

A base representation (H_ρ, π_ρ, D) of a C*-algebra A is called an ungraded Riemannian representation if it satisfies Definition 2.3.5 and conditions (i) and (ii) alone. We shall seldom use the ungraded notion until section 2.7.

Remark 2.3.6 Associated to each Riemannian representation of a C*-algebra A is the standard form (see Section 1.6.2)

$$\mathbf{Riem}(A, \rho) = (\Omega_D(\mathcal{A}_{\pi_\rho})'', H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho).$$

Hence Riemannian representations are intimately bound to the Tomita-Takesaki modular theory of von Neumann algebras[4].

2.3.5 Spin$_{\mathbb{C}}$ Representations

Let V be a n-dimensional vector space with inner product q and $\{v_i\}_{i=1}^n$ be an orthonormal basis of V. Define the function

$$m(n) = \begin{cases} 2^{\frac{n}{2}} & n \text{ even} \\ 2^{\frac{n-1}{2}} & n \text{ odd}. \end{cases}$$

Then the Clifford algebra Cliff(V, q) has a unique irreducible representation $(\mathbb{C}^{m(n)}, \phi)$ such that $\phi(\text{Cliff}(V, q)) = M_{m(n)}(\mathbb{C})$ [5]. Note ϕ is an isomorphism when n is even. The representation is not faithful in the odd case. Let n be odd. The volume element γ_V and the identity 1 generate the centre of Cliff(V, q) which is trivialised in any irreducible representation. Then Cliff$(V, q) \cong M_{m(n)}(\mathbb{C}) \otimes C_1$ where C_1 is the two-dimensional Clifford algebra.

These irreducible representations cannot always be transferred fibrewise to the Clifford bundle on a Riemannian manifold.

Let X be a n-dimensional Riemannian manifold. Then X is called a Riemannian spin$_{\mathbb{C}}$ manifold if there exists a Hermitian vector bundle $S \to X$ with a representation $\phi : C_0(X, \text{Cl}(X)) \to L(C_0(X, S), C_0(X, S))$ such that $C_0(X, \text{Cl}(X))) \cong K_{C_0(X)(\otimes C_1)}(C_0(X, S)(\otimes \mathbb{C}^2))$. Here C_1 is the graded two dimensional Clifford algebra and $(\otimes C_1)$ is added when $\dim X$ is odd. This implies $C_0(X, \text{Cl}(X)) \sim_M C_0(X)(\otimes C_1)$ where $\sim_M$ denotes strong Morita equivalence. The converse is also true, [HP, Lemma 3, Theorem 8(ii)].

[4]The relationship between the operators D, J_ρ, Δ_ρ is a very interesting question. It is a direction of further research.

[5]The details of the representation ϕ are not relevant here, see [LM].

Definition 2.3.7 *Let X be Riemannian manifold. We call X a Riemannian spin$_{\mathbb{C}}$ manifold if the C*-algebras $C_0(X)(\otimes C_1)$ and $C_0(X, \mathrm{Cl}(X))$ are strong Morita equivalent. Here C_1 is the graded two dimensional Clifford algebra and $(\otimes C_1)$ is added when $\dim X$ is odd.*

This definition of spin$_{\mathbb{C}}$ structure on a Riemannian manifold we can generalise to an arbitrary C*-algebra.

Definition 2.3.8 *Let A be a C*-algebra. Let (H_ρ, π_ρ, D) be a Riemannian representation of the C*-algebra A. Then we call (H_ρ, π_ρ, D) an even (resp. odd) spin$_R$ representation of A if $\mathcal{A}_\pi \sim_M \Omega_D(\mathcal{A}_\pi)$ (resp. $\mathcal{A}_\pi \otimes C_1 \sim_M \Omega_D(\mathcal{A}_\pi)$).*

Remark 2.3.9 Let X be a Riemannian manifold. The statements of this section (Section 2.3) can be applied to the pre-C*-algebras $C_0^\infty(X)$ and $C_0^\infty(X, \mathrm{Cl}(X))$ and the $C_0^\infty(X)$-module $C_0^\infty(X, \Lambda^* X)$ by replacing verbatim C^∞ for C, pre-Hilbert for Hilbert and pre-C* for C*.

2.4 Poincaré Duality in KK-theory

2.4.1 The elements of KK-theory

We refer the reader to the sources [Ks1] and [Ks2] for the detailed definitions and results of this theory. It is designed specifically for C*-algebras A and B (equivalently for pre-C*-algebras by using pre-Hilbert modules).

Definition 2.4.1 *A Kasparov A-B-bimodule is a triple (E, F, α) involving*

(i) *a countably generated graded Hilbert B-module (E, α) with a graded representation $\pi : A \to B_B(E)$,*

(ii) *an operator $F \in B_B(E)$ such that F is odd with respect to α and $\pi(a)(F - F^*), \pi(a)(F^2 - 1_E), [F, \pi(a)]_g$ are elements of $K_B(E)$ for all $a \in A$.*

Here $[\cdot,\cdot]_g$ is the graded commutator with respect to α. The Kasparov bimodule is called degenerate if $\pi(a)(F - F^*) = 0 = \pi(a)(F^2 - 1_E) = [F, \pi(a)]_g$. The collection $\mathcal{E}(A, B)$ of Kasparov A-B-bimodules and $\mathcal{D}(A, B)$ of degenerate Kasparov A-B-bimodules are closed under the direct sum $(E_1 \oplus E_2, F_1 \oplus F_2, \alpha_1 \oplus \alpha_2)$ of two Kasparov A-B-bimodules $(E_i, F_i, \alpha_i), i = 1, 2$.

Theorem 2.4.2 *Let $KK(A, B) = \Big(\mathcal{E}(A, B)/\mathcal{D}(A, B)\Big)/\sim$ where the equivalence relation $\sim$ is defined by homotopy equivalence. Then $(KK(A, B), \oplus)$ is an Abelian group.*

For separable C*-algebras A_1, A_2 and D there exists an associative bilinear product,

$$\otimes_D : KK(A_1, B_1 \hat{\otimes} D) \times KK(D \hat{\otimes} A_2, B_2) \to KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$$

called the intersection product. It is functorial in all possible senses given morphisms between C*-algebras, contravariant in A_1, A_2 and covariant in B_1, B_2. Note the presence of the graded tensor product $\hat{\otimes}$ of section 2.1.3. We recall from that section

$A\hat{\otimes} B \cong A \otimes B$ when A is a trivially graded C*-algebra. If we restrict to $B = \mathbb{C}$ and A always trivially graded Kasparov's KK-theory reduces to the K-theory and K-homology of the C*-algebra A,

$$K_0(A) = KK(\mathbb{C}, A) \ , \ K^0(A) = KK(A, \mathbb{C}).$$

Here $K_0(A)$ consists of stable classes of finite projective $\mathcal{A}$-modules and $K^0(A)$ consists of homotopy classes of Fredholm modules over A. For a pre-C*-algebra $\mathcal{A} \subset A$ we use the K-theory and K-homology of its norm closure. They are equivalent since the inclusion map $\iota : \mathcal{A} \to A$ provides an isomorphism in K-theory, $\iota_* : K_*(\mathcal{A}) \to K_*(A)$ and homotopy classes of Fredholm modules over $\mathcal{A}$ extend uniquely to A, [C6, App3]. Higher KK-groups are defined by $KK^n(A, B) = KK(A\hat{\otimes}C_n, B)$ and $KK_n(A, B) = KK^n(A, B\hat{\otimes}C_n)$. Here C_n is the graded Clifford algebra over $\mathbb{C}^n$ [6]. Then

$$K_*(A) = KK_*(\mathbb{C}, A) \ , \ K^*(A) = KK^*(A, \mathbb{C})$$

when A is trivially graded. Formal Bott periodicity,

$$KK_n(\mathbb{C}, A) \cong KK_{n \bmod 2}(\mathbb{C}, A) \ , \ KK^n(A, \mathbb{C}) \cong KK^{n \bmod 2}(A, \mathbb{C})$$

follows from functoriality of the KK-groups and the order 2 periodicity of the graded complexified Clifford algebras under graded tensor product. We note that KK-theory has the property of stability,

$$KK(A, \mathbb{C}) = KK(A \otimes K, \mathbb{C})$$

and hence

$$KK(A, \mathbb{C}) = KK(A \otimes K, \mathbb{C}) = KK(B \otimes K, \mathbb{C}) = KK(B, \mathbb{C})$$

for strong Morita equivalent C*-algebras A and B. The intersection product includes the usual K-theory cap product,

$$\otimes_A : KK_*(\mathbb{C}, A) \times KK^n(A \otimes B, \mathbb{C}) \to KK^{*+n \bmod 2}(B, \mathbb{C})$$

for C*-algebras A and B. Hence the intersection product enables a generalisation of the index theory of Atiyah-Singer and Poincaré Duality.

2.4.2 Poincaré Duality in KK-theory

Let A be a trivially graded unital C*-algebra, B a C*-algebra and $\nu \in KK(A \otimes B, \mathbb{C})$. The intersection product defines a group homomorphism,

$$\otimes_A \nu : KK_*(\mathbb{C}, A) \to KK^*(B, \mathbb{C}) \quad , \quad e \mapsto e \otimes_A \nu$$

The ν-index of $e \in KK_*(\mathbb{C}, A)$ is defined to be the index of the K-homology class $e \otimes_A \nu$ [7]. An isomorphism $\otimes_A \nu$ between the K-theory of A and the K-homology of B is an example of Poincaré duality in KK-theory.

[6] The grading of the Clifford algebras over $\mathbb{C}^n$ used here are detailed in [Ks1]

[7] See Section 7.2, Section 8.7, Section 9.7 and Section 11.4 of [HgR] to trace the ν-index as the generalisation of the Atiyah-Singer index theory.

Definition 2.4.3 *Let $\nu \in KK(A \otimes B, \mathbb{C})$ such that $\otimes_A \nu$ is an isomorphism. Then we say the C*-algebras A and B are Poincaré dual.*

See [C5], VI.4.β for a discussion of Poincaré Duality. The next example is of central importance in understanding the generalisation of Poincaré Duality to non-commutative geometry.

Example 2.4.4 Poincaré Duality for compact Riemannian manifolds

Let X be a compact n-dimensional Riemannian manifold. Let $(L^2(X, \Lambda^* X), \pi_l, d + d^*)$ be the $C_c^{n,\infty}$-representation of the C*-algebra $C(X)$ of Example 1.7.17. Let $f(x) = x(1 + x^2)^{-1/2}$ for $x \in \mathbb{R}$ and define $F_d = f(d + d^*)$. We recall from Section 2.3.2 that $(L^2(X, \Lambda^* X), \pi_l \otimes \pi_r, U_\epsilon)$ is a graded $C(X) \otimes C(X, \mathrm{Cl}(X))$-$\mathbb{C}$-C*-bimodule.

Theorem 2.4.5 (Kasparov) *The triple $(L^2(X, \Lambda^* X), F_d, U_\epsilon)$ as above constitutes a Kasparov $C(X) \otimes C(X, \mathrm{Cl}(X))$-$\mathbb{C}$-bimodule.*

Proof Definition-Lemma 4.2 of [Ks2]. □

We denote the homotopy class of the Kasparov $C(X) \otimes C(X, \mathrm{Cl}(X))$-$\mathbb{C}$-bimodule $(L^2(X, \Lambda^* X), F_d, U_\epsilon)$ by

$$[d] := [(L^2(X, \Lambda^* X), F_d, U_\epsilon)] \in KK(C(X) \otimes C(X, \mathrm{Cl}(X)), \mathbb{C}).$$

The class $[d]$ is called either the Dirac element or the fundamental class of the compact Riemannian manifold X.

Theorem 2.4.6 (Kasparov) *Let $[d]$ be the fundamental class of a compact Riemannian manifold X. Then we have the following isomorphism given by the intersection product,*

$$KK_*(\mathbb{C}, C(X)) \otimes_{C(X)} [d] = KK^*(C(X, \mathrm{Cl}(X)), \mathbb{C}).$$

Proof Theorem 4.8 or Corollary 4.11 of [Ks2]. □

The statement of Poincaré duality for a Riemannian manifold is the isomorphism in Theorem 2.4.6

$$\otimes_{C(X)}[d] : KK_*(\mathbb{C}, C(X)) \to KK^*(C(X, \mathrm{Cl}(X)), \mathbb{C}).$$

Hence we regard Poincaré duality on a compact Riemannian manifold as Poincaré duality of the C*-algebras $C(X)$ and $C(X, \mathrm{Cl}(X))$.

Remark 2.4.7 The situation when X is a Riemannian spin$_{\mathbb{C}}$ manifold is conceptually different. We detail this in Example 2.4.9 below.

2.4.3 KK-equivalence

The intersection product possesses an identity,

$$\mathfrak{i} : KK(A, \mathbb{C}) \otimes_{\mathbb{C}} KK(\mathbb{C}, \mathbb{C}) \to KK(A, \mathbb{C}).$$

The class $\mathfrak{i} = [H_i, T, \alpha]$ is described fully in Theorem 4.5 of [Ks1] or Proposition 9.3.1 of [HgR]. It involves a $\mathbb{Z}_2$-graded Hilbert space H_i and an odd operator T arising from an index 1 Fredholm operator $H_i^{ev} \to H_i^{odd}$. Take a class $[E, F] \in KK(A, \mathbb{C})$. The rank one projection aspect of T is used to transform $E\hat{\otimes}H$ into $(E\otimes\mathbb{C})\oplus(E\otimes H_i^{\perp})$ such that the Kasparov product $F\#_{\mathbb{C}}T$ becomes just $F\otimes 1$ on the first direct summand, and is degenerate on the second. Hence, we recover the same class as $(E, F) \cong (E\otimes\mathbb{C}, F\otimes 1)$. For a $\mathbb{Z}_2$-graded C*-algebra B we set

$$\mathfrak{i}_B = [B\hat{\otimes}H_i, 1\otimes T] \in KK(B, B).$$

This element provides the identity for

$$\mathfrak{i}_B : KK(A, B) \otimes_B KK(B, B) \to KK(A, B)$$

A KK-equivalence between $KK(A, \mathbb{C})$ and $KK(B, \mathbb{C})$ is given by elements $\alpha \in KK(A, B)$ and $\beta \in KK(B, A)$ such that $\alpha \otimes_B \beta = \mathfrak{i}_A$ and $\beta \otimes_A \alpha = \mathfrak{i}_B$. Then the intersection product provides the isomorphism $\alpha \otimes_B KK(B, \mathbb{C}) = KK(A, \mathbb{C})$ with inverse $\beta \otimes_A KK(A, \mathbb{C}) = KK(B, \mathbb{C})$.

Example 2.4.8 KK-equivalence of A and $M_2(A)$

Let A be a C*-algebra and u be a unitary in $M(A)$. We recall from Theorem 2.1.19 that $(A, \mathrm{ad}_u) \sim_M (M_2(A), \mathrm{ad}_{\gamma_u})$ as $\mathbb{Z}_2$-graded C*-algebras. We recall from Lemma 2.1.18 and the proof of Theorem 2.1.19 that (A^2, Γ'_u) is the graded Hilbert A-module such that $(M_2(A), \mathrm{ad}_{\gamma_u}) \cong (K_A(A^2), \mathrm{ad}_{\Gamma_u})$. Let

$$\kappa := (A^2, \Gamma'_u)$$

By similar considerations it is easily shown (A^2, Γ_u) is a graded Hilbert $M_2(A)$-module such that $(A, \mathrm{ad}_u) \cong (K_{M_2(A)}(A^2), \mathrm{ad}_{\Gamma_u})$. Let

$$\kappa^{\mathrm{op}} := (A^2, \Gamma_u).$$

Define

$$\mathfrak{i}(\kappa) = [A^2 \otimes H_i, 1\otimes T, \Gamma'_u \otimes \alpha] \in KK(M_2(A), A)$$

and

$$\mathfrak{i}(\kappa^{\mathrm{op}}) = [A^2 \otimes H_i, 1\otimes T, \Gamma_u \otimes \alpha] \in KK(A, M_2(A)).$$

Then

$$\mathfrak{i}(\kappa^{op}) \otimes_{M_2(A)} \mathfrak{i}(\kappa) = [A \otimes H_i, 1\otimes T, \mathrm{ad}_u \otimes \alpha] = \mathfrak{i}_A.$$

In reverse,

$$\mathfrak{i}(\kappa) \otimes_A \mathfrak{i}(\kappa^{op}) = [M_2(A)\hat{\otimes}H_i, 1\otimes T, \mathrm{ad}_{\gamma_u} \otimes \alpha] = \mathfrak{i}_{M_2(A)}.$$

Hence we have a KK-equivalence between $KK(A, \mathbb{C})$ and $KK(M_2(A), \mathbb{C})$.

Example 2.4.8 demonstrates the general procedure how every strong Morita equivalence between C*-algebras gives rise to a KK-equivalence. The next example completes the discussion of Poincaré duality on Riemannian manifolds in Example 2.4.4.

Example 2.4.9 Riemannian spin$_\mathbb{C}$ manifolds

Let X be a Riemannian manifold. Let $E \to X$ be a Hermitian vector bundle with $E_x \cong H$. Then there exists a Hermitian vector bundle $E^* \to X$, called the dual bundle of E, such that the fibres E_x^* are isomorphic to the dual Hilbert space H^* [Sr, III.4].

Let X be a Riemannian spin$_\mathbb{C}$ manifold as in section 2.3.5. Then $C_0(X)(\otimes C_1) \sim_M C_0(X, \mathrm{Cl}(X))$. Here C_1 is the graded two dimensional Clifford algebra and $(\otimes C_1)$ is added if $\dim X$ is odd. Let γ be the complex volume form of X. The complexified spinor bundle $S \to X$ provides a graded Hilbert $C_0(X)(\otimes C_1)$-module $(C_0(X,S)(\otimes\mathbb{C}^2), \phi(\otimes c_1), \phi(\gamma)(\otimes\beta))$ such that

$$C_0(X, \mathrm{Cl}(X)) \cong K_{C_0(X)(\otimes C_1)}(C_0(X,S)(\otimes\mathbb{C}^2)).$$

The dual of the complexified spinor bundle $S^* \to X$ provides a Hilbert $C_0(X, \mathrm{Cl}(X))$-module $(C_0(X,S^*)(\otimes\mathbb{C}^2), \phi^*(\otimes c_1), \phi^*(\gamma)(\otimes\beta))$ such that

$$C_0(X)(\otimes C_1) \cong K_{C_0(X,\mathrm{Cl}(X))}(C_0(X,S^*)(\otimes\mathbb{C}^2)).$$

Define

$$\mathfrak{i}(\kappa) = [C_0(X,S)(\otimes\mathbb{C}^2) \otimes H_i, 1(\otimes 1) \otimes T, \phi(\gamma)(\otimes\beta) \otimes \alpha].$$

Then $\mathfrak{i}(\kappa) \in KK(C_0(X, \mathrm{Cl}(X)), C_0(X)(\otimes C_1))$. Define

$$\mathfrak{i}(\kappa^{\mathrm{op}}) = [C_0(X,S^*)(\otimes\mathbb{C}^2) \otimes H_i, 1(\otimes 1) \otimes T, \phi^*(\gamma)(\otimes\beta) \otimes \alpha].$$

Then $\mathfrak{i}(\kappa^{\mathrm{op}}) \in KK(C_0(X)(\otimes C_1), C_0(X, \mathrm{Cl}(X)))$. The relations

$$\mathfrak{i}(\kappa^{op}) \otimes_{C_0(X,\mathrm{Cl}(X))} \mathfrak{i}(\kappa) = \mathfrak{i}_{C_0(X)(\otimes C_1)}$$

and

$$\mathfrak{i}(\kappa) \otimes_{C_0(X)(\otimes C_1)} \mathfrak{i}(\kappa^{op}) = \mathfrak{i}_{C_0(X,\mathrm{Cl}(X))}$$

follow. As in the last example, the strong Morita equivalence provides a KK-equivalence between $KK(C_0(X)(\otimes C_1), \mathbb{C})$ and $KK(C_0(X, \mathrm{Cl}(X)), \mathbb{C})$.

Poincaré Duality on a Riemannian spin$_\mathbb{C}$ manifold

Let X be a n-dimensional compact Riemannian spin$_\mathbb{C}$ manifold. Following Example 2.4.4 we can define, since X is a compact Riemannian manifold, the fundamental class

$$[d] = [(L^2(M, \Lambda^* X), F_d, U_\epsilon)] \in KK(C(X) \otimes C(X, \mathrm{Cl}(X)), \mathbb{C}).$$

We have the isomorphism

$$\otimes_{C(X)}[d] : KK(\mathbb{C}, C(X)) \longrightarrow KK(C(X, \mathrm{Cl}(X)), \mathbb{C})$$

from Theorem 2.4.6. A spin$_{\mathbb{C}}$ structure is a strong Morita equivalence $C(X) \sim_M C(X, \mathrm{Cl}(X))$ when n is even and $C(X) \otimes C_1 \sim_M C(X, \mathrm{Cl}(X))$ when n is odd. The KK-equivalence above provides the isomorphism

$$\mathrm{i}(\kappa^{\mathrm{op}}) \otimes_{C(X,\mathrm{Cl}(X))} \cdot \ : KK(C(X, \mathrm{Cl}(X)), \mathbb{C}) \longrightarrow KK(C(X)(\otimes C_1), \mathbb{C}).$$

Hence, if n is even the class

$$\mu = \mathrm{i}(\kappa^{\mathrm{op}}) \otimes_{C(X,\mathrm{Cl}(X))} [d] \in KK(C(X) \otimes C(X), \mathbb{C}) \cong K^0(C(X))$$

provides the isomorphism,

$$\begin{array}{ccl} K_0(C(X)) & \xrightarrow{[d]} & KK(C(X, \mathrm{Cl}(X)), \mathbb{C}) \\ & \searrow & \downarrow \mathrm{i}(\kappa^{\mathrm{op}}) \\ & \mu & KK(C(X), \mathbb{C}) = K^0(C(X)), \end{array}$$

in terms of the intersection product,

$$\otimes_{C(X)}\mu : K_0(C(X)) \to K^0(C(X)).$$

Similarly there is a triangle showing the isomorphism $\otimes_{C(X)}\mu : K_1(C(X)) \to K^1(C(X))$.

When n is odd, then

$$\mu = \mathrm{i}(\kappa^{\mathrm{op}}) \otimes_{C(X,\mathrm{Cl}(X))} [d] \in KK(C(X) \otimes C(X) \otimes C_1, \mathbb{C}) \cong K^1(C(X))$$

provides the isomorphism

$$\begin{array}{ccl} K_0(C(X)) & \xrightarrow{[d]} & KK(C(X, \mathrm{Cl}(X)), \mathbb{C}) \\ & \searrow & \downarrow \mathrm{i}(\kappa^{\mathrm{op}}) \\ & \mu & KK(C(X) \otimes C_1, \mathbb{C}) = K^1(C(X)), \end{array}$$

in terms of the intersection product,

$$\otimes_{C(X)}\mu : K_0(C(X)) \to K^1(C(X)).$$

Similarly there is a triangle showing the isomorphism $\otimes_{C(X)}\mu : K_1(C(X)) \to K^0(C(X))$.

Hence Poincaré duality on the compact Riemannian manifold X descends to the usual cap product statement in K-theory through the Morita equivalence given by the spin$_{\mathbb{C}}$ structure. Note that the class $\mu = \mathrm{i}(\kappa^{\mathrm{op}}) \otimes_{C(X,\mathrm{Cl}(X))} [d]$ is called the K-orientation (or sometimes called the fundamental class) of the Riemannian spin$_{\mathbb{C}}$ manifold X. The K-orientation is also known to be the homotopy class of the Fredholm module $(L^2(X, S), F_D, \phi(\gamma))$,

$$[D] := [(L^2(X, S), F_D, \phi(\gamma))],$$

where D is the Dirac operator on the complexified spinor bundle $S \to X$ and $F_D = D(1 + D^2)^{-1/2}$ [HgR, Definition 11.2.10]. Explicitly, the link between the fundamental class and the K-orientation is the relations

$$\mathrm{i}(\kappa) \otimes_{C(X)(\otimes C_1)} [D] = [d]$$
$$\mathrm{i}(\kappa^{\mathrm{op}}) \otimes_{C(M,\mathrm{Cl}(X))} [d] = [D].$$

2.4.4 Fundamental Class of a Riemannian Representation

A. Connes, in the foundation paper [C4], considered Poincaré Duality an essential indicator of smooth manifold structure. To generalise the statement of Poincaré Duality to non-commutative geometry involves generalising the notion of a fundamental class.

Graded Representations and Unbounded Kasparov Bimodules

Definition 2.4.10 *Let A be a C^*-algebra. Let (H, π, D) be a base representation of a C^*-algebra A with a selfadjoint operator $\Gamma \in B(H)$ such that $\Gamma^2 = 1$, $\{\Gamma, D\} = 0$, and $[\Gamma, \pi(a)] = 0$. Then (H, π, D, Γ) is called a graded base representation of A.*

Let $F_D := f(D)$ where $f(x) = x(1 + x^2)^{-1/2}$.

Theorem 2.4.11 *Let (H, π, D, Γ) be a graded C^1_c-representation of a C^*-algebra A such that $\pi(a)(D - \lambda)^{-1} \in K(H)$ for all $a \in A$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then (H, F_D, Γ) is a Kasparov A-$\mathbb{C}$-bimodule.*

Proof See [BJ] or [HgR, 10.9.15]. □

Corollary 2.4.12 *Let (H, π, D, Γ) be a graded $C^{n,\infty}_c$-representation of a C^*-algebra A. Then (H, F_D, Γ) is a Kasparov A-$\mathbb{C}$-bimodule.*

Proof Immediate from Definition 1.9.1, Theorem A.1.1 of the appendix and Theorem 2.4.11. □

The graded $C^{n,\infty}_c$-representation (H, π, D, Γ) of the C*-algebra A is an example of an unbounded Kasparov A-$\mathbb{C}$-bimodule [BJ].

The Index Algebra

Let (H_ρ, π_ρ, D) be a Riemannian representation of a C*-algebra A with associated standard form $\mathbf{Riem}(A, \rho) = (\Omega_D(\mathcal{A}_{\pi_\rho})'', H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$.

Definition 2.4.13 *Let $(H_\rho, \pi_\rho, D, \Gamma)$ be a graded Riemannian representation of a C^*-algebra A and $\mathbf{Riem}(A, \rho)$ be the standard form as above. Then Γ is called a real grading element if $[\Gamma, J_\rho] = 0$.*

Theorem 2.4.14 *Let (H_ρ, π_ρ, D) be a Riemannian representation of a C^*-algebra A. Then the real graded elements associated to this representation, if they exist, are parameterised by the group of unitaries $\{U \in U(Z(\Omega_D(\mathcal{A}_{\pi_\rho}))'') \mid [D, U] = 0\}$.*

Proof Let $R = \Omega_D(\mathcal{A}_{\rho_\pi})''$. Let Γ, Γ' be real grading elements. Consider the unitary $U = \Gamma\Gamma'$. Then $U \in R'$ such that $[U, J_\rho] = 0$. Hence $U \in Z(R)$ by (iv) Theorem 1.6.1. That $[D, U] = 0$ is immediate from Definition 2.4.10. □

Define the opposite action of the von Neumann algebra $\Omega_D(\mathcal{A}_{\pi_\rho})''$ by

$$w^{\mathrm{op}} = J_\rho w^* J_\rho.$$

Then $(\Omega_D(\mathcal{A}_{\pi_\rho})'')^{\text{op}} \cong \Omega_D(\mathcal{A}_{\pi_\rho})'$ by the Tomita-Takesaki theorem. Hence (H_ρ, π_ρ) is an $\Omega_D(\mathcal{A}_{\pi_\rho}) \otimes \Omega_D(\mathcal{A}_{\pi_\rho})^{\text{op}}$-$\mathbb{C}$-bimodule using the representation

$$w \otimes w^{\text{op}} \mapsto ww^{\text{op}}.$$

A grading element Γ implements the parity automorphism

$$\text{ad}_\Gamma(w) = \epsilon(w).$$

The opposite grading element $\Gamma_J := J_\rho \Gamma J_\rho$ implements the parity automorphism on the opposite representation

$$\text{ad}_{\Gamma_J}(w^{\text{op}}) = \epsilon(w)^{\text{op}}.$$

A real grading element $\Gamma = \Gamma_J$ implements the parity automorphism simultaneously

$$\text{ad}_\Gamma(ww^{\text{op}}) = \epsilon(w)\epsilon(w)^{\text{op}}$$

and hence allows $(H_\rho, \pi_\rho, \Gamma)$ the structure of a graded $\Omega_D(\mathcal{A}_{\pi_\rho}) \otimes \Omega_D(\mathcal{A}_{\pi_\rho})^{\text{op}}$-$\mathbb{C}$-bimodule.

Definition 2.4.15 *Let (H_ρ, π_ρ, D) be a Riemannian representation of a C^*-algebra A. Then we define the index algebra B_{π_ρ} as the uniform closure of the set $\mathcal{B}_{\pi_\rho} = \{w^{\text{op}} \in \Omega_D(\mathcal{A}_{\pi_\rho})^{\text{op}} \mid \|[D, w^{\text{op}}]\| < \infty\}$.*

Proposition 2.4.16 *Let (H_ρ, π_ρ, D) be a Riemannian representation of a separable C^*-algebra A. Then the index algebra B_{π_ρ} is a separable C^*-subalgebra of the von Neumann algebra $\Omega_D(\mathcal{A}_{\pi_\rho})'$ that contains $A \cap J_\rho A J_\rho$.*

Proof Let $\{a_i\}_{i=1}^{\infty}$ be the countable set of elements in $\mathcal{A}_{\pi_\rho}$ that generate A. Then $\Omega_D(\mathcal{A}_{\pi_\rho})$ contains the set $\{a_i, [D, a_i]\}_{i=1}^{\infty}$ that generates C, the uniform closure of $\Omega_D(\mathcal{A}_{\pi_\rho})$. Hence C is a separable C*-algebra and B_{π_ρ} is a C*-subalgebra of the separable C*-algebra $C^{\text{op}} = JC^*J$. As $\mathcal{A}_{\pi_\rho} \cap J_\rho \mathcal{A}_{\pi_\rho} J_\rho \subset \mathcal{B}_{\pi_\rho}$ then the norm closure $A \cap J_\rho A J_\rho \subset B_{\pi_\rho}$. □

This implies B_{π_ρ} is non-empty when A is unital.

