Some Results in the area of Generalized Convexity

and

Fixed Point Theory of Multi-valued Mappings

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SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text.

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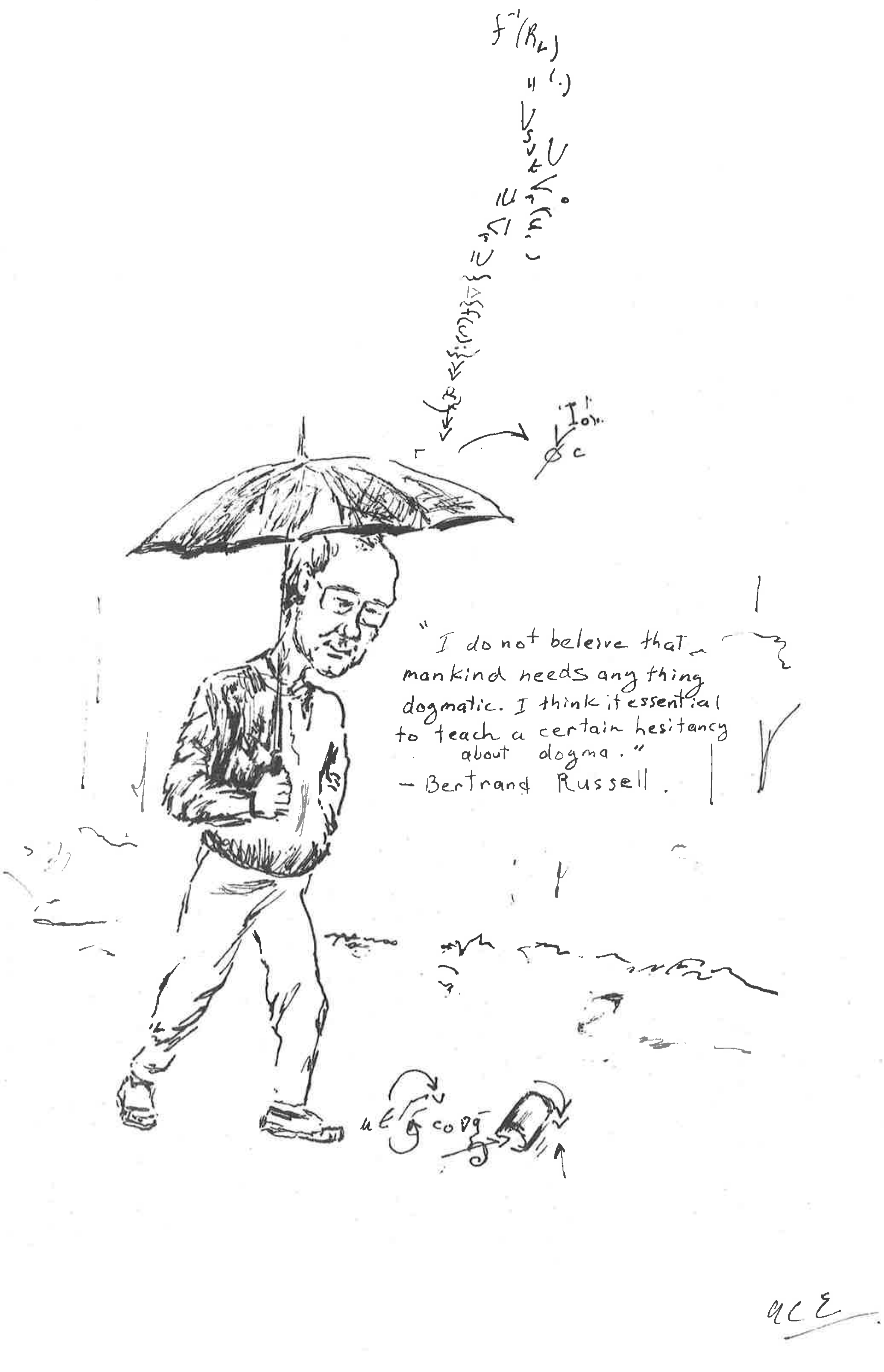
"I do not beleive that mankind needs anything dogmatic. I think it essential to teach a certain hesitancy about dogma."
– Bertrand Russell.

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INTRODUCTION

Since Kakutani it has been observed that certain multi-valued mappings admit fixed points. Convexity of image sets of these mappings has played an essential role in the development of such theorems. Continuity assumptions are also necessary. Unlike the topological properties, the role of convexity seems less obvious.

No totally geometric proof of Kakutani's theorem has been given. One notes that even in going from R to R^2 one loses the property that all continuous multi-valued mappings admit fixed points. This contrasts dramatically with single valued mappings. One needs to restrict the shape of the image set, or how it "changes", to provide an affirmative answer to the fixed point problem.

One wonders how the convexity assumptions may be altered and still allow the existence of fixed points. As a first step towards shedding light on this question, this thesis attempts to "decouple" the two concepts.

This approach proves to be rich in possibilities as it allows, within the context of first reflexive spaces and then R^n, to draw together a great variety of literature on seemingly unrelated topics, under a common theme. This includes literature on non-linear optimization, generalized Lagrangians, generalized derivatives, generalized convexity, continuous lattice theory and fuzzy topologies.

Chapter One is intended as an overview of basic definitions and theorems. It is in large intended for reference and the informed reader will probably find it more appropriate to begin with Chapter Two. It contains an account of various topological properties of multi-valued mappings and an account of basic continuous lattice theory. Within this context the lattice theoretic concept of "Scott continuity" provides an alternative charactization of the concept of inner semi-continuity for open set valued multi-functions. This approach, to the knowledge of the author, is probably new. Attempts at extending the usual concepts of lower and upper semi-continuity of single valued mappings using the lattice structure of 2^u occurred early in the development of multi-function theory. It was noted that these attempts could not, in general, be interpreted as continuity with respect to some topology on 2^u. Continuous lattice theory facilitates a similar approach devoid of this flaw.

Chapter Two develops various convexity concepts emphasizing the lattice nature of convexity. Its relevance to selection problems and the continuity of multi-valued mappings is explored. This culminates in the proof of a selection theorem for multi-valued mappings along the lines of the classical result demonstrating the existence of a continuous selection "separating" any two functions $f < g$ upper semi-continuous and lower seni-continuous respectively. Since any weakly compact convex set, in a reflexive Banach space, can be obtained by taking intersections of closed balls, the concept of strong convexity seems the most appropriate vehicle to obtain such a result.

Arrigo Cellina generalized the Kakutani theorem by approximating, in graph, upper semi-continuous convex set valued mappings with lower semi-continuous multi-valued mappings. In this thesis we address the question of whether one can approximate, in graph, upper semi-continuous multi-valued mappings with continuous multi-valued mappings. In Chapter Three we pursue this line of reasoning. The lattice theoretic nature of the approximation problem is further explored in an attempt to elucidate the nature of possible "convexity" generating subclasses. The lattice theoretic nature of the continuity properties of multi-valued mappings becomes more evident.

Using a continuous multi-valued mapping one is able to marry this approach much more strongly with the theory of non-linear optimization. The resultant continuity properties of the associated marginal and multi-valued mapping facilitates this approach. In Chapter Four we consider the role of constraint qualifications in this approach. Lipschitzness being equivalent to a generalized form of "differentiability" is of particular interest. Conditions are derived under which the solution set of a non-linear optimization problem, treated as a multi-function, is Lipschitz continuous. When this mapping is single valued, that is the constraint set is "selective", then this is equivalent to the existence of the Clarke derivative and its extensions. Lipschitzness of the marginal function implies the validity of the use of an augmented Lagrangian to solve such a problem. This is exploited to derive conditions under which such a marginal function has a gradient.

In Chapter Five the properties of continuous lattices are used to find equivalent characterizations of various classes of functions. This results in the proof that the lower semi-continuous strictly quasi-convex functions are "lower dense" in the class of lower semi-continuous quasi-convex functions (or in the terminology of Chapter Three, generates this class). That is, every quasi-convex function g is in the closure of the set $\{h \text{ strictly quasi-convex } h \leqslant g\}$.

When the convexity requirements of the image sets of multi-functions is weakened from a supremum complete lattice of sets to a topology, the resultant class of Scott continuous functions form a fuzzy topology. We relate theproperty of perfect normality of fuzzy topologies to the selection problem of Chapter Two. Perfect normality implies the "upper denseness" of continuous, open set valued, multi-valued mappings in the class of upper semi-continuous, closed set valued, multi-valued mappings. This shows an intimate relationship between topological properties and the ability to approximate with continuous multi-valued mappings. This, of course, does not imply the existence of a fixed point, except for when the image sets lie in R^1. One is not assured that the approximating continuous function admits a fixed point. To deduce the existence of a fixed point one needs to impose some sort of more stringent convexity concept to allow selectivity of the image sets. The convexity assumptions are not removed but their role redefined in this context.

In general, this thesis is concerned with conceptual and to a lesser extent, methodological concerns. It represents a preliminary exploration of these questions. If a complete theory was developed, it most probably would be cast in terms of continuous lattice theory. This would provide an overall structure in which such results could be placed in context. Proofs are given for all original results and appropriate references are given for all results present in current literature.

In particular, the following results are, to the author's knowledge, new:

Lemmas : 2.1, 2.4, 2.5, 2.6, 2.9, 2.10, 4.2;

Propositions : 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.16, 1.17, 2.2, 2.8, 2.9, 2.10, 3.2, 4.1, 4.11, 4.15, 5.1, 5.2, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10;

Theorems : 2.3, 2.6, 2.9, 3.9, 4.5, 4.6, 4.7, 4.10, 4.15, 4.18;

Corollaries : 2.2, 2.7, 2.91, 2.92, 3.2, 3.5, 3.9, 4.4, 4.5, 4.9, 4.14, 4.17, 5.2.

CHAPTER I

The extensive and varied nature of the literature relating to continuity concept of multi-valued mapping necessitates, I feel, some sort of summary, to familiarise the unaquainted. This chapter attempts to draw together that part of the literature related to the following chapters. In order to keep this chapter relatively self-contained as an overview, more detail than is probably necessary, has been presented. Various lower and upper semi-continuity concepts are defined and related to each other where possible. The topologies on 2^u which induce these concepts, are stated and the situations under which they become equivalent are noted.

The lattice structure of 2^u is insufficient in itself, to extend the usual concepts of lower and upper semi-continuity, of ordinary real functions, by the use of simple limsups and liminfs. Attempts early on were made in this direction, but it has been noted that the resulting concepts could not, in general, be related to some topology on 2^u. Continuous lattice theory appears to shed some light on this approach. A general introduction to concepts such as "way below" and "Scott continuity" is given. The relationship between these and the preceeding concepts is explored. We conclude by using "rate of continuity" to relate certain uniform semi-continuities and their local counterparts.

§1.1 Discussion of Semi-Continuity of Single and Multi-valued Mappings

In the following we take U_i ; i=1,2,..., to be topological spaces having topologies τ_i ; i=1,2,.. . If U_i is a metric we will denote its metric by $d_i(\cdot,\cdot) : U_i \times U_i \to R$. Lower or upper semi-continuity will be abbreviated to l.s.c. and u.s.c. respectively. We adopt wherever convenient, the usual abbreviations; "iff" for if and only if, "nbhd" for neighbourhood, "Top" for topology, "s.t." for such that, "m.v." for multi-valued and "w.r.t." for with respect to.

Definition 1.1 : A mapping $f : U \to R$ is called l.s.c. at $u \in U$ iff $\forall\ \varepsilon > 0\ \exists$ a neighbourhood of u, N say, s.t.

$$f(u) - \varepsilon \leqslant f(u');\ \forall\ u' \in N$$

and u.s.c. iff

$$f(u') \leqslant f(u) + \varepsilon;\ \forall\ u' \in N.$$

There have been numerous different approaches to extending these concepts to the class of mapping $\Gamma : U_1 \to 2^{U_2}$. A full account of such approaches can be found in references [1] (p.109-121), [2] (p.160-182), [3] and [6]. I will give here a quick survey of definitions and relationships.

If U_1 and U_2 are sets, a mapping Γ of U_1 to subsets of U_2 can be represented uniquely by its graph $G(\Gamma) = \{(u_1,u_2) : u_2 \in \Gamma(u_1)\}$. Conversely, any subset P of $U_1 \times U_2$ defines a multi-function $\Gamma\, u_1 = \{u_2 : (u_1,u_2) \in P\}$.

One can immediately see here the connection multi-functions have with relations.

If we define $\Gamma^{-1}u_2 = \{u_1 : u_2 \in \Gamma u_1\}$ then the usual convention is that when $B \subset U_1$ we have

$$\Gamma B = \bigcup_{u_1 \in B} \Gamma u_1 \text{ and for } A \subseteq U_2 \text{ we take } \Gamma^{-1}A = \bigcup_{u_2 \in A} \Gamma^{-1}u_2 = \{u_1 : \Gamma u_1 \cap A \neq \phi\}.$$

This is called the pre-image of A. The exponential pre-image* is defined by $\Gamma_{exp}^{-1}A = \{u_1 : \Gamma u_1 \subset A\}$ and we immediately have $\Gamma_{exp}^{-1}A = (\Gamma^{-1}A^c)^c$.

* using the notation of K. Kuratowski reference [2].

Definition 1.2 : Let U_1, U_2 be two top spaces. Then $\Gamma : U_1 \to 2^{u_2}$ is called upper (lower) semi-continuous if for each open (resp. closed) $A \subseteq U_2$ the set $\Gamma_{exp}^{-1}A$ is open (resp. closed) in U_1's topology τ_1.

Equivalently we have Γ is upper (lower) semi-continuous if for each closed (open) $A \subseteq U_2$ the set $\Gamma^{-1}A$ is closed (open in U_1).

Definition 1.3 : U_1, U_2 top spaces. Then $\Gamma : U_1 \to 2^{u_2}$ is said to be u.s.c. at u_1^0 if $u_1^0 \in \Gamma_{exp}^{-1}A \Rightarrow u_1^0 \in \mathrm{int}(\Gamma_{exp}^{-1}A)$ whenever A is open.

Similarly, Γ is lower semi-continuous at u_1^0 if

$u_1^0 \in \overline{\Gamma_{exp}^{-1}(A)} \Rightarrow u_1^0 \in \Gamma_{exp}^{-1}(A)$ whenever A is closed.

We note that Γ is upper (lower) semi-continuous iff Γ is upper (lower) semi-continuous at each $u_1 \in U_1$ (ref. [2],I, page 173).

Definition 1.4 : $\Gamma : U_1 \to 2^{u_2}$ is continuous at $u_1^0 \in U_1$ iff it is both upper and lower semi-continuous at u_1^0.

Consequently Γ is continuous iff it is both upper and lower semi-continuous.

Definition 1.5 : $\Gamma : U_1 \to 2^{u_2}$ is locally u.s.c. at (u_1^0, u_2^0) if for each neighbourhood N of u_2^0 there is a neighbourhood $M \subset N$ s.t. $\overline{M} \cap \Gamma$ is u.s.c. at u_1^0.

Definition 1.6 : A multi-valued mapping Γ is called δ-u.s.c. at (u_1^0, u_1^0) if $\exists$ a nbhd M of u_2^0 s.t.

$$\tilde{\Gamma} u_1 = \begin{cases} \overline{M} \cap \Gamma u_1^0 : & u_1 \neq u_1^0 \\ \Gamma u_1^0 & \text{otherwise} \end{cases}$$

is u.s.c. at u_1^0.

We note in passing that if U_2 is regular (T_3), then the u.s.c. of Γ implies local upper semi-continuity which becomes equivalent to δ- upper semi-continuity (Kuratowski [2],I, page 180).

Theorem 1.1 : Γ a m.v. fn from U_1 to U_2 we denote $\overline{\Gamma} u_1 = \overline{(\Gamma u_1)}$.

$\overline{\Gamma}$ is u.s.c. for each u.s.c. mult.fn Γ into the subsets of U_2 iff U_2 is normal (T_4).

Proof : Reference [3], p.8. □

Theorem 1.2 : If U_2 is regular and Γ is closed valued (i.e. $\overline{\Gamma} u_1 = \Gamma u_1$) and u.s.c., then the graph $G(\Gamma)$ is closed.

Proof : Reference [2], I, page 175. □

Theorem 1.3 : Let U_2 be T_2 and let $P \subseteq U_1 \times U_2$ be closed. Then $\Gamma u_1 = \{u_2 : (u_1,u_2) \in P\}$ satisfies the following for each compact $K \subseteq U_2$

$$u_1^0 \in \overline{\Gamma^{-1}K} \Rightarrow u_1^0 \in \Gamma^{-1}K.$$

We note that Γ is u.s.c. at u_1^0 iff Γ^{-1} is a closed mapping at u_1^0 (ie. for each closed set $K \subseteq U_2$ $u_1^0 \in \overline{\Gamma^{-1}K} \Rightarrow u_1^0 \in \Gamma^{-1}K$).

It follows that for U_2 compact and U_1 being T_2, the m.v. mapping Γ is u.s.c. closed valued iff $G(\Gamma)$ is closed (also ref. [2], II,p.57).

Definition 1.7 : Suppose that $\Gamma : U_1 \to 2^{u_2}$, $\mathbf{B}(u_2^0)$ is a basis of u_2^0 and U_1 and U_2 are topological spaces. Γ said to be l.s.c. at (u_1^0,u_2^0) if for each element B of $\mathbf{B}(u_2^0)$ there exists a nbhd W of u_1^0 s.t. $\Gamma^{-1}B \supseteq W$.

This definition doesn't depend on the basis used. We have Γ l.s.c. at $\forall(u_1^0,u_2^0) \in G(\Gamma)$ iff $\Gamma^{-1}A$ is open for open A (i.e. Γ is l.s.c. according to our previous definition 1.2). The local character of the definition 1.3 is expressed by the fact that Γ is l.s.c. at u_1^0 (viz. definition 1.3) iff it is l.s.c. at (u_1^0,u_2^0) for each $u_2^0 \in \Gamma u_1^0$.

Definition 1.8 : A multi-valued mapping Γ is said to be inner semi-continuous (i.s.c.) at u_1^0 if for each closed set $F \subseteq \Gamma u_1^0$, there is a neighbourhood (nbhd) W of u_1^0 such that for each $u_1 \in W$ we have $F \subseteq \Gamma u_1$.

Of course this is only an auxiliary notion as Γ is i.s.c. iff the complementary multifunction Γ^c is u.s.c. at a particular point u_1^0. On the other hand if the space U_2 is T_1-space i.s.c. entails l.s.c.

Theorem 1.4 : If U_2 is regular. Then if Γ_1 is l.s.c. at u_1^0 and Γ_2 is u.s.c. at u_1^0, the mapping

$$\Gamma = \overline{\Gamma_1 \setminus \Gamma_2} = \overline{\Gamma_1 \cap \Gamma_2^c}$$

is l.s.c. at u_1^0.

Proof : Reference [2], page 182. □

In the following (U_2,d_2) will denote a metric space, $N(u_2^0,\varepsilon)$ the ε neighbourhood $\{u_2 : d_2(u_2^0,u_2) < \varepsilon\}$ and for $A \subseteq U_2$

$$N(A,\varepsilon) = \bigcup_{u_2 \in A} N(u_2,\varepsilon); \; d(u_2,A) \equiv \inf\{r : N(u_2,r) \cap A \neq \phi\}.$$

Definition 1.9 : A multi-function $\Gamma : U_1 \to 2^{U_2}$ is called upper Hausdorff semi-continuous (u.H.s.c.) at u_1^0 if $\forall\, \varepsilon > 0, \exists$ a nbhd W of u_1^0 such that

$$\Gamma W \subseteq N(\Gamma u_1^0,\varepsilon)$$

Γ is called lower Hausdorff semi-continuous (l.H.s.c) at u_1^0 if

$\forall\ \varepsilon > 0\ \ \exists$ a nbhd W of u_1^0 s.t.

$$W \subseteq \{u_1 : \Gamma u_1^0 \subseteq N(\Gamma u_1,\varepsilon)\}.$$

We note that Γ is l.H.s.c. at u_1^0 iff it is l.s.c. at (u_1^0,u_2^0) uniformly for each $u_2^0 \in \Gamma u_1^0$ (in the sense of definition 1.3).

