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Riemannian Non-commutative Geometry

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Abstract

The elements of non-commutative geometry are presented from an operator algebraic viewpoint. Threaded through the presentation is the example of a spectral triple associated to a second countable metrisable locally compact oriented manifold without boundary and without the assumption of spin structure.

Generalisation of the spectral triple associated to such a manifold admits the new notion of a Riemannian representation of a C^* -algebra which directly links to the standard theory of von Neumann algebras. The involvement of the standard theory and the reformulation of the axioms of non-commutative geometry in the absence of spin structure are investigated and presented.

The construction of Riemannian representations of C^* -algebras is also considered. A new generalisation of a symmetric derivation on a von Neumann algebra R provides the means of constructing Riemannian representations of a C^* -subalgebra $A \subset R$ associated to a faithful finite trace on R . The interaction between the standard theory and the generalised symmetric derivation provides new analysis into the structure of K -cycles.

Contents

| | | |
|----------|-------------------------------------------------------------------------------|----------|
| 1 | Elements of Non-commutative Geometry | 2 |
| 1.1 | Review of Calculus and Differential Geometry | 2 |
| 1.1.1 | Basic Calculus | 2 |
| 1.1.2 | Multivariable Calculus | 4 |
| 1.1.3 | Differentiable Manifolds | 7 |
| 1.2 | Non-Commutative Topological Spaces | 12 |
| 1.2.1 | The GNS Construction | 12 |
| 1.2.2 | Topological Spaces associated to a C*-algebra | 13 |
| 1.2.3 | Commutative C*-algebras | 18 |
| 1.2.4 | Non-Commutative Topological Spaces | 20 |
| 1.3 | Exterior Derivation on Differentiable Manifolds | 21 |
| 1.3.1 | Exterior and Clifford Algebras | 23 |
| 1.3.2 | Vector Bundles | 24 |
| 1.3.3 | Exterior and Clifford Bundles | 26 |
| 1.3.4 | Covariant Derivatives and Exterior Differentiation | 27 |
| 1.3.5 | The Laplacian and the Signature operator | 30 |
| 1.3.6 | Summary of Riemannian Structure | 33 |
| 1.4 | Exterior Derivation on C*-algebras | 34 |
| 1.4.1 | Non-Commutative Differential Forms | 34 |
| 1.4.2 | Smooth Non-Commutative Differential Forms | 37 |
| 1.5 | Non-Commutative Measure Theory | 42 |
| 1.5.1 | Measure Theory on \mathbb{R} | 43 |
| 1.5.2 | Von Neumann Algebras and Weights | 44 |
| 1.5.3 | Remark - Structure of C_c^∞ -representations | 49 |
| 1.6 | Modular Theory and the Radon-Nikodym Theorem | 52 |
| 1.6.1 | Modular Theory | 53 |
| 1.6.2 | Generalised Radon-Nikodym Theorems | 54 |
| 1.6.3 | Modular Theory for von Neumann algebras with separable pre- dual | 55 |
| 1.7 | Non-Commutative Integral Calculus | 57 |
| 1.7.1 | Symmetric Norm Ideals | 57 |
| 1.7.2 | Symmetric Functionals | 58 |
| 1.7.3 | Symmetric Measures | 59 |
| 1.7.4 | Connes' Non-commutative Integral | 61 |
| 1.8 | The metric on pure states | 68 |
| 1.9 | Summary of Non-Commutative Calculus | 69 |

| | | |
|----------|---------------------------------------------------|------------|
| 2 | Riemannian Non-Commutative Geometry | 72 |
| 2.1 | Hilbert Modules | 73 |
| 2.1.1 | Definition of Hilbert Modules | 74 |
| 2.1.2 | \mathbb{Z}_2 -graded C^* -algebras | 78 |
| 2.1.3 | Tensor Products of Hilbert modules | 79 |
| 2.1.4 | Morita Equivalence | 81 |
| 2.2 | Non-commutative Vector Bundles | 82 |
| 2.3 | Graded Hilbert Modules in Riemannian Geometry | 82 |
| 2.3.1 | Structure and Gradings on the Clifford Bundle | 83 |
| 2.3.2 | Graded Representations | 84 |
| 2.3.3 | Riemannian Structure | 88 |
| 2.3.4 | Riemannian Representations | 90 |
| 2.3.5 | $\text{Spin}_{\mathbb{C}}$ Representations | 91 |
| 2.4 | Poincaré Duality in KK-theory | 92 |
| 2.4.1 | The elements of KK-theory | 92 |
| 2.4.2 | Poincaré Duality in KK-theory | 93 |
| 2.4.3 | KK-equivalence | 95 |
| 2.4.4 | Fundamental Class of a Riemannian Representation | 98 |
| 2.5 | Non-commutative Volume Form | 101 |
| 2.5.1 | Non-commutative De-Rham Complexes | 101 |
| 2.5.2 | Hochschild and Cyclic Homology | 103 |
| 2.5.3 | Volume Form | 107 |
| 2.5.4 | Riemannian Orientations and Gradings | 109 |
| 2.6 | Connes' Axioms of Non-commutative Geometry | 110 |
| 2.6.1 | Structure of Riemannian Representations | 111 |
| 2.6.2 | The Axioms of Riemannian Geometry | 113 |
| 2.6.3 | Reconstruction Theorem | 115 |
| 2.7 | Symmetric Derivations and Riemannian Cycles | 117 |
| 2.7.1 | Symmetric Derivations | 118 |
| 2.7.2 | Symmetric A -derivations | 121 |
| 2.7.3 | Abstract K-cycles | 125 |
| 2.7.4 | Riemannian cycles | 135 |
| 2.8 | Example - Riemannian Geometry of the Torus | 139 |
| 2.8.1 | The rotation algebra A_{θ} | 139 |
| 2.8.2 | A Riemannian Cycle on A_{θ} | 142 |
| 2.8.3 | The Riemannian Geometry of A_{θ} . | 145 |
| A | | 148 |
| A.1 | A Result used for the Fundamental Class | 148 |
| A.2 | Relation of the Fundamental and Signature Classes | 149 |
| A.2.1 | Basic Definitions | 149 |
| A.2.2 | The Signature Class | 149 |