Fundamental Class of a Riemannian representation

Theorem 2.4.17 *Let (H_ρ, π_ρ, D) be a n-dimensional Riemannian representation of a C^* algebra A with non-empty index algebra and real grading element Γ. Then (H_ρ, F_D, Γ) is an $A \otimes B_{\pi_\rho}$-$\mathbb{C}$-Kasparov bimodule.*

Proof It is immediate that $(H_\rho, \pi_\rho, \Gamma)$ is a graded $A \otimes B_{\pi_\rho}$-$\mathbb{C}$-C*-bimodule. Since $[D, \pi_\rho(a)w^{\text{op}}]$ is bounded for the norm dense set $a \otimes w^{\text{op}} \in \mathcal{A}_{\pi_\rho} \otimes \mathcal{B}_{\pi_\rho}$ it follows from [BJ] that $\pi_\rho(a)w^{\text{op}}(D-\lambda)^{-1} = w^{\text{op}}\pi_\rho(a)(D-\lambda)^{-1} \in K(H)$ for all $a \in A$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $w^{\text{op}} \in B_{\pi_\rho}$, that $w^{\text{op}}\pi(a)(1 - F_D^2), [F_D, \pi(a)w^{\text{op}}] \in K(H)$. To transfer to the graded commutator requires a trivial adjustment to the opposite representation. Let $\tilde{w}^{\text{op}} := w^{op}\Gamma$ define the right adjusted opposite representation for $w \in \Omega_D(A)''$. Then $\text{ad}_\Gamma(\tilde{w}^{\text{op}}) = \Gamma w^{\text{op}} = \epsilon(w^{\text{op}})\Gamma = \widetilde{\epsilon(w)}^{\text{op}}$. Hence the right adjusted opposite representation is still a graded representation such that $\tilde{w}^{\text{op}}\pi(a)(1-F_D^2) = w^{\text{op}}\Gamma\pi(a)(1-F_D^2) = w^{\text{op}}\pi(a)(1-F_D^2)\Gamma \in K(H)$ and $[F_D, \pi(a)\tilde{w}^{\text{op}}]_g = [F_D, \pi(a)w^{\text{op}}\Gamma]_g = [F_D, \pi(a)w^{\text{op}}]\Gamma \in K(H)$. □

By the notation $[(H, F_D, \Gamma)]$ we shall mean the K-homology class of the Kasparov $A \otimes B_{\pi_\rho}$-$\mathbb{C}$-bimodule (H, F_D, Γ) in $KK(A \otimes B_{\pi_\rho}, \mathbb{C})$. The following definition shall apply only to unital separable C*-algebras.

Definition 2.4.18 *Let (H_ρ, π_ρ, D) be a n-dimensional Riemannian representation of a unital separable C*-algebra A with real grading element Γ. Then the class*

$$\lambda^\rho_{-1} := [(H_\rho, F_D, \Gamma)] \in KK(A \otimes B_{\pi_\rho}, \mathbb{C})$$

is called the fundamental class associated to this representation.

Remark 2.4.19 Let Γ, Γ' be real grading elements for the n-dimensional Riemannian representation (H_ρ, π_ρ, D) of a unital separable C*-algebra A. Then $\Gamma = U\Gamma'$ for a central unitary such that $[D, U] = 0$ by Theorem 2.4.14. Hence $[F_D, U] = 0$ and there exists a homotopy $(H_\rho, F_D, \Gamma) \sim (H_\rho, F_D, \Gamma')$. Hence λ^ρ_{-1} is independent of the real grading element chosen.

Example 2.4.20 Riemannian Manifold

Let X be an n-dimensional compact Riemannian manifold. Let $C(X)$ be the separable C*-algebra of continuous functions on X. Let $(L^2(X, \Lambda^* X), \pi_l, d + d^*, U_\epsilon)$ be the graded $C(X, \mathrm{Cl}(X)) \otimes C(X, \mathrm{Cl}(X))^{\mathrm{op}}$-$\mathbb{C}$-C*-bimodule studied in Theorem 2.3.4 and Theorem 1.7.21.

Theorem 2.4.21 *Let X be a n-dimensional compact Riemannian manifold. Then*

(i) *$(L^2(X, \Lambda^* X), \pi_l, d + d^*, U_\epsilon)$ is an n-dimensional Riemannian representation of the C*-algebra $C(X)$ with real grading element U_ϵ,*

(ii) *the index algebra of this representation is*

$$B_{\pi_l} = C(X, \mathrm{Cl}(X))^{\mathrm{op}},$$

(iii) *the fundamental class λ_{-1} of this representation is the fundamental class of the Riemannian manifold X*

$$\lambda_{-1} = [d],$$

(iv) *the C*-algebras $C(X)$ and $C(X, \mathrm{Cl}(X))$ are Poincaré dual. In particular,*

$$\otimes_{C(X)} \lambda_{-1} : KK(\mathbb{C}, C(X)) \to KK(C(X, \mathrm{Cl}(X)), \mathbb{C})$$

is an isomorphism.

Proof (i) Immediate from Theorem 2.3.4 and Theorem 1.7.21. (ii) Let $D = d + d^*$. As $\pi_l(f)^{\mathrm{op}} = \pi_r(f) = \pi_l(f)$ then $[D, \pi_l(f)^{\mathrm{op}}]w = \pi_l(df)(w) = df \cdot w$ for all $f \in C^\infty(X)$ and $w \in C^\infty(X, \mathrm{Cl}(X))$. Similarly $[D, \pi_l(f)]^{\mathrm{op}} w = \pi_r(df)(w) = w \cdot df$. Then

$$[D, [D, \pi_l(f)]^{\mathrm{op}}]w = D(w \cdot df) - (Dw) \cdot df.$$

for $w \in C^\infty(X, \mathrm{Cl}(X))$. Working in a chart U with local tangent bundle basis $\{\partial_i(x)\}_{i=1}^n$ and local cotangent bundle basis $\{dx_i(x)\}_{i=1}^n$ for $x \in U$ then $D = \sum_i^n dx_i(x) \cdot \nabla_{\partial_i(x)}$. Hence $D(w \cdot df)(x) = (Dw)(x) \cdot df(x) + \sum_i dx_i(x) \cdot w(x) \cdot \nabla_{\partial_i(x)} df(x)$ and

$$[D, [D, \pi_l(f)]^{\mathrm{op}}]w(x) = \textstyle\sum_i dx_i(x) \cdot w(x) \cdot \nabla_{\partial_i(x)} df(x)$$

extends to a bounded operator (w, not df, is what we are acting upon). Then $[D, \pi_l(f)^{\mathrm{op}}]$ and $[D, [D, \pi_l(f)]^{\mathrm{op}}]$ are bounded for all $f \in C^\infty(X)$. Hence $C^\infty(X, \mathrm{Cl}(X))^{\mathrm{op}} \subset B_{\pi_l} \subset C(X, \mathrm{Cl}(X))^{\mathrm{op}}$. By norm density of $C^\infty(X, \mathrm{Cl}(X))$ in the C*-algebra $C(X, \mathrm{Cl}(X))$ the result follows.

(iii) follows from (ii) and (i)

(iv) the right adjusted right action, $\tilde{c_r}$, of the C*-algebra $C(X, \mathrm{Cl}(X))$ is the representation $\lambda. + \lambda^*_*$ used in Kasparov, [Ks2] Definition-lemma 4.2. Hence Theorem 2.4.6 applies. □

2.5 Non-commutative Volume Form

Basic Definitions

Let A be a unital associative algebra (over $\mathbb{C}$). We denote the k-fold algebraic tensor product

$$C_k(A) = A \odot \underbrace{A \odot \ldots \odot A}_{k}.$$

Let $\mathbb{C} \to A$ be the canonical inclusion $\lambda \to \lambda 1$ for all $\lambda \in \mathbb{C}$. Let $\overline{A} = \mathrm{coker}\{\mathbb{C} \to A\}$. Define

$$\overline{C}_k(A) = A \odot \underbrace{\overline{A} \odot \ldots \odot \overline{A}}_{k}.$$

We denote by $(\Omega(A), \delta)$ the universal graded differential algebra of section 1.4. We recall the A-bimodule $\Omega^1(A)$ with derivation $\delta : A \to \Omega^1(A)$ has the following property of universality. Let M be any A-bimodule with a derivation $\delta_M : A \to M$. Then there exists a unique element $\sigma \in E_A(\Omega^1(A), M)$ such that $\delta_M = \sigma\delta$.

Let A be a unital commutative associative algebra. We denote $\Omega^0_A = A$ and by Ω^1_A we denote the symmetric A-bimodule with derivation $\delta_A : A \to \Omega^1_A$ with the property that for any symmetric A-bimodule M with derivation $\delta_M : A \to M$ there exists a unique element $\sigma \in E_A(\Omega^1_A, M)$ such that $\delta_M = \sigma\delta_A$ [Ma, Pg 180]. Define the exterior algebra $\Omega_A := \Lambda(\Omega^1_A)$ with unique exterior derivative $\delta_A : \Lambda^k(\Omega^1_A) \to \Lambda^{k+1}(\Omega^1_A)$ extending $\delta_A : A \to \Omega^1_A$. Then the complex (Ω_A, δ_A) is called the complex of Kähler de-Rham differential forms on the unital commutative associative algebra A.

2.5.1 Non-commutative De-Rham Complexes

Let A be a unital associative algebra and $(\Omega(A), \delta)$ the universal graded differential algebra of A. Define

$$[\Omega(A), \Omega(A)] := \oplus_{k \geq 0}[\Omega(A), \Omega(A)]_k$$

where

$$[\Omega(A), \Omega(A)]_k := \sum_{p+q=k} \{\omega_p\omega_q - (-1)^{pq}\omega_q\omega_p \,|\, \omega_p \in \Omega^p(A), \omega_q \in \Omega^q(A)\}.$$

Define

$$\Lambda\Omega(A) := \Omega(A)/[\Omega(A), \Omega(A)].$$

As $\delta([\Omega(A), \Omega(A)]) \subset [\Omega(A), \Omega(A)]$ [Sb, JL] the quotient derivative $\delta : \Lambda^k\Omega(A) \to \Lambda^{k+1}\Omega(A)$ is well defined and we have the exact sequence of complexes [Sb, JL]

$$0 \to ([\Omega(A), \Omega(A)], \delta) \to (\Omega(A), \delta) \to (\Lambda\Omega(A), \delta) \to 0.$$

The complex $(\Lambda\Omega(A), \delta)$ is called the complex of non-commutative de-Rham exterior differential forms on A.

Remark 2.5.1 We remark on the distinction between the non-commutative exterior differential forms in the sense of Connes' non-commutative calculus[8] and non-commutative de-Rham exterior differential forms as above. The complex $(\Lambda\Omega(A), \delta)$ is a purely algebraic construction, independent of the representation theory of any C*-envelope of A, and is a differential complex that is not a representation of $\Omega(A)$ in general [Sb, Pg 94].

The first two terms in the non-commutative de-Rham complex of a unital associative algebra are: (1) The commutatisation or symmetrisation of A,

$$\Lambda^0\Omega(A) = A/\mathrm{Com}(A)$$

since

$$[\Omega(A), \Omega(A)]_0 = \{[a, b] \,|\, a, b \in A\} = \mathrm{Com}(A).$$

(2) The symmetric A-bimodule,

$$\Lambda^1\Omega(A) = \Omega^1(A)/\{a\delta(b) - \delta(b)a \,|\, a, b \in A\}$$

since

$$[\Omega(A), \Omega(A)]_1 = \{a\delta(b) - \delta(b)a \,|\, a, b \in A\}.$$

The symmetric A-bimodule $\Lambda^1\Omega(A)$ is viewed as the symmetrisation of $\Omega^1(A)$ as an A-bimodule.

Corollary 2.5.2 [II Remark 1.1.8, II Consequence 1.1.13, Sb] *Let A be a commutative unital associative algebra. Then*

$$A = \Lambda^0\Omega(A) = \Omega^0_A$$

[8]Let (H, π, D) be a C^1-representation of a unital C*-algebra B. Let A be any unital associative subalgebra of A^1. Then Connes, as in section 1.4, defines the differential representation $(\Lambda_D(A), \delta_D)$ of the universal differential algebra $(\Omega(A), \delta)$. The differential representation $(\Lambda_D(A), \delta_D)$ is also, confusingly, called the exterior differential forms on A.

and

$$\Omega^1(A)/[\Omega(A),\Omega(A)]_1 = \Lambda^1\Omega(A) = \Omega^1_A,$$

however

$$\Omega^2(A)/[\Omega(A),\Omega(A)]_2 = \Lambda^2\Omega(A) \neq \Omega^2_A$$

in general.

The complex of Kähler de-Rham exterior differential forms can only be defined on a commutative unital associative algebra. The consequence of Corollary 2.5.2 is that the non-commutative de-Rham complex of exterior differential forms $(\Lambda\Omega(A),\delta)$ over a unital associative algebra A is not the appropriate generalisation of the complex of Kähler de-Rham exterior differential forms. The search for the appropriate non-commutative generalisation of the Kähler de-Rham exterior differential complex and Kähler de-Rham cohomology led to cyclic homology and the situation as follows [Sb, JL] which we outline in the next section,

(i) The Hochschild homology $H_*(A)$ of a unital associative algebra A is the non-commutative generalisation of the Kähler de-Rham complex of exterior differential forms,

(ii) The *cohomology* $H^*(\Lambda\Omega(A),\delta)$ of the complex of non-commutative de-Rham differential forms is a component of the cyclic homology $HC_*(A)$ of a unital associative algebra A and is the non-commutative generalisation of the Kähler de-Rham cohomology.

2.5.2 Hochschild and Cyclic Homology

Let A be a unital associative algebra. Define the maps $b : C_{k+1}(A) \to C_k(A)$,

$$b(a_0 \otimes a_1 \otimes \ldots \otimes a_k) := \sum_{i=0}^{k-1}(-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k + (-1)^k a_k a_0 \otimes a_1 \otimes \ldots \otimes a_{k-1},$$

and $B : C_{k+1}(A) \to C_{k+2}(A)$,

$$\begin{aligned} B(a_0 \otimes a_1 \otimes \ldots \otimes a_k) := \; & \textstyle\sum_{i=0}^{k}(-1)^{ik} 1 \otimes a_i \otimes \ldots \otimes a_k \otimes a_0 \otimes \ldots \otimes a_{i-1} + \\ & \textstyle\sum_{i=0}^{k}(-1)^{ik} a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_k \otimes a_0 \otimes \ldots \otimes a_{i-1}. \end{aligned}$$

Then $(C_k(A), b, B)$ is a mixed complex[9] [Sb, I.2.3, I.2.4]. We define an associated chain complex $(\mathcal{C}_k(A), d_k)$ by

$$\mathcal{C}_k(A) := C_k(A) \oplus C_{k-2}(A) \oplus C_{k-4}(A) \oplus \ldots$$

$$d_k(c_k, c_{k-2}, c_{k-4}, \ldots) := (bc_k + Bc_{k-2}, bc_{k-2} + Bc_{k-4}, bc_{k-4} + Bc_{k-6}, \ldots).$$

The Hochschild homology $H_*(A)$ of a unital associative algebra A is defined to be the homology of the complex $(C_k(A), b)$. The cyclic homology $HC_*(A)$ of a unital associative algebra A is defined to be the homology of the complex $(\mathcal{C}_k(A), d_k)$.

[9] meaning $b^2 = B^2 = bB + Bb = 0$.

An explicit relationship between Hochschild and cyclic homology can be derived in terms of a long exact sequence as follows [C, JL, Sb]. We define the shift of chain complexes

$$S : (\mathcal{C}_k(A), d_k) \to (\mathcal{C}_{k-2}(A), d_{k-2})$$

by projecting out the first direct summand

$$S : C_k(A) \oplus C_{k-2}(A) \oplus C_{k-4}(A) \oplus \ldots \mapsto C_{k-2}(A) \oplus C_{k-4}(A) \oplus \ldots$$

We define the inclusion map of chain complexes

$$I : (C_k(A), b) \to (\mathcal{C}_k(A), d_k)$$

by

$$I : C_k(A) \to \mathcal{C}_k(A)\ ,\ c_k \to c_k \oplus 0 \oplus 0 \oplus \ldots$$

clearly with $Ibc_k = d_k I c_k$ for all chains $c_k \in C_k(A)$. Immediately from the above definitions of S and I we have the exact sequence of chain complexes

$$0 \to (C_k(A), b) \xrightarrow{I} (\mathcal{C}_k, d_k) \xrightarrow{S} (\mathcal{C}_{k-2}, d_{k-2}) \to 0.$$

This exact sequence of chain complexes induces a long exact sequence in homology where the connecting map is exactly the map induced in homology by B,

$$B : (\mathcal{C}_{k-2}(A), d_{k-2}) \to (C_{k-1}(A), b)$$

defined by

$$B : \mathcal{C}_{k-2}(A) \to C_{k-1}(A)\ ,\ c_{k-2} \oplus c_{k-4} \oplus \ldots \mapsto Bc_{k-2}.$$

Theorem 2.5.3 [CN,JL,Sb] *Let A be a unital associative algebra. Let $H_*(A)$ be the Hochschild homology of A, $HC_*(A)$ be the cyclic homology of A and the maps I, S and B be as above. Then there is a long exact sequence in homology*

$$\ldots \to H_k(A) \xrightarrow{I} HC_k(A) \xrightarrow{S} HC_{k-2}(A) \xrightarrow{B} H_{k-1}(A) \to \ldots$$

where the connecting map is induced by B.

A corollary to the theorem is that the combination

$$H_k(A) \xrightarrow{I} HC_k(A) \xrightarrow{B} H_{k+1}(A)$$

yields a cochain complex $(H_*(A), B)$. The resultant cohomology $H^*(H_*(A), B)$ can be seen to be a component of the reduced cyclic cohomology $HC_*^{\text{red}}(A)$ as follows.

Let $(D_k(A), b, B)$ be the mixed subcomplex defined by $D_k(A) = \{a_0 \otimes a_1 \otimes \ldots \otimes a_k \mid a_i = 1 \text{ some } i = 1, ..., k\}$ [Sb, I.2.5]. Define the quotient mixed complex

$$(\overline{C}_k(A), b, B) := (C_k(A), b, B)/(D_k(A), b, B).$$

The map $B : \overline{C}_k(A) \to \overline{C}_{k+1}(A)$ is simplified in the quotient to

$$B(a_0 \otimes a_1 \otimes \ldots a_k) = \sum_{i=0}^{k} (-1)^{ik} 1 \otimes a_i \otimes \ldots a_k \otimes a_0 \otimes \ldots \otimes a_{i-1}.$$

Let $\mathbb{C} \to A$ be the canonical inclusion $\lambda \to \lambda 1$ for all $\lambda \in \mathbb{C}$. Then we have an induced homomorphism of mixed complexes

$$\begin{array}{ccccccccc} \overline{C}_*(\mathbb{C}): & \ldots & \xrightarrow{b} & 0 & \to & 0 & \to & \mathbb{C} \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{C}_*(A): & \ldots & \xrightarrow{b} & \overline{C}_2(A) & \to & A \otimes \overline{A} & \to & A \end{array}$$

Define the quotient mixed complex

$$(C_k^{\mathrm{red}}(A), b, B) := (\overline{C}_k(A), b, B)/(\overline{C}_k(\mathbb{C}), b, B).$$

Then $C_0^{\mathrm{red}}(A) = \overline{A}$ and $C_k^{\mathrm{red}}(A) = \overline{C}_k(A)\ \forall\, k \geq 1$. The Hochschild homology $H_*^{\mathrm{red}}(A)$ and the cyclic homology $HC_*^{\mathrm{red}}(A)$ of the mixed complex $(C_k^{\mathrm{red}}(A), b, B)$ are called the reduced Hochschild homology and reduced cyclic homology of the unital associative algebra A respectively.

Theorem 2.5.4 *Let A be a unital associative algebra. Then there exists an inclusion map*

$$H(I) : H^k(H_k(A), B) \to HC_k^{\mathrm{red}}(A)$$

such that $B \circ H(I) = 0$ for all $k \geq 1$.

Proof The Hochschild homology $H_*(A)$ is the homology of the complex $(\overline{C}_k(A), b)$ [Sb, I Prop 2.5.3]. Let $Z_k = \{c_k \in \overline{C}_k(A) \,|\, bc_k = 0, Bc_k = bc_{k+2} \text{ some } c_{k+2} \in \overline{C}_{k+2}(A)\}/\{m_k \in \overline{C}_k(A) \,|\, m_k = bm_{k+1}\}$ and $B_k = \{Be_{k-1} \in \overline{C}_k(A) \,|\, be_{k-1} = 0, e_{k-1} \in \overline{C}_{k-1}(A)\}/\{m_k \in \overline{C}_k(A) \,|\, m_k = bm_{k+1}\}$. Let $M_k = \{c_k \in C_k^{\mathrm{red}}(A) \,|\, d_k c_k = 0\}/\{m_k \in C_k^{\mathrm{red}}(A) \,|\, m_k = d_k m_{k+1}\}$. Consider the map $\alpha_1 : \overline{C}_k(A) \to C_k^{\mathrm{red}}(A)$

$$\alpha_1 : c_k + m_k \mapsto (-c_{k+2} + Bm_{k+1}) \oplus (c_k + m_k) \oplus 0 \oplus \ldots \oplus 0$$

for $k \geq 1$. Then $d_{k+2}\alpha_1(c_k + m_k) = (-bc_{k+2} + Bc_k + bBm_{k+1} + Bm_k) \oplus (bc_k + bm_k) \oplus 0 \oplus \ldots \oplus 0 = (bB + Bb)m_{k+1} \oplus 0 \oplus \ldots \oplus 0 = 0$. And $\alpha_1(c_k + m_k) = -c_{k+2} \oplus c_k \oplus 0 \oplus \ldots \oplus 0 + Bm_{k+1} \oplus bm_{k+1} \oplus 0 \oplus \ldots \oplus 0 = -c_{k+2} \oplus c_k \oplus 0 \oplus \ldots \oplus 0 + d_{k+3}(0 \oplus m_{k+1} \oplus 0 \oplus \ldots \oplus 0)$. Hence the map $\alpha_2 : Z_k \to M_{k+2}$ given by $\alpha_2 : [c_k] \to [-c_{k+2} \oplus c_k \oplus 0 \oplus \ldots \oplus 0]$ is well defined. Now, suppose $c_k = Be_{k-1}$ for $k > 2$ such that $be_{k-1} = 0$. Then $Bc_k = B^2 e_{k-1} = 0$ and $\alpha_2 : [Be_{k-1}] \to [0 \oplus Be_{k-1} \oplus 0 \oplus \ldots \oplus 0] = [d_{k+3}(0 \oplus 0 \oplus e_{k-1} \oplus 0 \oplus \ldots \oplus 0)] = 0$. When $k = 1$ then $c_1 = Ba = (1, [a]) = B[a]$ and $[Ba] \to [d_3(0, 0, [a])] = 0$. Hence the map $\alpha_2 : Z_k/B_k \to M_{k+2}$ is well defined for $k \geq 1$. The combination $H(I) := S\alpha_2 : Z_k/B_k \to M_k$ for $k \geq 1$ is injective as $S\alpha_2[c_k] = 0$ if and only if $[c_k] = 0$. Clearly $BH(I) = BS\alpha_2 = 0$ by Theorem 2.5.3. □

Hence the cohomology $H^*(H_*(A), B)$ is a component of $\ker B \subset HC_*^{\mathrm{red}}(A)$. The final result that motivates Hochschild and cyclic homology is the identification of $\ker B$. Let θ be the map induced by the isomorphisms

$$\theta_k : \Omega^k(A) \to \overline{C}_k(A)$$

given by $\theta_k : a_0 \delta a_1 \ldots \delta a_k \mapsto a_0 \otimes a_1 \otimes \ldots a_k$.

Theorem 2.5.5 [Sb, II Thereom 1.1.18] *Let A be a unital associative algebra. Then there exists an exact sequence*

$$0 \to H^k(\Lambda\Omega(A), \delta) \xrightarrow{\theta} HC_k^{\mathrm{red}}(A) \xrightarrow{B} H_{k+1}^{\mathrm{red}}(A) \to 0$$

for all $k \geq 1$.

Example 2.5.6 Kähler De-Rham Differential Forms

Let A be a commutative unital associative algebra. Let (Ω_A, δ_A) be the cochain complex of Kähler de-Rham exterior differential forms on A and $H^*_{\mathrm{DR}}(A) := H^*(\Omega_A, \delta_A)$ the Kähler de-Rham cohomology of A. Define the map

$$\hat{\mu} : \overline{C}_k(A) \to \Omega_A \ , \ a_0 \otimes a_1 \otimes \ldots \otimes a_k \mapsto a_0 da_1 \wedge \ldots \wedge da_k.$$

Theorem 2.5.7 [Sb, II Complement 1.2.14] *Let A be a commutative unital associative algebra. Then we have the epimorphism of cochain complexes*

$$\mu : (H_*(A), B) \to (\Omega_A, \delta_A)$$

induced by the map $\hat{\mu}$ above.

Let A be a commutative unital associative algebra. Then A is called $\mathbb{C}$-smooth if for every prime ideal P of A the local ring A_P is formally smooth over $\mathbb{C}$.

Theorem 2.5.8 [Sb, II Corollary 1.2.17] *Let A be a commutative unital associative algebra. Then the following statements are equivalent.*

(i) $\mu : HC_k(A) \to \Omega_A^k / d\Omega_A^{k-1} \oplus H_{\mathrm{DR}}^{k-2}(A) \oplus H_{\mathrm{DR}}^{k-4}(A) \oplus \ldots$ *is an isomorphism for all $k \geq 0$,*

(ii) $\mu : H_k(A) \to \Omega_A^k$ *is an isomorphism for all $k \geq 0$,*

(iii) $H_*(A)$ *is an exterior algebra over $H_1(A)$,*

(iv) A *is $\mathbb{C}$-smooth.*

Corollary 2.5.9 *Let A be a commutative unital associative algebra that is a $\mathbb{C}$-smooth. Then $H^*_{DR}(A) = H^*(H_*(A), B) = H^*(\Lambda\Omega(A), \delta)$.*

Proof By Theorem 2.5.5 and Theorem 2.5.8 we have

$$H^k(\Lambda\Omega(A), \delta) = \ker B = \mu^{-1} H^k(\Omega_A^k, \delta_A) = \mu^{-1}\mu H^k(H_k(A), B).$$

□

Theorem 2.5.10 *Let X be an n-dimensional compact Riemannian manifold. Then we have the isomorphism of cochain complexes*

$$\mu : (H_*(C^\infty(X)), B) \to (\Lambda^* X, d)$$

where d is the exterior derivative and the isomorphism of cohomologies

$$\mu : H^*(H_*(C^\infty(X)), B) \to H^*_{\mathrm{DR}}(X)$$

where $H^*_{\mathrm{DR}}(X)$ is the de-Rham cohomology of the compact Riemannian manifold X.

Proof The result is immediate from Theorem 2.5.8, Corollary 2.5.9 and the definition of the Kähler de-Rham complex as $C^\infty(X)$ is $\mathbb{C}$-smooth. □

As a result of Example 2.5.6 the Hochschild homology $H_*(A)$ of a unital associative algebra A is considered the space of non-commutative Kähler de-Rham exterior differential forms on A and the cohomology $H^*(\Lambda\Omega(A), \delta)$ the non-commutative Kähler de-Rham cohomology of A.

2.5.3 Volume Form

Let (H_ρ, π_ρ, D) be a n-dimensional Riemannian representation of a unital C*-algebra A. Let $\mathcal{A}_{\pi_\rho}$ be the smooth unital pre-C*-algebra of A. The results of Section 2.5.2 indicate the Hochschild cycles $Z_n(\mathcal{A}_{\pi_\rho})$ are the non-commutative generalisation of the highest power smooth exterior differential forms. The Hochschild cycles $Z_n(\mathcal{A}_{\pi_\rho})$ are hence the candidates for non-commutative complex volume forms.

Define the linear representation $\pi_\rho := \pi_D\theta_k^{-1} : \overline{C}_k(\mathcal{A}_{\pi_\rho}) \to \Omega^k_D(\mathcal{A}_{\pi_\rho})$. Explicitly

$$\pi_\rho(a_0 \otimes a_1 \otimes \ldots \otimes a_k) = \pi_\rho(a_0)[D, \pi_\rho(a_1)] \ldots [D, \pi_\rho(a_k)]$$

for all $k \geq 1$. Define

$$\pi_\rho(a_0 \otimes a_1 \otimes \ldots \otimes a_k)^{\mathrm{op}} = J_\rho\Big(\pi_\rho(a_0)[D, \pi_\rho(a_1)] \ldots [D, \pi_\rho(a_k)]\Big)^* J_\rho.$$

Definition 2.5.11 *Let (H_ρ, π_ρ, D) be an n-dimensional Riemannian representation of a unital C*-algebra A. Then we call (H_ρ, π_ρ, D) an oriented n-dimensional Riemannian representation of the unital C*-algebra A if there exists a Hochschild cycle $c \in Z_n(\mathcal{A}_{\pi_\rho})$ such that*

(i) *$\pi_\rho(c)$ is a self-adjoint unitary,*

(ii) *$[\pi_\rho(c), \pi_\rho(a)] = 0$ for all $a \in A$, and*

(iii) *$D\pi_\rho(c) = (-1)^{n-1}\pi_\rho(c)D$.*

The element $\pi_\rho(c) \in \Omega^n_D(\mathcal{A}_{\pi_\rho})$ is called a (non-commutative) volume form for this representation.