Now if U_1 and U_2 are metric spaces the definition of lower semi-continuity (at (u_1,u_2)) of a multi-function $\Gamma : U_1 \to 2^{u_2}$ may be restated as follows: for $\varepsilon > 0$ there is a number $q(\varepsilon) > 0$ such that

$$\Gamma^{-1}N(u_2,\varepsilon) \supseteq N(u_1,q(\varepsilon)).$$

Similar definitions can be made for u.H.s.c. at u_1^0. If Γ is u.H.s.c. at u_1^0 and for each $\varepsilon \in (0,\varepsilon_0)$

$$\Gamma N(u_1^0,q(\varepsilon)) \subseteq N(\Gamma u_1^0,\varepsilon)$$

then q is the rate of u.H.s.c. at u_1^0.

Definition 1.10 : Γ is said to be l.s.c. uniformly at (u_1^0,u_2^0) if there are $\varepsilon > 0$, $\eta > 0$ and a function $q : (0,r_0) \to R_+$ such that for each $u_2 \in N(u_2^0,\varepsilon)$ and each $u_1 \in \Gamma^{-1}u_2 \cap N(u_1^0,\eta)$ we have

$$\Gamma^{-1}N(u_2,r) \supseteq N(u_1,q(r)).$$

Definition 1.11 : Γ is δ-u.H.s.c. uniformly at (u_1^0,u_2^0) if there are $\varepsilon > 0$, $\eta > 0$ and a function q such that for $u_1 \in N(u_1^0,\eta)$

$$\phi \neq \Gamma N(u_1,q(r)) \cap \overline{N(u_2^0,\varepsilon)} \subseteq N(\Gamma u_1,r).$$

Theorem 1.5 : Γ is l.s.c. uniformly at (u_1^0,u_2^0) iff Γ is δ-u.H.s.c. uniformly at (u_1^0,u_2^0). Besides the rates semi-continuity are the same on an interval $(0,r)$.

Proof : Reference [3], page 13. □

In the following section we will develop a consequence of this theorem.

§1.2 Relationship Between Various Semi-Continuity Concepts

In this section we will explore the situations in which various semi-continuity concepts become equivalent and relate this to the topologies one can create on 2^{U_2} to extend the concepts.

Theorem 1.6 : Let U_1 be a metrizable space and let U_2 be a topological space with a countable local basis $\mathbf{B}(u_1^0)$ at u_1^0. If Γ is u.s.c. at u_1^0, then Γ is u.H.s.c. (for each metric of U_2) at u_1^0.

It may be easily deduced the converse is true provided Γu_1^0 is closed. If one does not assumed closed image sets then one loses this simple correspondence between u.s.c. and u.H.s.c. even on very reasonable spaces.

Suppose that Γ is closed valued and not u.s.c. at u_1^0. This signifies the existence of an open set Q ($Q \supseteq \Gamma u_1^0$) such that $\Gamma W \cap Q^c$ is not empty for all neighbourhoods W at u_1^0. By the Urysohn Theorem, there is a continuous function d valued in [0,1] that vanishes on Γu_1^0 and is equal to 1 outside Q. Pick any metric ρ on U_2. Then $\rho(u_2,\bar{u}_2) + |d(u_2) - d(\bar{u}_2)|$ is an equivalent metric for which $N(\Gamma u_1^0,1) \subseteq Q$. This contradicts the u.H.s.c. of Γ for all metrics on U_2. See reference [6] for characterization theorems of u.H.s.c. for Γ which doesn't have closed values. We will quote the following characterization theorem.

Theorem 1.7 : Let U_2 be complete metric and let Γ be a closed-value u.H.s.c. (at u_1^0) multi-function. The following statements are equivalent:

(i) Γ is u.s.c. at u_1^0;

(ii) for each closed $K \subset U_2$; $K \cap \Gamma$ is u.H.s.c. at u_1^0;

(iii) for each open Q; $\overline{Q} \cap \Gamma$ is u.H.s.c. at u_1^0.

Hence the equivalence of u.s.c. and u.H.s.c. can be related to satisfactory local behaviour of u.s.c. multi-functions (see definition 1.6 and 1.7).

The discrepancies between Hausdorff semi-continuity and the previously defined concepts is seen to arise from the topologies need to be defined on the space 2^{u_2} (or suitable subspaces) to generate the continuity concepts. We will denote $P(U_2) \equiv 2^{u_2}$ when convenient and

$$C(U_2) = \{S \in 2^{u_2} | S \text{ is compact w.r.t. } \tau_2\}$$
$$K(U_2) = \{S \in 2^{u_2} | S \text{ is closed w.r.t. } \tau_2\}$$
$$O(U_2) = \{S \in 2^{u_2} | S \text{ is open w.r.t. } \tau_2\}$$
$$V(U_2) = \{S \in 2^{u_2} | S \text{ is convex}\}.$$

When necessary we will denote

$$KV(U_2) = K(U_2) \cap V(U_2)$$

the convex closed subsets of U_2 etc.

Definition 1.12 : The upper (lower) semi-finite topology on 2^{u_2} is generated by taking as a basis (resp. sub-basis) for the open collections in 2^{u_2} all collections of the form $\{E \in 2^{u_2} | E \subset S\}$ (resp. $\{E \in 2^{u_2} | E \cap S \neq \phi\}$) with S an open subset of U_2.

A multi-function Γ is lower (upper) semi-continuous in the sense of definitions 1.2 and 1.3 iff Γ is lower (upper) semi-continuous with the lower (upper) finite topology on 2^{u_2} (see reference [7]).

One can define a finer topology on 2^{u_2} by forming the join (or sup), in the lattice of all topologies on 2^{u_2}, of the upper and lower semi-finite topologies. The topology is known as the finite topology. A mapping continuous with respect to this topology is both upper and lower semi-continuous and hence continuous.

Definition 1.13 : Suppose U_2 is a uniform space. Then the upper (lower) semi-uniform structure on 2^{u_2} is generated by the index set

A (of the uniform structure on U_2) and the neighbourhoods

$$\overline{N}(E,\alpha) = \{F \mid F \subseteq V_\alpha(E)\} \text{ (resp. } \underline{N}(E,\alpha) = \{F \mid E \subseteq V_\alpha(F)\})$$

for $E \in 2^{u_2}$, where $V_\alpha(\cdot)$ refers to the uniform structure on U_2.

In the case of metric spaces, $V_\alpha(E)$ can be taken to be simply $N(E,\alpha)$.

The corresponding topologies are called the upper (lower) semi-uniform topologies. The upper (lower) semi-uniform topologies are coarser (finer) than the upper (lower) semi-finite topologies. In the case of a metric we can define upper (lower) Hausdorff semi-continuity with respect to the corresponding semi-uniform topologies. Hausdorff continuity can be defined with respect to the topology produced by the uniform structure formed by the join, in the lattice of all uniform structures on 2^{u_2}, of the semi-uniform structures. From reference [7] we quote :

Theorem 1.8 : If U_2 is a uniform space then the upper (lower) semi-uniform structures on 2^{u_2} coincide with the upper (lower) semi-finite topologies on the subspace $C(U_2)$ of 2^{u_2}.

Hence multi functions with compact image sets are very well behaved as all definitions of semi-continuity coincide.

Theorem 1.9 : If U_2 is normal, and if we induce a uniform structure on U_2 by the Stone-Čech compactification, then the corresponding uniform structure on 2^{u_2} agrees with the finite topology.

The other major problem is that the semi-uniform structure generated by a metric on U_2 generates a topology on 2^{u_2} which depends on the metric used. Fortunately, their restrictions to the family of non-empty compact subsets of U_2 is independent of the metric used,hence depending only on the topology τ_2 of U_2.

In the case of the uniform structure on the subspace $K(U_2)$ of 2^{u_2}, for a metric space U_2, one can generate the uniform structure with a metric σ. If the metric d_2 on U_2 is bounded, then σ can be taken as the ordinary Hausdorff metric on $K(U_2)$, defined by

$$\sigma(A,B) = \inf\{\varepsilon > 0 | A \subseteq N(B,\varepsilon),\ B \subseteq N(A,\varepsilon)\}.$$

If d_2 is not bounded, one can replace σ by a uniformly equivalent bounded metric and then use the new metric to generate a Hausdorff metric.

We have from various sources the following:

<u>Theorem 1.10</u> : If U_2 and U_1 are topological spaces and Γ_1, Γ_2 multi-valued mappings from U_1 to U_2 s.t. $\overline{\Gamma}_1 u_1 = \overline{\Gamma}_2 u_1$; $\forall u_1 \in U_1$, then we have

(i) Γ_1 is l.s.c. iff Γ_2 is l.s.c.

If we now suppose U_2 is metric, then

(ii) Γ_1 is l.H.s.c. iff Γ_2 is l.H.s.c.

(iii) Γ_1 is u.H.s.c. iff Γ_2 is u.H.s.c.

(iv) Γ_1 is H-continuous iff Γ_2 is H-continuous.

<u>Proof</u> :

(i) See reference [8], page 366. - Proposition 2.3.

(ii) The proposition is equivalent to; Γ is l.H.s.c. iff $\overline{\Gamma}$ is l.H.s.c..

We shall prove this instead.

Suppose $\overline{\Gamma}$ is a l.H.s.c. multi-valued mapping. Then for each $u_1^0 \in U_1$ all $\varepsilon > 0$ there exists a nbhd W of u_1^0 s.t. $\overline{\Gamma}u_1^0 \subseteq N(\overline{\Gamma}u_1,\varepsilon)$; $\forall u_1 \in W$, which implies

$$\Gamma u_1^0 \subseteq \overline{\Gamma}u_1^0 \subseteq N(\overline{\Gamma}u_1,\varepsilon) = N(\Gamma u_1,\varepsilon):\ \forall u_1 \in W$$

that is, Γ is l.H.s.c. at u_1^0.

Now suppose Γ is l.H.s.c. at $u_1^0 \in U_1$ and $\overline{\Gamma}$ is not l.H.s.c. at u_1^0. Then $\exists\ \varepsilon > 0$ s.t. $\forall$ nbhd W' of u_1^0 (say)

(a) $\overline{\Gamma}u_1^0 \backslash N(\overline{\Gamma}u_1,\varepsilon) \neq \phi$ for some $u_1 \in W'$. We take

(b) $W' \subseteq \{u_1 | \Gamma u_1^0 \subseteq N(\Gamma u_1, \varepsilon/2)\}$ a nbhd of u_1^0 which exists by virtue of the l.H.s.c. of Γ at u_1^0.

As U_2 is metric (a) implies $\exists\, u_2^n \to u_2$, $u_2 \notin N(\overline{\Gamma}u_1,\varepsilon) = N(\Gamma u_1,\varepsilon)$; $u_1 \in W'$, $u_2^n \in \Gamma u_1^0$.

If we choose n sufficiently large so that $d_2(u_2^n,u_2) < \varepsilon/2$ we find $u_2^n \notin N(\Gamma u_1, \varepsilon/2)$, and $u_2^n \in \Gamma u_1^0$, which contradicts (b).

(iii) Once again we may prove the equivalent statement that Γ is u.H.s.c. iff $\overline{\Gamma}$ is u.H.s.c. This is done in a similar manner to (ii).

(iv) See reference [9], Lemma 2.5, page 378. □

In (iii) we note that in the implication, $\overline{\Gamma}$ u.H.s.c. $\Rightarrow$ Γ u.H.s.c., we use the fact that if Q is a neighbourhood of Γu_1^0 then Q is a neighbourhood of $\overline{\Gamma}u_1^0$ in the corresponding upper semi-uniform structure, i.e.

$$\Gamma u_1^0 \subseteq N(\Gamma u_1^0,\varepsilon) \equiv N(\overline{\Gamma}u_1^0,\varepsilon)$$

and hence $\overline{\Gamma}u_1^0 \subseteq N(\Gamma u_1^0,\varepsilon)$.

This is not the case in the finite topologies on 2^{u_2}. We really would like to say that if $\Gamma u_1^0 \subseteq Q$ then $\overline{\Gamma}u_1^0 \subseteq Q$. In other words we would like Γu_1^0 to be inside Q in the sense that its boundary points avoid the boundary of Q.

In reference [6], S. Dolecki and S. Rolewicz have already noted the importance of the behaviour of certain boundary points of a multi-function in creating conditions for equivalence of u.H.s.c. and u.s.c.

We will not pursue this line of thought but return to the lattices of sets 2^{U_2} and its subclasses. It was noted in reference [7] that other approaches towards extending the concepts of lower and upper semi-continuity of ordinary real function to functions taking images in 2^{U_2}, were attempted, very early on, in terms of the lim sups and lim infs of sets in U_2 (that is using the lattice structure of 2^{U_2}). It was also noted that the results of these attempts could not, in general, be interpreted as continuity with respect to some topology on 2^{U_2}. Recently this approach has been revised and a new and rich are of mathematics has been created with the invention of continuous lattice theory. This has only occurred over the last twenty years and provides another method of extending continuity ideas to 2^{U_2} and its sub-lattices.

As it is well-known, one can rewrite the definition 1.1 to state that $f : U_1 \to R^*$ is lower semi-continuous iff $f^{-1}(\hat{\uparrow}c)$ is open for every $c \in R$ where $\hat{\uparrow}c = \{a \in R^* | a > c\}$ (ie. $\hat{\uparrow}c = (c,+\infty]$). As a consequence f is upper semi-continuous if -f is lower semi-continuous. To extend this type of definition to the lattice of subsets, we have to first define what we mean by $A \subseteq B$ but $A \neq B$, that is,define a "strictly less than" concept.

We could say that $A \ll B$ if we have $\overline{A} \subseteq B$. In other words, A avoids the boundary points of B even via limits. In this case of a compact Hausdorff space, this is a well-known and useful relation (even though for cl-open sets it is reflexive and doesn't imply $A \neq B$). If, on the other hand, the space is only locally compact, the relation is not as strong as it looks.

In order to say A is "well inside" we could require that $\overline{A} \subseteq B$ and $\overline{A}$ is compact. This means A avoids the boundary of B even in the compactification of the space. This relation, moreover, has a purely lattice theoretic definition since we can define it in $O(U_2)$ as meaning that every open cover of B has a finite sub-collection covering A (at least this works in the locally compact spaces). We can extend this relation from $O(U_2)$ to 2^{u_2} by saying A << B if there exists C, D ∈ $O(U_2)$ such that $A \subseteq C << D \subseteq B$.

Another way of defining a "way below" relation on a linear locally compact normed spaces (see Lemma 2.3) is to say that A << B if $N(A,\varepsilon) \subseteq B$ for some $\varepsilon > 0$. In this case if we let $C = N(A, \varepsilon/3)$ and $D = N(A, 2\varepsilon/3)$ then $A \subseteq C << D \subseteq B$. If A is relatively compact then so is C for ε sufficiently small and both relations coincide.

This relation has a purely lattice theoretic definition and we shall explore this definition and a few consequences before indicating its relevance to our discussion of lower (upper) semi-continuity.

Definition 1.14 : Let L be a complete lattice. We may say x is "way below y", in symbols x << y, iff for directed subsets $D \subseteq L$ (ie. every finite subset of D has an upper bound in D) the relation $y \leqslant \sup D$ always implies the existence of a d ∈ D with $x \leqslant d$. An element satisfying x << x is said to be "isolated from below" or compact.

Proposition 1.1 : In a complete lattice L one has the following statements holding true for all u, x, y, z ∈ L. We rotate V ≡ sup and Λ ≡ inf in L.

(i) x << y implies $x \leqslant y$

(ii) $u \leqslant x << y \leqslant z$ implies u << z (hence our extension of << from $O(U_2)$ to 2^{u_2} is consistent with the fact that $O(U_2)$ is a sub-lattice of 2^{u_2}).

(iii) x << z and y << z together imply x v y << z.

(iv) o << x (o the "smallest" element of L).

For a discussion of these implications and more see reference [10].

We will write

$$\Downarrow x = \{u \in L : u \ll x\} \text{ and } \Uparrow x = \{u \in L : u \gg x\}$$

in analogy to $\downarrow x = \{u \in L : u \leqslant x\}$. Combining all the above statements into one we get, for x in a complete lattice, that the set $\Downarrow x$ is an ideal contained in $\downarrow x$ which depends monotonically on x (ie. $x \leqslant y$ iff $\Downarrow x \subseteq \Downarrow y$). From reference [10] we have:

<u>Proposition 1.2</u> : Let U_2 be a topological space and let $L = \mathcal{O}(U_2)$.

(i) If $A, B \in L$ and if there is a quasicompact set $Q \subseteq U_2$ (ie. has the Heine-Borel property) with $A \subseteq Q \subseteq B$ then $A \ll B$.

(ii) Suppose U_2 is locally quasicompact (ie. every point in U_2 has a basis of quasicompact neighbourhoods). Then $A \ll B$ in L implies that there exists a quasicompact Q s.t. $A \subseteq Q \subseteq B$.

So in a Hausdorff space the relation $A \subseteq Q \subseteq B$ for a quasicompact Q is equivalent to $\overline{A} \subseteq B$ and $\overline{A}$ is compact. Once again from reference [10] we have the following.

<u>Definition 1.15:</u> A lattice L is called continuous if L is complete and satisfies the axiom of approximation,

$$x = \sup \{u \in L : u \ll x\}$$
$$= \bigvee\{u \in L : u \ll x\} = \bigvee \Downarrow x$$

for all $x \in L$.

<u>Proposition 1.3</u> : In a continuous lattice the way below relation satisfies the strong interpolation property, namely, for all $x, z \in L$

$$x \ll z \text{ and } x \neq z \text{ implies } \exists y$$
$$\text{s.t. } x \ll y \ll z \quad x \neq y.$$

See reference [10], chapters I and II for the following.

<u>Proposition 1.4</u> : In a continuous lattice the following conditions are equivalent,

(i) $x \ll y$

(ii) for each directed set D of L the relation $y \leqslant \bigvee D$ implies the existence of $d \in D$ with $x \ll d$.

<u>Example 1.1</u> : Let $LSC(U) \equiv LSC(U,R^*)$ denote the complete lattice of all lower semi-continuous functions on a topological space U with values in the extended real numbers R^*. For any function $f : U \to R^*$ we set $Gf = \{(u,r) : r < f(u)\}$. Then f is lower semi-continuous iff Gf is open in $U \times R^*$. We use the notion of $x \ll y$ in R^*, a continuous lattice itself, to mean $x < y$ or $x = -\infty$.

<u>Proposition 1.5</u> : Suppose U is compact space. Then the functions $f,g \in LSC(U)$ satisfy (i)-(v) equivalently.

(i) $f \ll g$ in $LSC(U)$.

(ii) There is an open cover $\{S_j : j \in J\}$ of U and a family $\{r_j : j \in J\}$ in R^* where $f(u) \leqslant r_j \ll g(u)$ for all $j \in J$ and $u \in S_j$.

(iii) For each element of $u \in U$ there is an open set S in U and an element $y \in R^*$ where $f(\bar{u}) \leqslant y \ll g(u)$ for all $\bar{u} \in S$.

(iv) $Gf \subseteq Gg$ in $U \times R^*$.

(v) There is a continuous function $h \in C(U,R^*)$ where for all $u \in U$ we have $f(u) \leqslant h(u) \ll g(u)$.

We note in passing that (v) implies that any $g \in LSC(U)$ can be approximated from below by continuous functions.

It has been known for many years that a l.s.c. function on a regular space can be written as a supremum of continuous functions. This sort of approximation problem will arise under the topic of continuous selection and generalized convexity.

Corollary 1.5 : If U is a compact space, then LSC(U) is a continuous lattice.

Definition 1.16 : A subset S of a complete lattice L is called Scott open (ie. $S \in \sigma(L)$) iff it satisfies the conditions

(i) $S = \uparrow S$ and

(ii) $\sup D \in S$ implies $D \cap S \neq \phi$ for all directed sets $D \subseteq L$.

We note that "directed" may be replaced by "ideals" in (ii).

Of course the complement of a Scott open set is Scott closed which is equivalent to being a lower set (ie. $S = \downarrow S$) closed under directed sups. Interestingly enough $\downarrow x \equiv \overline{\{x\}}$ (closure with respect to the Scott topology $\sigma(L)$ on L) for all $x \in L$.

Proposition 1.6 : Let L be a continuous lattice. Then

(i) each point $x \in L$ has a $\sigma(L)$ neighbourhood basis consisting of sets $\hat{\uparrow}u$ with $u \ll x$;

(ii) with respect to the Scott topology we have int $\uparrow x = \hat{\uparrow}x$;

(iii) with respect to $\sigma(L)$, we have for any subset $S \subseteq L$ int $S = \cup\{\hat{\uparrow}u : \hat{\uparrow}u \subseteq S\}$.

We note that a function $f : U_1 \to R^*$ from a topological space into the extended set of real numbers is lower semi-continuous iff it is continuous with respect to the Scott topology on R^*.

Definition 1.17 : For f taking a complete lattice U into a complete lattice T, the following are equivalent to Scott continuity of $f : U \to T$:

(i) f is continuous with respect to the Scott topology, that is $f^{-1}(S) \in \sigma(U)$ for all $S \in \sigma(T)$;

(ii) $f(\vee D) = \vee f(D)$, for all directed sets D of U ;

(iii) If we define $\underline{\lim}_j\ x_j = \sup_j\ \inf_{i \geq j}\ x_j$ we have $f(\underline{\lim}\ x_j) \leq \underline{\lim}\ f(x_j)$ for any net x_j in U.

If U and T are continuous lattices, then each of the above is equivalent to

(iv) $f(x) = \bigvee\{f(w): w \ll x\}$;

(v) $y \ll f(x)$ iff for some $w \ll x$ one has $y \ll f(w)$.

We note that Scott continuous functions are always monotone (not necessarily vice-versa). In the following we will use the notation:

$\Sigma(L) = (L,\sigma(L))$, an associated topological space, where L is a complete lattice. For U a T_0-space we can define a partial ordering for u, $\bar{u} \in U$ by letting

$u \leqslant \bar{u}$ iff $u \in S$ implies $\bar{u} \in S$ for all open sets S.

This is called the specialization order and we may associate with U the poset $(U,\leqslant) \equiv \Omega\, U$. As we have seen for a complete lattice $\Omega\, \Sigma\, L \equiv L$.

Definition 1.18 : For two T_0-spaces U_1 and U_2 let $[U_1,U_2]$ denote the poset defined on $TOP(U_1,U_2)$ (the continuous functions from U_1 to U_2) by the pointwise order induced by $\Omega\, U_2$.

Clearly $[\Sigma S, \Sigma T] \equiv [S \to T]$ is the complete lattice of Scott-continuous functions from S to T equipped with pointwise ordering induced by the order T.

Theorem 1.11 : Let U be a space and L a complete non-singleton lattice. Then the following are equivalent:

(i) $[U, \Sigma L]$ is a continuous lattice.

(ii) Both $O(U)$ and L are continuous lattices.

We will make use of the following canonical pair of mutually inverse bijections given by the formulae

$$\psi(f)(x)(y) = f(x,y)$$

$$\phi(g)(x,y) = g(x)(y)$$

where

$$(L^{Y})^{X} \underset{\psi}{\overset{\phi}{\rightleftarrows}} L^{X\times Y}.$$

Proposition 1.7 : Let U_2 be T_0 then the following statements are equivalent:

(i) $O(U_2)$ is a continuous lattice.

(ii) For all continuous $f : U_1 \to \Sigma\, O(U_2)$ the graph $Gf = \{(u_1,u_2) : u_2 \in f(u_1)\}$ is open in $U_1 \times U_2$.

(iii) For all spaces U and all continuous lattices L, the canonical pair ϕ,ψ induced by restriction order isomorphisms

$$[U_1,\Sigma[U_2,\Sigma L]] \rightleftarrows [U_1 \times U_2,\ \Sigma L]$$

As one can see, the Scott continuous functions from U_1 to $O(U_2)$ are associated with functions with open graphs. Upper semi-continuous functions are related to multi-valued mappings with closed graphs. We know that in a regular space every closed valued u.s.c. multi-function has a closed graph. Since the complementary multi-function has an open graph, we have when U_2 is regular and $O(U_2)$ a continuous lattice:

$$\{\Gamma^c : \Gamma \text{ is u.s.c. and } \overline{\Gamma} = \Gamma\} = \{\Gamma : \Gamma \text{ is i.s.c: open}\}$$
$$\subseteq [U_1,\ \Sigma\, \theta(U_2)].$$

For the case of U_2 a compact Hausdorff space we already know that $O(U_2)$ forms a continuous lattice. We also know that the class of compact valued u.s.c. multi-functions are exactly those with closed graphs in this case.

Proposition 1.8 : If U_2 is a compact Hausdorff space, then $[U_1,\Sigma\, O(U_2)]$ is equivalent to the class of open set valued i.s.c. multi-functions.

Proof : $\Gamma : U_1 \to O(U_2)$ is open valued i.s.c.

iff $\quad \Gamma^c : U_1 \to K(U_2)$ is closed valued u.s.c.

iff $\quad \{u_1 : \Gamma^c u_1 \subseteq A\}$ is open if A is open and Γu_1 open.

iff $\quad \{u_1 : \Gamma u_1 \supseteq A^c\}$ is open; A and Γu_1 open.

Now, as U_2 is Hausdorff compact we have

$$\Gamma u_1 \supseteq A^c \text{ iff } \Gamma u_1 \in \hat{\uparrow} A^c .$$

This follows from the fact that Γu_1 is open A^c is closed and hence there exists an open set C s.t. $\Gamma u_1 \supseteq C \supseteq A^c$ with $\Gamma u_1 \supseteq \bar{C}$; as U_2 is compact so is $\bar{C}$ hence $\Gamma u_1 >> A^c$. So

$$\Gamma : U_1 \to O(U_2) \text{ is open valued and is i.s.c.}$$

iff $\{u_1 : \Gamma u_1 >> B\}$ is open for B closed. We will complete the proof by showing

$$\{u_1 : \Gamma u_1 >> B\} \text{ is open for } B \text{ closed}$$

iff $\{u_1 : \Gamma u_1 >> C\}$ is open for C open.

Suppose is open. Then $C = \bigcap_{i \in I} B_i$ where the B_i are closed. Hence, if we have $\{u_1 : \Gamma u_1 >> B_i\}$ open for $i \in I$, then

$$\bigcup_{i \in I} \{u_1 : \Gamma u_1 >> B_i\} = \{u_1 : \Gamma u_1 >> \bigcap_{i \in I} B_i\}$$

$$= \{u_1 : \Gamma u_1 \in \hat{\uparrow} C\} \text{ is open for open } C.$$

Now suppose $\{u_1 : \Gamma u_1 >> C\}$ is open for open C and B is arbitrary closed. We have

$$\{u_1 : \Gamma u_1 \in \hat{\uparrow} B\} \equiv \{u_1 : \Gamma u_1 \in \hat{\uparrow} C : \text{for some open set } C \supseteq B\}$$

$$\equiv \bigcup_{\substack{C \geqslant B \\ C \text{ open}}} \{u_1 : \Gamma u_1 \in \hat{\uparrow} C\} \text{ which is open.} \qquad \square$$

We note that the compact Hausdorff property was used to show

$\Gamma u_1 \subseteq A^c$ iff $\Gamma u_1 \in \hat{\uparrow} A^c$. In a metric space $\Gamma u_1 \supseteq A^c$ iff $\Gamma u_1 \supseteq N(A^c,\varepsilon)$ for some $\varepsilon > 0$. One wonders whether the way below relation defined by $A \gg B$ iff $A \supseteq N(B,\varepsilon)$ for some $\varepsilon > 0$, which coincides with our previous definition on compact spaces, might be a tool for elucidating the differences between H-u.s.c. and u.s.c. in general. We will not pursue this line of thought here but finish off this section with the union and intersection properties of semi-continuous multi-functions.

Theorem 1.12 :

(i) The union of two u.s.c. mappings $\Gamma_1 \cup \Gamma_2$; $\Gamma_1, \Gamma_2 : U_1 \to K(U_2)$ is u.s.c.

(ii) The union of two l.s.c. functions at u_1^0 is l.s.c. at u_1^0. More generally, if each Γ_t $t \in T$ (T arbitrary) is l.s.c. at u_1^0 so is $\overline{U_t \Gamma_t}$.

(iii) If U_2 is normal $\Gamma_1, \Gamma_2 : U_1 \to K(U_2)$ u.s.c. at u_1^0 then $\Gamma_1 \cap \Gamma_2$ is u.s.c. at u_1^0. If U_2 is compact and if Γ_t is u.s.c. $t \in T$ (arbitrary) then $\bigcap_{t \in T} \Gamma_t$ is u.s.c.

(Proofs may be found in reference [2].)

It is noted in reference [1] that the intersection property for l.s.c. multi-functions does not hold. We may however deduce:

Corollary 1.12 :

The intersection of two i.s.c. functions at u_1^0 is i.s.c. at u_1^0.

Proof : Suppose Γ_1, Γ_2 are i.s.c. at u_1^0. Then Γ_1^c and Γ_2^c are u.s.c. at u_1^0 and $\Gamma^c = \Gamma_1^c \cup \Gamma_2^c$ is u.s.c. at u_1^0. Hence $\Gamma = (\Gamma_1^c \cup \Gamma_2^c)^c = \Gamma_1 \cap \Gamma_2$ is i.s.c. at u_1^0. □

We note of course that $\overline{\Gamma} = \overline{\Gamma_1 \cap \Gamma_2}$ is l.s.c. at u_1^0.

Proposition 1.9 : If U_2 is a locally quasicompact space, then the intersection of two Scott continuous functions is Scott continuous.

Proof : Γ_i is Scott continuous iff $\{u_1 : \Gamma_i u_1 \in \hat{\uparrow}A\}$ is open for open A. Hence we deduce that

$$\bigcap_{i=1,2} \{u_1 : \Gamma_i u_1 \in \hat{\uparrow}A\} = \{u_1 : \bigcap_{i=1,2} \Gamma_i u_1 \in \hat{\uparrow}A\}$$

is an open set by noting that

$$\Gamma_1 u_1 >> A \text{ iff } \exists\, Q_1 \text{ quasicompact s.t. } \Gamma u_1 \supseteq Q_1 \supseteq A$$

and similarly $\Gamma_2 u_1 >> A$ iff $\exists\, Q_2$ quasicompact with $\Gamma_2 u_1 \supseteq Q_2 \supseteq A$. Hence $(\Gamma_1 \cap \Gamma_2)u_1 \supseteq Q_1 \cap Q_2 \supseteq A$ where $Q \cap Q_2$ is quasicompact. □

Proposition 1.10 : Suppose $O(U_2)$ is a continuous lattice and $\Gamma_i \in [U_1, \Sigma\, O(U_2)], i \in I$ arbitrary then

$$\bigcup_{i \in I} \Gamma_i \in [U_1, \Sigma\, O(U_2)].$$

Proof: In a continuous lattice the graph G_i of Γ_i is an open set in $U_1 \times U_2$.

$$\begin{aligned} \text{Now } G_i^c &= \{(u_1,u_2) \in U_1 \times U_2 : u_2 \notin \Gamma_i u_1\} \\ &= \{(u_1,u_2) \in U_1 \times U_2 : u_2 \in \Gamma_i^c u_1\} \end{aligned}$$

is closed set in $U_1 \times U_2$

and hence Γ_i^c is a closed mapping. By reference [1] page 111 we have

$$\Gamma^c = \bigcap_{i \in I} \Gamma_i^c$$

is closed mapping. Hence

$$\Gamma = (\bigcap_{i \in I} \Gamma_i^c)^c = \bigcup_{i \in I} \Gamma_i$$

is open and therefore Scott continuous. □

Interestingly if we combine propositions 1.10 and 1.8 we can deduce the second statement of Theorem 1.12 (iii). One can see how u.s.c. mappings fit into this picture but how do l.s.c. mappings?

Proposition 1.11 : Suppose $\Gamma : {}_1 \to 2^{u_2}$ is l.H.s.c. at U_1^0. Then $\forall\, \varepsilon > 0$

$$N(\Gamma(u_1),\varepsilon) \text{ is i.s.c. at } u_1^0.$$

Proof : Let F be a closed set in U_2 and let $F \subseteq N(\Gamma u_1^0,\varepsilon)$. As Γ is closed $\exists\, \delta > 0$ s.t.

$$N(F,\delta) \subseteq N(\Gamma u_1^0,\varepsilon),$$

in other words

$$F \subseteq N(\Gamma u_1^0,\ \varepsilon - \delta/2),\ \text{for } \delta \text{ sufficiently small.}$$

By the l.H.s.c. of Γ at u_1^0 we have $\forall\, \bar{\varepsilon} > 0 \quad \exists\, \bar{\delta}(\bar{\varepsilon}) > 0$ s.t.

$$\Gamma(u_1^0) \subseteq N(\Gamma u_1,\bar{\varepsilon}),\ \forall\, u_1 \in N(u_1^0,\bar{\delta}(\bar{\varepsilon})).$$

If we choose $\bar{\varepsilon} = \delta/2 > 0$, then $\exists\, \bar{\delta} = \bar{\delta}(\delta/2) > 0$ s.t.

$$\begin{aligned} F &\subseteq N(\Gamma u_1^0,\ \varepsilon - \delta/2) \\ &\subseteq N(N(\Gamma u_1,\ \delta/2),\ \varepsilon - \delta/2) \\ &\equiv N(\Gamma u_1,\varepsilon) \end{aligned}$$

$$\forall\, u_1 \in N(u_1^0,\ \bar{\delta}),$$

which is the definition of i.s.c. □

As we have seen, if Γu_1 has compact image sets then we may replace l-H.s.c. by l.s.c.

It has been noted by many authors that one does not know in general whether the intersection of two l.s.c. multi-valued mappings is l.s.c. We can however, using the above, approximate l.s.c. multi-functions with Scott continuous multi-functions. This class is closed under intersection.

We have observed that when the image space is compact, a continuous multi-function $\Gamma : U_1 \to C(U_2)$, can be considered as a single valued mapping taking images in the metric space $C(U_2)$ (as long as U_2 is metric). If U_1 is compact this implies that Γ is uniformly continuous and hence Γ is both uniformly u.H.s.c. and l.H.s.c. We complete this section by noting some converse statements.

Proposition 1.12 : Suppose Γ is u.H.s.c. uniformly for $u_1 \in N(u_1^0,\eta)$ for some $\eta > 0$ and Diam $\Gamma(u_1^0) < \infty$. Then Γ is δ-u.H.s.c. uniformly at (u_1^0,u_2^0), $\forall\, u_2^0 \in \Gamma(u_1^0)$.

Proof : For $\forall\, r \in (0,r_0)$, $\exists\, q(r) > 0$ s.t. $\forall\, u_1 \in N(u_1^0,\eta)$ we have

$$\Gamma(N(u_1,q(r))) \subseteq N(\Gamma(u_1^0),r) .$$

Let

$$u_2^0 \in \Gamma(u_1^0) .$$

Then

$$\Gamma(N(u_1,q(r))) \cap \overline{N(u_2^0,\varepsilon)} \subseteq N(\Gamma(u_1^0),r) .$$

All we need to show to satisfy the definition 1.11 is that $\exists\, \varepsilon > 0$ s.t.

$$\Gamma(N(u_1,q(r))) \cap \overline{N(u_2^0,\varepsilon)} \neq \phi .$$

We choose ε sufficiently large to do this.

Since $u_1 \in N(u_1^0,\eta)$ we have

$$\Gamma(u_1) \subseteq N(\Gamma(u_1^0), r_1)$$

for some $r_1 > 0$. If not then simply let our η get smaller as to ensure this is so as Γ is u.H.s.c. at u_1^0. Thus

$$\Gamma(N(u_1, q(r))) \subseteq N(\Gamma(u_1), r)$$
$$\subseteq N(\Gamma(u_1^0), r_1 + r)$$

so we let $\varepsilon = r_0 + r_1 + \text{Diam}\,\Gamma(u_1^0)$, for $r \in (0, r_0)$. □

Proposition 1.13 : Let us suppose Γ is u.H.s.c. at u and has compact image sets. Then

Γ is δ-u.H.s.c. uniformly at $(u_1^0, u_2^0), \forall\, u_2^0 \in \Gamma(u_1^0)$

iff Γ is u.H.s.c. uniformly for $u_1 \in N(u_1^0, \eta)$ for some $\eta > 0$.

Proof : We need only prove necessity in view of the previous proposition.

Now $\forall\, u_2^0 \in \Gamma(u_1^0), \exists\, \varepsilon > 0,\ \eta > 0\ \ q : (0, r_0) \to R_+$ s.t. $\forall\, u_1 \in N(u_1^0, \eta)$ we have

(a) $\Gamma(N(u_1, q(r)) \cap N(u_2^0, \varepsilon) \subseteq N(\Gamma(u_1), r)$.

Γ has compact image sets and $\{N(u_2, \varepsilon(u_2)) \mid u_2 \in \Gamma(u_1^0)\}$ is a cover of $\Gamma(u_1^0), \forall\, \varepsilon(u_2) > 0$. We let $\varepsilon(u_2)$ be an $\varepsilon > 0$ which satisfies (a) at $u_2 \in \Gamma(u_1^0)$. Then $\exists$ a finite sub cover

$\{u_2^i : i=1,..,N\}$, $W = \bigcup_{i=1}^{N} N(u_2^i, \varepsilon_i) \supseteq \Gamma(u_1^0)$; $\varepsilon_i \equiv \varepsilon(u_2^i)$.

Now we let $\delta > 0$ be s.t.

$W \supseteq \Gamma(u_1)$; $\forall\, u_1 \in N(u_1^0, \delta)$.

This exists as W is a neighbourhood of $\Gamma(u_1^0)$ and Γ is u.H.s.c. at u_1^0. Let

$\eta = \min\{\delta, \eta_i : i=1,..,N\}$

$q(r) = \min\{q_i(r) : i=1,..,N\} : (0, r_0) \to R_+$

where $r_0 = \min\{r_0^i : i=1,..,N\} > 0$ and

η_i, q_i, r_0^i satisfy (a) for $u_2^i \in \Gamma(u_1^0)$. Then

$\Gamma(N(u_1,q(r)) \cap \overline{N(u_2^0,\varepsilon_i)}$

$\subseteq \Gamma(N(u_1,q_i(r))) \cap \overline{N(u_2^0,\varepsilon_i)}$

$\subseteq N(\Gamma(u_1),r), \forall\, u_1 \in N(u_1^0,\delta) \subseteq N(u_1^0,\eta_i)$; $i = 1, .., N$

(we may choose δ as small as we like). Further,

$\Gamma(N(u_1),q(r))) = \Gamma(N(u_1,q(r))) \cap W$

$= \bigcup_{i=1}^{N} [\Gamma(N(u_1,q(r))) \cap \overline{N(u_2^i,\varepsilon_i)}$

$\subseteq N(\Gamma(u_1),r), \forall\, u_1 \in N(u_1^0,\delta)$. □

<u>Proposition 1.14</u> : Suppose Γ is δ-u.H.s.c. uniformly at (u_1^0,u_2^0) $\forall\, u_2^0 \in \Gamma(u_1^0)$ and $\Gamma(\cdot)$ has compact image sets. Then Γ is u.H.s.c. at u_1^0.