Proposition 2.5.12 *Let (H_ρ, π_ρ, D) be a Riemannian representation of a C*-algebra A. Then there exist a selfadjoint unitary $\Gamma \in U(H_\rho)$ such that $\epsilon(w) = \Gamma w \Gamma$ for all $w \in \Omega_D(\mathcal{A}_{\pi_\rho})''$ and $[\Gamma, J_\rho] = 0$.*

Proof See Lemma 2.5.18 below. □

Let Parity(ρ, A, D) denote the non-empty set of selfadjoint unitaries $\Gamma \in U(H_\rho)$ as in Proposition 2.5.12. Recall the two-dimensional Clifford algebra C_1 is abstractly defined by

$$C_1 = \{\lambda_1 + h\lambda_2 \mid \lambda_1, \lambda_2 \in \mathbb{C}, h^2 = 1\}$$

with a $\mathbb{Z}_2$-grading $\beta : \lambda_1 + h\lambda_2 \to \lambda_1 - h\lambda_2$. Compare the following result with Theorem 2.3.1.

Theorem 2.5.13 (Riemannian Orientations and Gradings)
Let (H_ρ, π_ρ, D) be an oriented n-dimensional Riemannian representation of a unital C^*-algebra A with volume form $\pi_\rho(c)$. Let $\Gamma \in \mathrm{Parity}(\rho, A, D)$. Then, when n is even

(i) *$\pi_\rho(c)$ is a grading element for this Riemannian representation,*

(ii)
$$\Gamma \in \mathrm{Parity}(\rho, A, D) = \{\ V\pi_\rho(c)\pi_\rho(c)^{\mathrm{op}}\ |V^2 = 1, V \in Z(\Omega_D(\mathcal{A}_{\pi_\rho}))''\},$$

(iii) $\mathrm{ad}_\Gamma = \epsilon$ *is trivial on the centre $Z(\Omega_D(\mathcal{A}_{\pi_\rho}))''$,*

and when n is odd

(iv) *$\pi_\rho(c)$ is a central element of $\Omega_D(\mathcal{A}_{\pi_\rho})''$, hence not a grading element for this Riemannian representation,*

(v)
$$\mathrm{ad}_\Gamma(\pi_\rho(c)) = \Gamma\pi_\rho(c)\Gamma = -\pi_\rho(c),$$

hence $\mathrm{ad}_\Gamma = \epsilon$ is not trivial on the centre $Z(\Omega_D(\mathcal{A}_{\pi_\rho}))''$,

(vi) *the Hilbert space H_ρ is a graded $A \otimes C_1$-$\mathbb{C}$-C^*-bimodule. In particular we have the representation $A \otimes C_1 \to \Omega_D(\pi_\tau(\mathcal{A}))''$ given by,*

$$a_i \mapsto \pi_\rho(a_i)\ ,\ h \mapsto \pi_\rho(c)\ ,\ \beta \mapsto \mathrm{ad}_\Gamma$$

The proof of Theorem 2.5.13 shall be comprised of the lemmas and propositions of Section 2.5.4. That section collects related results on orientations and grading as well as provide the proof.

Theorem 2.5.14 *Let (H_ρ, π_ρ, D) be an oriented n-dimensional Riemannian representation of a unital C^*-algebra A with volume form $\pi_\rho(c)$. Then the following sets are equivalent:*

(i) *the set of $\Gamma \in \mathrm{Parity}(\rho, A, D)$ such that $[D^2, \Gamma] = 0$,*

(ii) *the set of $\Gamma \in \mathrm{Parity}(\rho, A, D)$ such that $\{D, \Gamma\} = 0$.*

Proof (ii) $\Rightarrow$ (i) is immediate. (i) $\Rightarrow$ (ii) Let $\Gamma \in \mathrm{Parity}(\rho, A, D)$. Hence $[\Gamma, \pi_\rho(a)] = 0$ and $\{\Gamma, [D, \pi_\rho(a)]\} = 0$ for all $a \in \mathcal{A}_{\pi_\rho}$. The relation $[\Gamma, |D|] = 0$ follows from $(\Gamma|D|\Gamma)^2 = \Gamma|D|^2\Gamma = \Gamma D^2\Gamma = D^2 = |D|^2$ and uniqueness of the positive square root of a positive operator. Hence Γ preserves the dense domains $Dom D^m = Dom|D|^m$ for $m \in \mathbb{N}$ and the selfadjoint operator $\{D, \Gamma\}$ has dense domain. Let $\xi \in \cap_m Dom D^m$. Then $[\{D, \Gamma\}, \pi_\rho(a)]\xi = \{\Gamma, [D, \pi_\rho(a)]\}\xi = 0$ and $[\{D, \Gamma\}, [D, \pi_\rho(a)]] = [[\Gamma, D^2], \pi_\rho(a)]\xi = 0$ for all $a \in \mathcal{A}_{\pi_\rho}$. Using density of the subset $\cap_m Dom D^m \subset H_\rho$, $[\{D, \Gamma\}, \pi_\rho(a)] = 0 = [\{D, \Gamma\}, [D, \pi_\rho(a)]]$ for all $a \in \mathcal{A}_{\pi_\rho}$. Hence $[\{D, \Gamma\}, \pi_\rho(c)] = 0$. However, $\{\{D, \Gamma\}, \pi_\rho(c)\} = 0$ as $\Gamma\pi(c) = (-1)^n\pi(c)\Gamma$ and $D\pi(c) = (-1)^{n-1}\pi(c)D$. Hence $\{D, \Gamma\} = 0$. □

Definition 2.5.15 *A Riemannian representation (H_ρ, π_ρ, D) of a C^*-algebra A is called a real Riemannian representation if there exists $\Gamma \in \mathrm{Parity}(\rho, A, D)$ such that $[D^2, \Gamma] = 0$.*

Corollary 2.5.16 *Let (H_ρ, π_ρ, D) be an oriented n-dimensional Riemannian representation of a unital C*-algebra A. Then there exists a real grading element Γ for this Riemannian representation if and only if the Riemannian representation is real.*

Proof Immediate from Theorem 2.5.14 and Definition 2.5.15. □

Remark 2.5.17 Let (H_ρ, π_ρ, D) be a real oriented n-dimensional Riemannian representation of a unital C*-algebra A. Then this representation has a fundamental class $\lambda_{-1} = [(H_\rho, F_D, \Gamma)] \in KK(A \otimes B_{\pi_\rho}, \mathbb{C})$ where B_{π_ρ} is the index algebra.

2.5.4 Riemannian Orientations and Gradings

Let (H_ρ, π_ρ, D) be a Riemannian representation of C*-algebra A. Let $\mathbf{Riem}(A,\rho) = (\Omega_D(\mathcal{A}_{\pi_\rho})'', H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be the associated standard form. Let $R = \Omega_D(\mathcal{A}_{\pi_\rho})''$ and $\iota_\rho(R_\rho)$ be the dense image of the ρ-finite elements R_ρ in H_ρ.

Lemma 2.5.18 *There exists a selfadjoint unitary $\Gamma \in U(H_\rho)$ such that $\epsilon(w) = \Gamma w \Gamma$ for all $w \in R$ and $[\Gamma, J_\rho] = 0$. Moreover, if ρ is a trace, there exists Γ such that $\Gamma\iota_\rho(r) = \iota_\rho(\epsilon(r))$ for all $r \in R_\rho$.*

Proof Let $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ and $(\epsilon(R), H_\rho, J_\rho, \Delta', \mathcal{P}')$ be two standard forms. Let Γ be the unitary given by Theorem 1.6.4. Existence follows from setting $\Delta' = \Delta_\rho$ and $\mathcal{P}' = \mathcal{P}_\rho$.

Let ρ be a trace. Define $\Gamma\iota_\rho(r) := \iota_\rho(\epsilon(r))$ for all $r \in R_\rho$. By density of $\iota_\rho(R)$ the linear operator Γ extends to a selfadjoint unitary operator on H_ρ. Moreover $\Gamma w \Gamma \iota_\rho(r) = \Gamma\iota_\rho(w\epsilon(r)) = \epsilon(w)\iota_\rho(r)$ for all $w \in R, r \in R_\rho$. Hence Γ implements the parity automorphism on R. As $\Delta_\rho = 1$ then $\mathcal{P}_\rho = \overline{\iota_\rho(R_\rho^+)}$ by Theorem 1.6.2. Hence $J_\rho\Gamma\iota_\rho(r^*r) = J_\rho\iota_\rho(\epsilon(r)^*\epsilon(r)) = \iota_\rho(\epsilon(r)^*\epsilon(r)) = \Gamma\iota_\rho(r^*r) = \Gamma J_\rho\iota_\rho(r^*r)$ for all $r \in R_\rho$ by Theorem 1.6.1 (vi). Then $[J_\rho, \Gamma]\mathcal{P}_\rho = 0$ and $[J_\rho, \Gamma]H_\rho = 0$ by linearity. Hence $[J_\rho, \Gamma] = 0$. □

Lemma 2.5.19 *Let (H_ρ, π_ρ, D) be an oriented odd-dimensional Riemannian representation of a unital C*-algebra A. Then each volume form $\pi(c)$ belongs to $Z(\Omega_D(A))$.*

Proof Follows as $[D, \pi(c)] = 0 = [\pi(c), \pi(a)]$ for all $a \in A$. □

Define the *-algebra closed under the holomorphic functional calculus, $\Omega_D^{\mathrm{even}}(\mathcal{A}_{\pi_\rho}) = \{w \in \Omega_D(\mathcal{A}_{\pi_\rho}) \mid \epsilon(w) = w\}$.

Proposition 2.5.20 *Let (H_ρ, π_ρ, D) be an oriented n-dimensional Riemannian representation of a unital C*-algebra A.*

(i) *The space of volume forms for this representation is parameterised by a subset of the group*

$$\{U \in U(Z(\Omega_D^{\mathrm{even}}(\mathcal{A}_{\pi_\rho}))) \mid U^2 = 1, [D, U] = 0\}.$$

(ii) *The volume form for this representation is unique if one of the following conditions hold*

(a) $Z(\Omega_D^{\rm even}(\mathcal{A}_{\pi_\rho}))$ *contains no proper projections,*

(b) $\mathrm{Prim}(\Omega_D^{\rm even}(\mathcal{A}_{\pi_\rho}))$ *is connected,*

(c) $Z(\Omega_D^{\rm even}(\mathcal{A}_{\pi_\rho})) = Z(\mathcal{A}_{\pi_\rho})$ *and* $Z(A)$ *contains no proper projections,*

(d) $Z(\Omega_D^{\rm even}(\mathcal{A}_{\pi_\rho})) = Z(\mathcal{A}_{\pi_\rho})$ *and* $\mathrm{Prim}(A)$ *is connected,*

(e) $Z(\Omega_D^{\rm even}(\mathcal{A}_{\pi_\rho}))'' = Z(A)''$ *and* (H_ρ, π_ρ, D) *is base irreducible.*

Proof (i) Let $R = \Omega_D(\mathcal{A}_{\pi_\rho})''$. Let $c, c' \in Z_n(\mathcal{A}_{\pi_\rho})$ such that $\pi_\rho(c), \pi_\rho(c')$ are volume forms. Let $U = \pi_\rho(c)\pi_\rho(c') \in \Omega_D^{2n}(\mathcal{A}_{\pi_\rho})$. Then $U^*U = UU^* = 1$ is unitary. Moreover, $[D, U] = [\pi_\rho(a), U] = 0$ for all $a \in A$. Hence $U \in R'$ and $U \in U(Z(R))$. This implies $[U, \pi_\rho(c)] = 0$ and hence $[\pi_\rho(c'), \pi_\rho(c)] = 0$. Then $U^2 = \pi_\rho(c)\pi_\rho(c')\pi_\rho(c)\pi_\rho(c') = 1$. So U is a selfadjoint unitary.

(ii) There is a bijective correspondence between self-adjoint unitaries U and projections P via the formula $U = 1 - 2P$. Since $[D, U] = 0 \Leftrightarrow [D, P] = 0$ then the statements imply the group in (i) is the trivial group $\{1\}$. This follows from the proof of Theorem 1.2.10 and Definition 1.5.17. □

Note that $Z(\Omega_D^{\rm even}(\mathcal{A}_{\pi_\rho})) = Z(\Omega_D(\mathcal{A}_{\pi_\rho}))$ under the hypothesis of Proposition 2.5.20 with n even. This follows from Theorem 2.5.13(iii).

Lemma 2.5.21 *Let* (H_ρ, π_ρ, D) *be an oriented even-dimensional Riemannian representation of a unital* C^**-algebra* A. *Then* Γ *is a selfadjoint unitary such as in Lemma 2.5.18 if and only if* $\Gamma = V\pi(c)\pi(c)^{\rm op}$ *for some selfadjoint unitary* $V \in U(Z(R)'')$. *If* ρ *is a trace, then* $\Gamma\iota_\rho(r) = \iota_\rho(\epsilon(r))$ *for all* $r \in R_\rho$ *if and only if* $\Gamma = \pi(c)\pi(c)^{\rm op}$.

Proof Let Γ be as in Lemma 2.5.18. Consider $U = \Gamma\pi(c)$. Then $U^* = \pi(c)\Gamma = \Gamma\pi(c) = U$. Hence U is a selfadjoint unitary. Moreover $[U, \pi_\rho(a)] = 0$ and $[U, [D, \pi_\rho(a)]] = 0$ for all $a \in \mathcal{A}_{\pi_\rho}$. Hence $U \in R'$. Then we have two selfadjoint unitaries $J\pi(c)J$ and U in R'. Let $V = J\pi(c)JU \in R'$ One checks that $[U, J\pi(c)J] = 0$. Hence $V^2 = 1$. Now $JVJ = \pi(c)\Gamma J\pi(c)J = UJ\pi(c)J = V$. Hence $V \in Z(R)''$. Finally $\Gamma = V\pi(c)\pi(c)^{\rm op}$. Conversely, let $V \in Z(R)''$ such that $V^2 = 1$. Since $[\pi(c), J\pi(c)J] = 0$ then $T = V\pi(c)\pi(c)^{\rm op}$ is a selfadjoint unitary that implements the parity automorphism on R. Moreover $[J, T] = 0$ as $[J, V] = 0$ and $J(\pi(c)J\pi(c)J) = J\pi(c)J\pi(c)J = (\pi(c)J\pi(c)J)J$.

Let ρ be a trace and $\Gamma\iota_\rho(r) = \iota_\rho(\epsilon(r))$. Consider $J\pi(c)J - \Gamma\pi(c) \in R'$. In particular $(J\pi(c)J - \Gamma\pi(c))\iota_\rho(r) = J\pi(c)\iota(r^*) - \pi(c)\Gamma\iota_\rho(r) = \iota(r\pi(c)) - \pi(c)\iota_\rho(\epsilon(r)) = \pi(c)(\iota(\epsilon(r)) - \iota(\epsilon(r))) = 0$ for all $r \in R_\rho$. Hence $\pi(c)J\pi(c)J - \Gamma = 0$. The reverse implication is obvious. □

2.6 Connes' Axioms of Non-commutative Geometry

We recall a faithful state ρ on a von Neumann algebra R is a faithful normal semifinite weight such that $\rho(1) = 1$.

2.6.1 Structure of Riemannian Representations

Let (H_ρ, π_ρ, D) be a Riemannian representation of a unital C*-algebra A with associated standard form $\mathbf{Riem}(A,\rho) = (\Omega_D(\mathcal{A}_{\pi_\rho})'', H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$, see Definition 2.3.5. Let ρ be a faithful state and $\iota_\rho : \Omega_D(\mathcal{A}_{\pi_\rho})'' \to H_\rho$ the injection given by the GNS construction.

Define

$$\Lambda_\rho := \iota_\rho(\Omega_D(\mathcal{A}_{\pi_\rho}))$$

which is a dense subset of H_ρ. We recall the results of Section 1.4.1. We have a concrete representation

$$(\Omega_D(\mathcal{A}_{\pi_\rho}), \delta_D) : (\Omega(\mathcal{A}_{\pi_\rho}), \delta) \to B(L^2(\Lambda_\rho))$$

and a graded differential representation

$$(\Lambda_D(\mathcal{A}_{\pi_\rho}), \delta_D) : (\Omega(\mathcal{A}_{\pi_\rho}), \delta) \to \Lambda_\rho$$

of the universal differential algebra. The unital *-algebra $\Omega_D(\mathcal{A}_{\pi_\rho})$ is naturally a $\Omega_D(\mathcal{A}_{\pi_\rho}) \otimes \Omega_D(\mathcal{A}_{\pi_\rho})^{\mathrm{op}}$-C*-bimodule, and hence an $\mathcal{A}_{\pi_\rho} \otimes \mathcal{A}_{\pi_\rho}^{\mathrm{op}}$-bimodule, by left and right multiplication. This structure is transferred faithfully to the set Λ_ρ by the representations π_ρ and π_ρ^{op} where

$$w^{\mathrm{op}} := J_\rho w^* J_\rho$$

for $w \in \Omega_D(\mathcal{A}_{\pi_\rho})''$. There exists a selfadjoint unitary parity grading Γ by Proposition 2.5.12 that grades Λ_ρ by parity of differential forms and ad_Γ implements the automorphism of parity of differential forms on $\Omega_D(\mathcal{A}_{\pi_\rho})$. This structure generalises the situation on a Riemannian manifold as in Theorem 2.3.3 and Theorem 2.3.4.

Let $H_\rho^\infty = \cap_{m\geq 1} Dom|D|^m$. These elements are considered the smooth elements of the Hilbert space H_ρ. A Riemannian representation is a C_c^∞-representation, see Definition 1.4.8, hence

$$\Omega_D(\mathcal{A}_{\pi_\rho}) : H_\rho^\infty \to H_\rho^\infty.$$

By the GNS construction

$$\Omega_D(\mathcal{A}_{\pi_\rho}) : \Lambda_\rho \to \Lambda_\rho.$$

There is no reason in general why $\Lambda_\rho \subset H_\rho^\infty$ and hence why Λ_ρ are non-commutative smooth exterior differential forms.

Remark 2.6.1 As ρ is a faithful state the vector $\iota_\rho(1)$ is a cyclic and separating vector for $\Omega_D(\mathcal{A}_{\pi_\rho})''$. A necessary and sufficient condition for $\Lambda_\rho \subset H_\rho^\infty$ is $\iota_\tau(1) \in \cap_m Dom D^m$.

We will assume $\Lambda_\rho = H_\rho^\infty$. We recall from section 1.4.2 the locally convex topology $\mathcal{S}_D$ generated by the seminorms p_m.

Proposition 2.6.2 *Let (H_ρ, π_ρ, D) be a Riemannian representation of a C*-algebra A such that ρ is a faithful state and $\Lambda_\rho = H_\rho^\infty$. Then*

$$\Omega_D(\mathcal{A}_{\pi_\rho}) = \{w \in \Omega_D(\mathcal{A}_{\pi_\rho})'' \,|\, p_m(w) < \infty, m \in \mathbb{N}\}$$

*and hence is a smooth *-algebra in the locally convex topology S_D.*

Proof Let $R = \Omega_D(\mathcal{A}_{\pi_\rho})$. Let $T \in R''$ such that $p_m(T) < \infty$ for all $m \in \mathbb{N}$. Then $T\iota_\rho(1) \in \cap_m Dom|D|^m$ by Proposition B.2. Hence $T\iota_\rho(1) = \iota_\rho(r) = r\iota_\rho(1)$ for some $r \in R$. Then $T = r$ as $\iota_\rho(1)$ is a separating and cyclic vector on R''. □

The space Λ_ρ is completely generalised as a space of non-commutative smooth sections of the Hermitian non-commutative vector bundle of non-commutative exterior differential forms when we assume the right $\mathcal{A}_{\pi_\rho}$-module Λ_ρ is finitely generated and projective, see Definition 2.1.4. We can define an $\mathcal{A}_{\pi_\rho}$-valued Hermitian structure

$$(\cdot,\cdot)_\rho : \Lambda_\rho \times \Lambda_\rho \to \mathcal{A}_{\pi_\rho}$$

by the equality [V2, 11.3],

$$\rho((\iota_\rho(w_1), \iota_\rho(w_2))_\rho) := \rho(w_1^* w_2)$$

for $w_1, w_2 \in \Omega_D(\mathcal{A}_{\pi_\rho})$. Through the isomorphism $\iota_\rho : \Omega_D(\mathcal{A}_{\pi_\rho}) \to \Lambda_\rho$ the right $\mathcal{A}_{\pi_\rho}$-module $\Omega_D(\mathcal{A}_{\pi_\rho})$ is finitely generated and projective with Hermitian structure $(w_1, w_2) = (\iota_\rho(w_1), \iota_\rho(w_2))_\rho$.

Let the Riemannian representation (H_ρ, π_ρ, D) of the unital C*-algebra A be n-dimensional. Let D_s be the set of dilation and translation invariant states on ℓ^∞. Then a non-commutative integral $\tau_\omega \in A^*$ is given by $\tau_\omega(a) := Tr_\omega(\pi(a) f_n(D))$ where $f_n(x) = (1 + x^2)^{-n/2}$ and $\omega \in D_s$, see Section 1.7.4. Assume that ρ provides the measure class of a non-commutative integral. In mathematics, $\rho \equiv \tau_\omega$ in A^* for some $\omega \in D_s$, see section 1.6.2. Then the inner product on the GNS Hilbert space H_ρ is identified on the dense subspace Λ_ρ as

$$\langle \eta_1, \eta_2 \rangle = \tau_\omega\Big((\eta_1, \eta_2)_\rho \, d_{\rho,\tau_\omega}\Big)$$

where d_{ρ,τ_ω} is the positive invertible Radon-Nikodym derivative $(\rho : \tau_\omega)$ of Corollary 1.6.7 and $\eta_i = \iota_\rho(w_i) \in \iota_\rho(\Omega_D(\mathcal{A}_{\pi_\rho}))$ for $i = 1, 2$. The Hilbert space H_ρ is completely generalised as a space of non-commutative L^2-sections of the Hermitian non-commutative vector bundle of non-commutative exterior differential forms $H_\rho = L^2(\Lambda_\rho, \tau_\omega)$ with the assumption $\rho \equiv \tau_\omega$ in A^* for some $\omega \in D_s$.

Compare the paragraphs above to the structure of a Riemannian manifold in Section 1.3.6 (i) and (ii).

Let $\pi_\rho(c)$ be a volume form for a real n-dimensional Riemannian representation (H_ρ, π_ρ, D) of the separable unital C*-algebra A, see Definition 2.5.11 and Definition 2.5.15. Then we recall there exists a non-empty separable C*-subalgebra B_{π_ρ} of the commutant $\Omega_D(\mathcal{A}_{\pi_\rho})'$ and a real grading element Γ such that $[(H_\rho, F_D, \Gamma)] \in KK(A \otimes B_{\pi_\rho}, \mathbb{C})$, see Proposition 2.4.6 and Theorem 2.5.7. We recall that the intersection product $\otimes_A$ induces a group homomorphism

$$\cdot \otimes_A [(H_\rho, F_D, \Gamma)] : KK(\mathbb{C}, A) \to KK(B_{\pi_\rho}, \mathbb{C}),$$

see section 2.4.1 and 2.4.2.

2.6.2 The Axioms of Riemannian Geometry

The following axioms, derived from those detailed by Connes in [C3] [C4], were put forward as determining the structure of (compact) Riemannian differential geometry.

Basic Definitions

Let A be a unital associative *-algebra. Then A is a C*-algebra if it is a Banach *-algebra with norm $\|\cdot\|$ that satisfies $\|a^*a\| = \|a\|^2$ for all $a \in A$, see the preamble to section 1.2. A C*-algebra A is called separable if it admits a countable basis. Let (H_ρ, π_ρ, D) be a real n-dimensional Riemannian representation of a separable unital C*-algebra A, see Definition 2.3.5. A Riemannian representation is called irreducible if it is base irreducible in the sense of Definition 1.5.7. Let $\mathcal{A}_{\pi_\rho}$ be the Frechet pre-C*-algebra of smooth elements, see Proposition 1.4.9. When ρ is a faithful state let Λ_ρ, ι_ρ, $\Omega_D(\mathcal{A}_{\pi_\rho})$, H_ρ^∞, τ_ω, D_s, $\pi_\rho(c)$, B_{π_ρ} and $\otimes_A$ be as described in section 2.6.1 above.

The Axioms of Compact Riemannian Geometry

Let A be a unital associative *-algebra.

R1. **Axiom of Second Countable Metrisable Compact Topology**
The unital *-algebra A is a separable C*-algebra.

R2. **Axiom of Riemannian Structure**
There exists an irreducible real Riemannian representation (H_ρ, π_ρ, D) of A such that ρ is a faithful state, and

R3. **Axiom of Symmetry**
The centre $\pi_\rho(Z(A))$ belongs to the centre $Z(\Omega_D(\mathcal{A}_{\pi_\rho}))''$.

R4. **Axiom of Finiteness and Smoothness**
The right $\mathcal{A}_{\pi_\rho}$-module $\Lambda_\rho := \iota_\rho(\Omega_D(\mathcal{A}_{\pi_\rho}))$ is finite projective and $\Lambda_\rho = H_\rho^\infty$.

R5. **Axiom of Absolute Continuity**
The Riemannian representation is n-dimensional and there exists $\omega \in D_s$ such that $\rho \equiv \tau_\omega$ in A^*.

R6. **Axiom of Orientation**
There exists a Hochschild cycle $c \in Z_n(\mathcal{A}_{\pi_\rho})$ such that $\pi_\rho(c)$ is a volume form.

R7. **Axiom of Poincaré Duality**
A and the index algebra B_{π_ρ} are Poincaré dual. In particular, the map

$$\cdot \otimes_A [(H_\rho, F_D', \Gamma)] : KK(\mathbb{C}, A) \to KK(B_{\pi_\rho}, \mathbb{C}),$$

is a group isomorphism.

Riemannian Geometries and Symmetry

The axioms constitute a formulation which at no point requires commutivity of the *-algebra A. The purpose of the axioms is this: a commutative unital *-algebra A

should satisfy the axioms of compact Riemannian geometry if and only if $A = C(X)$ where X is a compact Riemannian manifold[10].

Definition 2.6.3 *Let A be a unital *-algebra that satisfies the axioms R1, R2, R3, R4, R5, R6 and R7. We call*

(i) *the unital *-algebra A a unital Connes-Riemann- or CR-algebra,*

(ii) *the information $(A, H_\rho, \pi_\rho, D, c)_R$ a Riemannian geometry associated to the CR-algebra A,*

(iii) *the information $PS(A) \overset{[\cdot]_u}{\to} \hat{A} \overset{\ker}{\to} \mathrm{Prim}(A)$ a compact Riemannian manifold.*

Remark 2.6.4 A Riemannian geometry $(A, H_\rho, \pi_\rho, D, c)_R$ has an associated standard form

$$\mathbf{Riem}(A, \rho) = (\Omega_D(\mathcal{A}_{\pi_\rho})'', H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$$

by the Tomita-Takesaki theory.

Let R be a von Neumann algebra. Then $A \subset Z(R)$ implies A is commutative but the converse is false.

Proposition 2.6.5 *Let R be a von Neumann algebra. Then the following statements are equivalent*

(i) $A \subset Z(R) \Leftrightarrow A$ *is a commutative *-subalgebra of R,*

(ii) *R is commutative.*

Proof (ii) $\Rightarrow$ (i) is immediate. (i) $\Rightarrow$ (ii) Let $r \in R$. Then the C*-algebra generated by r, $C^*(r)$, is a commutative *-subalgebra. Hence $C^*(r) \subset Z(R)$ and $r \in Z(R)$. Then $Z(R) = R$. $\square$

This demonstrates that the axiom of symmetry is not a tautology. The next result demonstrates the necessity of the axiom of symmetry. Let $H_k(A)$ denote the k^{th} Hochschild homology group of a unital associative algebra A.

Proposition 2.6.6 *Let $(A, H_\rho, \pi_\rho, D, c)_R$ be a Riemannian geometry. Then the following statements are equivalent*

(i) $\pi(A) \subset Z(\Omega_D(\mathcal{A}_{\pi_\rho}))''$,

(ii) *the maps*

$$\pi_\rho : H_0(\mathcal{A}_{\pi_\rho}) \to \Omega_D^0(\mathcal{A}_{\pi_\rho})$$

[10] We revisit the opening discussion of this chapter. Compare the statement: a commutative unital *-algebra A should satisfy the axioms of compact Riemannian geometry if and only if $A = C(X)$ where X is a compact Riemannian manifold; which, with the terminology of definition 2.6.3, is stated: a commutative unital *-algebra A is a CR-algebra if and only if $A = C(X)$ where X is a compact Riemannian manifold; to the statement: a commutative unital *-algebra A is a C*-algebra if and only if $A = C(X)$ where X is a compact Hausdorff space; and the statement: a commutative unital *-algebra A is a von Neumann algebra if and only if $A = L^\infty(M, \mu)$ where M is a measure space and μ a finite regular Borel measure on M.

$$\pi_\rho : H_1(\mathcal{A}_{\pi_\rho}) \to \Omega^1_D(\mathcal{A}_{\pi_\rho})$$

exist and are isomorphisms.