<u>Proof</u> : Let $\varepsilon(u_2)$: $u_2 \in \Gamma(u_1^0)$ be an ε which satisfies the definition of δ-u.H.s.c. - (a).

We construct a cover of $\Gamma(u_1^0)$ $\{N(u_2^i,\varepsilon_i) : i=1,..,N\}$ as in the previous proposition and define η, q, r_0 as previously. Then as $u_1^0 \in N(u_1^0,\eta)$, we have

$\Gamma(N(u_1^0,q(r))) \cap N(u_2^i,\varepsilon_i)$

$\subseteq \Gamma(N(u_1^0,q_i(r))) \cap \overline{N(u_2^i,\varepsilon_i)}$

$\subseteq N(\Gamma(u_1^0),r)$.

Hence

$\Gamma(N(u_1^0,q(r))) \subseteq \Gamma(N(u_1^0,q_i(r))) \cap W$

$= \bigcup_{i=1}^{N} \Gamma(N(u_1^0,q_i(r)) \cap N(u_2^i,\varepsilon_i)$

$\subseteq N(\Gamma(u_1^0,r))$. □

<u>Proposition 1.15</u> : Suppose Γ has compact image sets. Then Γ is δ-u.H.s.c. uniformly at $(u_1^0,u_2^0), \forall\, u_2^0 \in \Gamma(u_1^0)$

iff Γ is u.H.s.c. uniformly for $u_1 \in N(u_1^0,\eta)$ for some $\eta > 0$.

Proof : This follows from the last two propositions noting that we used only the u.H.s.c. of Γ in the necessity of proposition 1.13. □

Proposition 1.16 : If Γ has compact image sets, then Γ is u.H.s.c. uniformly for $u_1 \in N(u_1^0,\eta)$ for some $\eta > 0$ iff Γ is l.s.c. uniformly at $(u_1^0,u_2^0), \forall\, u_2^0 \in \Gamma(u_1^0)$.

Proof : This follows from Theorem 1.15 and Proposition 1.15. □

Proposition 1.17 : Suppose Γ is uniformly u.H.s.c. and Γ has compact image sets and $\Gamma : U_1 \to C(U_2)$; U_1 compact. Then Γ is l.s.c. uniformly at (u_1^0,u_2^0) $\forall\, u_2^0 \in \Gamma(u_1^0)$; $\forall\, u_1^0 \in U_1$ iff Γ is l.H.s.c. at $u_1^0 \in U_1$ uniformly with respect to u_1^0.

Proof : Sufficiency : $\exists\, q(r) : (0,r_0) \to R_+$ s.t.

$$\Gamma^{-1}(N(u_2,r)) \supseteq N(u_1,q(r))$$

$$\forall\, u_1 \in U_1;\ \forall\, u_2 \in \Gamma(u_1) \text{ or } \forall\, u_1 \in \Gamma^{-1}(u_2).$$

Any ε, η will do to satisfy the definition of l.s.c. uniformly at $u_2 \in \Gamma(u_1), \forall\, u_1 \in u_1$.

Necessity : Let $\{N(u_2,\varepsilon(u_2)) : u_2 \in \Gamma(u_1)\}$ be a cover of $\Gamma(u_1)$ where $\varepsilon(u_2)$ satisfies the definition of l.s.c. uniformly at (u_1,u_2). There exists a finite subcover $\{u_2^i;\ i=1,..,N\}$.

Let $\bar{\eta} = \min\,\{\eta_i : i=1,\ldots,N\}$,

$$\bar{q}(r) = \min\,\{q_i(r) : i=1,..,N\},$$

$$\bar{r}_0 = \min\,\{r_0^i : i=1,\ldots,N\}.$$

Then $\forall\, u_2 \in N(\Gamma(u_1),\varepsilon) \subseteq \bigcup_{i=1}^{N} N(u_2^i,\varepsilon_i)$ (for some $\varepsilon > 0$ where $\varepsilon_i = \varepsilon(u_2^i)$)

and $\forall\, \bar{u}_1 \in \Gamma^{-1}(u_2) \cap N(u_1,\eta)$

$$0 < \eta < \bar{\eta} \qquad \text{(a)}$$

we have

$$\Gamma^{-1}(N(u_2,r)) \supseteq N(\bar{u}_1,\bar{q}(r)),$$

$$0 < r < \bar{r}_0.$$

Now U_1 is compact and $\{N(u_1,\eta) : u_1 \in U_1\}$ is a cover of U_1, $\forall\ \eta > 0$. Let η_1 be sufficiently small so that

$$\Gamma(\bar{u}_1) \subseteq N(\Gamma(u_1),\varepsilon);\ \forall\ \bar{u}_1 \in N(u_1,\eta_1)\ \forall\ u_1 \in U_1,$$

which is possible to find as Γ is uniformly u.H.s.c..

We let $\eta_2(u_1) = \min\{\eta_1,\eta(u_1)\}$ where $\eta(u_1)$ satisfies (a) at u_1. Now as $\{N(u_1,\eta_2(u_1)) : u_1 \in U_1\}$ covers U_1, $\exists$ a finite subcover $\{u_1^i : i=1,..,M\}$. Let

$$0 < q(r) = \min\{\bar{q}_i(r) : i=1,...,M\},$$

$$0 < r_0 = \min\{\bar{r}_0^i : i=1,...,M\},$$

where $\bar{q}_i(r)$, $\bar{r}_0^i$ satisfy (a) at u_1^i. Then if

$$\bar{u}_1 \in \Gamma^{-1}(u_2) \cap N(u_1^i,\eta)$$

for $u_2 \in N(\Gamma(u_1^i),\varepsilon)$, we have

$$\Gamma(\bar{u}_1) \subseteq N(\Gamma(u_1^i),\varepsilon)\ ;\ i=1,..,M.$$

If $\bar{u} \in U_1$, then $\exists$ i s.t.

$$\bar{u}_1 \in N(u_1^i,\eta).$$

As $\Gamma(\bar{u}_1) \subseteq N(\Gamma(u_1^i),\varepsilon)$ we have

$$\Gamma(U_1) \subseteq \bigcup_{i=1}^{M} N(\Gamma(u_1^i),\varepsilon). \qquad \text{(b)}$$

Noting that $\forall\ \bar{u}_1 \in \Gamma^{-1}(u_2) \cap N(u_1^i,r)$

$\forall\ u_2 \in N(\Gamma(u_1^i),\varepsilon)$

we have $\Gamma^{-1}(N(u_2,r)) \supseteq N(\bar{u}_1,q(r))$.

We can finally say

$$\Gamma^{-1}(N(u_2,r) \supseteq N(\bar{u}_1,q(r))$$

$\forall\ \bar{u}_1 \in \Gamma^{-1}(N(\Gamma(u_1^i),\varepsilon)) \cap N(u_1^i,\eta) \qquad i=1,\dots,M$. That is,

$$\bar{u}_1 \in \bigcup_{i=1}^{M} [\Gamma^{-1}(N(\Gamma(u_1^i),\varepsilon)) \cap N(u_1^i,\eta)$$

$$= \Gamma^{-1}(\bigcup_{i=1}^{M} N(\Gamma(u_1^i),\varepsilon)) \cap \bigcup_{i=1}^{M} N(u_1^i,\eta)$$

$$\supseteq \Gamma^{-1}(\Gamma(U_1)) \cap U_1 \qquad \text{(using (b))}$$

$$\supseteq U_1 \cap U_1 = U_1 .$$

Hence $\forall\ \bar{u}_1 \in U_1$ we have

$$\bar{u}_1 \in N(u_1^i,\eta) \text{ for some } i$$

and $\forall\ u_2 \in \Gamma(\bar{u}_1) \subseteq N(\Gamma(u_1^i),\varepsilon)$ we have

$$\Gamma^{-1}(N(u_2,r)) \supseteq N(\bar{u}_1,q(r)) ,$$

i.e. uniform l.H.s.c. . □

Theorem 1.13 : Let $\Gamma : U_1 \to C(U_2)$ and U_1 be compact. If Γ is uniformly u.H.s.c. on U_1 then Γ is uniformly l.H.s.c. on the interior of U_1.

Proof : This is a direct consequence of the Propositions 1.16 and 1.17. □

Corollary 1.13 : Suppose $\Gamma : U_1 \to C(U_2)$ and U is compact. If Γ is uniformly u.H.s.c. on U_1 then Γ is Hausdorff continuous on U_1.

Proof : We either use Theorem 1.13 and the uniformity or simply note that

$$N(u_1,q(r)) = \{\bar{u}_1 : u_1 \in N(\bar{u},q(r))\}.$$

Hence if $\forall\ u_1 \in N(\bar{u}_1,q(r))$, $q(r)$ independent of u_1, we have

$$N(\Gamma(\bar{u}_1),r) \supseteq \Gamma(u_1),$$

and it follows

$\forall\ \bar{u}_1 \in N(u_1,q(r)) = \{\bar{u}_1 : u_1 \in N(\bar{u}_1,q(r))\}$ it must be the case that

$N(\Gamma(\bar{u}_1,r) \supseteq \Gamma(u_1)$. □

So we see Dolecki's theorem on δ-u.H.s.c. and uniform l.s.c. can be related back to our initial comment about the uniform continuity of Hausdorff continuous functions. In a sense it is a localised version of a converse statement.

We finish by quoting a few theorems on composition of multi-valued mappings.

Theorem 1.14 : If Γ is u.s.c. (resp. l.s.c.) at u_1^0 and Λ is u.s.c. (l.s.c.) at each point in $\Gamma(u_1)$, then

$$\Lambda\Gamma(u_1) = \cup\{\Lambda(u_2) : u_2 \in \Gamma u_1\}$$

is u.s.c. (l.s.c.) at u_1^0.

Corollary 1. 14: If Γ is u.s.c. (resp. l.s.c.) at u_1^0 and $r > 0$ then

$N(\Gamma(u_1),r)$ is u.s.c. (l.s.c.) at u_1^0.

Proof : Reference [4] page 58, theorem 2.5. The corollary follows by letting

$$\Lambda(u_2) = \{\bar{u}_2 : d(\bar{u}_2,u_2) < r\}.$$ □

Theorem 1.15 : If Γ is l.s.c. at (u_1^0,u_2^0) at rate $q(\cdot)$ and Λ is l.s.c. at (u_2^0,u_3^0) at rate $p(\cdot)$, then $\Lambda\Gamma$ (as above) is l.s.c. at rate $q\circ p$.

Theorem 1.16 : Let Γ be u.H.s.c. at u_1^0 at a rate q and let Λ be u.H.s.c. on Γu_1^0 at a rate p. Then $\Lambda\Gamma$ is u.s.c. at u_1^0 at a rate $q\circ p$.

Proofs: Reference [4] page 58, theorems 2.6 and 2.7. □

A multi-valued mapping is said to be linearly continuous if it is upper and lower semi-continuous at a linear rate.

For a multi-valued mapping $\Gamma(\cdot) : U_1 \to K(U_2)$, the existence of a $K > 0$ s.t.

$$d_2(u_2,\Gamma(\bar{u}_1)) \leqslant kd_1(u_1,\bar{u}_2)$$

for $\forall\, u_2 \in \Gamma(u_1)$ and $u_1,\ \bar{u}_1 \in U$, is equivalent to $\Gamma(\cdot)$ being uniformly linearly continuous.

Finally we note that for closed set valued mappings we can define the following

Definition 1.19 :

(a) $\Gamma(\cdot)$ is closed at $\bar{u}_1$ iff $\forall \{u_1^n\} \subseteq U_1$, $u_1^n \to \bar{u}_1$ and $\forall\, u_2^n \in \Gamma(u_1^n)$ s.t. $u_2^n \to \bar{u}_2$, we have $\bar{u}_2 \in \Gamma(\bar{u}_1)$.

(b) $\Gamma(\cdot)$ is open at $\bar{u}_1$ iff for $\{u_1^n\} \subseteq U_1$; $u_1 \to \bar{u}_1$ and $\bar{u}_2 \in \Gamma(\bar{u}_1)$ implies $\exists \{u_2^n\} \subseteq U_2$ s.t. $u_2^n \in \Gamma(u_1^n)$ and $u_2^n \to \bar{u}_2$.

A number of theorems are related to the continuity of "marginal" functions and the associated set valued mappings. These will be used in Chapters 3 and 4, so we give a brief survey here.

Theorem 1.17 : Assume U_2 is metrizable and complete and let U_1 fulfill the first countability axiom.

Let Γ be u.s.c. at $\bar{u}_1$. There is a compact subset K_0 of $\Gamma(\bar{u}_1)$ such that if

$f : U_2 \to R$ is l.s.c. on K_0, then

$$m(u_1) = \inf\{f(u_2) : u_2 \in \Gamma(u_1)\}$$

is l.s.c. at $\bar{u}_1$.

Proof : Reference [6] Theorem 10. □

Proof : Reference [33] theorem 6. □

Theorem 1.21 : Suppose U_1 and U_2 are complete metric spaces. If $\Gamma : U_1 \to P(U_2)$ is continuous at $\bar{u}_1$ and if $f : U_1 \times U_2 \to R$ is continuous on $\bar{u}_1 \times \Gamma(\bar{u}_1)$, then $\alpha(u_1)$ is closed at $\bar{u}_1$.

Proof : Reference [33] theorem 8. □

Definition 1.20 : A mapping $\Gamma : U_1 \to P(U_2)$ is said to be uniformly compact near $\bar{u}_1$ iff there is a neighbourhood N of $\bar{u}_1$ s.t. the closure of $\cup\{\Gamma(u_1) : u_1 \in N\}$ is compact.

Theorem 1.22 : Suppose U_1 and U_2 are complete metric and $\Gamma : U_1 \to P(U_2)$ is continuous at $\bar{u}_1$, f is continuous on $\bar{u}_1 \times \Gamma(\bar{u}_1)$, $\alpha(\cdot)$ is non-empty and uniformly compact near $\bar{u}_1$. Then if $\alpha(\bar{u}_1)$ is single valued it is also continuous at $\bar{u}_1$.

Proof : Reference [33] corollary 8.1. □

CHAPTER II

The lattice structure of "classical" convexity has been noted and exploited by many authors. Convex functions can be generated by taking the supremum of a class of affine functions. In view of the Hahn-Banach theorem this class consists of the proper, lower semi-continuous convex functions, the function $+\infty$ and the function $-\infty$. The properties of closed, weakly compact sets in reflexive Banach spaces are considered. ~~In a reflexive Banach space the convex sets are weakly closed (and hence strongly closed)~~. We begin Chapter Two by showing that the weakly compact convex sets in a reflexive Banach space can be generated by taking arbitrary intersections of closed balls. The corresponding class of functions, generated by the class of mappings

$$\Phi_c = \{\psi : \psi(u_2) = c\|u_2 - \bar{u}_2\| - a : \bar{u}_2 \in U_2;\ a \in R\}.$$

by taking arbitrary supremums, we call strongly convex* and denote by $SC_c(U_2)$.

We pursue the line of reasoning of S. Dolecki and S. Kurcyusz (reference [11]) and consider convexity as a general lattice property. We say $f(\cdot)$ is Φ-convex, for some very general class of mappings, if

$$f(u_2) = \sup\{\psi(u_2) : \psi \in \Phi' \subseteq \Phi\},$$

for some sub-collection Φ' of Φ. We show that as long as U_2 is compact and such a class Φ (a supremum complete lattice) consists of l.s.c. functions $\psi(\cdot) : U_2 \to R$ then we can consider the convex functions to be a continuous lattice.

For any given mapping $h(\cdot) : U_2 \to R$ we can generate a multi-function

$$\Gamma(b) = \{u_2 : h(u_2) \leqslant b\} : R \to \mathcal{P}(U_2).$$

We show that the strongly convex functions generate such multi-functions

$$\Gamma(\cdot) : B \to KV(U_2),$$

* unfortunately this name is used elsewhere for a different class.

which possess a very strong type of linear continuity. Conditions for various types of continuity of such multi-valued mappings have been previously derived. Quasi-convex functions (denoted $QC(U_2)$) possess an ability to generate u.s.c. multi-functions. Both strictly-convex (denoted $SQC(U_2)$) and pseudo-convex (denoted $PC(U_2)$) functions possess an ability to generate such multi-functions which are continuous. Taking care of continuity assumptions we can obtain the inclusion

$$SC_c(U_2) \subseteq PC(U_2) \subseteq SQ(U_2) \subseteq QC(U_2).$$

Corresponding to these classes of functions we have various classes of multi-functions possessing various degrees of continuity.

The classes $SC_c(U_2)$ and $QC(U_2)$ are sup-complete but the classes $PC(U_2)$ and $SQ(U_2)$ are not. We can generate any convex, weakly compact set by taking level sets of any of the functions from these classes (i.e. $\Gamma(h)$). Since the class $SC_c(U_2)$ was arrived at by using the separation properties of affine functions, it is conjectured that an equivalent expression of this property would be the ability of $SQ(U_2)$ (or $PC(U_2)$) to generate $QC(U_2)$, by taking arbitrary supremums. This is in fact shown to be achievable, later in Chapter Five. In order to show this we need to consider the following.

Suppose we are given an u.s.c. multi-function $\Gamma_\varepsilon(\cdot)$ ℓ.s.c., approximating $\Gamma(\cdot)$ an u.s.c. multi-function, both with convex image sets, for which

(i) $\Gamma_\varepsilon(\cdot), \Gamma(\cdot) : U_1 \to KV(U_2)$, and

(ii) $\Gamma_\varepsilon(u_1) \supseteq \Gamma(u_1)$ for all $u_1 \in U_1$.

When we can "squeeze" a continuous multi-function inbetween these two multi-functions. That is, does there exist a continuous and convex imaged $\Gamma_\varepsilon(\cdot) : U_1 \to KV(U_2)$ s.t.

$$\Gamma_\varepsilon(u_1) \supseteq \Gamma_\varepsilon(u_1) \supseteq \Gamma(u_1) \text{ for all } u_1 \in U_1.$$

We show that if U_1 is a compact subset of a metric space and U_2 is a weakly compact subset of a reflexive Banach space, in which the weakly compact subsets are "locally F-normed", we can in fact show the existence of $\Gamma_\varepsilon(\cdot)$ s.t.

$$\Gamma_\varepsilon((u_1),\varepsilon) \supseteq T_\varepsilon(u_1) \supseteq \Gamma(u_1) \text{ for all } u_1 \in U_1.$$

In fact this can be achieved for a mapping $f(u_1,u_2) : U_1 \times U_2 \to R$ s.t. $f(u_1, .)$ is strongly convex $SC_1(U_2)$. We use the continuous lattice structure of $SC_1(U_2)$ in order to show this.

Combining these results with the work of A. Cellina (reference [14]) we can obtain various statements about our ability to approximate u.s.c. multi-functions. This work has relevance to some aspects of fixed point theory which are explored in the following chapter.

§2.1 Generalized Convexity

We consider the following characterization of classical convexity. Let Φ stand for the set of affine functions on U_2. Then each convex function f on U_2 can be obtained by

(a) taking $f(u_2) = \sup\{\psi(u_2) : \psi \in \Phi' \subseteq \Phi\}$ for some sub-collection Φ' of affine functions.

This formulation of convexity has been explored by many authors. We will pursue the line of reasoning of S. Dolecki and S. Kureyusz (reference [11]) in their paper on Φ convexity in which they generalize the convexity generating class Φ. We also have the equivalent statement.

A subset $A \subseteq U_2$ is called convex (or Φ-convex) whenever $A = \cap_{\psi\in\Phi'}\{u_2 \in U_2 : \psi(u_2) \leqslant a\}$, that is:

(b) $A = \cap_{\psi\in\Phi'} \sigma_a(\psi)^c$

where $\sigma_a(\psi) = \{u_2 \in U_2 : \psi(u_2) > a\}$, $\Phi' \subseteq \Phi$ and $a \in R$.

We may generalise convexity by simply allowing Φ to be a family of arbitrary real functions which satisfy $\Phi + c = \{\psi+c : \psi \in \Phi\} = \Phi$. In this situation f is Φ-convex if (a) holds (if $\Phi' = \phi$ then $f \equiv -\infty$) and A is Φ-convex if (b) holds (if $\Phi = \phi$ then $A \equiv U_2$).

When Φ is the set of affine functions on U_2 we can deduce that Φ-convex functions are just those for which

$$\lambda f(u_2) + (1-\lambda)f(\bar{u}_2) \geqslant f(\lambda u_2 + (1-\lambda)\bar{u}_2).$$

Let $\tau(\Phi)$ be the coarsest topology on U_2 s.t. the Φ-convex sets are closed. The following set is closed:

$$\{u_2 : f(u_2) \leqslant a\} = \{u_2 : \sup \psi(u_2) \leqslant a;\ \psi \in \Phi'\}$$

$$= \cap_{\psi\in\Phi'} \{u_2 : \psi(u_2) \leqslant a\}.$$

That is,

$$\sigma_a(f) \equiv \cup_{\psi\in\Phi'} \sigma_a(\psi) \text{ is closed if}$$

f is Φ-convex. Thus f is l.s.c. with respect to the topology $\tau(\Phi)$.

Since u_2 may be viewed as a finite real function on the set Φ by noting

$$u_2(\psi) = \psi(u_2)\ ;\ u_2 \in U_2,$$

we may say that a function $g : \Phi \to R^*$ is U_2 convex whenever

$$g(\psi) = \sup\{\psi(u_2) : u_2 \in U_2' \subseteq U_2\}.$$

Analogously we may define U_2-convex subsets of Φ and so on. The roles of U_2 and Φ are fully symmetric.

In the case when Φ consists of the affine functions, in view of the Hahn-Banach theorem, the Φ-convex functions are exactly those which are convex, proper, lower semi-continuous functions, the function $+\infty$ and the function $-\infty$. The Φ-convex sets are those which are closed-convex with respect to $\tau(\Phi)$ the weak topology. The topological dual U_2^* is a layer of Φ (a subset of those which vanish at zero). The U_2-convex sets are in the case of U_2 reflexive, weakly closed as $\tau(U_2)$ is the weak * topology which coincides with the weak topology.

There may be more than one class Φ which generate identical convex functions. A class $\mathcal{L}$ which generates the Φ-convex functions is called a basis. Let us suppose we are dealing with a reflexive Banach space U_2. From reference [12] page 36, we have

Definition 2.1 : A space is called smooth if there is at most one supporting plane through every boundary point of the closed unit ball.

Definition 2.2 : A Banach space is called strictly convex if any non-identically zero continuous linear functional takes a maximum value on the closed unit ball at one point.

In a reflexive space we always have a maximum in this case as the closed unit ball is weakly compact (which is equivalent to being weakly sequentially compact).

Theorem 2.1 : Let U_2 be a reflexive Banach space. Then there exists an equivalent norm on U_2, such that under the new norm U_2 and U_2^* are strictly convex.

Proof : See reference [12], page 36. □

Theorem 2.2 : A reflexive normed space is smooth (strictly convex) iff its dual is strictly convex (smooth).

Proof : reference [12], page 36. □

Corollary 2.2 : If U_2 is a reflexive Banach space, then there exists an equivalent norm under which U_2 is simultaneously smooth and strictly convex.

Proof : This is a consequence of Theorems 2.1 and 2.2. □

If we let $\|\cdot\|$ be this norm we may define the norm one duality mapping on U_2 for each $u_2 \in U_2$ by

$$J(u_2) = \{u_2^* \in \bar{N}^*(0,1) \subseteq U_2^* : \langle u_2^*, u_2 \rangle = \|u_2\|\}$$

where $\langle \cdot , \cdot \rangle$ is the duality pairing.

From Corollary 2.2 we know that $J(u_2)$ is single-valued for all non zero u_2 and in each case

$$\lim_{t\to 0} \frac{\|u_2 + t\hat{u}_2\| - \|u_2\|}{t} = \langle J(u_2), \hat{u}_2 \rangle$$

(ie. grad $\|u_2\| = J(u_2)$), where $J(\cdot)$ is continuous from $(U_2, \|\cdot\|)$ into U_2^* with the weak topology.

Theorem 2.3 : Suppose U_2 is a reflexive Banach space, $C_1 \neq \phi$ is a closed bounded convex set of U_2, C_2 is a closed convex set of U_2 s.t.

$$C_1 \cap C_2 = \phi.$$

Then $\exists\ \bar{u}_2$ and $c > 0$ s.t.

$$C_1 \subseteq K = \{u_2 : \|u_2 - \bar{u}_2\| \leqslant c\}$$

and

$$C_2 \subseteq K^c .$$

Proof : As C_1 and C_2 are convex sets, by the Hahn-Banach theorem $\exists$ a linear function f on U_2 s.t.

$$C_1 \subseteq \{u_2 | f(u_2) < a\} = H,$$
$$C_2 \subseteq H^c ,$$

as $C_1 \cap C_2 = \phi$.

Now let $\bar{u}_2 \in L = \{u_2 | f(u_2) = a\}$ attain

$$\inf_{u_2 \in C_1} d(u_2, L) = \inf_{u_2 \in C_1} d(u_2, \bar{u}_2),$$

which exists due to the convexity of H^c, C_1 and the compactness of C_1.

Let $u_2^1 \in H$ be s.t.

$$N(u_2^1, 1) \cap L = \{\bar{u}_2\}.$$

This exists and is uniquely defined by $\bar{u}_2$ as $N(u_2^1,1)$ has only one support plane at each point of its boundary.

We note that $\| u_2^1 - \bar{u}_2 \| = 1$.

We define inductively

$$u_2^n \in \{u_2 : u_2 = n\, u_2^1 + (1-n)\bar{u}_2\} \cap H$$

s.t.

$$N(u_2^n, n) \cap L = \{\bar{u}_2\}.$$

We note that $\| u_2^n - \bar{u}_2 \| = \| n\, u_2^1 + (1-n)\bar{u}_2 - \bar{u}_2 \|$
$= n \| u_2^1 - \bar{u}_2 \| = n.$

We wish to show $\bigcup_{n=1}^{\infty} N(u_2^n, n) = H$: (S).

If we can show this then we can complete our proof as follows.

Suppose our theorem is false. Then $\forall$ n, $\exists$ $\hat{u}_2^n \in C_1$ s.t. $\hat{u}_2^n \notin N(u_2^n, n)$, that is $\forall$ n, $\exists$ $\hat{u}_2^n \in N^c(u_2^n, n)$ s.t. $\hat{u}_2^n \in C_1$. Now as C_1 is weakly sequentially compact, $\exists$ a convergent sub-sequence of u_2^n converging to $\hat{u}_2$. Hence after renumbering we can say $\hat{u}_2^n \rightharpoonup \hat{u}_2 \in C_1$. As

$$N(u_2^n, n) \supseteq N(u_2^m, m) \text{ for } n \geq m$$

we have

$$\hat{u}_2^n \in N^c(u_2^n, n) \subseteq N^c(u_2^m, m) \text{ for } n > m$$

and

$$\hat{u}_2^n \rightharpoonup \hat{u}_2 \in N^c(u_2^m, m) \ : \ \forall\, m.$$

Therefore

$$\hat{u}_2 \in \bigcap_{m=1}^{\infty} N^c(u_2^m, m)$$

$$= \left(\bigcup_{m=1}^{\infty} N(u_2^m, m)\right)^c$$

$$= H^c$$

and hence $\exists$ $\hat{u}_2 \in C_1$ s.t. $\hat{u}_2 \notin H$, which contradicts our choice of H.

We now finish by proving the statement (S). As

$$N(u_2^n, n) \subseteq H \quad \forall\, n, \text{ we have}$$

$$\bigcup_{n=1}^{\infty} N(u_2^n, n) \subseteq H.$$

To prove the reverse inequality we note that L is the tangent plane to $\overline{N(u_2^n, n)}$ at $\bar{u}_2$, $\forall$ n. Now as

$$\overline{N(u_2, n)} = \{u_2 : \|u_2 - (n\, u_2^1 + (1-n)\bar{u}_2)\| \leq n\}$$

$$= \{u_2 : \frac{1}{n}\|u_2 - (nu_2^1 + (1-n)\bar{u}_2)\| \leq 1\}$$

and

$$\text{grad}\,\frac{1}{n}\|u_2 - (n\,u_2^1 + (1-n)\bar{u}_2)\|_{u_2 = \bar{u}_2}$$

$$= J(\bar{u}_2 - (n\,u_2^1 + (1-n)\bar{u}_2))/n)$$

$$= \quad J(u_2^1 - \bar{u}_2),$$

by letting

$$<J(u_2^1-\bar{u}_2),\bar{u}_2> = b,$$

we have

$$L = \{u_2 : < J(u_2^1 - \bar{u}_2),u_2> = b\}$$

and

$$H = \{u_2 :< J(u_2^1 - \bar{u}_2),u_2> <b\}.$$

Now $J(\cdot)$ is demi-continuous (ie. if $\bar{u}_2^n \to u_2$ then $J(\bar{u}_2^n) \rightharpoonup J(u_2)$).

Let us suppose $\bar{u}_2^n \in$ bdd $N(u_2^n,n)$ and $\|\bar{u}_2^n-\bar{u}_2\| \leqslant K : \forall\, n$. Then

$$\frac{1}{n}(\bar{u}_2^n - u_2^n) \to (\bar{u}_2 - u_2^1)$$

as

$$0 \leqslant \|(\bar{u}_2 - u_2^1) - \frac{1}{n}(\bar{u}_2^n - u_2^n)\|$$

$$= \|(\bar{u}_2 - u_2^1) - \frac{1}{n}(\bar{u}_2^n - u_2^n + \bar{u}_2 - \bar{u}_2)\|$$

$$= \|(\bar{u}_2 - u_2^1) - \frac{1}{n}(\bar{u}_2 - u_2^n) - \frac{1}{n}(\bar{u}_2^n - \bar{u}_2)\|$$

$$\leqslant \frac{1}{n}\|\bar{u}_2^n - \bar{u}_2\| \leqslant \frac{1}{n}K \to 0$$

as $n \to \infty$, noting

$$n(\bar{u}_2 - u_2^1) = (\bar{u}_2 - u_2^n).$$

Now

$$\hat{u}_2^n = \frac{1}{n}(u_2^n - \bar{u}_2^n) - (u_2^1 - \bar{u}_2)$$

is the "direction" in which $\bar{u}_2^n$ lies with respect to $\bar{u}_2$, that is

$$\bar{u}_2^n = \bar{u}_2 + C^n \frac{\hat{u}_2^n}{\|\hat{u}_2^n\|}; \quad (C^n = -n\|\hat{u}_2^n\|).$$

Since $\|\bar{u}_2^n - \bar{u}_2\| \leq K$ we must have $0 < |C^n| \leq K$. We have $\|\hat{u}_2^n\| \to 0$ as $n \to \infty$ and by weak sequential compactness there must exist a subsequence of both $\{\bar{u}_2^n\}$ and $\{\hat{u}_2^n\}$ s.t. both $\bar{u}_2^n \rightharpoonup u_2$ and $\frac{\hat{u}_2^n}{\|\hat{u}_2^n\|} \rightharpoonup \hat{u}_2$, where $\hat{u}_2 = \dfrac{u_2 - \bar{u}_2}{\|u_2 - \bar{u}_2\|}$.

As a consequence the following limit exists;

$$\lim_{n\to\infty} \left\langle J(u_2^1 - \bar{u}_2), \frac{\hat{u}_2^n}{\|\hat{u}_2^n\|} \right\rangle$$

$$= \lim_{n\to\infty} \lim_{t\to 0} \frac{\left\|(u_2^1 - \bar{u}_2) + t\left(\frac{\hat{u}_2^n}{\|\hat{u}_2^n\|}\right)\right\| - \|u_2^1 - \bar{u}_2\|}{t}$$

$$= \lim_{n\to\infty} \frac{\left\|(u_2^1 - \bar{u}_2) + \|\hat{u}_2^n\|\left(\frac{\hat{u}_2^n}{\|\hat{u}_2^n\|}\right)\right\| - \|u_2^1 - \bar{u}_2\|}{\|\hat{u}_2^n\|}$$

$$= \lim_{n\to\infty} \frac{\|(u_2^1 - \bar{u}_2) + \hat{u}_2^n\| - \|u_2^1 - \bar{u}_2\|}{\|\hat{u}_2^n\|}$$

$$= \lim_{n\to\infty} \frac{\left\|\frac{1}{n}\|(u_2^n - \bar{u}_2^n)\|\right\| - \|u_2^1 - \bar{u}_2\|}{\|\hat{u}_2^n\|}$$

$$= \lim_{n\to\infty} \frac{1-1}{\|\hat{u}_2^n\|} = 0.$$

Hence we have

$$|\langle J(u_2^1-\bar{u}_2),\bar{u}_2^n-\bar{u}_2\rangle|$$

$$= |\langle J(u_2^1-\bar{u}_2),\ C^n\frac{\hat{u}_2^n}{||\hat{u}_2^n||}\rangle|$$

$$= |C^n||\langle J(u_2^1-\bar{u}_2),\ \frac{\hat{u}_2}{||\hat{u}_2^n||}\rangle|$$

$$= K\ \ \langle J(u_2^1-\bar{u}_2),\ \frac{\hat{u}_2^n}{||\hat{u}_2^n||}\rangle \to 0,\text{ as } n\to\infty,$$

that is

$$\langle J(u_2^1-\bar{u}_2),\bar{u}_2^n\rangle \to \langle J(u_2^1-\bar{u}_2),\bar{u}_2\rangle \text{ as } n\to\infty.$$

We let $b_n = J(\frac{1}{n}(u_2^n-\bar{u}_2^n),\bar{u}_2^n\rangle$ and note that since, $(u_2^1-\bar{u}_2) = \frac{1}{n}(u_2^n-\bar{u}_2)$ we have,

$$b_n \to b = \langle J(u_2^1-\bar{u}_2),\bar{u}_2\rangle = b.$$

The half space at $\bar{u}_2^n$

$$T_n(u_2^n) = \{u_2 : \langle J(\tfrac{1}{n}(\bar{u}_2^n - \bar{u}_2^n)),u_2\rangle < b_n\},$$

has the property that if $u_2 \in H$, then $u_2 \in T_n(\bar{u}_2^n)$ for n sufficiently large. If $u_2 \in H$, then $\langle J(u_2^1 - \bar{u}_2),u_2\rangle < b$. Since we have a strict inequality, $\exists\ \delta > 0$ s.t.

$$\langle J(u_2^1 - \bar{u}_2),u_2\rangle + \delta < b.$$

For n sufficiently large we have $b - \frac{\delta}{2} \leq b_n$ and

$$\langle J(\tfrac{1}{n}(\bar{u}_2^n - \bar{u}_2^n)),u_2\rangle$$

$$\leq\ |\langle J(\tfrac{1}{n}(\bar{u}_2^n - \bar{u}_2^n) - J(\bar{u}_2^1 - u_2),u_2\rangle| + \langle J(u_2^1 - \bar{u}_2),u_2\rangle$$

$$\leq \langle J(u_2^1 - \bar{u}_2),u_2\rangle + \frac{\delta}{2} < b - \frac{\delta}{2} \leq b_n,$$

using once again the demi-continuity of $J(\cdot)$.

Now if we suppose $\bigcup_{n=1}^{\infty} N(u_2^n,n) \not\supseteq H$, then $\exists\ u_2 \in H$ s.t.

$$u_2 \notin N(u_2^n,n); \forall\ n.$$

We arrive at a contradiction as follows. Let $\bar{u}_2^n$ be the closest point in $\overline{N(u_2^n,n)}$ to u_2, i.e.

$$d(u_2,\overline{N(u_2^n,n)}) = \|u_2 - \bar{u}_2^n\|.$$

This point is unique as $\overline{N(u_2^n,n)}$ is strictly convex closed and

$$\begin{aligned}
&\|\bar{u}_2^n - \bar{u}_2\| \\
&\quad \leqslant \|u_2 - \bar{u}_2^n\| + \|\bar{u}_2 - u_2\| \\
&\quad = d(u_2,N(u_2^n,n)) + \|\bar{u}_2 - u_2\| \\
&\quad \leqslant d(u_2,N(u_2^1,1)) + \|\bar{u}_2 - u_2\| \\
&\quad \leqslant \|u_2 - u_2^1\| + \|\bar{u}_2 - u_2\| \\
&\quad = K < \infty.
\end{aligned}$$

Since $u_2 \notin T_n(\bar{u}_2^n)\ \forall\ n$, from the above we have $u_2 \notin H$, which is a contradiction. □

An immediate consequence of this Theorem is that the weakly compact convex sets in a reflexive Banach space are generated by the class

$$\Phi = \{\psi : \psi(u_2) = \|u_2 - \bar{u}_2\| - a;\ \bar{u}_2 \in U_2;\ a \in R\}$$

we define for $c \in R^+$

$$\Phi_c = \{\psi : \psi(u_2) = c\|u_2 - \bar{u}_2\| - a;\ \bar{u}_2 \in U_2;\ a \in R\}.$$

As one may have noted by now, convexity in this context has a definite lattice structure. We can for a general class Φ define the convex hull of a set A to be the intersection of all convex sets containing A. In terms of Φ-convex functions, the convex hull of a function f is the supremum of all Φ-convex functions majorized by f. This can be reinterpreted according to the basis, to be the supremum of all the members of the basis Φ (say) which f majorizes. Correspondingly when we discuss ordinary convexity this corresponds to the fact that the convex hull of A is equivalent to the intersection of all half spaces containing A. The above theorem indicates that when we wish to define the closed convex hull of a bounded set, in a reflexive Banach space, we may define it to be the intersection of all closed balls containing the set.

Proposition 2.1 : (separation property)

(i) A function $f : U_2 \to R^*$ is Φ convex iff for each $u_2 \in U_2$ and $r < f(u_2)$ there is a ψ majorized by f s.t. $\psi(u_2) > r$.

(ii) A set A is Φ-convex iff for each $u_2^0 \notin A$ there is a function $\psi \in \Phi$ s.t.

$$\sup_{u_2 \in A} \psi(u_2) < \psi(u_2^0).$$

Proof : See reference [11] page 279. □

Lemma 2.1 : Suppose f is Φ-convex, all the $\psi \in \Phi$ are l.s.c. with respect to the topology on U_2 and $g : U_2 \to R$ is u.s.c.

If $g(u_2) < f(u_2)$, $\forall\, u_2 \in U_2$, then $\exists$ a neighbourhood N of $\bar{u}_2$ and $\psi \in \Phi$ for each $\bar{u}_2 \in U_2$ s.t.

$$g(u_2) < \psi(u_2) \quad \forall\, u_2 \in N\,,$$
$$\psi(u_2) < f(u_2) \quad \forall\, u_2 \in U_2.$$

Proof : If $\psi \in \Phi$ are l.s.c. then

$$\sup_{\psi \in \Phi} \psi = f \text{ is l.s.c.}$$

Now if we define

$$\rho(\bar{u}_2) = \sup\{\delta : g(\bar{u}_2) < f(\bar{u}_2) - \delta\} > 0\,,$$

we can show that $\rho(\bar{u}_2)$ is bounded away from zero on U_2.