Proof Let B be a unital associative algebra. The homology group $H_0(B) = B/\mathrm{Com}(B)$. Then $B \cong H_0(B)$ iff B is commutative. Let $Z_1 = \{a \otimes b \,|\, a, b \in B, [a, b] = 0\}$. Then $Z_1 = C_1(B)$ iff B is commutative. Let B be commutative. Let $I_1 = \{b_0 b_1 \otimes b_2 - b_0 \otimes b_1 b_2 + b_2 b_0 \otimes b_1 \,|\, b_0, b_1, b_2 \in B\}$. Then $H_1(B) = C_1(B)/I_1$. Let $B = \mathcal{A}_{\pi_\rho}$ and $\pi_\rho = \pi$. The linear map $\pi : C_1(B) \to \Omega^1_D(B)$ given by $\pi(a \otimes b) = \pi(a)[D, \pi(b)]$ is an isomorphism. This follows as the Riemannian representation is irreducible and π is faithful representation of B. Hence $\pi : H_1(B) \to \Omega^1_D(B)$ exists and is an isomorphism iff $\pi(I_1) = \{0\}$. Consider $\pi(b_0 b_1 \otimes b_2 - b_0 \otimes b_1 b_2 + b_2 b_0 \otimes b_1) = \pi(b_0)\pi(b_1)[D, \pi(b_2)] - \pi(b_0)[D, \pi(b_1)\pi(b_2)] + \pi(b_2)\pi(b_0)[D, \pi(b_1)] = \pi(b_0)\Big(\pi(b_2)[D, \pi(b_1)] - [D, \pi(b_1)]\pi(b_2)\Big)$. Hence $\pi(I_1) = \{0\}$ iff $\pi(a)[D, \pi(b)] = [D, \pi(b)]\pi(a)$ for all $a, b \in B$.

In summary $B \cong H_0(B)$ iff B is commutative and when B is commutative $\pi : H_1(B) \to \Omega^1_D(B)$ exists and is an isomorphism iff $\pi(a)[D, \pi(b)] = [D, \pi(b)]\pi(a)$ for all $a, b \in B$.

(i) $\Rightarrow$ (ii) The hypothesis implies B is commutative and $\pi(a)[D, \pi(b)] = [D, \pi(b)]\pi(a)$ for all $a, b \in B$.

(ii) $\Rightarrow$ (i) The hypotheses imply B is commutative and $\pi(a)[D, \pi(b)] = [D, \pi(b)]\pi(a)$ for all $a, b \in B$. Hence $\pi(a)w = w\pi(a)$ for all $a \in B$ and $w \in \Omega_D(B)$. Hence $\pi(B) \in Z(\Omega_D(B))''$. The result follows as B is norm dense in A. □

Remark 2.6.7 The term symmetry comes from the fact

$$\Omega^1_{Z(\mathcal{A}_{\pi_\rho})} \cong H_1(Z(\mathcal{A}_{\pi_\rho})) \cong \Omega^1_D(Z(\mathcal{A}_{\pi_\rho}))$$

is the universal symmetric $Z(\mathcal{A}_{\pi_\rho})$-bimodule with a derivation $\delta : Z(\mathcal{A}_{\pi_\rho}) \to \Omega^1_{Z(\mathcal{A}_{\pi_\rho})}$.

Spin$_R$ Geometries

Let C_1 denote the two-dimensional Clifford algebra. We recall section 2.3.5 and Definition 2.3.8 of spin$_R$ structure. This allows a definition of spin$_R$ geometries.

Definition 2.6.8 *Let $(A, H_\rho, \pi_\rho, D, c)_R$ be a Riemannian geometry of dimension n. Then $(A, H_\rho, \pi_\rho, D, c)_R$ is a Riemannian spin$_R$ geometry if $\mathcal{A}_{\pi_\rho}(\otimes C_1) \sim_M \Omega_D(\mathcal{A}_{\pi_\rho})$ where $(\otimes C_1)$ is added if n is odd.*

2.6.3 Reconstruction Theorem

The following results demonstrate the sufficiency of the axioms. We recall that we use the term compact Riemannian manifold to denote a metrisable, compact, connected, orientated smooth manifold without boundary.

Theorem 2.6.9 *Let (X, g) be an n-dimensional compact Riemannian manifold, $C(X)$ be the unital *-algebra of continuous functions on X, the essentially selfadjoint operator $d + d^* : C^\infty(X, \Lambda^* X) \to C^\infty(X, \Lambda^* X)$ be the signature operator and $\gamma \in C^\infty(X, \Lambda^n(X))$ be the complex volume form. Then*

(i)
$$\rho(w) := \int_X q_g(1,w)(x)\sqrt{\det g}\,dx,$$
where q_g is the metric, is a faithful state on the von Neumann algebra $L^\infty(X, \mathrm{Cl}(X))$,

(ii) *$(L^2(X, \Lambda^* X), \pi_l)$ is the faithful GNS representation of $L^\infty(X, \mathrm{Cl}(X))$ associated to ρ where π_l is the left multiplication representation,*

(iii) *$C(X)$ is a unital CR-algebra,*

(iv) *$(C(X), L^2(X, \Lambda^* X), \pi_l, d + d^*, \gamma)_R$ is a compact Riemannian geometry.*

Proof (i) Theorem 1.7.21(iv). (ii) Theorem 1.7.21(iv),(v). (iii) (R1) As X is a second countable metrisable compact space, $C(X)$ is a separable unital C*-algebra. (R2) Theorem 2.4.21, (ii) and (i) imply R2. Irreducibility follows from Proposition 2.6.2 and Proposition 1.1.14 as any central projection $p \in L^\infty(X)$ must lie in $C^\infty(X)$. As X is connected by hypothesis $C^\infty(X)$ has no projections. Reality follows as the Laplacian Δ is an even differential operator, hence commutes with the parity grading U_ϵ. (R3) It follows from Theorem 2.3.1 that $Z(L^\infty(X, \mathrm{Cl}(X))) = L^\infty(X)$ when n is even or $Z(L^\infty(X, \mathrm{Cl}(X))) = L^\infty(X) \otimes C_1$ when n is odd. (R4) Theorem 1.7.18 and Theorem 2.1.9. (R5) Theorem 1.7.21(ii) (R6) Theorem 2.5.10 implies $\gamma \in Z_n(C^\infty(X))$. The selfadjoint properties and commutivity with $C^\infty(X)$ are immediate from Section 2.3.1. For the relation with $d + d^*$, see [LM]. (R7) Theorem 2.4.21. (iv) Follows from the proof of (iii). □

We recall the spectrum $\Sigma(A)$ of a commutative C*-algebra A from section 1.2.3 and Theorem 1.2.12.

Theorem 2.6.10 (Connes' Reconstruction Theorem [C3])

Let A be a unital commutative CR-algebra with associated n-dimensional Riemannian geometry $(A, H_\rho, \pi_\rho, D, c)_R$. Then

(i) *$\Sigma(A)$ is a compact n-dimensional Riemannian manifold with geodesic metric*
$$d_\pi(\phi, \psi) = \sup_{a \in A}\{|\phi(a) - \psi(a)| \mid \|[D, \pi_\rho(a)]\| \le 1\}$$
for $\phi, \psi \in \Sigma(A)$, and

we have the identifications

(ii) *$A = C(\Sigma(A))$, $\mathcal{A}_{\pi_\rho} = C^\infty(\Sigma(A))$ and $\Omega_D(\mathcal{A}_{\pi_\rho}) = C^\infty(\Sigma(A), \mathrm{Cl}(\Sigma(A)))$ acting by left multiplication on the Hilbert space $H_\rho \cong L^2(\Sigma(A), \Lambda^*\Sigma(A))$,*

(iii) *the selfadjoint operator D is given by $d + d^* + E$ where E is an operation by one forms. When n is even (resp. odd) then $E \in C^\infty(\Sigma(A), \mathrm{Cl}^1(\Sigma(A)))$ (resp. $E = E_1 + JE_2^* J$ where $E_1, E_2 \in C^\infty(\Sigma(A), \mathrm{Cl}^1(\Sigma(A)))$),*

(iv) *the Hochschild cycle $c \in Z_n(\mathcal{A}_{\pi_\rho})$ is the complex volume form γ for $\Sigma(A)$.*

[[(v) *Let $D(E)$ (resp. $D(E_1, E_2)$) denote the space of selfadjoint operators given by $d + d^* + E$ as in* (iii) *above. Let $n > 2$. Then* $\mathrm{WRes}(|D|^{2-n})$, *where*

WRes *denotes the Wodzicki residue, is a positive quadratic form on* $D(E)$ *(resp.* $D(E_1, 0)$*) with unique minimum* $D_\sigma = d + d^*$*. In particular, for all* $\omega \in D_s$*,*

$$Tr_\omega(D_\sigma^2(1+D_\sigma^2)^{-n/2}) = \mathrm{Wres}(|D_\sigma|^{2-n}) = -c(n/2)\int_X R\sqrt{\det g}\,dx$$

where R *is the scalar curvature of* $\Sigma(A)$ *and*

$$c(x) = \frac{1}{6}(x-1)(x\pi^x\Gamma(x))^{-1}.\ ]]$$

Proof The proof of this theorem is a thesis in itself. The extension and proof of Connes' original formulation for spin geometry was undertaken in [Re, Re2]. We do not attempt the proof here. □

We shall not discuss the Wodzicki residue here. Hence the brackets around [[(v)]] to indicate this result is included for completeness and intended only for the specialist reader.

Corollary 2.6.11 *Let* A *be a unital* CR*-algebra. Then* A *is commutative if and only if* $A = C(X)$ *where* X *is a compact Riemannian manifold.*

Proof Theorem 2.6.9 and Theorem 2.6.10. □

Corollary 2.6.12 *Let* A *be a unital* CR*-algebra that admits a Riemannian* $spin_{\mathbb{R}}$ *geometry. Then* A *is commutative if and only if* $A = C(X)$ *where* X *is a compact Riemannian* $spin_{\mathbb{C}}$ *manifold.*

Proof Theorem 2.6.9, Theorem 2.6.10 and Definition 2.3.7. □

2.7 Symmetric Derivations and Riemannian Cycles

The theory of C*-algebras and von Neumann algebras has an abstract basis independent of their concrete representation on Hilbert space[11]. The relation between abstract and concrete is the GNS construction, which is a function from the state space of a C*-algebra or the pre-dual of a von Neumann algebra to concrete representations.

The algebraic core of compact Riemannian structure is a Riemannian representation (H_ρ, π_ρ, D) of a unital C*-algebra A where ρ is a faithful state on a von Neumann algebra R with $A \subset R$. The representation (H_ρ, π_ρ) is the GNS representation of R associated to ρ but the selfadjoint operator D is concrete. It is natural to consider the question of construction of a Riemannian representation of a C*-algebra A from abstract considerations on a von Neumann algebra R that contains A as a C*-subalgebra.

[11] Abstract von Neumann algebras were studied under the title of W^*-algebras by Sakai, [Sak2] [Sak1].

2.7.1 Symmetric Derivations

The study of derivations on von Neumann algebras is extensive [Sak1] [Kad] [BR]. The relevance of the theory to our situation is the existence of a selfadjoint unbounded operator on a concrete representation that spatially implements an unbounded derivation. This is detailed as follows.

Definition 2.7.1 (BR, 3.2.21, 3.2.54) **(i)** *A symmetric derivation δ of a C^*-algebra A with domain $Dom\delta \subset A$ is a linear operator $\delta : Dom\delta \to A$ such that $\delta(a)^* = \delta(a^*)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in Dom\delta$.*

(ii) *A symmetric derivation δ is spatially implemented by a symmetric operator D on a Hilbert space H if there exists a representation $\pi : A \to B(H)$ such that $\pi(Dom\delta)DomD \subset DomD$ and $\pi(\delta(a)) = i[D, \pi(a)]$.*

A symmetric derivation $\delta : Dom\delta \to A$ is called bounded if there exists $M < \infty$ such that $\|\delta(a)\| \leq M\|a\|$ for all $a \in Dom\delta$.

Theorem 2.7.2 (BR, 3.2.47) *Let δ be a bounded symmetric derivation of a von Neumann algebra R such that $Dom\delta$ is norm dense in R. Then there exists a self-adjoint operator $D \in R$ such that $\delta(r) = i[D, r]$ for all $r \in R$.*

Hence bounded symmetric derivations correspond to bounded spatial implementers. A symmetric derivation $\delta : Dom\delta \to R$ of a von Neumann algebra R is called σ-weak closed if $r_i \to r$ and $\delta(r_i) \to t$ converge σ-weakly in R implies $r \in Dom\delta$ and $\delta(r) = t$. A symmetric derivation $\delta : Dom\delta \to R$ is called σ-weak closable if there exists a closed symmetric derivation $\overline{\delta} : Dom\overline{\delta} \to R$ such that $Dom\delta \subset Dom\overline{\delta}$ and $\delta(r) = \overline{\delta}(r)\ \forall r \in Dom\delta$.

Theorem 2.7.3 (BR, 3.2.27, 3.2.28, 3.2.61) *Let $\delta : Dom\delta \to R$ be a symmetric derivation of a von Neumann algebra R such that $Dom\delta$ is σ-weak dense in R. Let ρ be a faithful state of R and (H_ρ, π_ρ) be the GNS representation of R associated to ρ. Assume (ρ, δ) satisfies the condition*

$$\rho(\delta(a)) = 0$$

for all $a \in Dom\delta$. Then

(i) *δ is σ-weak closable,*

(ii) *there exists a self-adjoint operator D on H_ρ such that*

  (a) *$\iota_\rho(Dom\delta) \subset DomD$ is a core for D,*

  (b) *$\pi_\rho(\delta(a)) = i[D, \pi_\rho(a)]\ \forall a \in Dom\delta$,*

  (c) *if $1 \in Dom\delta$, $D\iota_\rho(1) = 0$,*

(iii) *the following statements are equivalent*

  (a) *$e^{itD}\pi_\rho(R)e^{-itD} = \pi_\rho(R)$ for all $t \in \mathbb{R}$*

  (b) *D and Δ_ρ commute strongly, that is $\Delta^{is}D\Delta^{-is} = D$ for all $s \in \mathbb{R}$.*

Remark 2.7.4 We remark that $\Delta_\rho = 1$ if ρ is a trace. Hence the result (iii)(a) is automatic and the self-adjoint operator D is the generator of a σ-weak-continuous one-parameter family of automorphisms of R.

A relevant notion is the analytic elements of a derivation.

Definition 2.7.5 (BR, 3.1.17, 3.1.5) *The analytic elements of a symmetric derivation $\delta : Dom\delta \to R$ are those $a \in R$ such that $a \in Dom\delta^m$ for all m and the function*

$$f_a : \mathbb{C} \to \mathbb{C} \ , \ z \mapsto \sum_{m \in \mathbb{N}} \frac{z^m}{m!} \|\delta^m(a)\|$$

is entire. Let R_δ denote the analytic elements in R for δ.

Corollary 2.7.6 *Let the triple (R, ρ, δ) satisfy the conditions of Theorem 2.7.3. such that D and Δ_ρ strongly commute. Then R_δ is σ-weak dense in R.*

Proof Since δ generates a σ-weak-continuous family of automorphisms, then the analytic elements are σ-dense by [BR] Proposition 2.5.22. □

Definition 2.7.7 *Let δ be a symmetric derivation of a von Neumann algebra R such that $Dom\delta$ is σ-weak dense and ρ a faithful state on R. We call* (1) *the triple (R, ρ, δ) an inner K-cycle if it satisfies the condition*

$$\rho(\delta(r)) = 0$$

for all $r \in Dom\delta$, and (2) *the triple $(H_\rho, \pi_\rho, D_\delta)$ the GNS representation associated to an inner K-cycle (R, ρ, δ) where D_δ is the selfadjoint operator given by Theorem 2.7.3.*

Let (R, ρ, δ) be an inner K-cycle and $A \subset R$ a C*-subalgebra. Define $\mathcal{A} := A \cap R_\delta$ where R_δ are the analytic elements for δ. Define $\Omega_\delta(\mathcal{A}) :=< \mathcal{A}, \delta(\mathcal{A}) >$.

Lemma 2.7.8 *Let (R, ρ, δ), A and R_δ be as above. Then* (1) *$\mathcal{A}$ is a *-algebra closed under the holomorphic functional calculus, and* (2) *$\Omega_\delta(\mathcal{A})$ is a *-subalgebra of $\cap_m Dom\delta^m$.*

Proof Follows from calculations involving linearity, symmetry and the derivation property of δ. □

Definition 2.7.9 *Let (R, ρ, δ) be an inner K-cycle and $A \subset R$ be a C*-subalgebra. Then (R, ρ, δ) is called an inner Riemannian cycle over the C*-algebra A if the following conditions are satisfied* (1) *$\mathcal{A} = A \cap R_\delta$ is norm dense in A, and* (2) *$\Omega_\delta(\mathcal{A})$ is σ-weak dense in R.*

The following theorem establishes a construction function with domain an inner Riemannian cycle (R, ρ, δ) of a C*-algebra A and range an ungraded Riemannian representation of A.

Theorem 2.7.10 [GNS Construction]
Let (R, ρ, δ) be an inner Riemannian cycle over a C^*-algebra A. Then the associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$ is an ungraded Riemannian representation of the C^*-algebra A such that

(i) $[D_\delta, \pi_\rho(a)] = -i\pi_\rho(\delta(a))$ *for all* $a \in \mathcal{A}$,

(ii) $\Omega_{D_\delta}(\pi_\rho(A)) = \pi_\rho(\Omega_\delta(\mathcal{A}))$,

(iii) $\Lambda_\rho := \iota_\rho(\Omega_\delta(\mathcal{A})) \subset H_\rho^\infty := \cap_m Dom|D_\delta|^m$.

Proof (i) and (iii) follow immediately from Theorem 2.7.3. (ii) follows from (i). All properties are immediate from (i), (ii) and Theorem 2.7.3 except for the C^∞-property of the representation. Let $D := D_\delta$, $\iota := \iota_\rho$, and for simplicity let r stand for $\pi_\rho(r)$ where $r \in R$. By construction $D\iota(r) := -i\delta(r)\iota(1)$ for all $r \in \cap_m Dom\delta^m$. Note that $\mathcal{D} = \iota(Dom\delta^m) = DomD^m$ is dense in H_ρ and $\iota(1)$ is a separating and cyclic vector. Define $\nabla(r) := [D^2, r]$ for $r \in \cap_m Dom\delta^m$. Then $\nabla(r)\iota_\rho(s) = D^2\iota(rs) - rD^2\iota(s) = -(\delta^2(rs) + r\delta^2(s))\iota(1) = -(\delta^2(r)s + 2\delta(r)\delta(s))\iota(1) = -\delta^2(r)\iota(s) - 2i\delta(r)D\iota(s) = -(\delta^2(r) + 2i\delta(r)D)$. Further $[D, \delta(r)] = -i\delta^2(r)$ as D is the spatial implementer. Hence on $\mathcal{D}$

$$\nabla(r) = -\delta^2(r) - 2i\delta(r)D = \delta^2(r) - 2iD\delta(r).$$

Let $f(x) = (1+x^2)^{-1/2}$. Then $\|f(D)\nabla(r)\| + \|\nabla(r)f(D)\| \leq 2\|\delta^2(r)\| + 4\|\delta(r)\| < \infty$. Hence $\|[|D|, r]\| < \infty$ by Proposition 1.4.7. One continues in this method to find $r \in \cap_{m=1}^{2n} Dom\delta^m$ implies $\|\delta_{|D|}^m(r)\| < \infty$ for $m = 1, ..., n$. Hence, as by hypothesis and the previous lemma $\mathcal{A}, \Omega_\delta(\mathcal{A}) \subset \cap_m Dom\delta^m$, the representation is C^∞. □

Remark 2.7.11 It is immediate that a graded Riemannian representation, one such that $\Omega_{D_\delta}(\pi_\rho(A))$ admits the order 2 automorphism of parity of differential forms, can be recovered by adding to Definition 2.7.9 the condition: (3) the map $\epsilon(a) = a$, $\epsilon(\delta(a)) \to -\delta(a)$ is well defined on $\Omega_\delta(\mathcal{A})$ for all $a \in \mathcal{A}$.

Remark 2.7.12 Ultimately we are searching for the 'geometric pre-dual' of a C*-algebra A and the 'GNS function' from the geometric pre-dual to Riemannian representations. The pre-dual of a von Neumann algebra R contains the information necessary to construct all the concrete representations of R via the GNS construction and decomposition theory. In analogy we are searching for the abstract information and the process necessary to construct all the Riemannian representations of a C*-algebra A.

The theory of unbounded derivations on von Neumann algebras is insufficient to provide the 'geometric pre-dual'. One need only consider the example of the Riemannian representation $(L^2(X, \Lambda^* X), \pi_l, d+d^*)$ of the C*-algebra $C(X)$ where X is a compact Riemannian manifold to find a representation that is not constructed from an inner Riemannian cycle. This follows as $\delta^2(f)$ is unbounded where $f \in C^\infty(X)$ and $\delta(f) = i[d+d^*, f]$. Hence $Dom\delta$ is not σ-weak dense in the von Neumann algebra $L^\infty(X, \mathrm{Cl}(X))$.

The more general construction involves, surprisingly, not symmetric derivations on the C*-algebra A contained in a von Neumann algebra R, but symmetric derivations from R^{op} to an A-linear R^{op}-bimodule.

2.7.2 Symmetric A-derivations

Basic Definitions

Let R be a von Neumann algebra. Let R^{op} denote the opposite algebra of R. Then R^{op} is a von Neumann algebra[12].

Let A and B be a topological *-algebras. We recall the definitions of section 2.1.1. Let (W, π^{op}) be a right A-module. Then $E_A(W)$ are the elements of $E \in L(W, W)$ such that $E \circ \pi^{\mathrm{op}}(a) = \pi^{\mathrm{op}}(a) \circ E$ for all $a \in A$. Then $(W, \pi, \pi^{\mathrm{op}})$ is an B-A-bimodule if there exists a representation $\pi : B \to L(W, W)$ satisfying $[\pi(b), \pi^{\mathrm{op}}(a)] = 0 \ \forall a \in A, b \in B$. Then $\pi(B) \subset B_A(W)$.

Let R be a von Neumann algebra. Then R is a C*-algebra. In the theory of Hilbert modules $L(R, R)$ denotes the continuous linear operators $R \to R$ where the topology on R in consideration is the uniform topology. This implies $E \in L(R, R)$ is bounded in the norm

$$\|E\| := \sup_{\|r\| \leq 1} \|E(r)\|.$$

As an example let $s \in R$ and define $m_s(r) := sr \ \forall r \in R$ and $m_s^{\mathrm{op}}(r) := rs \ \forall r \in R$. Then $m : R \to L(R, R)$ and $m^{\mathrm{op}} : R^{\mathrm{op}} \to L(R, R)$ such that $\|m_s\| = \|m_s^{\mathrm{op}}\| = \|s\|$. An element $E \in L(R, R)$ is called adjointable if there exists $E^* \in L(R, R)$ such that $(E^*(r))^* s = r^* E(s) \ \forall r, s \in R$. This condition implies E is adjointable if and only if $E = m_s$ for some $s \in R$.

We diverge from the above treatment.

Linear Operators on von Neumann algebras

Let R be a von Neumann algebra. Let $\mathrm{Lin}(R, R)$ denote the linear operators $R \to R$. Let $\rho \in R_*$ be a faithful state.

Definition 2.7.13 *An element $E \in \mathrm{Lin}(R, R)$ is called*

(i) *ρ-adjointable if there exists $E^* \in \mathrm{Lin}(R, R)$ such that*

$$\rho((E^*(r))^* s) = \rho(r^* E(s)) \ \forall r, s \in R,$$

(ii) *ρ-selfadjoint if E is ρ-adjointable and $E = E^*$,*

(iii) *ρ-positive if E is ρ-selfadjoint and $\rho(r^* E(r)) \geq 0 \ \forall r \in R$,*

(iv) *ρ-bounded if $\|E\|_\rho := \sup\{\rho(|E(r)|^2)^{\frac{1}{2}} \mid r \in R, \rho(|r|^2) \leq 1\}$ is finite.*

Definition 2.7.14 *Denote the linear operators $R \to R$ that are*

(i) *ρ-adjointable by $C^\rho(R, R)$,*

(ii) *ρ-selfadjoint by $D^\rho(R, R)$,*

(iii) *ρ-bounded and ρ-adjointable by $B^\rho(R)$.*

We define on $C^\rho(R, R)$ the weakest topology such that $\rho(E_\alpha(1)) \to \rho(E(1))$ is continuous. Define a state on $C^\rho(R, R)$ to be a ρ-positive continuous linear functional τ on $C^\rho(R, R)$ such that $\tau(1) = 1$.

[12] Every von Neumann algebra admits a faithful normal semi-finite weight, hence admits a standard form $(R, H, J, \Delta, \mathcal{P})$ and hence an isomorphism between R^{op} and the commutant R'.

Proposition 2.7.15 *Let $\rho \in R_*$ be a faithful state. Then*

(i) *Let $E \in C^\rho(R, R)$. Then E^*E and EE^* are ρ-positive,*

(ii) *A state $\hat{\rho}$ on $C^\rho(R, R)$ is defined by $\hat{\rho}(E) := \rho(E(1))$,*

(iii) *$\|.\|_\rho$ is a C^*-norm.*

Proof (i) As E and E^* are adjointable, $\rho((E^*E(r))^*s) = \rho(E(r)^*E(s)) = \rho(r^*E^*E(s))$. Hence E^*E is ρ-selfadjoint and $\rho(r^*E^*E(r)) = \rho(E(r)^*E(r)) \geq 0$ as ρ is positive on R. Similar argument for EE^*. (ii) Define $\hat{\rho}(E) := \rho(E(1)) = \rho(1E(1)) \geq 0$ for all ρ-positive E. Let $E_\alpha \to E$, then $\hat{\rho}(E_\alpha - E) = \rho(E_\alpha(1) - E(1)) \to 0$. The other properties are immediate. (iii) Positivity and scalar properties of a norm are immediate. For instance $\|E\|_\rho = 0$ if and only if $\rho(|E(r)|^2) = 0$ for $r \neq 0$ as $E(0) = 0$ by linearity which occurs if and only if $E(r) = 0 \ \forall r \in R$. The triangle inequality follow from the Cauchy-Schwartz inequality. The submultiplicative property follows as $\|AB\|_\rho = sup_{r\in R}\frac{\rho(|A(B(r))|^2)^{\frac{1}{2}}}{\rho(|B(r)|^2)^{\frac{1}{2}}}\frac{\rho(|B(r)|^2)^{\frac{1}{2}}}{\rho(|r|^2)^{\frac{1}{2}}} \leq \sup_{r\in R}\frac{\rho(|A(r)|^2)^{\frac{1}{2}}}{\rho(|r|^2)^{\frac{1}{2}}}\sup_{r\in R}\frac{\rho(|B(r)|^2)^{\frac{1}{2}}}{\rho(|r|^2)^{\frac{1}{2}}} = \|A\|_\rho\|B\|_\rho$. As $\rho(E(r)^*E(r)) = \rho(r^*E^*E(r))$ we have $|\rho(E(r)^*E(r))|^2 \leq \rho(r^*r)\rho((E^*E(r))^*E^*E(r))$ by the Cauchy-Schwartz inequality. Hence $\frac{\rho(|E(r)|^2)^2}{\rho(|r|^2)^2} \leq \frac{\rho(|E^*E(r)|^2}{\rho(|r|^2)}$ and $\|E\|_\rho^2 \leq \|E^*E\|_\rho$. Similarly $\|E^*\|_\rho^2 \leq \|EE^*\|_\rho$. Combining this with the submultiplicative property (1) $\|E\|_\rho^2 \leq \|E^*\|_\rho\|E\|_\rho$ and hence $\|E\|_\rho \leq \|E^*\|_\rho$, and (2) $\|E^*\|_\rho^2 \leq \|E\|_\rho\|E^*\|_\rho$ and hence $\|E^*\|_\rho \leq \|E\|_\rho$. This finally proves $\|E\|_\rho = \|E^*\|_\rho$ and $\|E\|_\rho^2 = \|E^*E\|_\rho$. □

Let H be a Hilbert space. Denote by $C(H)$ the closable linear operators on H and $D(H)$ the essentially selfadjoint linear operators on H.

Proposition 2.7.16 *Let (H_ρ, π_ρ) be the GNS representation of R associated to ρ. Then there exists a faithful *-representation*

(i) *$\hat{\pi}_\rho : C^\rho(R, R) \to C(H_\rho)$,*

(ii) *$\hat{\pi}_\rho : D^\rho(R, R) \to D(H_\rho)$,*

(iii) *$\hat{\pi}_\rho : B^\rho(R, R) \to B(H_\rho)$ such that $\|E\|_\rho = \|\hat{\pi}_\rho(E)\|$.*

Proof Let $\iota : R \to H_\rho$ be the dense linear injection given by the GNS construction and $\pi : R \to B(H_\rho)$ the GNS representation. As ρ is a faithful state then $\iota(1)$ is a cyclic and separating vector. Let $E \in C^\rho(R, R)$. Define $\pi(E)\iota(r) := \iota(E(r))$ for all $r \in R$. Then $\pi(E)$ is a densely define linear operator on H_ρ such that $\pi(E) : \iota(R) \to \iota(R)$. Note that $\langle\iota(r_1), \pi(E)\iota(r_2)\rangle = \langle\iota(r_1), \iota(E(r_2))\rangle = \rho(r_1^*E(r_2))$ for all $r_1, r_2 \in R$ by the GNS construction. Hence $\langle\iota(r_1), \pi(E)\iota(r_2)\rangle = \rho(r_1^*E(r_2)) = \rho((E^*(r_1)r_2) = \langle\pi(E^*)\iota(r_1), \iota(r_2)\rangle$ for all $r_1, r_2 \in R$. Hence $\pi(E^*)$ is closable with $\pi(E^*) = \pi(E)^*$ on $\iota(R)$ and $\pi(E)$ is closable with $\pi(E) = \pi(E)^{**}$ on $\iota(R)$. This follows from [RS, Theorem VIII.1] as $\iota(R)$ is dense. Let $E = E^*$. The domain of $\pi(E)^{**}$ is the completion of $\iota(R)$ in the norm $\|\iota(r)\|_E = \|\iota(r)\| + \|\iota(E(r))\|$. Similarly, as $\pi(E) : \iota(R) \to \iota(R)$, the domain of $\pi(E)^*$ is the completion of $\iota(R)$ in the norm $\|\iota(r)\|_E$. Hence the domains are equal and $\pi(E)^* = \pi(E)^{**}$.

Note that $\|\pi(E)\iota(r)\|^2 = \rho(|E(r)|^2)$. Hence $\|\pi(E)\iota(r)\| = 0$ for all r iff $E(r) = 0 \ \forall r$. Hence π is faithful by density of $\iota(R)$. The product homomorphism is given by $\pi(EF)\iota(r) = \iota(EF(r)) = \iota(E(F(r))) = \pi(E)\iota(F(r)) = \pi(E)\pi(F)\iota(r)$. The identification of the norms $\|.\|_\rho$ and the operator norm $\|.\|$ on $B(H)$ is immediate from the construction of H_ρ from ρ. □

Remark 2.7.17 Let $\iota_\rho : R \to H_\rho$ be the dense linear injection given by the GNS construction. Let $E \in C^\rho(R, R)$. Then the definition

$$\hat{\pi}_\rho(E)\iota_\rho(r) := \iota(E(r))$$

implies that

$$\hat{\rho}(E) = \langle \iota_\rho(1), \hat{\pi}(E)\iota_\rho(1)\rangle$$

where $\iota_\rho(1) \in H_\rho$ is the canonical separating and cyclic vector for R. This construction of closable linear operators on a GNS Hilbert space H_ρ is an extended GNS construction. The construction was used in Proposition 3.2.28 of [BR] to obtain the selfadjoint operator of Theorem 2.7.3.

Remark 2.7.18 It is immediate that $\iota(R)$ is a core for the selfadjoint closure $\overline{\hat{\pi}_\rho(E)}$ of $E \in D^\rho(R, R)$ as $\overline{\overline{\hat{\pi}_\rho(E)}|_{\iota(R)}} = \hat{\pi}_\rho(E)$.

Example 2.7.19 Let $s \in R$. Define $m_s(r) := sr$ and $m_s^{\text{op}}(r) := rs \;\forall r \in R$. Then $m_s, m_s^{\text{op}} \in \text{Lin}(R, R)$ and the following properties can be derived (1) $(m_s)^* = m_{s^*} \;\forall s \in R$, (2) m_s is ρ-positive if and only if s is positive, and (3) $\|m_s\|_\rho = \|m_s^{\text{op}}\|_\rho = \|s\| \;\forall s \in R$.

The following definition is independent of the normal state ρ on R.

Definition 2.7.20 *Let R be a von Neumann algebra and $S_*(R) \subset R_*$ the set of normal faithful states on R. Then we call a linear operator $E : R \to R$ (1) R_*-adjointable if $E \in C_*(R, R)$ where $C_*(R, R) := \cap_{\rho \in S_*(R)} C^\rho(R, R)$, and (2) R_*-bounded if $E \in B_*(R)$ where $B_*(R) := \cap_{\rho \in S_*(R)} B^\rho(R)$.*

Remark 2.7.21 A R_*-adjointable (resp. R_*-bounded) linear operator $E : R \to R$ has a GNS representative $\hat{\pi}_\rho(E)$ in $C(H_\rho)$, (resp. $B(H_\rho)$) for every faithful state $\rho \in R_*$. Hence it is clearly stronger to be R_*-adjointable (resp. R_*-bounded) than ρ-adjointable (resp. ρ-bounded).

Remark 2.7.22 Let R_w be a σ-weak dense unital *-subalgebra of R. Then one may replace R by R_w verbatim in the results of this section. This is possible since $\iota(R_w)$ is dense in H_ρ for any σ-weak algebra R_w and any state $\rho \in R_*$.

Symmetric A-derivations

Let R be a von Neumann algebra and $\rho \in R_*$ be a faithful state. Let R_w be a σ-weak dense unital *-subalgebra of R and A be a *-subalgebra of R_w. Then define

$$C^\rho_A(R_w, R_w) := \{E \in C^\rho(R_w, R_w) \,|\, E(a) = aE(1) \;\; \forall a \in A\}$$

with subspace

$$B^\rho_A(R_w) := \{E \in B^\rho(R_w) \,|\, E(a) = aE(1) \;\; \forall a \in A\}.$$

An element $E \in C^\rho_A(R_w, R_w)$ is called left A-linear. Define the natural multiplication map ${}^o : R_w^{\mathrm{op}} \to C^\rho(R_w, R_w)$ by $r^o(s) = m_r^{\mathrm{op}}(s) = sr \ \forall s \in R_w, r \in R_w$. Define $C_*(R_w, R_w, A) = \cap_{\rho \in S_*(R)} C^\rho_A(R_w, R_w)$ and $B_*(R_w, A) = \cap_{\rho \in S_*(R)} B^\rho_A(R_w)$.

Lemma 2.7.23 *Let R_w, ρ, $B_*(R_w, A)$ and ^{o} be as above. Then ${}^o : R_w^{\mathrm{op}} \to B_*(R_w, A)$.*

Proof Let $a \in A, r, s \in R_w$. Then $r^o(a) = ar = ar^o(1)$. Hence r^o is A-linear. It follows from the Cauchy-Schwartz inequality that $\rho(|sr|^2) \leq \rho(|r|^2)\rho(|s|)^2$. Hence $\|r\|_\rho < \infty \ \forall \rho \in S_*(R)$. □

We note that $m_s \in B_*(R_w, A)$ for $s \in R_w$ if and only if $[s, a] = 0$ for all $a \in A$.

Definition 2.7.24 *Let R_w be a σ-weak dense unital *-subalgebra of a von Neumann algebra R and $\rho \in R_*$ be a faithful state. Let A be a *-subalgebra of R. Then a bounded symmetric A-derivation on R_w^{op} is a linear map*

$$\delta : R_w^{\mathrm{op}} \to B^\rho_A(R_w)$$

such that (1) $\delta(r^{\mathrm{op}} s^{\mathrm{op}})(t) = (r^o \delta(s^{\mathrm{op}}) + \delta(r^{\mathrm{op}}) s^o)(t)$ for all $r, s, t \in R_w$, and (2) $\delta((r^{\mathrm{op}})^*) = \delta(r^{\mathrm{op}})^*$ for all $r \in R_w$.

Definition 2.7.25 *Let R_w be a σ-weak dense unital *-subalgebra of a von Neumann algebra R and $\rho \in R_*$ be a faithful state. Let A be a *-subalgebra of R_w. Then a symmetric A-derivation on R_w^{op} is a linear map*

$$\delta : R_w^{\mathrm{op}} \to C^\rho_A(R_w, R_w)$$

such that (1) $\delta(r^{\mathrm{op}} s^{\mathrm{op}})(t) = (r^o \delta(s^{\mathrm{op}}) + \delta(r^{\mathrm{op}}) s^o)(t)$ for all $r, s, t \in R_w$, and (2) $\delta((r^{\mathrm{op}})^*) = \delta(r^{\mathrm{op}})^*$ for all $r \in R_w$.

Lemma 2.7.26 *Let R_w be a σ-weak dense unital *-subalgebra of a von Neumann algebra R, $\rho \in R_*$ be a faithful state, A be a *-subalgebra of R_w and δ be a symmetric A-derivation on R_w^{op}. Then $\delta(1)(s) = 0$ for all $s \in R$.*

Proof Let $s \in R$. Then $\delta(1)(s) = \delta(1.1)(s) = (1^o\delta(1) + \delta(1)1^o)(s) = \delta(1)(s)1 + \delta(1)(s1) = 2\delta(1)(s)$. Hence $\delta(1)(s) = 0$. □

Remark 2.7.27 The definition of an A-linear element of $C^\rho(R_w, R_w)$ can be modified to $E(ar) = aE(r) \ \forall a \in A, r \in R_w$ when R_w is non-unital. This allows a definition of a symmetric A-derivation $\delta : R_w^{\mathrm{op}} \to C^\rho_A(R_w, R_w)$ when R_w is non-unital. The definitions can also be made independent of ρ, but more restrictive, by replacing $C^\rho(R_w, R_w)$ by the R_*-adjointable elements $C_*(R_w, R_w)$.

Remark 2.7.28 Let $\delta : R_w^{\mathrm{op}} \to C^\rho_A(R_w, R_w)$ be an A-symmetric derivation. Let $E \in C^\rho_A(R_w, R_w)$ and define the evaluation map $\iota_1 : C^\rho_A(R_w, R_w) \to R_w$ by $\iota_1(E) = E(1)$. We can repeat the derivation in the following manner

$$\begin{tikzcd} R_w^{\mathrm{op}} \arrow[r, "\delta"] & C^\rho_A(R_w, R_w) \arrow[d, "\iota_1"] \\ & R_w \arrow[ul, "\mathrm{op}"] \end{tikzcd}$$

Let $\gamma = \mathrm{op} \circ \iota_1 \circ \delta$. Define $\delta^m(r^{\mathrm{op}}) := \delta(\gamma^{m-1}(r^{\mathrm{op}}))$. As an example $\delta^2(r^{\mathrm{op}})(s) := \delta(\delta(r^{\mathrm{op}})(1)^{\mathrm{op}})(s)$ for all $s \in R_w$. We remark that we have defined symmetric A-derivations δ such that R_w^{op} is the invariant domain for δ^m, $m \in \mathbb{N}$. The definition of an A-symmetric derivation can be generalised by considering δ as a map $\delta : R^{\mathrm{op}} \to C^\rho(R_w, R)$ where $C^\rho(R_w, R)$ are the ρ-adjointable linear maps $R_w \to R$ [13]. Then one defines $r^{\mathrm{op}} \in Dom\delta^m$ if $\gamma^j(r^{\mathrm{op}}) \in R_w$ for $j = 1, ..., m-1$.

We highlight that the condition $\delta(r^{\mathrm{op}}) \in C_A'^\rho(R_w, R_w)$ for all $r \in R_w$ implies $\delta(r^{\mathrm{op}})(a) = a\delta(r^{\mathrm{op}})(1)$ for all $a \in A$.

2.7.3 Abstract K-cycles

The definition of a symmetric A-derivation on the von Neumann algebra R^{op} allows us to abstractly classify Riemannian representations arising from a faithful trace.

Definition 2.7.29 *Let R be a von Neumann algebra with faithful state ρ. Let A be a *-subalgebra of a σ-weak dense unital *-subalgebra R_w of R. Let $\delta : R_w^{\mathrm{op}} \to C_A'^\rho(R_w, R_w)$ be an A-symmetric derivation on R_w^{op}. Then we call the triple (R_w, ρ, δ) an abstract K-cycle over A if it satisfies the condition*

$$\hat{\rho}(\delta(r^{\mathrm{op}})) = 0$$

for all $r \in R_w$.

Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$. Let (H_ρ, π_ρ) be the GNS representation of R associated to ρ with dense linear injection $\iota_\rho : R_w \to H_\rho$. On the dense subspace $\iota_\rho(R)$ define

$$\tilde{D}_\delta \iota(r) := -i\hat{\pi}_\rho(\delta(r^{\mathrm{op}}))\iota_\rho(1)$$

Lemma 2.7.30 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$. Let $\tilde{D}_\delta$ be defined as above. Then $\tilde{D}_\delta$ is an essentially selfadjoint operator $\tilde{D}_\delta : \iota(R_w) \to \iota(R_w)$ such that $[\tilde{D}_\delta, \pi_\rho(a)] : \iota(R_w) \to \iota(R_w)$ is norm bounded on $\iota(R_w)$.*

Proof Let $\pi := \pi_\rho$, $\iota := \iota_\rho$ and $D := \tilde{D}_\delta$. We first note that $D : \iota(R_w) \to \iota(R_w)$ by

[13] and 'ρ-closable'. In general the operators $\hat{\pi}_\rho(D^\rho(R_w, R))$ would be a space of symmetric linear operators on H_ρ instead of essentially selfadjoint operators.

construction. Secondly, that $D\iota(1) = 0$. Let $s, r \in R_w$. Then

$$\begin{aligned}
\langle \iota(s), D\iota(r)\rangle = &\ \langle \iota(s), \iota(-i\delta(r^{\mathrm{op}})(1))\rangle \\
= &\ \rho(s^*(-i\delta(r^{\mathrm{op}})(1))) \\
\overset{\text{(i)}}{=} &\ \rho((-i\delta(r^{\mathrm{op}})^*(s))^*1) \\
\overset{\text{(ii)}}{=} &\ i\rho(\delta((r^{\mathrm{op}})^*)(1s)^*) \\
= &\ \overline{i\rho((\delta((r^{\mathrm{op}})^*)s^o(1))} \\
\overset{\text{(iii)}}{=} &\ \overline{i(-\rho((r^o)^*\delta(s^{\mathrm{op}})(1)) + \delta((r^{\mathrm{op}})^*s^{\mathrm{op}})(1)))} \\
\overset{\text{(iv)}}{=} &\ -i\overline{\rho(r^*\delta(s^{\mathrm{op}})(1))} + i\overline{\rho(\delta(((sr^*)^{\mathrm{op}})^*)(1))} \\
= &\ -i\rho(\delta(s^{\mathrm{op}})(1)^*r) + i\hat{\rho}(\delta((sr^*)^{\mathrm{op}})) \\
\overset{\text{(v)}}{=} &\ \rho(-i\delta(s^{\mathrm{op}})(1)^*r) \\
= &\ \langle \iota(-i\delta(s^{\mathrm{op}})), \iota(r)\rangle \\
= &\ \langle D\iota(s), \iota(r)\rangle.
\end{aligned}$$

Where at (i) we used the definition of ρ-adjoint, at (ii) the symmetry of δ, at (iii) we used the definition of derivation, at (iv) we used the definition of ρ-adjoint to obtain $((r^o)^*(s))^* = ((r^o)^*(s))^*1 = s^*r^o(1) = s^*1r = s^*r$ and hence $(r^o)^*(s) = (s^*r)^* = r^*s$, and at (v) closure of the derivation with respect to the state $\hat{\rho}$ on $C^\rho(R_w, R_w)$. Hence D is symmetric, with a closed adjoint with dense domain. As in Proposition 2.7.17 invariance of $D : \iota(R_w) \to \iota(R_w)$ on the dense domain $\iota(R_w)$ provides the equality of $D^* = D^{**}$ on the closure of $\iota(R)$ in the norm $\|\iota(r)\| = \rho(|r|^2)^{1/2} + \hat{\rho}(|\delta(r^{\mathrm{op}})|^2)^{1/2}$.

As $\pi(a)\iota(r) = \iota(ar)$ then $\pi(a) : \iota(R_w) \to \iota(R_w)$ for all $a \in A$. Consider

$$\begin{aligned}
\pi(a)D\iota(r) = &\ -i\pi(a)\pi(\delta(r^{\mathrm{op}}))\iota(1) \\
= &\ \iota(-ia\delta(r^{\mathrm{op}})(1)) \\
\overset{(*)}{=} &\ \iota(-i\delta(r^{\mathrm{op}})(a1)) \\
= &\ \iota(-i\delta(r^{\mathrm{op}})(1a)) \\
= &\ \iota(-i\delta(r^{\mathrm{op}})a^o(1)) \\
= &\ \iota(-i\delta(r^{\mathrm{op}}a^{\mathrm{op}})(1)) + i\iota(r^o\delta(a^{\mathrm{op}})(1))
\end{aligned}$$

where we used the A-linearity property at $(*)$, and

$$D\pi(a)\iota(r) = D\iota(ar) = \iota(-i\delta((ar)^{\mathrm{op}})(1)) = \iota(-i\delta(r^{\mathrm{op}}a^{\mathrm{op}})(1)).$$

Hence

$$[D, \pi(a)]\iota(r) = -i\iota(\delta(a^{\mathrm{op}})(1)r)$$

Then

$$\|[D, \pi(a)]\iota(r)\| = \rho(r^*|\delta(a^{\mathrm{op}}))(1)|^2r)^{1/2} \leq \|\delta(a^{\mathrm{op}})(1)\|\|\iota(r)\|$$

by a consequence of the Cauchy-Schwartz inequality [BR, Prop 2.3.11(c)]. Hence $\|[D, \pi(a)]\| \leq \|\delta(a^{\mathrm{op}})(1)\|$ when taken over a supremum of $r \in R_w$. □

Let D_δ denote the unique selfadjoint closure of $\tilde{D}_\delta$. Note that $\iota(R_w)$ is an invariant core for D_δ and $\pi_\rho(r) : \cap_m DomD^m \to \cap_m DomD^m$ for all $r \in R_w$ and $m \in \mathbb{N}$ as a consequence of the proof and Remark 2.7.28.

Definition 2.7.31 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$. Then $(H_\rho, \pi_\rho, D_\delta)$ is called the GNS representation associated to (R_w, ρ, δ).*

Remark 2.7.32 We remark that the construction involves a faithful state ρ on the von Neumann algebra R_w''. Hence each GNS representation has an associated standard form $(R_w'', H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ as in Section 1.6.1.

Let $C^*(A)$ denote the C*-closure of A in R. Let $\Omega_\delta(A) :=< A^o(1), \delta(A^o)(1) >$ denote the *-algebra of R generated by the operators $\iota_1(a^o) = a$, $\iota_1(\delta(b^{\text{op}})) = \delta(b^{\text{op}})(1)$ for $a, b \in A$.

Corollary 2.7.33 *Let (R_w, ρ, δ) be an abstract K-cycle over a unital *-algebra $A \subset R_w$ with associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$. Then*

(i) *$(H_\rho, \pi_\rho, D_\delta)$ is a C_c^1-representation of the unital C^*-algebra $C^*(A)$, and*

(ii) $\Omega_{D_\delta}(\pi_\rho(A)) \cong \pi_\rho(\Omega_\delta(A))$

Proof (i) Immediate from Lemma 2.7.30. (ii) Let $D := D_\delta$, $\pi := \pi_\rho$, $\iota := \iota_\rho$. Then from the proof of Lemma 2.7.30 we have $[D, \pi(a)]\iota_\rho(r) = \iota(\delta(a^{\text{op}})(1)r)$ for all $r \in R$. This provides a linear isomorphism and $\pi(a)[D, \pi(b)]\iota_\rho(r) = \pi(a)\iota(\delta(b^{\text{op}})(1)r) = \iota(a\delta(b^{\text{op}})(1)r)$ with $[D, \pi(a)][D, \pi(b)]\iota_\rho(r) = [D, \pi(a)]\iota(\delta(b^{\text{op}})(1)r) = \iota(\delta(a^{\text{op}})\delta(b^{\text{op}})(1)r)$ for all $a, b \in A$ and $r \in R$ extends the linear isomorphism to $\Omega_{D_\delta}(\pi_\rho(A))$. □

Remark 2.7.34 Let (R, δ, ρ) be a inner K-cycle such that the derivation δ has a σ-weak dense invariant domain $R_w = \cap_m Dom\delta^m$. Let $(H_\rho, \pi_\rho, D_\delta)$ be the GNS representation associated to the *inner* K-cycle (R, δ, ρ), see Definition 2.7.7. Define the map $\delta^o : R_w^{\text{op}} \to C^\rho(R_w, R_w)$ by $\delta^o(r^{\text{op}})(s) = s\delta(r)$. Then it is easily verified that (R_w, δ^o, ρ) is an abstract K-cycle with identical GNS representation $(H_\rho, \pi_\rho, D_\delta)$. Hence abstract K-cycles extend inner K-cycles.

Remark 2.7.35 Outer sheer of a symmetric A-derivation

We highlight a distinction between abstract and inner K-cycles. In section 2.7.1 we saw an inner K-cycle (R, ρ, δ) with associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$ has, in general, an unbounded coderivation $\nabla(r) = \frac{1}{2}[D_\delta^2, \pi_\rho(r)]$ however $[D_\delta, [D_\delta, \pi_\rho(r)]] = \pi(\delta^2(r))$ was bounded for all $r \in Dom\delta^2$. This restricted the application of inner K-cycles in the theory of generalised differential geometry. This is not the case for an abstract K-cycle (R_w, δ, ρ) over a *-algebra A and we identify the obstruction to $[D_\delta, [D_\delta, \pi_\rho(a)]]$ being bounded for $a \in A$.

Proposition 2.7.36 *Let (R_w, δ, ρ) be an abstract K-cycle over a *-algebra A with associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$. Then*

$$[D_\delta, [D_\delta, \pi_\rho(a)]]\iota_\rho(r) = -\iota_\rho(\delta^2(a^{\text{op}})(1)r) - \iota_\rho\Big(\delta(r^{\text{op}})(\delta(a^{\text{op}})(1)) - \delta(a^{\text{op}})(1)\delta(r^{\text{op}})(1)\Big)$$

and

$$[D_\delta^2, \pi(a)]\iota_\rho(r) = -\iota_\rho(\delta^2(a^{\rm op})(1)r) - \iota_\rho\Big(\delta(r^{\rm op})(\delta(a^{\rm op})(1)) + \delta(a^{\rm op})(1)\delta(r^{\rm op})(1)\Big)$$

for all $r \in R_w$.

Proof Let $\pi_\rho := \pi$, $\iota := \iota_\rho$ and $D := D_\delta$. Then

$$D[D, \pi(a)]\iota(r) = -iD\iota(\delta(a^{\rm op})(1)r) = -\iota(\delta((\delta(a^{\rm op})(1)r)^{\rm op})(1)).$$

We have $\delta(r^{\rm op}\delta(a^{\rm op})(1)^{\rm op})(1) = r^o\delta(\delta(a^{\rm op})(1)^{\rm op})(1) + \delta(r^{\rm op})(\delta(a^{\rm op})(1))$. Hence

$$D[D, \pi(a)]\iota(r) = -\iota(\delta(\delta(a^{\rm op})(1)^{\rm op})(1)r) - \iota(\delta(r^{\rm op})(\delta(a^{\rm op})(1))).$$

In the other direction, $[D, \pi(a)]D\iota(r) = -i[D, \pi(a)]\iota(\delta(r^{\rm op})(1)) = -\iota(\delta(a^{\rm op})(1)\delta(r^{\rm op})(1))$ as $\delta(r^{\rm op})(1) \in R_w$. The second formula follows from $[D^2, \pi(a)] = D[D, \pi(a)] + [D, \pi(a)]D$. □

The first term in the expressions in Proposition 2.7.36 is uniformly bounded

$$\|\iota_\rho(\delta^2(a^{\rm op})(1)r)\| \le \|\delta^2(a^{\rm op})(1)\|\|\iota_\rho(r)\|.$$

Hence the second terms in the expression in Proposition 2.7.36 contain the obstructions of interest.

Definition 2.7.37 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$. We call the map $I_\delta : A \times R_w \to R_w$ given by*

$$I_\delta(a, r) := \frac{1}{2}\Big(\delta(r^{\rm op})(\delta(a^{\rm op})(1)) - \delta(a^{\rm op})(1)\delta(r^{\rm op})(1)\Big)$$

the outer sheer of the symmetric A-derivation δ. We call the map $S_\delta : A \times R_w \to R_w$ given by

$$S_\delta(a, r) := \frac{1}{2}\Big(\delta(r^{\rm op})(\delta(a^{\rm op})(1)) + \delta(a^{\rm op})(1)\delta(r^{\rm op})(1)\Big)$$

the metric sheer of the symmetric A-derivation δ.

On the dense subspace $\iota_\rho(R_w) \subset H_\rho$ the operators in Proposition 2.7.36 have the form

$$[D_\delta, [D_\delta, \pi_\rho(a)]] = -\pi_\rho(\delta^2(a^{\rm op})(1)) - 2\iota_\rho(I_\delta(a, \cdot))$$

and

$$[D_\delta^2, \pi(a)] = -\pi_\rho(\delta^2(a^{\rm op})(1)) - 2\iota_\rho(S_\delta(a, \cdot))$$

for $a \in A$. Hence the first operator is uniformly bounded if and only if

$$\|\iota_\rho(I_\delta(a, r))\| \sim O(\|\iota_\rho(r)\|)$$

for all $r \in R_w$. We note that $I_\delta(a, r) = 0$ for all $r \in R_w$ if δ is $\delta(A^{\rm op})(1)$-linear, meaning $\delta(r^{\rm op})(s) = s\delta(r^{\rm op})(1)$ for all $s \in \delta(A^{\rm op})(1)$. This condition of linearity is satisfied when $\delta = \tilde{\delta}^0$ for a symmetric derivation $\tilde{\delta}$ on the von Neumann algebra R. There is a partial converse.

Theorem 2.7.38 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra A such that $\Omega_\delta(A) := < A, \delta(A^{\rm op})(1) > \subset R_w$ is σ-weak dense in R_w. Then the following statements are equivalent*

(i) *$(R_w, \rho, \tilde{\delta})$ is an inner K-cycle over A where $\delta = \tilde{\delta}^o$.*

(ii) *$I_\delta(a, r) = 0$ for all $a \in A$, $r \in R_w$.*

Proof (i) $\Rightarrow$ (ii) Let $\delta = \tilde{\delta}^o$. Then $\delta(r^{\rm op})(s) = s\tilde{\delta}(r) = s1\tilde{\delta}(r) = s\delta(r^{\rm op})(1)$. Then $I_\delta(a, r) = 0$ using $s = \delta(a^{\rm op})(1)$.

(ii) $\Rightarrow$ (i) Let $I_\delta(a, r) = 0$. Then $\delta(r^{\rm op})(\delta(a^{\rm op}(1)) = \delta(a^{\rm op})(1)\delta(r^{\rm op})(1)$. Hence $\delta(r^{\rm op})(a\delta(a^{\rm op})(1)) = a\delta(a^{\rm op})(1)\delta(r^{\rm op})$ using A-linearity and the derivation property of δ. Similarly $\delta(r^{\rm op})(s) = s\delta(r^{\rm op})(1)$ for all $s \in \Omega_\delta(\mathcal{A})$. The result follows since $\tilde{\delta}(r) := \delta(r^{\rm op})(1)$ defines an inner derivation on R with domain R_w and $\Omega_\delta(A)$ is σ-weak dense in R. □

To conclude the remark we note the previous result indicates essentially three categories of abstract K-cycles over a unital *-algebra A that are distinguished by the outer sheer I_δ of the symmetric A-derivation δ.

abstract K-cycle	**outer sheer**	$[D_\delta, [D_\delta, \pi_\rho(a)]] = -\pi_\rho(\delta^2(a^{\rm op})(1)) + M$
inner	$I_\delta \equiv 0$	$M = 0$
essentially inner	$\iota_\rho(r) \to \iota_\rho(I_\delta(a, r))$ bounded	M bounded
outer	$\iota_\rho(r) \to \iota_\rho(I_\delta(a, r))$ unbounded	M unbounded

Remark 2.7.39 Reality of a symmetric A-derivation

Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$ with associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$. Define the positive elements of R_w as the space $R_w^+ = \{r \in R_w \,|\, r = s^*s, s \in R_w\}$. We recall a linear mapping $\alpha : R_w \to R_w$ is called positive definite if $\alpha : R_w^+ \to R_w^+$.