Suppose not, then $\exists\ u_2^n \in U_2$ s.t.

$$\rho(u_2^n) < \frac{1}{n};\ \forall\ n \in Z^+ .$$

As U_2 is compact there is a sub-sequence convergent to $\bar{u}_2$ (say). After renumbering we can say $u_2^n \to \bar{u}_2$,

$$\rho(u_2^n) \leqslant \frac{1}{m_n} \to 0;\ n \to \infty.$$

We know that $\forall\ o < \delta < \rho(\bar{u}_2)$ we have

$$g(\bar{u}_2) < f(\bar{u}_2) - \delta.$$

Let $0 < \varepsilon < \delta < \rho(\bar{u}_2)$ and as g is u.s.c. $\exists$ a neighbourhood N_1 of $\bar{u}_2$ s.t.

$$g(u_2) \leqslant g(\bar{u}_2) + \varepsilon;\forall\ \ u_2 \in N_1.$$

Let $\varepsilon' = \frac{1}{2}(\delta-\varepsilon) > 0$. Then $\exists\ N_2$ s.t.

$$f(\bar{u}_2) - \varepsilon' \leqslant f(u_2);\ \forall\ u_2 \in N_2 .$$

Hence

$$\forall\ u_2 \in N_1 \cap N_2$$

we have

$$\begin{aligned} g(u_2) &\leqslant g(\bar{u}_2) + \varepsilon \\ &< f(\bar{u}_2) + \varepsilon - \delta \\ &= f(\bar{u}_2) - 2\varepsilon' \\ &\leqslant f(u_2) - \varepsilon'. \end{aligned}$$

For n sufficiently large we have

$$\bar{u}_2^n \in N_1 \cap N_2 ,$$

as $N_1 \cap N_2$ is a neighbourhood of $\bar{u}_2$ and $\rho(\bar{u}_2^n) \geqslant \varepsilon'\ \forall$ n sufficiently large, a contradiction.

As $f-\delta$ for $0 < \delta < \inf_{u_2 \in U_2} \rho(u_2)$ is Φ-convex and $g(u_2) < f(u_2) - \delta$ $\forall\, u_2 \in U_2$, then f satisfies the separation property at all $\bar{u}_2 \in U_2$.

Hence $\exists\, \psi \in \Phi$ s.t.

$$g(\bar{u}_2) < \psi(\bar{u}_2)$$

and

$$\psi(u_2) \leqslant f(u_2) - \delta;\ \forall\, u_2 \in U_2.$$

Now as g is u.s.c. $\exists$ a neighbourhood N_3 of $\bar{u}_2$ s.t.

$$g(u_2) \leqslant g(\bar{u}_2) + \varepsilon;\ \forall\, u_2 \in N_3,$$

where

$$0 < \varepsilon = \frac{1}{m}(\psi(\bar{u}_2) - g(\bar{u}_2)) < \inf_{u_2} \rho(u_2)$$

for some $m \in Z^+$.

Similarly $\exists$ a neighbourhood N_4 of $\bar{u}_2$ s.t.

$$\psi(\bar{u}_2) - \varepsilon \leqslant \psi(u_2);\ \forall\, u_2 \in N_4.$$

So, if we let $N = N_4 \cap N_3$ a neighbourhood of $\bar{u}_2$, then $\forall\, u_2 \in N$

$$\begin{aligned} g(u_2) \leqslant g(\bar{u}_2) + \varepsilon &= [g(\bar{u}_2) + m\varepsilon] - (m-1)\varepsilon \\ &= g(\bar{u}_2) + \psi(\bar{u}_2) - g(\bar{u}_2) - (m-1)\varepsilon \\ &= \psi(\bar{u}_2) - (m-1)\varepsilon < \psi(\bar{u}_2) - \varepsilon \leqslant \psi(u_2) \\ &\leqslant f(u_2) - \varepsilon < f(u_2). \end{aligned}$$

□

<u>Proposition 2.2</u> : Suppose f is Φ-convex, all the $\psi \in \Phi$ are l.s.c. with respect to the topology on U_2, U_2 is compact and g is u.s.c. on U_2. If

$$g(u_2) < f(u_2) \quad \forall\, u_2 \in U_2$$

then $\exists\, \{\psi_i : i=1,..,n\} \subseteq \Phi$ and a Φ-convex function

$$h(u_2) = \sup\{\psi_i(u_2) : i=1,..,n\}$$

s.t.

$$g(u_2) < h(u_2) < f(u_2); \ \forall\, u_2 \in U_2.$$

Proof : This follows immediately from the previous lemma and the compactness of U_2. □

Corollary 2.2 : Suppose $\forall\, \psi \in \Phi$ are l.s.c. with respect to the topology on U_2. Suppose U_2 is compact, f, g Φ-convex and g continuous where

$$g(u_2) < f(u_2); \ \forall\, u_2 \in U_2.$$

Then $g << f$ in the lattice of Φ-convex functions and if $\forall\, \psi \in \Phi$ are continuous, then this is a continuous lattice.

Proof : This is straightforward when one notes that for any directed set D in the lattice of convex functions we can produce a corresponding directed set in Φ which has the same supremum, namely,

$$\Phi' = \cup\{\Phi'' : \sup\{\psi \in \Phi''\} = h \in D\},$$

where $\sup \Phi' \geqslant f$.

Now if we suppose $\psi \in \Phi$ is s.t.

$$g(u_2) < \psi(u_2) < f(u_2), \ \forall\, u_2 \in U_2$$

does $\exists\, \psi' \in \Phi'$ s.t. $\psi(\cdot) \leqslant \psi'(\cdot)$? Suppose not. Then $\exists\, \bar{u}_2 \in U_2$ s.t.

$$\forall\, \psi' \in \Phi', \ \psi'(\bar{u}_2) < \psi(\bar{u}_2).$$

If this is so, then

$$f(\bar{u}_2) \leqslant \sup\{\psi'(\bar{u}_2) : \psi' \in \Phi'\} \leqslant \psi(\bar{u}_2),$$

contradicting our choice of ψ.

Now this particular $\psi' \in \Phi''$ where $\sup \Phi'' = h \in D$. Hence

$$g(u_2) < \psi'(u_2) \leqslant h(u_2), \text{ implying } g << f.$$

The remark follows from the fact that if $\psi \in \downarrow f$ then $\psi - \frac{1}{n} \in \underset{\vee}{\downarrow} f$; $\forall\, n \in Z^+$, so

$$f \geqslant \sup\{\psi \in \Phi : \psi << f\}$$

$$\geqslant \sup\{\psi - \frac{1}{n} : n \in Z^+;\ \psi \in \downarrow f\}$$

$$= \sup \downarrow f = f,$$

that is, $\sup \underset{\vee}{\downarrow} f = f$. □

This has some relationship to the topic of continuous selection. We state some well-known concepts and theorems by Ernest Michael which can all be found in reference [8].

The central concept of E. Michael's work is that of continuous selection. If $\Gamma : U_1 \to 2^{u_2}$ is a multifunction, then a selection Γ is a continuous function $f : U_1 \to U_2$ s.t.

$$f(u_1) \in \Gamma(u_1) \text{ for every } u_1 \in U_1.$$

It can easily be shown that if $S \subseteq 2^{u_2}$ contains all one-point subsets of elements of S, then the following are equivalent;

(a) Every l.s.c. $\Gamma : U_1 \to S$ admits a selection.

(b) If $\Gamma : U_1 \to S$ is l.s.c., then every selection of $\Gamma|A$ (for $A \subseteq U_1$ closed) can be extended to a selection for Γ.

Both of these imply

(c) U_2 is an extension space with respect to U_1, ie. every continuous $g : A \to U_2$ can be extended to a continuous $f : U_1 \to U_2$. We note in passing that Urysohn's theorem was concerned with the extension of continuous functions.

<u>Theorem 2.4</u> : The following properties of a T_1 space are equivalent:

(a) U_1 is normal (perfectly normal).

(b) Every l.s.c. multifunction $\Gamma : U_1 \to CV(R)$ $(\Gamma : U_1 \to V(R))$ admits a continuous selection.

(c) If $\Gamma : U_1 \to CV(U_2)$ $(\Gamma' : U_1 \to V(U_2))$ is a l.s.c. multifunctions in U_2, a separable Banach space, then there exists a continuous selection.

<u>Corollary 2.4</u> : Suppose U_2 is normal (perfectly normal). Then for $g : U_2 \to R$ u.s.c., $f : U_2 \to R$ l.s.c., $\exists$ a continuous function $h : U_2 \to R$ s.t.

$$g(u_2) \leq h(u_2) \leq f(u_2); \; \forall \, u_2$$

$$(g(u_2) < h(u_2) < f(u_2); \; \forall \, u_2).$$

<u>Proof</u> : This follows immediately from the fact that $\Gamma(u_2) = \{x \in R : g(u_2) \leq x \leq f(u_2)\}$ is l.s.c. whenever g is u.s.c. and f is l.s.c. Similarly for

$$\Gamma(u_2) = \{x \in R : g(u_2) < x < f(u_2)\}$$

the above observation holds and one only needs to apply Theorem 2.4. □

One can deduce the Urysohn Theorem from this. We will revisit this in the context of "Fuzzy Topologies".

<u>Proposition 2.3</u> : The space LSC (U_2) consists of all convex function with respect to $\Phi = C(U_2)$, the space of continuous functions, if U_2 is normal. It is a continuous lattice if U_2 is compact.

<u>Proof</u> : Follows immediately from what has been covered. □

Definition 2.3 : For an arbitrary class Φ a Φ-convex function f is said to be Φ-sub-differentiable at $\bar{u}_2 \in U_2$ if $\exists\ \psi \in \Phi$ s.t.

$$f(\bar{u}_2) = \psi(\bar{u}_2)$$

and

$$f(u_2) \geq \psi(u_2);\ \forall\ u_2.$$

We note in passing that a function $h(u_2) = \sup\{\psi_i(u_2) : i=1,..,N\}$ defined by $\psi_i \in \Phi$ is Φ-sub-differentiable everywhere in U_2 since if $\bar{u}_2 \in U_2$ then

$$h(\bar{u}_2) = \psi_i(u_2) \text{ for some } i=1,..,N$$

and

$$h(u_2) \geq \psi_i(u_2);\ \forall\ u_2 .$$

We say that $h(\cdot)$ a Φ-convex function is strictly sub-differentiable at $\bar{u}_2$ if the second inequality holds strictly, namely,

$$h(u_2) > \psi_i(u_2);\ \forall\ u_2 \neq \bar{u}_2.$$

Definition 2.4 : A function $h : U_2 \to R$ is called strictly quasi-convex if

$$h(u_2) < h(\bar{u}_2) \Rightarrow h(\lambda u_2 + (1-\lambda)\bar{u}_2) < h(\bar{u}_2), \forall\ \lambda \in (0,1).$$

Definition 2.5 : A convex subset S of a reflexive Banach space is said to locally F-normed if a translation-invarient metric

$$d(u_2,\bar{u}_2) = d(u_2 - \bar{u}_2, 0) \equiv \|u_2 - \bar{u}_2\|^*$$

can be defined satisfying

(i) $\|u_2\|^* \geq 0 \qquad u_2 \in S$

(ii) $u_2 \equiv 0$ iff $\|u_2\|^* = 0$

(iii) $\|u_2+\bar{u}_2\|^* \leq \|u_2\|^* + \|\bar{u}_2\|^*;\ u_2,\bar{u}_2 \in S$

(iv) $\|\lambda_n u_2\|^* \to 0$ if $\lambda_n \to 0;\ u_2 \in S - S$

which generates the topology of S.

In the case of those reflexivespaces for which the dual space has an orthonormal set, we can immediately define such a norm. Let

$$\|u\|^* = \sum_{i=1}^{\infty} \frac{1}{2^i} |\langle u, u_i^* \rangle|$$

where $\{u_i^*\}_{i=1}^{\infty}$ is an orthonormal spanning set. The norm obviously defines the weak topology on the compact sets. The compactness of the set is essential as this makes sure $\|u\|^* \leq K$. We note in passing that

$$\|u\|^* \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|u\|$$

$$\leq \|u\| ,$$

where $\|\cdot\|$ is the usual norm in the Banach space. In these situations Hausdorff continuity of $\Gamma(\cdot)$ with respect to $\|\cdot\|$ would obviously imply Hausdorff continuity with respect to $\|\cdot\|^*$. This is in general true as the weak topology is coarser than the strong topology on U_2.

We note in passing that local F-norms are similar to the para norms of reference [15]. They differ in that they only define the relevant topology locally (on the compact sets) but they still reflect a compatibility with the linear structure within this local context.

The condition (iv) is obviously satisfied by any para normed space for which the compact subsets satisfy the Zima condition,namely:

$(U, \|\cdot\|^*)$ a para normed space with $S \subseteq U$ and $\exists\ c > 0$ s.t.

$\|\lambda u\| \leq c\lambda\|u\|^*$, for every $0 \leq \lambda \leq 1$ and every $u \in S - S$.

These structures were used in reference [15] to deduce a fixed point theorem for convex-valued multi-valued mappings on certain topological linear space.

Proposition 2.4 : Let U_2 be a convex subset of a Banach space which is locally F-normable. Suppose $h : U_2 \to R$ is strictly quasi-convex and continuous with respect to the same topology. Then if

$$I(b) = \{u_2 : h(u_2) < b\} \neq \phi \text{ and } \Gamma(b) = \{u_2 : h(u_2) \leqslant b\}$$

we have $\text{cl}\, I(b) = \Gamma(b)$ and $\Gamma(b)$ is convex.

Proof : We argue similarly to the proof of Lemma 5 of reference [13]. We note that $u_2(\theta) = \theta(\hat{u}_2) + (1-\theta)\bar{u}_2 \to \bar{u}_2$ in the local F-norm since,

$$\|u_2(\theta) - \bar{u}_2\| = \|\theta(\hat{u}_2 - \bar{u}_2)\| \to 0$$

as $\theta \to 0$ due to condition (iv) of the definition 2.6.

In this way we establish $\text{cl}\, I(b) = \Gamma(b)$ which implies

$$\text{bdd}\, \Gamma(b) = \Gamma(b) \setminus I(b)$$

$$= \{u_2 : f(u_2) = b\}.$$

So if $\bar{u}_2, u_2 \in \Gamma(b)$

$$h(\bar{u}_2) \leqslant b; \quad h(u_2) \leqslant b,$$

then

$$h(\lambda\bar{u}_2 + (1-\lambda)u_2) \leqslant \max(h(u_1), h(\bar{u}_2)) = b$$

and hence $\lambda u_2 + (1-\lambda)u_2 \in \Gamma(b)$. □

Proposition 2.5 : If U_2 is a reflexive Banach space and the unit ball's weak topology is metrizable, then h strictly convex, weakly continuous and $\Gamma(\bar{b})$ weakly compact imply $\exists\, b^* > \bar{b}$ s.t. $\Gamma(b^*)$ is weakly compact.

Proof : Identical to Lemma 6 of reference [13] using the equivalence of boundedness, weak compactness and sequentially weak compactness. □

Definition 2.6 : $f : U_2 \to R$ is called quasi convex iff the sets $\Gamma(b)$ are convex $\forall\, b \in R$.

Proposition 2.6 : If Γ is compact valued and u.s.c., the image sets of a compact set K in U_1 is also compact.

Proof : Reference [1] page 110. □

If we assume $f : U_2 \to R$ is weakly continuous and U_2 is weakly compact then f will be bounded. For

$b^* = \sup\{f(u_2) : u_2 \in U_2\}$ we can define

$B^* = \{b \leqslant b^* : \Gamma(b) \neq \phi\}$,

where

$\Gamma(b) = \{u_2 \in U_2 : f(u_2) \leqslant b\}$,

which is weakly compact.

We note the following:

(i) B^* is bounded if f is bounded.

(ii) $I(b) = \{u_2 \in \Gamma(b) : f(u_2) < b\} \neq \phi$ if $b \in \text{Int } B^*$.

(iii) If $\Gamma(b)$ is u.s.c. and U_2 weakly compact, then $\Gamma(B^*) = \cup_{b \in B^*}\Gamma(b)$ is weakly compact (this follows from Proposition 2.6).

(iv) If the space U_2 is separable reflexive then the weakly compact sets are metrizable.

Theorem 2.5 :

(i) Suppose U_2 is a metrizable weakly compact subset of a reflexive Banach space and $f : U_2 \to R$ is weakly continuous then $\Gamma(b)$ is u.H.s.c. at $\forall\, b \in B^*$ with respect to the induced metric.

(ii) Suppose U_2 is a subset of a reflexive Banach space and $f : U_2 \to R$ is (weakly) strongly continuous, $\Gamma(\bar{b})$ is (weakly) strongly compact then the mapping $\Gamma(b)$ is l.H.s.c. at $\bar{b}$ iff $\text{cl}\, I(\bar{b}) = \Gamma(\bar{b})$, where in the case of the weak topology the Hausdorff continuities refer to those on some weakly compact metrized space U_2 containing $\Gamma(\bar{b})$.

Proof : Direct adaptation of the proofs of reference [13]. These originally were only proved for $U_2 = R^n$ but go across to the case of a reflexive Banach space. In (i) we use the equivalence of weak compactness (ie. closed boundedness) and sequential weak compactness. In (ii) we use the metrizability of U_2 and the linear structure on the reflexive space. □

So if U_2 is (weakly) strongly compact and $f : U_2 \to R$ is (weakly) strongly continuous then B^* is bounded and we need only deal with the compact metric space $\Gamma(B^*)$, in which case $\Gamma(b)$ is l.H.s.c. iff cl $I(b) = \Gamma(b)$. Now if we suppose that $\Gamma(b)$ is always convex then the strong and weak closures of $I(b)$ will coincide. Since a strongly compact set is weakly compact we have strong l.H.s.c. implying weak l.H.s.c. This is so even if we remove the necessity that $\Gamma(b)$ is strongly compact.

The join semi-lattice $SQC(U_2)$ of l.s. continuous strictly quasi-convex functions from U_2 to R contains the convex continuous function and the classes Φ_c $(c \in R^+)$. The classes Φ_c $(c \in R^+)$ generate the lattices of strongly convex functions $SCc(U_2)$ $(c \in R^+)$ which are contained in the class of l.s.c. quasi-convex function $QC(U_2)$.

Definition 2.7 : A function $\psi(\cdot) : U_2 \to R$ is called pseudo-convex at $\bar{u}_2 \in U_2$ if it is differentiable at $\bar{u}_2$

(ie. $\lim_{t \to 0} \frac{f(\bar{u}_2 + tu_2) - f(\bar{u}_2)}{t} = \langle \nabla f(\bar{u}_2), u_2 \rangle$ exists $\forall\, u_2 \in U_2$)

$\nabla f(\bar{u}_2) \in U_2^*$

and $\forall\, u_2 \in U_2$

$\langle \nabla f(\bar{u}_2), (u_2 - \bar{u}_2) \rangle \geqslant 0$ implies $f(u_2) \geqslant f(\bar{u}_2)$.

We let $PC(U_2)$ be the class of such functions.

A full discussion of these concepts in the case $U_2 = R^n$ is given in reference [19]. As usual many of the proofs go over to the case of U_2 reflexive and $\psi(\cdot)$ weakly continuous. Taking care with the continuity assumption on the classes one obtains the following inclusions.

$$SCc(U_2) \subseteq PC(U_2) \subseteq SQC(U_2) \subseteq QC(U_2).$$

We obtained the class $SCc(U_2)$ by considering a separation theorem of the same type as the Hahn-Banach theorem. One wonders if the lattice $PC(U_2)$ generate the lattice $QC(U_2)$. The proof of this would correspond to a "generalization" of the Hahn Banach theorem. There may in fact be generating classes which are theoretically more accessible than these for some purposes. Possibly the class of functions $\psi(u_2) = \eta(\|u_2 - \bar{u}_2\|) - a$ where $\eta(\cdot) : R^+ \to R^+$ is monotonically increasing which are once again in $SQC(U_2)$ might generate $QC(U_2)$.