Lemma 2.7.40 *Let (R_w, ρ, δ), A and $(H_\rho, \pi_\rho, D_\delta)$ be as above. Then the following conditions are equivalent*

(i) *$\mathcal{L}_\delta : r \to -\delta^2(r^{\rm op})(1)$ is positive definite,*

(ii) *$\rho\Big(\delta(r^{\rm op})(\delta(s^{\rm op})(1)) - \delta(r^{\rm op})(1)\delta((s^*)^{\rm op})(1)^*\Big) = 0$ for all $r, s \in R_w$.*

Proof Let $\pi_\rho := \pi$, $\iota := \iota_\rho$, $S := S_\rho$ and $D := D_\delta$. Let $r, s \in R_w$. We have

$$\begin{aligned}
\rho(\delta^2(r^{\rm op})(1)s) &= \rho(s^{\rm op}\delta^2(r^{\rm op})(1)) \\
&\overset{(i)}{=} \rho(\delta(s^{\rm op}\delta(r^{\rm op})(1)^{\rm op})(1) - \delta(s^{\rm op})(\delta(r^{\rm op})(1)) \\
&= \rho(\delta(s^{\rm op}\delta(r^{\rm op})(1)^{\rm op})(1)) - \rho(\delta(s^{\rm op})(\delta(r^{\rm op})(1))) \\
&\overset{(ii)}{=} -\rho(\delta(s^{\rm op})(\delta(r^{\rm op})(1))
\end{aligned}$$

where (i) used the derivation property of δ and (ii) used the cycle condition $\rho(\delta(s^{\mathrm{op}})(1)) = 0$ for all $s \in R_w$. Moreover

$$\begin{aligned}
\rho(\delta^2((r^*)^{\mathrm{op}})(1)^*s) &= \overline{\rho(s^*\delta^2((r^*)^{\mathrm{op}})(1))} \\
&\overset{(iii)}{=} \overline{\rho(\delta(\delta((r^*)^{\mathrm{op}})^{\mathrm{op}})^*(s)^*1)} \\
&\overset{(iv)}{=} \rho(\delta((\delta(r^{\mathrm{op}})(1))(1)^{\mathrm{op}})^*)(s)) \\
&\overset{(i),(ii)}{=} -\rho((\delta((r^*)^{\mathrm{op}})(1)^*)^{\mathrm{op}}\delta(s^{\mathrm{op}})(1)) \\
&= -\rho(\delta(s^{\mathrm{op}})(1)\delta((r^*)^{\mathrm{op}})(1)^*)
\end{aligned}$$

where (iii) used the fact δ is ρ-adjointable and (iv) used symmetry of δ. Let $r \in R_w$. Hence (*)

$$\rho(\delta^2(r^{\mathrm{op}})(1)s) = \rho(\delta^2((r^*)^{\mathrm{op}})(1)^*s)$$

if and only if (**)

$$\rho(\delta(s^{\mathrm{op}})(\delta(r^{\mathrm{op}})(1)) = \rho(\delta(s^{\mathrm{op}})(1)\delta((r^*)^{\mathrm{op}})(1)^*).$$

for all $s \in R_w$. As ρ is a faithful state then (*) is equivalent to $\delta^2((r^*)^{\mathrm{op}})(1)^* = \delta^2(r^{\mathrm{op}})(1)$ by setting $s = (\delta^2((r^*)^{\mathrm{op}})(1)^* - \delta^2(r^{\mathrm{op}})(1))^*$. We note that $\delta^2((\lambda r + \mu s)^{\mathrm{op}})(1) = \lambda\delta^2(r^{\mathrm{op}})(1) + \mu\delta^2(s^{\mathrm{op}})(1)$ for all $\lambda, \mu \in \mathbb{C}$ and $r, s \in R_w$ from linearity of δ. Hence the condition $\delta^2(r^{\mathrm{op}})(1) \geq 0$ for all $r \geq 0, r \in R_w$ is necessary and sufficient for the result $\delta^2((r^*)^{\mathrm{op}})(1)^* - \delta^2(r^{\mathrm{op}})(1)$. □

Let $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be a standard form associated to $R_w, \rho, \delta)$ as in Remark 2.7.32. We recall from the Tomita-Takesaki theory, see Theorem 1.6.2(ii), that the modular conjugation J_ρ and the modular operator Δ_ρ are derived from the unbounded anti-linear operator $S_\rho := J_\rho\Delta_\rho^{1/2} : \iota_\rho(r) \to \iota_\rho(r^*)$ for all $r \in R_w$.

Theorem 2.7.41 *Let (R_w, ρ, δ) be an abstract K-cycle over a unital *-algebra $A \subset R_w$. Let $(H_\rho, \pi_\rho, D_\delta)$ be the associated GNS representation and $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be an associated standard form. Then the following conditions are equivalent*

(i) $\mathcal{L}_\delta : r \to -\delta^2(r^{\mathrm{op}})(1)$ *is positive definite,*

(ii) $\rho\Big(\delta(r^{\mathrm{op}})(\delta(s^{\mathrm{op}})(1)) - \delta(r^{\mathrm{op}})(1)\delta((s^*)^{\mathrm{op}})(1)^*\Big) = 0$ *for all* $r, s \in R_w$,

(iii) $[D_\delta^2, S_\rho] = 0$.

Proof Let $\iota := \iota_\rho$, $S := S_\rho$ and $D := D_\delta$. Let $r \in R_w$ and note $S\iota(r) = \iota(r^*)$. Then

$$(D^2S - SD^2)\iota(r) = D^2\iota(r^*) - S\iota(-\delta^2(r^{\mathrm{op}})(1)) = -\iota\Big(\delta^2((r^*)^{\mathrm{op}})(1) - \delta^2(r^{\mathrm{op}})(1)^*\Big).$$

Since ι is an isomorphism $(D^2S - SD^2)\iota(r) = 0$ for all $r \in R_w$ if and only if $\delta^2((r^*)^{\mathrm{op}})(1) = \delta^2(r^{\mathrm{op}})(1)^*$. This is equivalent to the statements in Lemma 2.7.40 by the proof of Lemma 2.7.40. Note that S preserves Cauchy sequences in the graph norm of D^2. Hence S preserves $DomD^2$ and we take $[D^2, S]$ defined on $DomD^2$. □

Corollary 2.7.42 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$ such that $\rho \in R_*$ is a faithful trace. Let $(H_\rho, \pi_\rho, D_\delta)$ be the associated GNS representation and $(R, H_\rho, J_\rho, 1, \overline{\iota_\rho(R^+)})$ the associated standard form. Then the following conditions are equivalent*

(i) $\mathcal{L}_\delta : r \to -\delta^2(r^{\text{op}})(1)$ *is positive definite,*

(ii) $\rho\Big(\delta(r^{\text{op}})(\delta(s^{\text{op}})(1)) - \delta((s^*)^{\text{op}})(1)^*\delta(r^{\text{op}})(1)\Big) = 0$ *for all* $r, s \in R_w$,

(iii) $[D_\delta^2, J_\rho] = 0$.

Proof (ii) Follows by using the trace property. (iii) Immediate from Theorem 2.7.41 as $\Delta_\rho = 1$. □

Remark 2.7.43 We remark on the content of the equivalent conditions. Example 2.7.61 suggests the linear map $\mathcal{L}_\delta$ is the appropriate generalisation of the Laplacian operator. Hence the results suggest a deep relationship between the positivity of the Laplacian and the modular theory. This is demonstrated further by the next example.

Remark 2.7.44 Modular Dynamics of a symmetric A-derivation

Let (R_w, ρ, δ) be an abstract K-cycle over a unital *-algebra $A \subset R_w$.
Let (H_ρ, π_ρ, D) be the GNS representation associated to (R_w, ρ, δ). Define

$$\nabla_\delta(\cdot) := [D_\delta^2, \cdot]$$

as the covariant derivation associated to the abstract K-cycle (R_w, ρ, δ) (c.f. Section 1.4.2). Associated to the derivation ∇_δ is a one-parameter family of unitaries $e^{-isD_\delta^2}$: $H_\rho \to H_\rho$ that induce a one-parameter family of automorphisms on $\cap_m Dom \nabla^m$

$$\text{geo}_\delta^s(T) := e^{-isD_\delta^2} T e^{isD_\delta^2}$$

for $s \in \mathbb{R}$ and $T \in \cap_m Dom \nabla^m$. We call the family geo_δ the geodesic flow associated to (R_w, ρ, δ). This name is derived from parallels with geodesic flow on a manifold as discussed in [C5, Section 6].

Let $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be the standard form associated to (R_w, ρ, δ) as in Remark 2.7.32. Then there exists another one-parameter family of automorphisms given by the modular flow

$$\sigma_\rho^t(r) = \Delta_\rho^{-it} r \Delta_\rho^{it}$$

for $t \in \mathbb{R}$ and $r \in R$, see Section 1.6.1. The modular flow has been associated to time flow in thermodynamic systems [C11].

Given the physical context of the two one-parameter families, the coupling of the geodesic flow geo_δ and the modular flow σ_ρ is of great interest. In particular they are decoupled if $[D_\delta^2, \Delta_\rho] = 0$ and intertwined if $[D_\delta^2, \Delta_\rho] \neq 0$. We remark here on some details of the coupling as a consequence of Theorem 2.7.41.

Lemma 2.7.45 *Let (R_w, ρ, δ), $(H_\rho, \pi_\rho, D_\delta)$ and $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be as above such that $\mathcal{L}_\delta : r \to -\delta^2(r^{\mathrm{op}})(1)$ is positive definite. Then (1) if $J_\rho : DomD_\delta^{2m} \to DomD_\delta^{2(m-1)}$ then $\Delta_\rho^{1/2} : DomD_\delta^{2m} \to DomD_\delta^{2(m-1)}$ for any $m \in \mathbb{N}$. Suppose $J_\rho : DomD_\delta^4 \to DomD_\delta^2$ and define $\mathcal{R}_{\delta,\rho} := [D_\delta^2, J_\rho]J_\rho$ in this case. Then (2) $[D_\delta^2, \Delta_\rho^{1/2}] = \mathcal{R}_{\delta,\rho}\Delta_\rho^{1/2}$ and $\{\mathcal{R}_{\delta,\rho}, \Delta_\rho^{1/2}\} = 0$ on $\iota(R_w)$.*

Proof Let $D := D_\delta$, $J := J_\rho$, $\Delta := \Delta_\rho$, $S := J\Delta^{1/2}$ and $H := H_\rho$.

(1) Note $\Delta^{1/2} = JS$ as $J^1 = 1$. By Theorem 2.7.41 $[D^2, S] = 0$. Hence $S : DomD^{2m} \to DomD^{2m}$ for any $m \in \mathbb{N}$. Then $\Delta^{1/2} : DomD^{2m} \to JDomD^{2m}$.

With the supposition $[D^2, J]J = JD^2J - D^2J^2 = JD^2J - D^2$ is a densely defined linear operator $DomD^4 \to H$. Note it is immediate $J : DomD^2 \to H$ as J is a bounded anti-linear operator. Hence $\mathcal{R}_{\delta,\rho}$ is well defined. With (1) and the supposition the densely defined linear operator $[D^2, \Delta^{1/2}] : DomD^4 \to H$ is well defined as $\Delta^{1/2} : DomD^4 \to DomD^2$ and $DomD^2 \to H$.

(2) Note $\Delta^{1/2} = JS$ as $J^2 = 1$. Hence $[D^2, \Delta^{1/2}] = [D^2, JS] = J[D^2, S] + [D^2, J]S = [D^2, J]S$ from Theorem 2.7.41. Note that $[D^2, S] = 0$ by Theorem 2.7.41 and $JS = S^*J$. Hence $S^*[D^2, J] = [D^2, J]S$. This implies $\Delta^{1/2}J[D^2, J] = [D^2, J]J\Delta^{1/2}$. Hence $\Delta^{1/2}[D^2, J]J = -[D^2, J]J\Delta^{1/2}$ as $J[D^2, J] = -[D^2, J]J$ from $J^2 = 1$. □

Lemma 2.7.46 *Let (R_w, ρ, δ), $(H_\rho, \pi_\rho, D_\delta)$ and $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be as above. Let $P_{\rho,\delta} = DomD_\delta^2 \cap \mathcal{P}_\rho$. Then $J_\rho : DomD_\delta^2 \to DomD_\delta^2$ if and only if $P_{\rho,\delta}$ is a closed positive cone of $DomD_\delta^2$.*

Proof Let $D := D_\delta$, $J := J_\rho$, $\pi := \pi_\rho$, $\iota := \iota_\rho$ and $\mathcal{P} := \mathcal{P}_\rho$. We recall from Theorem 1.6.2 that $\mathcal{P}$ is the closure of the set $P = \{\pi(r)J\iota(r) \,|\, r \in R_w\}$ and from Theorem 1.6.1 that $JP = P$. Here R_w can be used in place of R due to σ-weak density.

($\Rightarrow$) Let $r \in R_w$. Then $\iota(r) \in DomD_\delta^2$ and $\pi(r) : DomD_\delta^2 \to DomD_\delta^2$ by construction. Hence $\pi(r)J\iota(r) \in DomD_\delta^2$ as $J : DomD_\delta^2 \to DomD_\delta^2$. Hence $P \subset P_{\rho,\delta}$. As the linear span of P is densely defined, then the linear span of $P_{\rho,\delta}$ is densely defined. Hence the closure of the linear span of $P_{\rho,\delta}$ is $DomD_\delta^2$ by uniqueness of the closure of a selfadjoint operator.

($\Leftarrow$) Suppose the linear span of $P_{\rho,\delta} = \mathcal{P} \cap DomD_\delta^2$ is $DomD_\delta^2$. Then $JDomD_\delta^2 = DomD_\delta^2$ as $JP_{\rho,\delta} = P_{\rho,\delta}$ by Theorem 1.6.1 and anti-linearity of J. □

Theorem 2.7.47 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$. Let $(H_\rho, \pi_\rho, D_\delta)$ be the associated GNS representation and $(R, H_\rho, J_\rho, \Delta_\rho, \mathcal{P}_\rho)$ be an associated standard form such that*

(i) *$\mathcal{L}_\delta : r \to -\delta^2(r^{\mathrm{op}})(1)$ is positive definite, and*

(ii) *J_ρ preserves the domain of D_δ^2.*

Then

$$[D_\delta^2, \Delta_\rho] = 0$$

on $\iota_\rho(R_w)$.

Proof Let $D := D_\delta$ and $\Delta := \Delta_\rho$. The equivalence of the condition follows from Lemma 2.7.46. By Lemma 2.7.45 $\Delta : DomD^2 \to DomD^2$ and the linear operator $[D^2, \Delta] : DomD^4 \to H$ is well defined. Moreover $[D^2, \Delta] = \Delta^{1/2}[D^2, \Delta^{1/2}] + [D^2, \Delta^{1/2}]\Delta^{1/2} = \{[D^2, \Delta^{1/2}], \Delta^{1/2}\}$. Hence by Lemma 2.7.45 we have on $\iota(R_w)$, $\{[D^2, \Delta^{1/2}], \Delta^{1/2}\} = \{\mathcal{R}_{\delta,\rho}\Delta^{1/2}, \Delta^{1/2}\} = \{\mathcal{R}_{\delta,\rho}, \Delta^{1/2}\}\Delta^{1/2} = 0$. □

Remark 2.7.48 Real Gradings of a symmetric A-derivation

Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$. Define $\Omega_\delta(A) :=< A, \delta(A^{\mathrm{op}})(1) >$. We say (R_w, ρ, δ) is an abstract K-cycle over A with parity ϵ if the von Neumann algebra R admits an order two *-automorphism $\epsilon \in \mathrm{Aut}(R)$ such that $\epsilon(a) = a$ and $\epsilon(\delta(a^{\mathrm{op}}(1)) = -\delta(a^{\mathrm{op}})(1)$ for all $a \in A$. Compare the following result with Theorem 2.5.14.

Lemma 2.7.49 *Let (R_w, ρ, δ), A and $\Omega_\delta(A)$ be as above where (R_w, ρ, δ) has parity ϵ. Then the following conditions are equivalent*

(i) $\epsilon(\delta(s^{\mathrm{op}})(1)) = -\delta(\epsilon(s)^{\mathrm{op}})(1)$ *for all* $s \in \Omega_\delta(A)$,

(ii) $\epsilon(\delta^2(a^{\mathrm{op}})(s)) = \delta^2(a^{\mathrm{op}})(\epsilon(s))$ *for all* $s \in \Omega_\delta(A)$.

Proof Let $s \in A$. By construction $\delta(\epsilon(a)^{\mathrm{op}})(1) = \delta(a^{\mathrm{op}})(1) = -(-\delta(a^{\mathrm{op}})(1)) = -\epsilon(\delta(a^{\mathrm{op}})(1))$. Now consider

$$\delta(\epsilon(a\delta(b^{\mathrm{op}})(1))^{\mathrm{op}})(1) = -\delta(\delta(b^{\mathrm{op}})(1)^{\mathrm{op}}a^{\mathrm{op}})(1) = -\delta(b^{\mathrm{op}})(1)\delta(a^{\mathrm{op}})(1) - \delta^2(b^{\mathrm{op}})(a)$$

and

$$\epsilon(\delta((a\delta(b^{\mathrm{op}})(1))^{\mathrm{op}})(1)) = \delta(b^{\mathrm{op}})(1)\delta(a^{\mathrm{op}})(1) + \epsilon(\delta^2(b^{\mathrm{op}})(a)).$$

If statement (i) is true for $s \in \Omega^1_\delta(A)$ then (ii) is true for $t \in A$. If statement (ii) is true for $t \in A$ then (i) is true for $s \in \Omega^1_\delta(A)$. Similarly

$$\begin{aligned}\delta(\epsilon(a\delta(b^{\mathrm{op}})(1)\delta(c^{\mathrm{op}})(1))^{\mathrm{op}})(1) = \; & \delta(c^{\mathrm{op}})(1)\delta(b^{\mathrm{op}})(1)\delta(a^{\mathrm{op}})(1) \\ & +\delta(c^{\mathrm{op}})(1)\delta^2(b^{\mathrm{op}})(a) + \delta^2(c^{\mathrm{op}})(a\delta(b^{\mathrm{op}})(1))\end{aligned}$$

and

$$\begin{aligned}\epsilon(\delta((a\delta(b^{\mathrm{op}})(1)\delta(c^{\mathrm{op}})(1)^{\mathrm{op}})(1)) = \; & -\delta(c^{\mathrm{op}})(1)\delta(b^{\mathrm{op}})(1)\delta(a^{\mathrm{op}})(1) \\ & -\delta(c^{\mathrm{op}})(1)\delta^2(b^{\mathrm{op}})(a) + \epsilon(\delta^2(c^{\mathrm{op}})(a\delta(b^{\mathrm{op}})(1))).\end{aligned}$$

If statement (i) is true for $s \in \Omega^2_\delta(A)$ then (ii) is true for $t \in \Omega^1_\delta(A)$. If statement (ii) is true for $t \in \Omega^1_\delta(A)$ then (i) is true for $s \in \Omega^2_\delta(A)$. The proof proceeds by induction. □

Definition 2.7.50 *Let (R_w, ρ, δ) be an abstract K-cycle over a *-algebra $A \subset R_w$ with parity ϵ. Then we say (R_w, ρ, δ) is a real abstract K-cycle over $A \subset R_w$ if $\epsilon(\delta^2(a^{\mathrm{op}})(s)) = \delta^2(a^{\mathrm{op}})(\epsilon(s))$ for all $a \in A$ and $s \in \Omega_\delta(A)$.*

Theorem 2.7.51 *Let (R_w, ρ, δ) be a real abstract K-cycle over a *-algebra $A \subset R_w$ with parity ϵ such that (1) $\rho \in R_*$ is a trace and (2) $\Omega_\delta(A)$ is σ-weak dense in R_w. Let $(H_\rho, \pi_\rho, D_\delta)$ be the associated GNS representation. Then there exists a selfadjoint unitary $\Gamma \in B(H_\rho)$ such that*

(i) $\pi_\rho(\epsilon(s)) = \Gamma\pi_\rho(s)\Gamma$ *for all* $s \in \Omega_\delta(A)$,

(ii) D_δ *and* Γ *anticommute.*

Proof Let $D := D_\delta$ and $\iota := \iota_\rho$. The existence of Γ and the proof of (i) follows from Lemma 2.5.18. Let $s \in \Omega_\delta(A)$. Then $D\Gamma\iota(s) = D\iota(\epsilon(s)) = -i\iota(\delta(\epsilon(s)^{\mathrm{op}})(1)) \stackrel{(a)}{=} +i\iota(\epsilon(\delta(s^{\mathrm{op}})(1))) = \Gamma i\iota(\delta(s^{\mathrm{op}})(1)) = -\Gamma D\iota(s)$. Here (a) used the hypothesis of reality contained in Definition 2.7.50 and Lemma 2.7.49. The result follows from density of $\iota(\Omega_\delta(A))$ in H_ρ. □

In the terminology of Section 2.4.4 the unitary Γ is a real grading for the GNS representation $(H_\rho, \pi_\rho, D_\delta)$ associated to (R_w, ρ, δ).

The next result culminates Section 2.7.3. It classifies the Riemannian representations of a C*-algebra arising from a trace by abstract K-cycles over the C*-algebra.

Theorem 2.7.52 *Let* (H_ρ, π_ρ, D) *be a Riemannian representation of a C*-algebra* A *such that*

(i) $\rho \in R_*$ *is a faithful trace on* $\Omega_D(\mathcal{A}_{\pi_\rho})''$,

(ii) $\iota_\rho(\Omega_D(\mathcal{A}_{\pi_\rho})) \subset H_\rho^\infty$ *is an invariant core for* D.

Then there exists an abstract K-cycle $(\Omega_D(\mathcal{A}_{\pi_\rho}), \rho, \delta)$ *over* $\mathcal{A}_{\pi_\rho}$ *with associated GNS representation* $(H_\rho, \pi_\rho, D_\delta)$ *such that*

$$D = D_\delta + \omega$$

where $\omega \in \Omega_D(\mathcal{A}_{\pi_\rho})$ *and*

$$\Omega_D(\mathcal{A}_{\pi_\rho}) = \Omega_{D_\delta}(\mathcal{A}_{\pi_\rho}) = \pi_\rho(\Omega_\delta(\mathcal{A}_{\pi_\rho})).$$

Proof Let $R_w := \Omega_D(\mathcal{A}_{\pi_\rho})$, $\iota := \iota_\rho$ and $A = \mathcal{A}_{\pi_\rho}$. Let $r^{\mathrm{op}} \in R_w^{\mathrm{op}}$. Define $\delta(r^{\mathrm{op}})(s) := \iota_\rho^{-1}(i[D, r^{\mathrm{op}}]\iota(s))$ for all $s \in R_w$. This is well defined since R_w is an invariant subspace for the operators r^{op} and D. Hence $\delta(r^{\mathrm{op}}) : R_w \to R_w$. Note that $[D, r^{\mathrm{op}}]\iota(a) = [D, r^{\mathrm{op}}]\pi(a)\iota(1) = \pi(a)[D, r^{\mathrm{op}}]\iota(1) + [\pi(a), [D, r^{\mathrm{op}}]]\iota(1) = \pi(a)[D, r^{\mathrm{op}}]\iota(1) + [[D, \pi(a)], r^{\mathrm{op}}]\iota(1) = \pi(a)[D, r^{\mathrm{op}}]\iota(1)$ as $r^{\mathrm{op}} \in R'$. Hence $\delta(r^{\mathrm{op}})(a) = \pi(a)\delta(r^{\mathrm{op}})(1)$ is A-linear. It is clear that δ is a derivation and linear. The symmetric property follows as D is selfadjoint. Since $D : \iota(R_w) \to \iota(R_w)$ there exists $\omega \in R_w$ such that $D\iota(1) = \omega\iota(1)$. Consider $\rho(\delta(r^{\mathrm{op}})(1)) = \langle \iota(1), i[D, r^{\mathrm{op}}]\iota(1)\rangle = \langle D\iota(1), ir^{\mathrm{op}}\iota(1)\rangle - \langle r^{\mathrm{op}}\iota(1), D\iota(1)\rangle = \langle \iota(1), i(\omega r - r\omega)\iota(1)\rangle = 0$ as ρ is a trace. Hence (R_w, ρ, δ) is an abstract K-cycle. Let D_δ be the associated selfadjoint operator. The result follow from the identification of D. Let $r \in \iota(R_w)$ which is an invariant core for D. Then, using the tracial property, $D\iota(r) = Dr^{\mathrm{op}}\iota(1) = [D, r^{\mathrm{op}}]\iota(1) + r^{\mathrm{op}}D\iota(1)$. Hence $D\iota(r) = [D, r^{\mathrm{op}}]\iota(1) + r^{\mathrm{op}}\omega\iota(1) = \iota(-i\delta(r^{\mathrm{op}})(1)) + \omega r^{\mathrm{op}}\iota(1) = D_\delta\iota(r) + \omega\iota(r) = (D_\delta + \omega)\iota(r)$ for all $r \in \iota(R_w)$. □

Remark 2.7.53 The conditions of Theorem 2.7.52 require that (H_ρ, π_ρ, D) be a Riemannian representation of a C*-algebra A such that $\rho \in R_*$ is a faithful trace. We remark on the difficulty when $\rho \in R_*$ is not a trace. Let (H_ρ, π_ρ, D) be a Riemannian representation of a C*-algebra A and $R_w := \Omega_D(\mathcal{A}_{\pi_\rho})$. Let $s \in R_w$. When ρ is not a trace then there exists $s_\lambda \in R$ such that $s^{\text{op}}\iota_\rho(r) = (\lambda + \Delta_\rho)^{-1}\iota_\rho(rs_\lambda)$ for any $\lambda > 0$ [BR, Lemma 2.5.12]. The resolvent of the modular operator Δ_ρ 'twists' the opposite representation. Let $a \in \mathcal{A}_{\pi_\rho}$. Then $s^{\text{op}}[D_\delta, \pi_\rho(a)]\iota_\rho(r) = (\lambda + \Delta_\rho)^{-1}[D_\delta, \pi_\rho(a)](\lambda + \Delta_\rho)s^{\text{op}}\iota_\rho(r)$. This prevents the identification of $[D_\delta, \pi_\rho(a)] \in R$ without further assumptions and hence prevents the construction of a symmetric A-linear derivation.

2.7.4 Riemannian cycles

Let $C^*(A)$ denote the C*-closure of a unital *-subalgebra A of a von Neumann algebra R. Let (R_w, ρ, δ) be an abstract K-cycle and $A \subset R$ a C*-subalgebra. The result of Theorem 2.7.52 introduces the following definition.

Definition 2.7.54 *Let (R_w, ρ, δ) be an abstract K-cycle over a unital *-algebra $\mathcal{A} \subset R_w$. Then we call (R_w, ρ, δ) a Riemannian cycle over the C*-algebra $C^*(\mathcal{A})$ if the associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$ is a Riemannian representation of $C^*(\mathcal{A})$.*

Remark 2.7.55 We note that Theorem 2.7.38 provides the result that a Riemannian cycle (R_w, ρ, δ) over a C*-algebra A is an inner Riemannian cycle if and only if the outer sheer $I_\delta(a, r)$ vanishes for all $a \in \mathcal{A}$ and $r \in R_w$.

To define a Riemannian cycle abstractly requires the converse of Theorem 2.7.52. At present, however, the converse of Theorem 2.7.52 is beyond our treatment. The following results culminate in a partial converse to Theorem 2.7.52.

Theorem 2.7.56 *Let (R_w, ρ, δ) be an abstract K-cycle over a unital *-algebra $A \subset R_w$ with the hypothesis*

(i) *$\rho \in R_*$ is a faithful trace,*

(ii) *$\mathcal{L}_\delta : r \to -\delta^2(r^{\text{op}})(1)$ is positive definite,*

(iii) *$\delta^m|_{A^{\text{op}}} : A^{\text{op}} \to B^\rho_A(R_w)$ for all $m \in \mathbb{N}$, and*

(iv) *$\Omega_{D_\delta}(\pi_\rho(A)) = \pi_\rho(\Omega_\delta(A))$*

Then the GNS representation $(H_\rho, \pi_\rho, D_\delta)$ associated to (R_w, ρ, δ) is a C^∞_c representation of the unital C-algebra $C^*(A)$.*

Proof That the conditions of Definition 1.4.8 are satisfied by $(H_\rho, \pi_\rho, D_\delta)$ follows from Lemma 2.7.30 with the exception of Definition 1.4.8(iii). Let $\pi_\rho := \pi$, $\iota := \iota_\rho$, $J := J_\rho$, $D := D_\delta$, and $f_n(x) := (1 + x^2)^{-n/2}$. We recall that $\pi(s)\iota(r) = \iota(sr)$ and $\pi^{\text{op}}(s)\iota(r) = \iota(rs)$ for all $s, r \in R_w$ as ρ is a trace.

We are required to show $[|D|, t]$ is bounded where $t = \pi(a)$ or $[D, \pi(a)]$.

1) We obtain an equivalent statement. By Proposition 1.4.7 we are required to prove $f_1(D)[D^2, t]$ is bounded where $t = \pi(a)$ or $[D, \pi(a)]$. We note $[D, \pi(a)] = -i\pi(\delta(a^{\text{op}})(1))$ by condition (iii) of the hypothesis and Corollary 2.7.33. We also note $f_1(D)[D^2, s^*]$ is bounded if and only if $J(f_1(D)[D^2, s^*])^*J \overset{(a)}{=} [D^2, s^{\text{op}}]f_1(D)$ is bounded where $s \in R_w$. The equality (a) is given by Corollary 2.7.42 since hypothesis

(i) and (ii) result in $[D^2, J] = 0$. Combining the above remarks, we are reduced to showing $[D^2, t^{\mathrm{op}}]f_1(D)$ is bounded where $t = \pi(a)$ or $\pi(\delta(a^{\mathrm{op}})(1))$.

2) Let $r, s \in R_w$. We show that $[D^2, \pi^{\mathrm{op}}(s)] = -i\{D, \pi_\rho(\delta(s^{\mathrm{op}}))\}$. Consider

$$[D^2, \pi^{\mathrm{op}}(s)]\iota(r) = D^2\iota(rs) - \iota(-\delta^2(r^{\mathrm{op}})(1)s) = -\iota\Big(\delta^2((rs)^{\mathrm{op}})(1) - \delta^2(r^{\mathrm{op}})(1)s\Big).$$

We have

$$\begin{aligned}
\delta^2((rs)^{\mathrm{op}})(1) = &\ \delta(\delta(s^{\mathrm{op}}r^{\mathrm{op}})(1)^{\mathrm{op}})(1)\\
= &\ \delta\Big((\delta(r^{\mathrm{op}})(1)s + \delta(s^{\mathrm{op}})(r))^{\mathrm{op}}\Big)(1)\\
= &\ \delta(s^{\mathrm{op}}\delta(r^{\mathrm{op}})(1)^{\mathrm{op}})(1) + \delta(\delta(s^{\mathrm{op}})(r)^{\mathrm{op}}(1)\\
= &\ \delta^2(r^{\mathrm{op}})s + \delta(s^{\mathrm{op}})(\delta(r^{\mathrm{op}})(1)) + \delta(\delta(s^{\mathrm{op}})(r)^{\mathrm{op}}(1)
\end{aligned}$$

by repeated use of the derivation property of δ. Hence

$$\delta^2((rs)^{\mathrm{op}})(1) - \delta^2(r^{\mathrm{op}})(1)s = \delta(s^{\mathrm{op}})(\delta(r^{\mathrm{op}})(1)) + \delta(\delta(s^{\mathrm{op}})(r)^{\mathrm{op}})(1)$$

and

$$\begin{aligned}
[D^2, \pi^{\mathrm{op}}(s)]\iota(r) = &\ -\iota\Big(\delta(s^{\mathrm{op}})(\delta(r^{\mathrm{op}})(1)) + \delta(\delta(s^{\mathrm{op}})(r)^{\mathrm{op}})(1)\Big)\\
= &\ -i\pi_\rho(\delta(s^{\mathrm{op}}))D\iota(r) - iD\pi_\rho(\delta(s^{\mathrm{op}}))\iota(r)\\
= &\ -i\{D, \pi_\rho(\delta(s^{\mathrm{op}}))\}\iota(r).
\end{aligned}$$

3) Let T be a bounded operator such that $[D, T]$ is bounded. Then $\{D, T\} = 2DT + [T, D] = 2TD + [D, T]$. Hence $f_1(D)\{D, T\} = 2f_1(D)DT + f_1(D)[T, D]$ and $\{D, T\}f_1(D) = 2TDf_1(D) + [T, D]f_1(D)$ are bounded.

4) We show that $[D, \pi(\delta(s^{\mathrm{op}}))]$ is bounded where $s = a$ or $\delta(a^{\mathrm{op}})(1)$. Consider

$$\begin{aligned}
[D, \pi^{\mathrm{op}}(s)]\iota(r) = &\ D\iota(rs) - \pi^{\mathrm{op}}(s)D\iota(r)\\
= &\ \iota\Big(- i\delta(s^{\mathrm{op}}r^{\mathrm{op}})(1) + i\delta(r^{\mathrm{op}})s\Big)\\
= &\ \iota\Big(- i\delta(r^{\mathrm{op}})(1)s - i\delta(s^{\mathrm{op}})(r) + i\delta(r^{\mathrm{op}})s\Big)\\
= &\ -i\iota(\delta(s^{\mathrm{op}})(r)) = -i\pi_\rho(\delta(s^{\mathrm{op}}))\iota(r).
\end{aligned}$$

Hence $[D, \pi^{\mathrm{op}}(s)]$ is bounded iff $\delta(s^{\mathrm{op}}) \in B^\rho_A(R_w)$. By hypothesis (iii) $\delta(a^{\mathrm{op}})$ and $\delta^2(a^{\mathrm{op}}) = \delta(\delta(a^{\mathrm{op}})(1)^{\mathrm{op}})$ belong to $B^\rho_A(R_w)$.

The combination of 1,2,3,4 prove that $[|D|, \pi(a)]$ and $[|D|, [D, \pi(a)]]$ is bounded for all $a \in A$. One continues in this method, with increasingly complicated calculations, to find $\delta^m(a^{\mathrm{op}}) \in B^\rho_A(R_w)$ for $m = 1, ..., 2n$ implies $\delta^m_{|D|}(\pi(a))$ and $\delta^m_{|D|}([D, \pi(a)])$ are bounded for $m = 1, ..., n$ for all $a \in A$. □

Let (R_w, ρ, δ) be an abstract K-cycle and $A \subset R$ a C*-subalgebra. Define $\mathcal{A} := \{a \in A \cap R_w \,|\, \delta^m(a^{\mathrm{op}}) \in B^\rho_A(R_w)\ \forall m \in \mathbb{N}\}$ and $\Omega_\delta(\mathcal{A}) :=< \mathcal{A}, \delta(\mathcal{A}^{\mathrm{op}})(1) >\subset R_w$.

Definition 2.7.57 *Let (R_w, ρ, δ) be an abstract K-cycle over the unital *-algebra $\mathcal{A} \subset A$. Then (R_w, ρ, δ) is called a uniform positive Riemannian cycle over the C*-algebra A if the following conditions are satisfied* (1) *$\mathcal{A}$ is norm dense in A,* (2) *$\Omega_\delta(\mathcal{A}) = R_w$ and* (3) *$\mathcal{L}_\delta : r \to -\delta^2(r^{\mathrm{op}})(1)$ is positive definite.*

Remark 2.7.58 We remark that the index algebra B_{π_ρ} of the GNS representation $(H_\rho, \pi_\rho, D_\delta)$ associated to a tracial Riemannian cycle (R_w, ρ, δ) over the unital C*-algebra A is the C*-algebra

$$B_{\pi_\rho} = C^*(\{\pi_\rho^{\mathrm{op}}(r) \,|\, r \in R_w, \delta(r^{\mathrm{op}}) \in B_A^\rho(R_w)\}).$$

In particular for a uniform positive Riemannian cycle, from the proof of Theorem 2.7.56, $B_{\pi_\rho} = C^*(\pi_\rho(\Omega_\delta(\mathcal{A})))$.

The following theorem establishes a construction function with domain a tracial uniform positive Riemannian cycle (R, ρ, δ) of a C*-algebra A and range an ungraded Riemannian representation of A. It extends Theorem 2.7.10.

Theorem 2.7.59 [GNS Construction]
Let (R_w, ρ, δ) be a uniform positive Riemannian cycle over a C-algebra A such that $\rho \in R_*$ is a faithful trace. Then the associated GNS representation $(H_\rho, \pi_\rho, D_\delta)$ is an ungraded Riemannian representation of the C*-algebra A such that*

(i) $[D_\delta, \pi_\rho(a)] = -i\pi_\rho(\delta(a^{\mathrm{op}})(1))$ *for all* $a \in \mathcal{A}$,

(ii) $\Omega_{D_\delta}(\pi_\rho(\mathcal{A})) = \pi_\rho(\Omega_\delta(\mathcal{A})) = \pi_\rho(R_w)$,

(iii) $\Lambda_\rho := \iota_\rho(\Omega_\delta(\mathcal{A})) \subset H_\rho^\infty := \cap_m Dom|D_\delta|^m$.

Proof Let $D := D_\delta$, $\pi = \pi_\rho$ and $\iota := \iota_\rho$. We prove the representation is Riemannian. Combining Theorem 2.7.36 and Corollary 2.7.32 leaves only the condition $\Omega_{D_\delta}(\pi_\rho(A))'' = R$. From Lemma 2.7.30 and Corollary 2.7.32 $[D, \pi(a)]\iota(r) = -i\iota(\delta(a^{\mathrm{op}})(1)r)$. Let $s^{\mathrm{op}} \in R'$. Then

$$\begin{aligned} s^{\mathrm{op}}[D, \pi(a)]\iota(r) &= -is^{\mathrm{op}}\iota(\delta(a^{\mathrm{op}})(1)r) \\ &\overset{(i)}{=} -i\iota(\delta(a^{\mathrm{op}})(1)rs) \\ &= [D, \pi(a)]\iota(rs) \overset{(ii)}{=} [D, \pi(a)]s^{\mathrm{op}}\iota(r) \end{aligned}$$

where (i) and (ii) used the tracial property. This implies $[D, \pi(a)] \in R''$. Hence $[D, \pi(a)] = -i\pi(\delta(a^{\mathrm{op}})(1))$ as $\iota_\rho(1)$ is a separating and cyclic vector for R''. This proves statement (i) and statements (ii) and (iii) are immediate. As $\Omega_D(\pi(\mathcal{A})) = \pi(R_w)$ then $\Omega_D(\pi(\mathcal{A}))$ is σ-weak dense in $\pi(R) \cong R$. □

Remark 2.7.60 We remark that one can amend the prefix ungraded in Theorem 2.7.38 by assuming the uniform positive Riemannian cycle is with parity as an abstract K-cycle. Similarly one can add the prefix real by assuming the uniform positive Riemannian cycle is real as an abstract K-cycle.

Example 2.7.61 Riemannian Manifold

Let X be an n-dimensional compact Riemannian manifold. We apply Theorem 2.7.40 to the Riemannian representation $(L^2(X, \Lambda^* X), \pi_l, d + d^*)$ of the C*-algebra $C(X)$, see Theorem 2.4.21 and Theorem 2.6.9. Let

$$C^\infty(\mathrm{Cl}) := C^\infty(X, \mathrm{Cl}(X)),$$

$$\lambda(w) := \int_X q_g(1, w)(x)\sqrt{\det g}\, dx,$$

where q_g is the metric, and

$$\delta_d(w^{\mathrm{op}})(u) := \iota_\lambda^{-1}(-i[d + d^*, \pi_l^{\mathrm{op}}(w)]\iota_\lambda(u))$$

for all $w, u \in C^\infty(\mathrm{Cl})$ where $\iota_\lambda : C^\infty(\mathrm{Cl}) \to L^2(X, \Lambda^* X)$ is the linear injection of Theorem 2.3.4(v).

Theorem 2.7.62 *Let $C^\infty(\mathrm{Cl})$, λ and δ_d be as above. Then $(C^\infty(\mathrm{Cl}), \lambda, \delta_d)$ defines a Riemannian cycle on the smooth unital *-subalgebra $C^\infty(X) \subset L^\infty(X, \mathrm{Cl}(X))$ with associated GNS representation $(L^2(X, \Lambda^* X), \pi_l, d + d^*)$.*

Proof By construction of the selfadjoint operator $d+d^*$ and Theorem 2.4.21 the hypothesis of Theorem 2.7.52 is satisfied for $A = C(X)$. The triple $(C^\infty(\mathrm{Cl}), \lambda, \delta_d)$ is exactly the triple constructed by Theorem 2.7.52. We note $(d + d^*)\iota_\lambda(1) = 0$. Hence $\omega = 0$ in the statement of Theorem 2.7.52. □

We examine the explicit form of the symmetric $C^\infty(X)$-derivation δ_d and the metric sheer for this example.

We recall from the proof of Theorem 2.4.21 that

$$\pi_l^{\mathrm{op}}(f)\iota_\lambda(u) = \iota_\lambda(uf),$$

$$\pi_l^{\mathrm{op}}(df)\iota_\lambda(u) = \iota_\lambda(u \cdot df),$$

$$-i[d + d^*, \pi_l^{\mathrm{op}}(f)]\iota_\lambda(u) = \pi_l(df)\iota_\lambda(u) = \iota_\lambda(df \cdot u)$$

for all $f \in C^\infty(X)$ and $u \in C^\infty(\mathrm{Cl})$. Hence $\delta_d : C^\infty(\mathrm{Cl})^{\mathrm{op}} \to B^\lambda_{C^\infty(X)}(C^\infty(\mathrm{Cl}))$ restricted to zero- and one-forms is given by

$$\delta_d^0(f)(u) = uf,$$

$$\delta_d(f)(u) = df \cdot u$$

for all $f \in C^\infty(X)$ and $u \in C^\infty(\mathrm{Cl})$. In particular

$$\delta_d^0(f)(1) = f,$$

$$\delta_d(f)(1) = df$$

for all $f \in C^\infty(X)$. In a chart U with local tangent bundle basis $\{\partial_i(x)\}_{i=1}^n$ and local cotangent bundle basis $\{dx_i(x)\}_{i=1}^n$ for $x \in U$ we have

$$-i[d + d^*, -i\pi_l^{\mathrm{op}}(df)(x)]\iota_\lambda(u(x)) = -\iota_\lambda(\sum_i \partial_{i,j} f(x) dx_i(x) \cdot u(x) \cdot dx_j(x))$$

for all $f \in C^\infty(X)$ by the proof of Theorem 2.4.21(ii). Hence

$$\delta_d^2(f)(u)(x) := \delta_d(\delta_d(f)^{\mathrm{op}})(u)(x) = -\sum_i \partial_{i,j} f(x) dx_i(x) \cdot u(x) \cdot dx_j(x)$$

and

$$\delta_d^2(f)(1)(x) = -\sum_i \partial_{i,j} f(x) dx_i(x) \cdot dx_j(x) = \sum_i \partial_{i,i} f(x) = -\Delta f(x)$$

where Δ is the generalised Laplacian given by the Levi-Civita connection on X. We note we could continue to higher powers of the symmetric $C^\infty(X)$-derivation δ_d and demonstrate that $(C^\infty(\mathrm{Cl}), \lambda, \delta_d)$ infact defines a uniform positive Riemannian cycle over $C(X)$. The differentiability of functions on the manifold is hence related to the domain of $d + d^*$ as a spatial implementer of a derivation with domain in the *commutant* $C^\infty(\mathrm{Cl})'$. This is conceptually and in practical calculations more appealing than dealing with the derivations implemented by $|d + d^*|$.
Finally we consider the metric sheer $S_d := S_{\delta_d}$ associated to the symmetric $C^\infty(X)$-derivation δ_d as in Definition 2.7.37. We identify a restriction of the metric sheer $S_d : C^\infty(X) \times C^\infty(X) \to \Omega^2_{d+d^*}(C^\infty(X))$. Let $f, h \in C^\infty(X)$. Then we have

$$\begin{aligned} S_d(f,h) &= \tfrac{1}{2}\Big(\delta_d(h)(\delta_d(f)) + \delta_d(f)(1)\delta_d(h)(1)\Big) \\ &= \tfrac{1}{2}(dh \cdot df + df \cdot dh) \\ &= -g(df, dh) \end{aligned}$$

where g is the metric of the Riemannian manifold X.

2.8 Example - Riemannian Geometry of the Torus

We provide an example of a Riemannian geometry $(A_\theta, H_\tau, \pi_\tau, D, c)_R$ where A_θ is an irrational rotation algebra.

2.8.1 The rotation algebra A_θ

The u, v be unitary operators such that $vu = e^{2\pi i\theta} uv$ for some $\theta \in \mathbb{R}$. Let $F(\mathbb{Z}^2)$ denote the set of double sequences $\{a_{r,s}\}$ such that $(r,s) \in F$ where F is a finite subset of $\mathbb{Z}^2$. Define the unital *-algebra

$$F_\theta := \{a = \sum_{r,s} a_{r,s} u^r v^s \mid \{a_{r,s}\} \in F(\mathbb{Z}^2)\}.$$

with product and involution

$$\begin{aligned} ab &:= \sum\nolimits_{r,s} \Big(\sum\nolimits_{n,m} a_{r-n,m} \lambda^{mn} b_{n,s-m}\Big) u^r v^s \\ a^* &:= \sum\nolimits_{r,s} (\lambda^{rs} \overline{a}_{-r,-s}) u^r v^s \end{aligned}$$

where $\lambda := e^{2\pi i\theta}$. Define a linear functional $\tau' : F_\theta \to \mathbb{C}$ by

$$\tau'(a) := a_{0,0}$$

Lemma 2.8.1 *Let F_θ and τ' be as above. Then $\tau'(a^*a) > 0$ for all $a \neq 0$.*

Proof Let $a \in F_\theta$. Then $\tau'(a^*a) = \sum_{n,m} \lambda^{-nm} \overline{a_{n,-m}} \lambda^{nm} a_{n,-m} = \sum_{n,m} |a_{n,-m}|^2 > 0$ and is equal to zero iff $|a_{n,-m}| = 0$ $\forall n, m$ which occurs iff $a = 0$. $\square$

Hence $\langle a, b\rangle = \tau'(a^*b)$ defines an inner product on F_θ. Denote by H_τ the closure of the pre-Hilbert space $(F_\theta, \langle ., .\rangle)$. Let $a \in F_\theta$. Let $\pi(a) : F_\theta \to F_\theta$ denote the linear operator defined by $\pi(a)b = ab$ for all $b \in F_\theta$.

Lemma 2.8.2 *The faithful representation $\pi : F_\theta \to L(F_\theta, F_\theta)$ extends to a faithful *-representation $\pi_\tau : F_\theta \to B(H_\tau)$.*

Proof Define $\pi_\tau(a)b = ab$ for the dense subset $F_\theta \subset H_\tau$. Clearly $\pi_\tau(ab) = \pi_\tau(a)\pi_\tau(b)$ and $\langle \pi_\tau(a^*)b, c\rangle = \tau'((a^*b)^*c) = \tau'(b^*ac) = \langle b, \pi_\tau(a)c\rangle$ for all $b, c \in F_\theta$. If $\pi_\tau(a)b = 0$ for all $b \in F_\theta$ then $ab = 0$ for all $b \in F_\theta$. Letting $b = 1 \in F_\theta$ proves $a = 0$. Hence the proof is complete by density of F_θ in H_τ once we demonstrate $\pi_\tau(a)$ is uniformly bounded in norm. This follows as $\|\pi_\tau(a)b\|_\tau^2 = \tau'((ab)^*ab) \leq \tau'(a^*a)\tau'(b^*b) = \|a\|_\tau^2\|b\|_\tau^2$ by the Cauchy-Schwartz inequality. □

Let A_θ be the closure of F_θ in the uniform topology of $B(H_\tau)$. Let A_θ'' be the closure of F_θ in the weak topology of $B(H_\tau)$.

Corollary 2.8.3 *Let F_θ and τ' be as above. Then*

(i) *A_θ is a separable unital C^*-algebra,*

(ii) *A_θ'' is a von Neumann algebra with separable pre-dual,*

(iii) *F_θ is norm dense in A_θ and σ-weak dense in A_θ'',*

(iv) *there exists a normal faithful trace state τ on A_θ'' such that $\tau|_{F_\theta} = \tau'$,*

(v) *(H_τ, π_τ) is the GNS representation of A_θ'' associated to τ.*

Proof (i) and (ii) are immediate as F_θ is countably generated. (iii) Follows from the von Neumann density theorem. (iv) Define $\tau'(a) = \langle 1, \pi_\tau(a)1\rangle$ for $a \in A_\theta''$. Normality is immediate from construction. The properties of state, trace, and faithfulness are then immediate by σ-weak density of F_θ in A_θ''. (v) Immediate from the GNS construction. □

Let $\theta \in \mathbb{Q}$. Then the spectrum $\hat{A}_\theta$ is isomorphic to the torus $\mathbb{T}^2$ and $A_\theta \sim_M C(\mathbb{T}^2)$ [Ri2]. Hence, in terms of the non-commutative topological features, the rational and integer case are equivalent.

Let $\theta \notin \mathbb{Q}$. Then A_θ is called an irrational rotation algebra. We recall a simple C*-algebra is a C*-algebra that has no proper two-sided closed ideals.

Theorem 2.8.4 *The following statements are equivalent.*

(i) *θ is irrational,*

(ii) *A_θ is a simple C^*-algebra,*

(iii) *τ is the unique normal faithful trace state on A_θ'',*

(iv) *A_θ'' is isomorphic to the unique hyperfinite type II_1 factor.*

Proof Theorem 1.10 and Corollary 1.16 [Bo]. □

We define two symmetric derivations on A''_θ by the linear extension and closure of the maps $\delta_u, \delta_v : F_\theta \to F_\theta$ defined by the assignments

$$\delta_u(a_{r,s}u^r v^s) := 2\sqrt{\pi} i r a_{r,s} u^r v^s$$
$$\delta_v(a_{r,s}u^r v^s) := 2\sqrt{\pi} i s a_{r,s} u^r v^s.$$

Note that $\delta_u(1) = \delta_v(1) = 0$.

Lemma 2.8.5 *Let $\alpha = u, v$ and $(A''_\theta, \tau, \delta_\alpha)$ be as above. Then $(A''_\theta, \tau, \delta_u)$ and $(A''_\theta, \tau, \delta_v)$ are inner Riemannian cycles over the unital C^*-algebra A_θ.*

Proof We check the conditions of Definition 2.7.9. A simple calculation reveals δ_u and δ_v are symmetric derivations. As $F_\theta \in Dom\delta_u^i \cap Dom\delta_v^j$ for all $i, j \in \mathbb{N}$, then the derivations have σ-weak dense domains by Corollary 2.8.3(iii). Let $a \in F_\theta$. Clearly $(\delta_\alpha(a))_{0,0} = 0$ for $\alpha = u, v$. Hence $\tau(\delta_\alpha(a)) = 0$ for $\alpha = u, v$. Let $a \in F_\theta$ with support $f \in F(\mathbb{Z}^2)$. Let $k = \sup_{(i,j)\in f} \|(i,j)\|$. Then $\|\delta_\alpha^m(a)\| \le k^m\|a\|$ for all $m \in N$ and $\alpha = u, v$. Hence F_θ contains analytic elements for δ_u and δ_v. Hence $A_\theta \cap F_\theta = F_\theta$ is a subset of analytic elements norm dense in A_θ. Finally $\Omega_\delta(F_\theta) = F_\theta$ is σ-weak dense in A''_θ. □

Let $S(\mathbb{Z}^2)$ denote the double sequences of rapid decay. Define

$$\mathcal{A}_\theta := \{a = \sum_{r,s} a_{r,s}u^r v^s \mid \{a_{r,s}\} \in S(\mathbb{Z}^2)\}.$$

Lemma 2.8.6 *Let $\mathcal{A}_\theta$ be as above. Then $\mathcal{A}_\theta$ is a unital *-algebra such that*

(i) *$\mathcal{A}_\theta$ is a Frechet pre-C^*-algebra of A_θ*

(ii) $\mathcal{A}_\theta = A''_\theta \cap \left(\cap_m Dom\delta_u^m\right) \cap \left(\cap_m Dom\delta_u^m\right)$, *and*

(iii) *$\mathcal{A}_\theta$ is σ-weak dense in A''_θ.*

Proof Define the seminorms $p_\alpha^m(a) := \|\delta_\alpha^m(a)\|$ for $m = 0, 1, 2, ..., \alpha = u, v$.

(ii) Let $a = \sum_{r,s} a_{r,s}u^r v^s$ be of rapid decay. Then for each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ and $n_k \in \mathbb{N}$ such that $|a_{r,s}| \le C_k(1 + |r| + |s|)^{-k}$ for all $|r|, |s| > n_k$. Let $m \in \mathbb{N} \cup \{0\}$. Consider $\delta_u^m(a) = \sum_{r,s} r^m a_{r,s}u^r v^s$. Then $\|\delta_u^m(a)\| \le C_{m+2}\sum_{(r,s)} |r|^m(1+|r|+|s|)^{-m-2} + \sum_{|r|,|s|\le n_{m+2}} |a_{r,s}| < \infty$. Similarly for δ_v^m. Hence $a \in \left(\cap_m Dom\delta_u^m\right) \cap \left(\cap_m Dom\delta_u^m\right)$.

Let $b \in A''_\theta$ such that that $b_{r,s}$ is not of rapid decay. Then there exists k such that $|r^k b_{r,s}| > |r|$ for all $|r| > r_0$ or $|s^k b_{r,s}| > |s|$ for all $|s| > s_0$. Hence $p_u^k(b)$ or $p_v^k(b)$ is not finite. Then $b \notin \left(\cap_m Dom\delta_u^m\right) \cap \left(\cap_m Dom\delta_u^m\right)$.

(i) Result (ii) implies $\mathcal{A}_\theta$ is a *-algebra closed in the metrisable locally convex topology generated by the countable family of seminorms p_α^m and in the holomorphic functional calculus. Let $a \in \mathcal{A}_\theta \setminus F_\theta$. Let $F_k := \{(r,s) \in \mathbb{Z}^2 \mid |a_{s,r}| \ge (1 + |s| + |r|)^{-k-2}\}$. Then F_k is a finite subset of $\mathbb{Z}^2$ as a is of rapid decay. Define $a_k := \sum_{(r,s)\in F_k} a_{r,s}u^r v^s \in F_\theta$. Then $\|a - a_k\| \le \sum_{(r,s)\notin F_k} |a_{r,s}| \le \sum_{(r,s)\notin F_k} (1 + |s| + |r|)^{-k-2} \le M \max_{(r,s)\notin F_k}(1+|s|+|r|)^{-k}$ for a constant $M > 0$. Hence $\lim_k \|a - a_k\| =$

0 by the condition $a \notin F_\theta$. The inclusion $F_\theta \subset \mathcal{A}_\theta$ then implies $\mathcal{A}_\theta$ is norm dense in A_θ.

(iii) follows from $F_\theta \subset \mathcal{A}_\theta$ and Corollary 2.8.3(iii). □

2.8.2 A Riemannian Cycle on A_θ

Let A be a unital *-algebra. Define $M_2(A) := A \otimes M_2(\mathbb{C})$. Then $M_2(A)$ is a unital *-algebra. We canonically identify A with $A \otimes I_2$ where I_2 is the identity of $M_2(\mathbb{C})$. Let Tr denote the normalised A-valued matrix trace on $M_2(A)$. We recall that $M_2(\mathbb{C})$ has a basis given by the Pauli-sigma matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Define the linear functional $\rho : M_2(A''_\theta) \to \mathbb{C}$ by

$$\rho(E) := \tau \circ Tr(E) \quad \forall\, E \in M_2(A''_\theta).$$

Define the opposite representation $\pi^{\mathrm{op}} : M_2(A''_\theta) \to L(M_2(A''_\theta), M_2(A''_\theta))$ by

$$E^{\mathrm{op}}(F) = FE \quad \forall\, E, F \in M_2(A''_\theta).$$

Define linear maps $d_u, d_v : M_2(\mathcal{A}_\theta) \to M_2(\mathcal{A}_\theta)$ by

$$\begin{aligned} d_u(\textstyle\sum_{i=0} a_i\sigma_i) &:= \textstyle\sum_{i=0} \delta_u(a_i)\sigma_i \\ d_v(\textstyle\sum_{i=0} a_i\sigma_i) &:= \textstyle\sum_{i=0} \delta_v(a_i)\sigma_i. \end{aligned}$$

Formally we may consider these maps as $d_u = \delta_u \otimes I_2$ and $d_v = \delta_v \otimes I_2$. Then define the linear map $\delta : M_2(\mathcal{A}_\theta)^{\mathrm{op}} \to L(M_2(\mathcal{A}_\theta), M_2(\mathcal{A}_\theta))$ by

$$\delta(E^{\mathrm{op}})(F) := \sigma_1 F d_u(E) + \sigma_2 F d_v(E)$$

for $E, F \in M_2(\mathcal{A}_\theta)$. We note that

$$\delta(a^{\mathrm{op}})(1) = \delta_u(a)\sigma_1 + \delta_v(a)\sigma_2$$

for all $a \in \mathcal{A}_\theta$.

Lemma 2.8.7 *Let $\mathcal{A}_\theta$, ρ and δ be as above. Then $(M_2(\mathcal{A}_\theta), \rho, \delta)$ is an abstract K-cycle over $\mathcal{A}_\theta$ such that $\delta^m(E^{\mathrm{op}}) \in B^\rho_{\mathcal{A}_\theta}(M_2(\mathcal{A}_\theta)) \quad \forall m \in \mathbb{N} \quad \forall E \in M_2(\mathcal{A}_\theta)$.*

Proof We check the conditions of Definition 2.7.29. (1) $M_2(\mathcal{A}_\theta)$ is a unital σ-weak dense *-subalgebra of the von Neumann algebra $M_2(\mathcal{A}''_\theta)$ such that $\mathcal{A}_\theta \otimes I_2 \subset M_2(\mathcal{A}_\theta)$ by Lemma 2.8.6. Moreover ρ is a normal faithful trace state on $M_2(\mathcal{A}''_\theta)$ by construction.

(2) Let $E = \sum_i a_i\sigma_i \in M_2(\mathcal{A}_\theta)$. Then $\rho(\delta(E^{\mathrm{op}})(1)) = \frac{1}{2}\tau(\delta_u(a_1) + \delta_v(a_1)) = \frac{1}{2}\tau(\delta_u(a_1)) + \frac{1}{2}\tau(\delta_v(a_1)) = 0$.

(3) It is an exercise to check that $d_\alpha(EF) = Ed_\alpha(F) + d_\alpha(E)F$ for $E, F \in M_2(\mathcal{A}_\theta)$ and $\alpha = u, v$. Hence

$$\begin{aligned}\delta(E_1^{\mathrm{op}}E_2^{\mathrm{op}})(F) &= \sigma_1 F d_u(E_2E_1) + \sigma_2 F d_v(E_2E_1)\\ &= \sigma_1 F E_2 d_u(E_1) + \sigma_2 F E_2 d_v(E_1) + \sigma_1 F d_u(E_2)E_1 + \sigma_2 F d_v(E_2)E_1\\ &= \delta(E_1^{\mathrm{op}})(E_2^{\mathrm{op}}F) + E_1^{\mathrm{op}}\delta(E_2^{\mathrm{op}})(F).\end{aligned}$$

We have $\delta(E^{\mathrm{op}})(a\sigma_0) = \sigma_1 a\sigma_0\delta_u(E) + \sigma_2 a\sigma_0\delta_v(E) = a\sigma_0(\sigma_1\delta_u(E) + \sigma_2\delta_v(E)) = a\sigma_0\delta(E^{\mathrm{op}})(1)$ as $[\sigma_1, \sigma_0] = 0 = [\sigma_2, \sigma_0]$. This has shown δ is an $\mathcal{A}_\theta$-derivation. We have

$$d_\alpha(E^*) = d_\alpha(\sum_i a_i^*\sigma_i) = \sum_i \delta_\alpha(a_i^*)\sigma_i = \sum_i \delta_\alpha(a_i)^*\sigma_i = (\sum_i \delta_\alpha(a_i)\sigma_i)^* = d_\alpha(E)^*$$

for all $E \in M_2(\mathcal{A}_\theta)$ and $\alpha = u, v$. Hence

$$\begin{aligned}\rho(\delta((E^{\mathrm{op}})^*)(F_1)^*F_2) &= \rho((\sigma_1 F_1 d_u(E)^* + \sigma_2 F_1 d_v(E)^*)^*F_2)\\ &= \rho(d_u(E)F_1^*\sigma_1 F_2 + d_v(E)F_1^*\sigma_2 F_2)\\ &\overset{(i)}{=} \rho(F_1^*(\sigma_1 F_2 d_u(E) + \sigma_2 F_2 d_v(E)))\\ &= \rho(F_1^*\delta(E^{\mathrm{op}})(F_2))\end{aligned}$$

where we used the tracial property of ρ at (i). Hence δ is ρ-adjointable and symmetric.