Definition 2.8 : A set S is called strongly convex iff $\forall\ \hat{u}_2 \in \text{bdd}\ S$ $\exists\ \bar{u}_2 \in U_2 : r \in R^+$ s.t.

$$S \subseteq \bar{N}(\bar{u}_2,r) = \text{cl}\,N(\bar{u}_2,r)$$

and

$$\hat{u}_2 \in \text{bdd}\ \bar{N}(\bar{u}_2,r).$$

It is easily seen that a strongly convex set is strictly convex in a Banach space which is strictly convex. This definition is prompted by the knowledge that if f is Φ_c sub-differentiable then $\Gamma(b)$ is strictly convex $\forall\ b \in B^*$.

Definition 2.9 : A multi-valued mapping $\Gamma : R \to K(U_2)$ is said to be metrically increasing with a rate $\eta(\cdot)$ if $\exists\ \eta(\cdot)$ s.t. $\eta(0) = 0$ and

$$\eta(\cdot) : R^+ \to R^+ \text{ iff for } b \leqslant \bar{b}$$

$$\text{cl}\,N(\Gamma(b),\eta(\bar{b}-b)) = \bar{N}(\Gamma(b),\eta(\bar{b}-b)) = \Gamma(\bar{b}).$$

<u>Theorem 2.6</u> : Let U_2 be a weakly compact convex subset of a reflexive space U on which we have a local F-norm. Suppose $f : U_2 \to R$ is strongly continuous. Then

(a) f is Φ_c sub-differentiable on U_2 iff

(b) $\Gamma(b)$ is strongly convex and $b \to \Gamma(b)$ is metrically increasing with a rate $\eta(x) = \frac{x}{c}$.

<u>Proof</u> : Suppose (a) holds then $\exists\, \bar{u}_2 \in U$, $a \in R$ s.t. for any $\hat{u}_2 \in U_2$

(i) $f(\hat{u}_2) = c\|\hat{u}_2 - \bar{u}_2\| - a$

(ii) $f(u_2) \geq c\|u_2 - \bar{u}_2\| - a, \ \forall\, u_2 \in U_2.$

As f is sub-differentiable it is Φ_c convex and hence strictly quasi-convex. As $U_2 \subseteq U$ is a Banach space we may consider our topology on U_2 being given by the norm of the strong topology. As f is strongly continuous Proposition 2.4 tells us $\text{cl}\, I(b) = \Gamma(b)$. Hence $\text{bdd}\, \Gamma(b) = \{u_2 : f(u_2) = b\}$. If we let $\hat{u}_2 \in \text{bdd}\, \Gamma(b)$ then (i) and (ii) above become equivalent to;

(i)' $\hat{u}_2 \in \text{bdd}\, N(\bar{u}_2, \frac{b+a}{c})$

(ii)' $\Gamma(b) = \{u_2 \in U_2 : f(u_2) \leq b\}$

$$\subseteq N(\bar{u}_2, \tfrac{b+a}{c}).$$

That is $\Gamma(b)$ is strongly convex.

Now if we let

$$D = \{(\bar{u}_2, a) \in U \times R : \psi(u_2) = c\|u_2 - \bar{u}_2\| - a \quad \text{is a sub-derivative of } f(\cdot)\},$$

then

$$f(u_2) = \sup\{c\|u_2 - \bar{u}_2\| - a;\ (\bar{u}_2, a) \in D\}.$$

Now $\Gamma(b) = \cap_D \bar{N}(\bar{u}_2, \frac{b+a}{c})$, so

$$d(u_2,\Gamma(b)) = \sup_D d(u_2,\bar{N}(\bar{u}_2,\frac{b+a}{c})).$$

If we choose $u_2 \notin \Gamma(b)$ and let $b \geqslant \bar{b}$, then

$$d(u_2,\Gamma(b)) = \sup_D d(u_2,\bar{N}(\bar{u}_2,\frac{b+a}{c})).$$

If we let $D' = D(u_2,b)$ and $D'' = D(u_2,\bar{b})$ where

$D(u_2,b) = \{(\bar{u}_2,a) \in D : u_2 \notin \bar{N}(\bar{u}_2,\frac{b+a}{c})\}$, then

$$\begin{aligned} d(u_2,\Gamma(b)) &= \sup_{D'} \{\|u_2 - \bar{u}_2\| - (\tfrac{b+a}{c})\} \\ &= \sup_{D'} \{\|u_2 - \bar{u}_2\| - (\tfrac{b+a}{c}) + \tfrac{(b-\bar{b})}{c}\} - \tfrac{1}{c}(b-\bar{b}) \\ &\leqslant \sup_{D''} \{\|u_2 - \bar{u}_2\| - (\tfrac{\bar{b}+a}{c})\} - \tfrac{1}{c}(b-\bar{b}) \\ &= \sup_{D''} d(u_2,\bar{N}(\bar{u}_2,\tfrac{\bar{b}+a}{c})) - \tfrac{1}{c}(b-\bar{b}) \\ &= d(u_2,\Gamma(\bar{b})) - \tfrac{1}{c}(b-\bar{b}) \end{aligned}$$

the inequality following from $D(u_2,b) \subseteq D(u_2,\bar{b})$. Hence

$d(u_2,\Gamma(b)) \leqslant d(u_2,\Gamma(\bar{b})) - \frac{1}{c}(b-\bar{b})$ and

$\frac{1}{c}(b-\bar{b}) \leqslant d(u_2,\Gamma(\bar{b})) - d(u_2,\Gamma(b)).$

As $u_2 \notin \Gamma(b)$ we have $d(u_2,\Gamma(b)) > 0$, implying

$$\frac{1}{c}(b-\bar{b}) \leqslant d(u_2,\Gamma(\bar{b})).$$

Hence

$$\frac{1}{c}(b-\bar{b}) \leqslant \inf\{d(u_2,\Gamma(\bar{b})) : u_2 \notin \Gamma(b)\},$$

ie.,

$$\bar{N}(\Gamma(\bar{b}), \frac{1}{c}(b-\bar{b})) \subseteq \Gamma(b).$$

Still supposing $b \geqslant \bar{b}$ and supposing $u_2 \notin \Gamma(\bar{b})$, we have

$$d(u_2,\Gamma(\bar{b})) = d(u_2,\hat{u}_2)$$

for some $\hat{u}_2 \in \text{bdd}\ \Gamma(\bar{b})$.

As $\Gamma(\bar{b})$ is closed and convex, if we let $\psi(u_2) = c\|u_2-\bar{\bar{u}}_2\| - a'$ be the sub-derivative of f at $\hat{u}_2$, then

$$\text{(iii)}\quad d(u_2,\Gamma(\bar{b})) = \|u_2 - \bar{\bar{u}}_2\| - (\frac{\bar{b}+a'}{c})$$

$$= d(u_2,\bar{N}(\bar{\bar{u}}_2,\frac{\bar{b}+a'}{c})) .$$

It follows that

$$d(u_2,\Gamma(b)) = \sup_D (u_2,\bar{N}(\bar{u}_2,\frac{b+a}{c}))$$

$$\geqslant d(u_2,\bar{N}(\bar{\bar{u}}_2,\frac{b+a'}{c}))$$

$$= \{\|u_2 - \bar{\bar{u}}_2\| - (\frac{\bar{b}+a'}{c})\} - \frac{1}{c}(b-\bar{b}).$$

We then have via (iii) that

$$d(u_2,\Gamma(b)) \geqslant d(u_2,\Gamma(\bar{b})) - \frac{1}{c}(b-\bar{b})$$

or

$$d(u_2,\Gamma(\bar{b})) - d(u_2,\Gamma(b)) \leqslant \frac{1}{c}(b-\bar{b}).$$

Hence

$$\sup\{d(u_2,\Gamma(\bar{b})) : u_2 \in \Gamma(b)\}$$

$$= \sup\{d(u_2,\Gamma(\bar{b})) : u_2 \in \Gamma(b)/\Gamma(\bar{b})\}$$

$$\leqslant \frac{1}{c}(b-\bar{b}).$$

That is, $\Gamma(b) \subseteq \bar{N}(\Gamma(\bar{b}), \frac{1}{c}(b-\bar{b}))$ and hence is l.s.c. at a rate $\eta(x) = \frac{1}{c}\cdot x$, so that

$$\Gamma(b) = \bar{N}(\Gamma(\bar{b}), \frac{1}{c}(b-\bar{b})) \quad \text{for } b \geqslant \bar{b}.$$

Now, suppose (b) holds.

As $\Gamma(b)$ is strongly convex it is weakly compact and as

$$\Gamma(b) = \bar{N}(\Gamma(\bar{b}), \frac{1}{c}(b-\bar{b})) \quad \text{for } b \geqslant \bar{b}$$

the mapping $b \to \Gamma(b)$ is l.s.c. with respect to the strong topology.

Since l.s.c. with respect to the strong topology implies l.s.c. with respect to the weak topology, Theorem 2.5 (ii) tells us, as $\Gamma(\cdot)$ is weakly l.s.c. that $\text{cl}\, I(b) = \Gamma(\bar{b})$; $\forall\, b \in B^*$. As $I(b)$ is convex this holds in both the strong and weak topologies.

If we choose $\hat{u}_2 \in U_2$ and let $f(\hat{u}_2) = \bar{b}$ then $\hat{u}_2 \in \text{bdd}\, \Gamma(\bar{b})$.

As $\Gamma(\bar{b})$ is strongly convex then $\exists\, r \in R^+$ and $\bar{u}_2 \in U$ s.t.

(i) $\quad \Gamma(\bar{b}) \subseteq \bar{N}(\bar{u}_2, r)$

(ii) $\quad \hat{u}_2 \in \text{bdd}\, \bar{N}(\bar{u}_2, r)$.

We let $r = (\bar{b}+a)/c$ or $a = rc - \bar{b}$. We have
$\Gamma(\bar{b}) = \{u_2 \in U_2 : f(u_2) \leqslant \bar{b}\} \subseteq N(\bar{u}_2, \frac{\bar{b}+a}{c})$.

As $\text{bdd}\, \bar{N}(\bar{u}_2, \frac{a+\bar{b}}{c}) = \{u_2 : c\|u_2 - \bar{u}_2\| - a = \bar{b}\}$, then

$$\hat{u}_2 \in \text{bdd}\, \bar{N}(\bar{u}_2, \frac{a+\bar{b}}{c}) \cap \text{bdd}\, \Gamma(\bar{b})$$

implies

$$\bar{b} = f(\hat{u}_2) = c\|\hat{u}_2 - \bar{u}_2\| - a.$$

All we need to show to complete our proof is

$$\Gamma(b) \subseteq \bar{N}(\bar{u}_2, \frac{a+b}{c}) : \forall\, b \in B^*,$$

for if we assume this and suppose that

$$f(u_2) < c\|u_2 - \bar{u}_2\| - a$$

for some $u_2 \in U_2$, then

$$b = f(u_2) \in B^*$$

and

$$u_2 \in \Gamma(b).$$

But

$$u_2 \notin \bar{N}(\bar{u}_2, \frac{a+b}{c}),$$

a contradiction. As a consequence

$$c\|u_2 - \bar{u}_2\| - a \leqslant f(u_2) \ : \ \forall\, u_2 \in U_2$$

and f is Φ_c sub-differentiable on U_2. So to round off the proof we note

$$b \to \Gamma(b)$$

is metrically increasing rate $\frac{x}{c}$; $\Gamma(b) = \bar{N}(\Gamma(\bar{b}), \frac{1}{c}(b-\bar{b}))$ for $b \geqslant \bar{b}$. As $\Gamma(\bar{b}) \subseteq \bar{N}(\bar{u}_2, \frac{\bar{b}+a}{c})$ we have

$$\Gamma(b) \subseteq \bar{N}(\bar{u}_2, \frac{\bar{b}+a}{c} + \frac{1}{c}(b-\bar{b}))$$

$$= \bar{N}(\bar{u}_2, \frac{b+a}{c}).$$

For $b < \bar{b}$ we have

$$\bar{N}(\Gamma(b), \frac{1}{c}(\bar{b}-b)) = \Gamma(\bar{b})$$

$$\subseteq N(\bar{u}_2, \frac{a+\bar{b}}{c}).$$

Hence

$$\Gamma(b) \subseteq N(\bar{u}_2, \frac{a+\bar{b}}{c} - \frac{1}{c}(\bar{b}-b))$$

ie.

$$\Gamma(b) \subseteq \bar{N}(\bar{u}_2, \frac{a+b}{c}).$$

□

<u>Corollary 2.7</u> : Let U_2 be a weakly compact subset of a reflexive Banach space which is locally F normed. Then

(a) $f : U_2 \to R$ is Φ_c convex iff

(b) $\Gamma(b)$ is convex and $b \to \Gamma(b)$ is metrically increasing at a rate $\eta(x) = x/c$.

Proof : Suppose statement (a) holds. As U_2 is compact and all $\psi \in \Phi_c$ are continuous, we have from Corollary 2.2 that

$$f(u_2) = \sup\{h(u_2) : h; \Phi_c \text{ subdiff. and } h << f\}.$$

In fact Proposition 2.2 tells us, along with the separability of U_2 (as it is compact metric), that $\exists\ h_i : U_2 \to R$; $i \in I$, Φ_c- sub-differentiable and continuous s.t.

$$f(u_2) = \sup\{h_i (u_2) = \sup\{\psi_j (u_2) : j=1,..,N(i)\}: i \in I\}.$$

Thus

$$\begin{aligned}\Gamma(b) &= \{u_2 : f(u_2) \leqslant b\} \\ &= \{u_2 : \sup_I h_i (u_2) \leqslant b\} \\ &= \bigcap_{i \in I} \{u_2 : h_i (u_2) \leqslant b\} \\ &= \bigcap_{i \in I} \bigcap_{j=1}^{N(i)} \{u_2 : \psi_i (u_2) \leqslant b\},\end{aligned}$$

which, from Theorem 2.3, is convex weakly compact. We note also in passing that any convex set may be produced in this fashion. Now as $h_i(\cdot)$ is Φ_c sub-differentiable $\forall\ i \in I$ we can say

$$\bar{N}(\Gamma_i (b), \frac{1}{c}(\bar{b}-b)) = \Gamma_i (\bar{b});\ \bar{b} \geqslant b$$

where $\Gamma_i (b) = \{u_2 : h_i (u_2) \leqslant b\}$. Hence

$$\begin{aligned}\bar{N}(\Gamma(b), \frac{1}{c}(\bar{b}-b)) &= \bar{N}(\bigcap_{i \in I} \Gamma_i (b), \frac{1}{c}(\bar{b}-b)) \\ &= \bigcap_{i \in I} \bar{N}(\Gamma_i (b), \frac{1}{c}(\bar{b}-b)) \\ &= \bigcap_{i \in I} \Gamma_i (\bar{b}) = \Gamma(\bar{b}).\end{aligned}$$

Now suppose (b) holds. As $\Gamma(\bar{b})$ is also weakly compact convex, the reflexivity of U_2 and Theorem 2.3 imply $\exists\ r_i \in R^+$ s.t.

$$\Gamma(\bar{b}) = \bigcap_{i \in I} \bar{N}(\bar{u}_2^i, r_i).$$

By letting $r_i = \dfrac{a_i + \bar{b}}{c}$ we get

$$\Gamma(\bar{b}) = \bigcap_{i \in I} \{u_2 : c\|u_2 - \bar{u}_2^i\| - a_i \leqslant \bar{b}\}$$

$$= \{u_2 : \sup_{i \in I} \psi_i(u_2) \leqslant \bar{b}\}.$$

Now as

$$\bar{N}(\Gamma(b), \tfrac{1}{c}(\bar{b}-b)) = \Gamma(\bar{b}); \ \bar{b} \geqslant b,$$

$b \in B^*$, then $\Gamma(b) \neq \phi$ implies

$$\bar{N}(\Gamma(b), \tfrac{1}{c}(\bar{b}-b)) = \bigcap_{i \in I} \bar{N}(\bar{u}_2^i, \frac{a_i + \bar{b}}{c}).$$

Hence

$$\Gamma(b) = \bigcap_{i \in I} \bar{N}(\bar{u}_2^i, \frac{a_i + \bar{b}}{c} - \tfrac{1}{c}(\bar{b}-b))$$

$$= \bigcap_{i \in I} \bar{N}(u_2^i, \frac{a_i + b}{c}).$$

Now if we suppose $\bar{b} < b$ we have

$$\bar{N}(\Gamma(\bar{b}), \tfrac{1}{c}(b-\bar{b})) = \Gamma(b)$$

or

$$\bar{N}(\bigcap_{i \in I} \bar{N}(\bar{u}_2^i, \frac{a_i + \bar{b}}{c}), \tfrac{1}{c}(b-\bar{b})) = \Gamma(b)$$

$$= \bigcap_{i \in I} \bar{N}(\bar{u}_2^i, \frac{a_i + \bar{b}}{c} + \tfrac{1}{c}(b-\bar{b})) = \Gamma(b),$$

so

$$\bigcap_{i \in I} \bar{N}(\bar{u}_2^i, \frac{a_i + b}{c}) = \Gamma(b).$$

We have $\Gamma(b) = \{u_2 \in U_2 : \sup_{i \in I} c\|\bar{u}_2^i - u_2\| - a_i \leqslant b\}$.

Arguing as before, this holding for all $b \in B^*$ implies

$$f(u_2) = \sup_{i \in I} c\|\bar{u}_2^i - u_2\| - a_i$$

and hence that f is Φ_c convex. □

§2.2 Approximation of Multi-valued Mappings

To complete this chapter we turn to the topic of approximation of multi-valued mappings. This has relation to fixed point theorems for multi-valued mappings. We begin with some notation and definitions. If (U_1,d_1) is a metric space and (U_2,d_2) is a metric we know that $U_1 \times U_2$ is a metric space with a metric

$$d((u_1,u_2),(\bar{u}_1,\bar{u}_2)) = \max\{d_1(u_1,\bar{u}_1),\ d_2(u_2,\bar{u}_2)\}.$$

As usual we define, for $A \subseteq U_1 \times U_2$,

$$d((u_1,u_2),A) = \inf\{d(u_1,u_2),(\bar{u}_1,\bar{u}_2));\ (\bar{u}_1,\bar{u}_2) \in A\}.$$

The separation of two subsets $A,\ B \subseteq U_1 \times U_2$ is given by

$$d^*(B,A) = \sup\{d(u_1,u_2),A);\ (u_1,u_2) \in B\}.$$

These sets may be graphs of multi-valued mappings, ie.,

$$G = \{(u_1,u_2) : u_1 \in U_1,\ u_2 \in \Gamma(u_1)\}.$$

We state a slightly reworded statement of part of the content of Theorem 1 of reference [14].

Theorem 2.7 : Suppose (U_1,d_1) is a compact metric space and (U_2,d_2) is a metric space. If $\Gamma : U_1 \to K(U_2)$ is u.s.c. (or equivalently u.H.s.c.), then we can approximate Γ from above by l.s.c. multi-valued mappings

$$\Gamma_\varepsilon : U_1 \to K(U_2) \text{ s.t. } \bigcap_{\varepsilon>0} \Gamma_\varepsilon(u_1) = \Gamma(u_1)$$

and

$$d^*(F_\varepsilon,G) \leqslant \varepsilon \quad \forall\ \varepsilon > 0\ ,$$

where F_ε is the graph of Γ_ε ,

G is the graph of Γ.