(4) Let $E = \sum_i a_i\sigma_i \in M_2(\mathcal{A}_\theta)$. Then $d_u^m d_v^n(E) \in M_2(\mathcal{A}_\theta)$ for all $m, n \in \mathbb{N}$ as $a_i \in \mathcal{A}_\theta$. Now $\delta(E^{\mathrm{op}})(F) = \sigma_1 F d_u(E) + \sigma_2 F d_v(E)$ is clearly a bounded operation as $d_u(E), d_v(E) \in M_2(\mathcal{A}_\theta)$. We have

$$\begin{aligned}\delta(\delta(E^{\mathrm{op}})(1)^{\mathrm{op}})(F) &= \sigma_1 F d_u(\sigma_1 d_u(E) + \sigma_2 d_v(E)) + \sigma_2 F d_v(\sigma_1 d_u(E) + \sigma_2 d_v(E))\\ &= \sigma_1 F(\sigma_1 d_u^2(E) + \sigma_2 d_u d_v(E)) + \sigma_2 F(\sigma_1 d_v d_u(E) + \sigma_2 d_v^2(E))\end{aligned}$$

for all $E, F \in M_2(\mathcal{A}_\theta)$. Hence $\delta^2(E^{\mathrm{op}})$ is a bounded operator as $d_u^2(E), d_v^2(E), d_u d_v(E) = d_v d_u(E) \in M_2(\mathcal{A}_\theta)$. The proof $E^{\mathrm{op}} \in Dom\delta^m$ for all $m \in \mathbb{N}$ follows from induction. We note from above that

$$\begin{aligned}\delta^2(E^{\mathrm{op}})(1) &= \sigma_1(\sigma_1 d_u^2(E) + \sigma_2 d_u d_v(E)) + \sigma_2(\sigma_1 d_v d_u(E) + \sigma_2 d_v^2(E))\\ &= d_u^2(E) + d_v^2(E).\end{aligned}$$

Hence $\delta^2((\cdot)^{\mathrm{op}})(1)$ is the natural positive Laplacian on $M_2(\mathcal{A}_\theta)$. □

Theorem 2.8.8 *Let A_θ, ρ and δ be as above. Then $(M_2(\mathcal{A}_\theta), \rho, \delta)$ is a uniform positive Riemannian cycle over the C^*-algebra A_θ.*

Proof Follows from Lemma 2.8.7 with the exception of the condition $\Omega_\delta(\mathcal{A}_\pi) = M_2(\mathcal{A}_\theta)$. This condition follows from [C, Section VI.3 Lemma 12] with the identification $\delta(a^{\mathrm{op}})(1) = \delta_u(a)\sigma_1 + \delta_v(a)\sigma_2$. □

Let $(H_\rho, \pi_\rho, D_\delta)$ be the GNS representation associated to the Riemannian cycle $(M_2(\mathcal{A}_\theta), \rho, \delta)$ over A_θ.

Corollary 2.8.9 *Let $(M_2(\mathcal{A}_\theta), \rho, \delta)$ be the Riemannian cycle over the C^*-algebra A_θ as above. Then $(H_\rho, \pi_\rho, D_\delta)$ is an irreducible Riemannian representation of the C^*-algebra A_θ such that*

(i) $[D_\delta, \pi_\rho(a)] = -i(\pi_\rho(\delta_u(a))\sigma_1 + \pi_\rho(\delta_v(a))\sigma_2)$ *for all* $a \in \mathcal{A}_\theta$,

(ii) $\Omega_{D_\delta}(\pi_\rho(\mathcal{A}_\theta)) = \pi_\rho(M_2(\mathcal{A}_\theta))$,

(iii) $\Lambda_\rho := \iota_\rho(M_2(\mathcal{A}_\theta)) = H_\rho^\infty := \cap_m Dom|D_\delta|^m$.

Proof Let $\iota := \iota_\rho$ and $D := D_\delta$. The result follows from Theorem 2.8.8 and Theorem 2.7.59 as ρ is a trace, with the exceptions of irreducibility and $H_\rho^\infty \subset \Lambda_\rho$. Let p be a central projection in $M_2(A_\theta'')$. Let $\theta \in \mathbb{Q}$. Then $[D,p] = 0$ implies $p \in \mathcal{A}_\theta \subset A_\theta$. However A_θ has no proper central projections. Let $\theta \notin \mathbb{Q}$. Then $M_2(A_\theta'')$ is a factor. Hence there exist no proper central projections. This proves the representation is base irreducible in the sense of Definition 1.5.17.

Let $E = \sum_i a^i\sigma_i \in M_2(\mathcal{A}_\theta)$. Consider $\|\iota(E)\|_D = \|\iota(E)\| + \|D\iota(E)\|$. Notice that (*) $\|\iota(E)\|^2 = \rho(E^*E) = \sum_i \frac{1}{2}\tau((a^i)^*a^i) = \sum_{r,s,i} \frac{1}{2}|a^i_{r,s}|^2$. Moreover (**)

$$\|D(E)\|^2 = \rho(E^*D^2(E)) = -\rho(E^*(d_u^2(E) + d_v^2(E)))$$

by definition of D and the proof of Lemma 2.8.7. Now $d_u^2(E) = \sum_i \delta_u^2(a_i)\sigma_i$ and $d_v^2(E) = \sum_i \delta_u^2(a_i)\sigma_i$. Hence

$$Tr(E^*(d_u^2(E) + d_v^2(E))) = \frac{1}{2}\sum_i \left((a^i)^*\delta_u^2(a_i) + (a^i)^*\delta_v^2(a_i)\right)$$

Then (***)

$$\begin{aligned} -\rho(E^*(d_u^2(E) + d_v^2(E))) &= -\tfrac{1}{2}\textstyle\sum_i \tau((a^i)^*\delta_u^2(a_i) + (a^i)^*\delta_v^2(a_i)) \\ &= -\tfrac{1}{2}\textstyle\sum_i((a^i)^*\delta_u^2(a_i))_{0,0} + ((a^i)^*\delta_v^2(a_i))_{0,0} \\ &= 2\pi\textstyle\sum_{r,s,i}(r^2+s^2)|a_{r,s}|^2 \end{aligned}$$

as $\delta_u(a^i) = -4\pi\sum_{r,s} r^2 a^i_{r,s}u^r v^s$ and $\delta_v(a^i) = -4\pi\sum_{r,s,i} s^2 a^i_{r,s}u^r v^s$. Hence if $E(n)$ is a sequence in $M_2(\mathcal{A}_\theta)$ such that $\lim_{p,n}\|\iota(E(p) - E(n))\|_D = 0$ then $|a(p)^i_{r,s} - a(n)^i_{r,s}| \to 0$ and $(r^2+s^2)^{1/2}|a(p)^i_{r,s} - a(n)^i_{r,s}| \to 0$ by (*) (**) and (***) for all $i = 1,2,3,4$ and $r,s \in \mathbb{Z}^2$. One continues by induction on the graph norms $\|\iota(E)\|_{D^m} = \|\iota(E)\| + \|D^m\iota(E)\|$ to find that that $\|\iota(E(p) - E(n))\|_{D^m} \to 0$ implies $|a(p)^i_{r,s} - a(n)^i_{r,s}| \to 0$ and $(r^2+s^2)^{m/2}|a(p)^i_{r,s} - a(n)^i_{r,s}| \to 0$ for all $i = 1,2,3,4$ and $r,s \in \mathbb{Z}^2$. Hence, if $\iota(E(n))$ is Cauchy in the graph norms $\|.\|_{D^m}$ for all $m \in \mathbb{N}\cup\{0\}$ then $E(n)$ is Cauchy in the seminorms of rapid decay, $(r^2+s^2)^{m/2}|a^i(p)_{r,s} - a(n)^i_{r,s}| \to 0$ for all $r,s \in \mathbb{Z}^2$ and $m \in \mathbb{N}\cup\{0\}$. Hence, by closure of $\mathcal{A}_\theta$ in the seminorms of rapid decay, $a^i(n) \to a^i \in \mathcal{A}_\theta$ for all $n \in \mathbb{N}, i = 1,2,3,4$. Hence $E(n) \to E \in M_2(\mathcal{A}_\theta)$ where $E = \sum_i a^i\sigma_i$. In summary, for any Cauchy sequence $\iota(E(n)) \subset \iota(M_2(\mathcal{A}_\theta))$ in the locally convex topology determined by the graph norms $\|.\|_{D^m}$ for all $m \in \mathbb{N}\cup\{0\}$ there exists $E \in M_2(\mathcal{A}_\theta)$ such that $\iota(E(n)) \to \iota(E)$. Hence H_ρ^∞, which is the closure of the invariant core $M_2(\mathcal{A}_\theta)$ in the locally convex topology of graph norms, is contained in $\iota(M_2(\mathcal{A}_\theta))$. □

2.8.3 The Riemannian Geometry of A_θ.

Let θ be irrational. The following lemmas will enable us to prove the irrational rotation C*-algebra A_θ satisfies the axioms of compact Riemannian geometry. They are basic modifications of the statements in [C3] and establish the GNS representation $(H_\rho, \pi_\rho, D_\delta)$ of the Riemannian cycle $(M_2(\mathcal{A}_\theta), \rho, \delta)$ over A_θ is 2-dimensional.

Lemma 2.8.10 *Let $(H_\rho, \pi_\rho, D_\delta)$ and A_θ be as above and $f(x) = (1+x^2)^{-1}$. Then*

(i) *$f(D_\delta) \in L_{1,\infty}$, and*

(ii) *$Tr_\omega(af(D_\delta)) = \tau(a)$ for all $a \in A_\theta$ and all $\omega \in D_s$.*

Proof Let $A := A_\theta$, $D := D_\delta$ and $\iota_\rho := \iota$. Let $E \in M_n(\mathcal{A})$. Consider $D^2\iota(E) = -\iota((\delta_u\sigma_1 + \delta_v\sigma_2)^2 E) = \iota(-(\delta_u^2 I_2 + \delta_v^2 I_2)E)$. The spectrum of D^2 is hence reduced to the spectrum of $\Delta = -\delta_u^2 - \delta_v^2$. We follow [Vr, 4]. It is easily computed that, for each pair $(r,s) \in \mathbb{Z}$, Δ has discrete eigenvalues $4\pi(r^2+s^2)$ with eigenelements $c_1u^rv^s + c_2u^{-r}v^s + c_3u^rv^{-s} + c_4u^{-r}v^{-s}$ for constants c_1, c_2, c_3, c_4. Note that the zero value has eigenspace spanned by 1. Hence Δ^{-1}, defined as 0 on the finite dimensional kernel of Δ, is compact as it has discrete eigenvalues of multiplicity 4 with limit point zero. Then D^{-2}, defined as zero on the finite dimensional kernel of D^2, has the same discrete eigenvalues with multiplicity 16. Let λ_N be the eigenvalues of D^{-2} listed in decreasing order with multiplicity. Let S_R be the circle of radius R. Let N_R be the number of eigenvalues, listed with multiplicity, such that $r^2 + s^2 \leq R^2$. Then $4R^2 \leq N_R \leq 4(2R)^2$. Hence $2\log(R) + 4\log(2) \leq \log(N_R) \leq 2\log(R) + 8\log(2)$. Then

$$\begin{aligned}\lim_{N\to\infty} \tfrac{1}{\log N} \textstyle\sum^N \lambda_N &= 4\lim_{R\to\infty} \tfrac{1}{2\log R} \textstyle\sum_{r^2+s^2\leq R^2} (4\pi)^{-1}(r^2+s^2)^{-1} \\ &= (2\pi)^{-1} \lim_{R\to\infty} \tfrac{1}{\log R} \textstyle\sum_{r^2+s^2\leq R^2} (r^2+s^2)^{-1} \\ &= (2\pi)^{-1} 2\pi = 1\end{aligned}$$

by the calculation in [Vr, 4]. Hence $D^{-2} \in L_{1,\infty}$ and $Tr_\omega(D^{-2}) = 1$ for all $\omega \in D_s$. As $(1+x^2)^{\ 1} - x^{-2} = (1+x^2)^{-2}x^{-2}$ then $f(D) \in L_{1,\infty}$ by the spectral theorem. Moreover $f(D) - D^{-2} \in L_1$ by the Hölder inequality for $L_{p,\infty}$-spaces [S]. Hence $Tr_\omega(aD^{-2}) = Tr_\omega(af(D))$ for all $a \in \mathcal{A}$ and $\omega \in D_s$. Let $\kappa(a) := Tr_\omega(af(D))$ for all $a \in \mathcal{A}$ and some $\omega \in D_s$. Then κ is a continuous trace on A by [CGS] and Lemma 1.7.14. We now show $\kappa = \tau$. Consider $\kappa(u^rv^s) = \kappa(v^*vu^rv^s) = \kappa(v^*u^rv^{s+1}) = e^{2\pi i\theta r}\kappa(v^*vu^rv^s) = e^{2\pi i\theta r}\kappa(u^rv^s)$. Hence $\kappa(u^rv^s)(1 - e^{2\pi i\theta r}) = 0$. However, $e^{2\pi i\theta r} \neq 1$ for all $r \in \mathbb{Z}\setminus\{0\}$ as θ is irrational. Hence $\kappa(u^rv^s) = 0$ for all $r \in \mathbb{Z}\setminus\{0\}$ and $s \in \mathbb{Z}$. Similarly $\kappa(u^sv^s) = 0$ for all $s \in \mathbb{Z}\setminus\{0\}$ and $r \in \mathbb{Z}$. Let $a \in F_\theta$. Then $\kappa(a) = \kappa(\sum_{(r,s)\in f} a_{r,s}u^rv^s) = \sum_{(r,s)\in f} a_{r,s}\kappa(u^rv^s) = a_{0,0}\kappa(1) = a_{0,0}$. Hence $\kappa = \tau$ restricted to F_θ and $\kappa = \tau$ on A by continuity. □

The other factor to dimension is the orientation. There exists a Hochschild cycle $c \in Z_2(\mathcal{A}_\theta)$ given by [C3]

$$c := \frac{1}{8\pi i}(v^{-1}u^{-1} \otimes u \otimes v - u^{-1}v^{-1} \otimes v \otimes u)$$

that provides a volume form.

Lemma 2.8.11 *Let $c \in Z_2(\mathcal{A}_\theta)$ be as above. Then $\pi_\rho(c) = \sigma_3$.*

Proof Let $D := D_\delta$ and $\pi_\rho := \pi$. Then $\pi(u)^*[D, \pi(u)] = 2\sqrt{\pi} i\sigma_1$ and $\pi(v)^*[D, \pi(v)] = 2\sqrt{\pi} i\sigma_2$. Hence $\pi(v)^*\pi(u)[D, \pi(u)][D, \pi(v)] - \pi(u)^*\pi(v)[D, \pi(v)][D, \pi(u)] = -4\pi\sigma_1\sigma_2 + 4\pi\sigma_2\sigma_1 = 8\pi i\sigma_3$. □

Theorem 2.8.12 *Let A_θ be the irrational rotation algebra as above. Then A_θ satisfies the axioms of compact Riemannian geometry.*

Proof (R1,R2) Follow from Corollary 2.8.3, Corollary 2.8.9 and Lemma 2.8.11. We note that $\Gamma \in U(H_\rho)$ defined densely by $\Gamma E = \pi_\rho(c)\pi_\rho(c)^{\mathrm{op}} E = \sigma_3 E \sigma_3$ for all $E \in M_2(A_\theta)$ is a real grading element. (R3) Immediate as A_θ'' is a factor. (R4) Follows as $\Lambda_\rho = \iota(M_2(\mathcal{A}_\theta))$ is a rank 4 free $\mathcal{A}_\theta$-module such that $\Lambda_\rho = H_\rho^\infty$ by Corollary 2.8.9. (R5) Lemma 2.8.10. (R6) Lemma 2.8.11. (R7) Lemma 2.8.7 and Remark 2.7.58 establish that the index algebra $B = M_2(A_\theta)$. Hence $A_\theta \sim_M B$ by Example 2.1.17. The result follows as the C*-algebra A_θ is Poincaré dual to itself [C3]. □

Corollary 2.8.13 *The information $(A_\theta, H_\rho, \pi_\rho, D_\delta, c)$ constitutes a compact Riemannian spin$_R$ geometry.*

Proof Immediate from Theorem 2.8.12 and Corollary 2.8.9(ii) as $M_2(A_\theta) \sim_M A_\theta$. □

As a final remark we compute the restriction of the metric sheer S_δ for the symmetric $\mathcal{A}_\theta$-derivation δ as defined in Definition 2.7.37. Let $a, b \in \mathcal{A}_\theta$. Then we denote by

$$g_\delta(da, db) := -\frac{1}{2}(\delta(a^{\mathrm{op}})(1)\delta(b^{\mathrm{op}})(1) + \delta(b^{\mathrm{op}})(1)\delta(a^{\mathrm{op}})(1))$$

the 'commutative metric'.

Theorem 2.8.14 *Let $(A_\theta, H_\rho, \pi_\rho, D_\delta, c)$ be the compact Riemannian geometry of the irrational rotation C*-algebra as above. Then*

$$S_\delta(a, b) = -g_\delta(da, db) + C_\delta(a, b)$$

where

$$C_\delta(a, b) = \frac{1}{2}\sigma_0\Big([\delta_u(a), \delta_u(b)] + [\delta_v(a), \delta_v(b)]\Big) + \frac{1}{2} i\sigma_3\Big([\delta_u(a), \delta_v(b)] - [\delta_v(a), \delta_u(b)]\Big).$$

for all $a, b \in \mathcal{A}_\theta$.

Proof We have $\delta(b^{\mathrm{op}})(\delta(a^{\mathrm{op}})(1)) = \sigma_1(\sigma_1\delta_u(a) + \sigma_2\delta_v(a))\delta_u(b) + \sigma_2(\sigma_1\delta_u(a) + \sigma_2\delta_v(a))\delta_v(b)$. The commutation factors arise by commuting $\delta_\alpha(b)$ through $\delta_\beta(a)$ for $\alpha, \beta = u, v$. □

Let $a = u$ and $b = v$. Then $g_\delta(du, dv) = -2\pi i\sigma_3[u, v]$. Hence $g_\delta(du, dv) \neq 0$ for the irrational rotation algebra. We compute the metric sheer. From $\delta_u(u) = 2\sqrt{\pi} iu$, $\delta_v(u) = 0$, $\delta_u(v) = 0$ and $\delta_v(v) = 2\sqrt{\pi} iv$ we obtain $2C_\delta(a, b) = i\sigma_3[\delta_u(a), \delta_v(b)] = i\sigma_3[2\sqrt{\pi} iu, 2\sqrt{\pi} iv] = i(2\sqrt{\pi} i)^2\sigma_3[u, v] = -4\pi i\sigma_3[u, v]$. Let $a, b = u$ or $a, b = v$. Then

$g_\delta(du,du) = -2\pi = g_\delta(dv,dv)$ and $C_\delta(u,u) = 0 = C_\delta(v,v)$. Hence the restriction of the metric sheer provides a bilinear form

$$S_\delta : \mathcal{A}_\theta \times \mathcal{A}_\theta \to \pi_\rho(\Omega^2_\delta(\mathcal{A}_\theta))$$

such that

$$S_\delta(u,v) = 2\pi i\sigma_3[u,v] - 2\pi i\sigma_3[u,v] = 0$$

and

$$S_\delta(u,u) = S_\delta(v,v) = 2\pi.$$

Appendix A

A.1 A Result used for the Fundamental Class

Let (H, π, D) be a base representation of a C*-algebra A. Let A_c be the norm dense ideal of finite supported elements in A of Theorem 1.4.2. Then we defined p-integrability of a base representation in Definition 1.7.7 and Definition 1.9.1 as the condition $\pi(a)(1+D^2)^{-p/2} \in L_{1,\infty}$ for all $a \in A_c$. In this small section we shall prove the following result.

Theorem A.1.1 *Let $p \geq 1$. Let (H, π, D) be a base representation of a C*-algebra A such that $\pi(a)(1+D^2)^{-p/2} \in L_{1,\infty}$ for all $a \in A_c$. Then $\pi(a)(D-\lambda)^{-1} \in K(H)$ for all $a \in A$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

The proof consists of the following propositions and lemmas.

Lemma A.1.2 *Let I be a two-sided *-ideal of $K(H)$. Let $D : DomD \to H$ be selfadjoint and $S \in B(H)$. Then the following statements are equivalent*

(i) $S(D+i)^{-1} \in I$,

(ii) $S(D-\lambda)^{-1} \in I$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof (ii) $\Rightarrow$ (i) is obvious. (i) $\Rightarrow$ (ii) Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By the spectral theorem $(D-\lambda)^{-1}$ is bounded. Moreover $(D-\lambda)^{-1} = (D+i)^{-1} + (i+\lambda)(D+i)^{-1}(D-\lambda)^{-1}$. The result follows from left multiplication by S and the ideal properties of I. □

Proposition A.1.3 *Let $p > 0$. Let $S(1+D^2)^{-p/2} \in K(H)$ where $S \in B(H)$ and $D : DomD \to H$. Then $S(D-\lambda)^{-1} \in K(H)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof Let n be the least integer greater than p. By hypothesis $S(1+D^2)^{-n} = S(1+D^2)^{-p}(1+D^2)^{-(n-p)}$ is compact. Hence $S(D-i)^{-n}(D+i)^{-n} = S(1+D^2)^{-n}$ is compact. Now $(D-i)(D+i)^{-1} = 1 + 2i(D+i)^{-1} \in B(H)$ by the proof of the Lemma A.1.2. Hence $(D-i)^n(D+i)^{-n} = ((D-i)(D+i)^{-1})^n$ is bounded and $S(1+D^2)^{-n}(D-i)^n(D+i)^{-n} = S(D+i)^{-2n}$ is compact. By multiplication by phases $|S||D+i|^{-2n}$ is compact. The square root of a compact operator is compact hence $||S||D+i|^{-n}| = \sqrt{|S||D+i|^{-2n}|S|}$ is compact. Multiplication by phases implies $|S||D+i|^{-n}$ is compact. The square root of a compact operator is compact hence $||S||D+i|^{-n/2}| = \sqrt{|S||D+i|^{-n}|S|}$ is compact. Multiplication by phases implies $|S||D+i|^{-n/2}$ is compact. We continue by induction to the smallest m such that

$2^m > n$. Then $|S||D+i|^{-\frac{n}{2m}}$ is compact. Multiplication on the right by the bounded operator $|D+i|^{-1+\frac{n}{2m}}$ implies $|S||D+i|^{-1}$ is compact. Multiplication by phases provides $S(D+i)^{-1}$ is compact. The result follows from Lemma A.1.2. $\square$

The proof of Theorem A.1.1 now follows from Proposition A.1.3 and norm density of A_c in A.

A.2 Relation of the Fundamental and Signature Classes

A.2.1 Basic Definitions

We recall the Clifford algebra C_1 is defined by

$$C_1 = \{\lambda_1 + h\lambda_2 \,|\, \lambda_1, \lambda_2 \in \mathbb{C}, h^2 = 1\}$$

with a $\mathbb{Z}_2$-grading $\beta : \lambda_1 + h\lambda_2 \to \lambda_1 - h\lambda_2$. A concrete representation of C_1 in $B(\mathbb{C}^2) = M_2(\mathbb{C})$ is provided by

$$\lambda_i \mapsto \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}, \quad h \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \beta \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that β is implemented by the grading element

$$V := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let A be a unital C*-algebra. Define

$$F_c := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Let (H, F) be the pair of a concrete representation (H, π) of A and an operator $F \in B(H)$ such that $F - F^*, F^2 - 1, [F, \pi(a)] \in K(H)$. Then $(H \otimes \mathbb{C}^2, F \otimes F_c, 1 \otimes V)$ defines a Kasparov $A \otimes C_1$-$\mathbb{C}$-bimodule. By default $(H \otimes \mathbb{C}^2, F \otimes F_c, 1 \otimes V)$ defines a Kasparov A-$\mathbb{C}$-bimodule by the inclusion $A \to A \otimes 1 \subset A \otimes C_1$.

A.2.2 The Signature Class

Let (H_ρ, π_ρ, D) be a real oriented n-dimensional Riemannian representation of a unital C*-algebra A with volume form $\pi_\rho(c)$. We recall there exists a K-homology class $\lambda^\rho_{-1} := [(H_\rho, F_D, \Gamma)] \in KK(A, \mathbb{C})$ called the fundamental class of this representation, see Remark 2.5.17. As a result of Definition 2.5.11 a volume form $\pi(c)$ provides a grading element in the sense of Definition 2.4.10 when n is even and a trivial grading when n is odd, see Theorem 2.5.13.

Definition A.2.1 *Let (H_ρ, π_ρ, D) be an oriented n-dimensional Riemannian representation of a C*-algebra A with volume form $\pi_\rho(c)$. Then define the signature class $\sigma^\rho \in KK(A, \mathbb{C})$ by $\sigma^\rho := [(H_\rho, D, \pi_\tau(c))]$ when n is even and $\sigma^\rho := [(H_\rho \otimes \mathbb{C}^2, D \otimes F_c, 1 \otimes V)]$ when n is odd.*

Remark A.2.2 That the signature class σ^ρ is independent of the choice of volume form for this representation is immediate from Proposition 2.5.20(i).

Theorem A.2.3 *Let (H_ρ, π_ρ, D) be a real oriented n-dimensional Riemannian representation of a C^*-algebra A. Then $\sigma^\rho = (1 + (n \bmod 2))\lambda^\rho_{-1}$.*

The proof shall consist of the following two propositions.

Proposition A.2.4 *Let n be odd. Then $\sigma^\rho = 2\lambda^\rho_{-1}$.*

Proof Let $a \in A$. Let $\pi_\rho(c) = W$ and $\pi_\rho = \pi$. Then $[W, \pi(a)] = 0$. Let Γ be a real grading element. Then $\{\Gamma, W\} = 0$. Hence (H_ρ, W, Γ) with representation π forms a degenerate Kasparov A-$\mathbb{C}$-bimodule.

Let $h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the concrete representation of C_1 into $M_2(\mathbb{C})$. From degeneracy $[(H_\rho \otimes \mathbb{C}^2, W \otimes h, \Gamma \otimes 1)]$ is the identity of $KK(A, \mathbb{C})$. Hence

$$[(H_\rho \otimes \mathbb{C}^2, F_D \otimes F_c, 1 \otimes V)] = [(H_\rho \otimes \mathbb{C}^2, F_D \otimes F_c, 1 \otimes V)] \oplus [(H_\rho \otimes \mathbb{C}^2, W \otimes h, \Gamma \otimes 1)].$$

It is more convenient to write the RHS in matrix form as $[(H_\rho \otimes \mathbb{C}^4, \mathcal{F}, U)]$ where

$$\mathcal{F} = \begin{bmatrix} 0 & iF_D & 0 & 0 \\ -iF_D & 0 & 0 & 0 \\ 0 & 0 & 0 & W \\ 0 & 0 & W & 0 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & \Gamma \end{bmatrix}$$

with representation ρ of $a \in A$ given by $\operatorname{diag}(\pi(a))$. Let $S_t = iF_D \cos t + W \sin t$ and $\hat{S}_t = W \cos t + iF_D \sin t$. Now consider the norm continuous map $[0, \frac{\pi}{2}] \to B(H_\tau \otimes \mathbb{C}^4)$ given by

$$\mathcal{F}_t = \begin{bmatrix} 0 & S_t & 0 & 0 \\ S_t^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_t \\ 0 & 0 & \hat{S}_t^* & 0 \end{bmatrix}.$$

Clearly we have $\mathcal{F}_0 = \mathcal{F}$ with $\mathcal{F}_t - \mathcal{F}_t^*, [\mathcal{F}_t, \rho(a)] \in K(H_\rho \otimes \mathbb{C}^4)$ and $\{\mathcal{F}_t, U\} = 0$. One checks that $\mathcal{F}_t^2 - 1 \in K(H_\tau \otimes \mathbb{C}^4)$ using $S_t S_t^* = F_D^2 \cos^2 t + \sin^2 t$ as $[W, F_D] = 0$. Similarly for $\hat{S}_t$. The result is we have an operator homotopy from $(H_\rho \otimes \mathbb{C}^4, \mathcal{F}, U)$ to $(H_\rho \otimes \mathbb{C}^2, W \otimes h, 1 \otimes V) \oplus (H_\rho \otimes \mathbb{C}^2, F_D \otimes F_c, \Gamma \otimes 1)$ where the first direct summand is degenerate. Then

$$[(H_\rho \otimes \mathbb{C}^2, F_D \otimes F_c, 1 \otimes V)] = [(H_\rho \otimes \mathbb{C}^2, F_D \otimes F_c, \Gamma \otimes 1)] = [(H_\rho, F_D, \Gamma)] \oplus [(H_\rho, F_D, \Gamma)]$$

by a trivial homotopy of F_c to 1 since the grading $\Gamma \otimes 1$ is independent of F_c. □

Proposition A.2.5 *Let n be even. Then $\sigma^\rho = \lambda^\rho_{-1}$.*

Proof Let $\pi(c) = W$. Then $[\Gamma, W] = 0$ and $\{\Gamma, D\} = 0 = \{W, D\}$. Hence define the Kasparov A-$\mathbb{C}$-bimodule (H_t, F_t, U_t) with representation $\pi_t = \pi_\rho$ by

$$H_t = H_\rho \ , \ F_t = F_D \ , \ U_t = \cos(t)\Gamma + \sin(t)W$$

for $t \in [0, \frac{\pi}{2}]$. This provides an operator homotopy between (H_ρ, F_D, Γ) and (H_τ, F_D, W).
□

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