Proof : We argue identically to the first part of Theorem 1 of reference [14]. In doing so, we define

$$\rho(u_1,\varepsilon) = \sup\{\delta \leqslant \varepsilon/2 : \exists\, u_1' \in N(u_1,\delta)$$

s.t.

$$\Gamma(N(u_1,\delta)) \subseteq N(\Gamma(u_1'), \varepsilon/2)\}$$

and show it is bounded away from zero on U_1. We then go on to show that the mapping $\Gamma_\varepsilon(u_1) = \mathrm{cl}\ \Gamma(N(u_1,\xi_1))$, where $0 < \xi_1 < \inf\{\rho(u_1,\varepsilon) : u_1 \in U_1\}$ is l.s.c. on U_1. We finish by noting that, $\forall\ u_1 \in U_1$, by the definition of $\rho(u_1,\varepsilon)$ we have that $\exists\ u_1' \in N(u_1,\xi_1)$ s.t.

$$\Gamma_\varepsilon(u_1) = \mathrm{cl}\,\Gamma(N(u_1,\xi_1)) \subseteq N(\Gamma(u_1'), \varepsilon/2)$$

and as $\xi_1 < \varepsilon/2$ we have

$$d_1(u_1,u_1') < \varepsilon/2.$$

This implies

$$d^*(F_\varepsilon,G) = \sup_{F\varepsilon} \inf_G \max\{d_1(u_1,\bar{u}_1),d_2(u_2,\bar{u}_2)\}$$

$$\leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ □

We note in passing that, if we assume U_1 is compact, by our definition of $\rho(u_1,\varepsilon)$ we have $\Gamma_\varepsilon(u_1) \subseteq N(\Gamma(u_1'), \varepsilon/2)$; $u_1' \in N(u_1,\xi_1)$. Hence

$$\Gamma_\varepsilon(U_1) \subseteq N(\Gamma(N(U_1,\xi_1)), \varepsilon/2)$$

$$\subseteq \bar{N}(\Gamma(\bar{N}(U_1, \varepsilon/2), \varepsilon/2) \quad : (S).$$

This in general does not tell us whether $\Gamma_\varepsilon(U_1)$ itself is compact. We need the following.

Lemma 2.2 : Suppose U is a linear, locally compact normed space. Then $\exists\ r \in R^+;\ r > 0$ s.t. $\forall\ 0 \leqslant \varepsilon < r;\ \forall\ u \in U$ we have $\bar{N}(u,\varepsilon)$ compact.

Proof : There exists a basis of pre-compact neighbourhoods of zero which generates the topology of the space. Let V be a compact neighbourhood of zero. Then for r sufficiently small $N(0,r) \subseteq V$ and hence is relatively compact. So for $0 < \varepsilon < r$

$\bar{N}(0,\varepsilon)$ is compact and as U is normed and linear,

$\bar{N}(u,\varepsilon) = u + \bar{N}(0,\varepsilon)$ is compact. □

Lemma 2.3 : Suppose U is a linear, locally compact, normed space and $S \subseteq U$ is compact. Then for $0 < \varepsilon < r$, r sufficiently small, $\bar{N}(S,\varepsilon)$ is compact.

Proof : If we can show sequential compactness of $\bar{N}(S,\varepsilon)$ we have shown compactness. Let $\{u_n\}_{n=1}^{\infty} \subseteq \bar{N}(S,\varepsilon)$. Then $\exists\ \bar{u}_n \in S$ s.t. $\|u_n - \bar{u}_n\| \leqslant \varepsilon$ $\forall$ n. By the compactness of S, $\exists$ a convergent subsequence, converging to $\bar{u} \in S$, $\{\bar{u}_n\}$ (say) after renumbering. Now for $n \geqslant N(\delta)$ we have

$$\|u_n - \bar{u}\| \leqslant \|u_n - \bar{u}_n\| + \|\bar{u}_n - \bar{u}\| < \varepsilon + \delta < r$$

for δ sufficiently small and hence $\{u_n : n \geqslant N(\delta)\} \subseteq \bar{N}(\bar{u},\varepsilon+\delta)$, a compact set. As a consequence a convergent subsequence exists, which is, of course, a convergent subsequence of our original sequence $\{u_n\}_{n=1}^{\infty}$. □

When the conditions of this Lemma hold for the spaces U_1 and U_2 we can from statement (S) deduce that the range of Γ_ε is contained in a compact subset of U_2. If we introduce F-norms we can say a little more.

Proposition 2.7 : Suppose U and $\bar{U}$ are Banach spaces each of which satisfy one of the following

(i) the conditions of Lemma 2.3

(ii) is reflexive and the weakly compact sets are locally F-normable.

We let $U_1 \subseteq U$ and $\Gamma : U_1 \to CV(\bar{U})$.

When (i) holds for either, or both, of U_1 and $\bar{U}$ we consider that the corresponding space(s) U_1 and/or $\bar{U}$ are endowed with the strong topology.

When (ii) holds for either, or both, of U and $\bar{U}$ we consider that the corresponding space(s) U_1 and/or $\bar{U}$ are endowed with the weak topology.

Suppose

(a) U_1 is compact, and

(b) $\Gamma : U_1 \to CV(\bar{U})$ is an u.s.c. multi-function.

Then there exists a multi=function $\Gamma_\varepsilon : U_1 \to CV(\bar{U})$, l.s.c. with respect to the above topologies on U_1 and $\bar{U}$, which approximates Γ is the sense of Theorem 2.7.

Proof : If (i) holds we let $d_1(u_1,\bar{u}_1) = \|u_1 - \bar{u}_1\|$ and if (ii) holds for U we let $d_1(u_1,\bar{u}_1) = \|u_1 - \bar{u}_1\|^*$. In any case, since Γ is u.s.c. from proposition 2.4, we have $\Gamma(U_1) \subseteq \Gamma_\varepsilon(U_1) \subseteq U_2$, a convex subset of $\bar{U}$ compact with respect to the relevant topology.

This follows from the statement (S) and Lemma 2.3 in the case of (i) and in the case of (ii) from the fact that any closed bounded set is compact. In the case of (i) U_2 is already a metric space and in case (ii) we may make it metric by imposing an F-norm on it since it is weakly compact. We ensure the mapping Γ_ε produced via this process using Theorem 2.7 is convex closed valued by taking the convex closure of it, the resultant being once again l.s.c.. It is easily seen that this does not upset the approximation properties as $\Gamma(\cdot)$ is convex closed valued as well. □

Now take these mappings and rewrite our approximation problem as follows. As $\Gamma(u_1) = \{u_2 : d(u_2,\Gamma(u_1)) \leqslant 0\}$ we say equivalently

$$g_\varepsilon(u_1,u_2) = d(u_2,\Gamma_\varepsilon(u_1)) \leqslant d(u_2,\Gamma(u_1)) = f(u_1,u_2)$$

and

$$g_\varepsilon \uparrow f \text{ as } \varepsilon \to 0.$$

We note the following.

Theorem 2.8 : Suppose U_2 is a metric space. Let 2^{u_2} have the topology generated by this uniform structure (see Definition 1.13). Then a necessary and sufficient condition that $\Gamma : U_1 \to 2^{u_2}$ is continuous is that the family of mappings $\{u_1 \to d(u_2,\Gamma(u_1)) : u_2 \in U_2\}$ be equi-continuous.

Proof : Theorem 2.1 of reference [16]. □

This opens up the question of whether we can select an $f_\varepsilon : U_1 \times U_2 \to R^n$ s.t. f_ε looks sufficiently like $d(u_2,T_\varepsilon(u_1))$ and

$$g_\varepsilon(u_1,u_2) \leqslant f_\varepsilon(u_1,u_2) \leqslant f(u_1,u_2),$$

where the family

$$\{u_1 \to f_\varepsilon(u_1,u_2) : u_2 \in U_2\}$$

is equi-continuous. If we can do this, then we can say $\Gamma(u_1)$ can be approximated above, in the same sense that Γ_ε does, by a continuous multi-valued mapping. It turns out for the case when $\Gamma(\cdot)$ is convex valued that the Φ_c convex mappings are those which look sufficiently like $d(u_2,T_\varepsilon(u_1))$.

Lemma 2.4 : Suppose $C \subseteq U_2$ is Φ_c convex set, U_2 being a Banach space. Then $\bar{N}(C,\varepsilon)$ is Φ_c convex $\forall\ \varepsilon > 0$.

Proof : As C is Φ_c convex, $\exists$ a set $D \subseteq R \times U_2$ s.t.

$$C = \cap\{u_2 : c\|u_2 - \bar{u}_2\| - a \leqslant \alpha;\ (a,\bar{u}_2) \in D\}.$$

Now

$$\bar{N}(C,\varepsilon) = \{u_2 : d(u_2, \cap_D\{\hat{u}_2 : c\|\hat{u}_2 - \bar{u}_2\| - a \leq \alpha\}) \leq \varepsilon\}$$

$$= \{u_2 : \sup_D d(u_2,\{\hat{u}_2 : c\|\hat{u}_2 - \bar{u}_2\| - a \leq \alpha\}) \leq \varepsilon\}$$

$$= \{u_2 : \sup_D d(u_2,\bar{N}(\bar{u}_2,(\alpha+a)/c)) \leq \varepsilon\}$$

$$= \cap_D\{u_2 : d(u_2,\bar{N}(\bar{u}_2,(\alpha+a)/c) \leq \varepsilon \}$$

$$= \cap_D \bar{N}(\bar{u}_2,(\alpha+a)/c + \varepsilon)$$

$$= \{u_2 : \sup_D c\|u_2 - \bar{u}_2\| - a \leq \alpha + \varepsilon \},$$

a Φ_c convex set. □

We now formulate our problem stated above as a selection problem.

If we define a multi-valued mapping

$$\psi(u_1,u_2) = \{x \in R : d(u_2,\Gamma_\varepsilon(u_1)) - 2\varepsilon < x < d(u_2,\Gamma(u_1)) - \varepsilon\}$$

$$= \{x \in R : d(u_2,\bar{N}(\Gamma_\varepsilon(u_1),2\varepsilon)) < x < d(u_2,\bar{N}(\Gamma(u_1),\varepsilon))\}$$

can we select from this an appropriate function?

<u>Lemma 2.5</u> : Suppose U_2 is a Banach space. If C is a Φ_c convex set, then $u_2 \to cd(u_2,C)$ is Φ_c convex function.

<u>Proof</u> : Since C is Φ_c convex, $\exists\ D \subseteq R \times U_2$ s.t.

$$\{u_2 : \sup_D c\|u_2 - \bar{u}_2\| - a \leq \alpha\} = C .$$

Hence letting $D' = D(u_2,\alpha)$ we have

$$d(u_2,C) = \sup_{D'} d(u_2,\bar{N}(\bar{u}_2,\tfrac{\alpha+a}{c}))$$

$$= \sup_{D'}\|u_2 - \bar{u}_2\| - (\tfrac{\alpha+a}{c}) .$$

Thus

$$c.d(u_2,C) = \sup_{D'} c\|u_2 - \bar{u}_2\| - (\alpha+a).$$ □

Hence the natural choice of convexity is Φ_1 convexity which would make

$$u_2 \to d(u_2,\Gamma_\varepsilon(u_1)) - 2\varepsilon;\ \Phi_1 \text{ convex.}$$

If we fix u_1 and suppose the conditions of proposition 2.7 hold, then we may restrict the above function of u_2 to a compact domain, since $\Gamma(\cdot)$ would have a compact range for ε sufficiently small. Supposing this, then proposition 2.2 tells us we can, for each u_1, select mappings of the sort

$$\sup_{i=1,..,n} \|u_2 - \bar{u}_2^i\| - a_i = h(u_2)$$

s.t. $g_\varepsilon(u_1,u_2) - \varepsilon < h(u_1,u_2) < f(u_1,u_2)$. This prompts us to define a new multi-valued mapping

$$\psi(u_1) = \{h(u_2) : h(u_2) = \sup_{i=1,...,n} \|u_2 - \bar{u}_2^i\| - a_i$$

s.t. $h(\cdot)$ is a selection of $\psi(u_1,\cdot)\}$.

Definition 2.10 : Let U_1,U_2 be metric spaces, $\psi(\cdot,u_2) : U_1 \to V(R)$. Then $\{\psi(\cdot,u_2) : u_2 \in U_2\}$ is said to be equi-lower semi-continuous iff $\forall\ \varepsilon > 0, \exists\ \delta(u_1^0)$ s.t. if $y \in \phi(u_1^0,u_2)$ then

$$\psi(u_1,u_2) \cap N(y,\varepsilon) \neq \phi$$

$$\forall\ (u_1,u_2) \in N(u_1^0,\delta) \times U_2.$$

Proposition 2.8 : Let U, $\bar{U}$ satisfy the conditions of proposition 2.7 and let U_1, U_2 be compact sets $U_1 \subseteq U$ and $U_2 \subseteq \bar{U}$ s.t.

$\Gamma : U \to KV(\bar{U})$ is u.s.c.

$\Gamma(U_1) \subseteq U_2$ and

$\Gamma_\varepsilon(U_1) \subseteq U_2$,

where Γ_ε is the l.s.c. approximation in graph of Γ.

If ϕ is defined as above, then

$\{\psi(\cdot,u_2) : u_2 \in U_2\}$ is an equi-l.s.c. family.

Proof : As Γ is u.s.c. on U_1, $\forall\ \bar{\varepsilon} > 0\ \exists\ \delta(u_1^0) > 0$ s.t. $\forall\ u_1 \in N(u_1^0,\delta)$ $\Gamma(u_1) \subseteq N(\Gamma(u_1^0),\bar{\varepsilon}) \subseteq \bar{N}(\Gamma(u_1^0),\bar{\varepsilon})$. Hence

$$d(u_2,\Gamma(u_1)) \geqslant d(u_2,\bar{N}(\Gamma(u_1^0),\bar{\varepsilon})$$

$$\geqslant d(u_2,\Gamma(u_1^0)) - \bar{\varepsilon}$$

$\forall\ u_2 \in U_2$.

Thus

$$\{d(u_2,\Gamma(u_1)) : u_2 \in U_2\}$$

is an equi-l.s.c. family of single valued mappings. We can show by an identical argument that for $\Gamma_\varepsilon(\cdot)$ l.s.c.

$$\{d(u_2,\Gamma(u_1)) : u_2 \in U_2\}$$

is an equi-upper semi continuous family of single valued mappings, ie. $\forall\ \varepsilon > 0$ and $u_2 \in U_2$; $u_1^0 \in U_1, \exists\ \delta(u_1^0) > 0$ s.t.

$$d(u_2,\Gamma_\varepsilon(u_1^0)) + \bar{\varepsilon} \geqslant d(u_2,\Gamma_\varepsilon(u_1)), \forall\ u_1 \in N(u_1^0,\delta).$$

If we let $\delta^*(u_1^0) = \min(\delta(u_1^0),\bar{\delta}(u_1^0)) > 0$, then $\forall\ \bar{\varepsilon} > 0\ \exists\ \delta^*(u_1^0) > 0$ s.t. for $u_1 \in N(u_1^0,\delta^*)$

$$\{x : d(u_2,\Gamma_\varepsilon(u_1^0)) - \varepsilon < x < d(u_2,\Gamma(u_1^0))\}$$

$$\subseteq \{x : d(u_2,\Gamma_\varepsilon(u_1^0)) - \varepsilon - \bar{\varepsilon} < x < d(u_2,\Gamma(u_1)) + \bar{\varepsilon}\}$$

$$= N(\psi(u_1,u_2),\bar{\varepsilon}).$$

Namely

$$\psi(u_1^0,u_2) \subseteq N(\psi(u_1,u_2),\bar{\varepsilon});\ \forall\ u_2 \in U_2,$$

or for $y \in \psi(u_1^0,u_2)$ we have

$$\psi(u_1,u_2) \cap N(y,\bar{\varepsilon}) \neq \phi. \qquad \square$$

<u>Lemma 2.6</u> : Let $\bar{U}$ be a reflexive Banach space and $\{f_i : i=1,2,3\}$ are Φ_c convex function where the Dom $f_i = U_2$ (compact) i=1,2,3. Suppose f_3 is continuous and $f_3(u_2) < \min\{f_1(u_2),f_2(u_2)\} = h(u_2)$. Then $\exists\, D \subseteq R \times \bar{U}$ s.t.

$$f_3(u_2) < \sup_{D} c\|u_2 - \bar{u}_2\| - a \leqslant h(u_2),$$

and in fact as U_2 is compact we may choose the set D to be finite.

<u>Proof:</u> As f_1, f_2 are Φ_c convex $\exists\, D_1, D_2 \subseteq R \times \bar{U}$ s.t.

$$f_1(u_2) = \sup_{D_1} c\|u_2 - \bar{u}_2\| - a,$$

$$f_2(u_2) = \sup_{D_2} c\|u_2 - u_2'\| - a'.$$

Hence

$$\{u_2 \in U_2 : h(u_2) \leqslant b\};\ b \in B^*$$

$$= \{u_2 \in U_2 : \min\{\sup_{D_1} c\|u_2 - \bar{u}_2\| - a,\ \sup_{D_2} c\|u_2 - \bar{u}_2'\| - a'\} \leqslant b\}$$

$$= \{u_2 \in U_2 : \sup_{D_1,D_2} \min\{c\|u_2 - \bar{u}_2\| - a,\ c\|u_2 - \bar{u}_2'\| - a'\} \leqslant b\}$$

$$= \cap_{D_1,D_2} \bar{N}(\bar{u}_2,\tfrac{b+a}{c}) \cup \bar{N}(u_2',\tfrac{b+a'}{c}).$$

Let

$$D = \{(a^*,u_2^*) \in R \times \bar{U} : \forall\, b \in B^*;\ (a,\bar{u}_2) \in D;$$

$$(a',u_2') \in D_2;\ \bar{N}(u_2^*,\tfrac{a^*+b}{c}) \supseteq \bar{N}(\bar{u}_2,\tfrac{a+b}{c}) \cup \bar{N}(u_2',\tfrac{a'+b}{c})\}.$$

As f_3 is Φ_c convex, $\exists\, D_3 \subseteq R \times \bar{U}$ s.t.

$$f_3(u_2) = \sup_{D_3} c\|u_2 - u_2''\| - a''.$$

Since

$$f_3(u_2) < \min\{f_1(u_2),f_2(u_2)\},\ \forall\, u_2 \in U_2,$$

$\forall\, (a'',u_2'') \in D_3$ we have $\forall\, b \in B^*$

$$N(u_2'',\frac{a''+b}{c}) \supseteq \bar{N}(\bar{u}_2,\frac{a+b}{c}) \cup \bar{N}(u_2',\frac{a'+b}{c})$$

for all $(a,\bar{u}_2) \in D_1$ and $(a',u_2') \in D_2$. Hence

$$D_3 \subseteq D$$

and

$$f_3(u_2) \leqslant \sup_D c\|u_2 - u_2^*\| - a^*.$$

Furthermore let us suppose that

$$f_3(\hat{u}_2) = \sup_D c\|\hat{u}_2 - u_2^*\| - a^*$$

for some $\hat{u}_2 \in U_2$. We let

$$f_3(\hat{u}_2) = b$$

$$\hat{u}_2 \in N(u_2^*,\frac{a^*+b}{c}); \; \forall\, (a^*,u_2^*) \in D$$

iff

$$\sup_D c\|\hat{u}_2 - u_2^*\| - a^* \leqslant b = f_3(\hat{u}_2).$$

Since the assertion that $\exists\ (a^*,u_2^*) \in D$ s.t. $\hat{u}_2 \notin \bar{N}(u_2^*,\frac{a^*+b}{c})$ is equivalent to $\sup_D c\|u_2 - u_2^*\| - a^* > b = f_3(\hat{u}_2)$ which we assume doesn't happen, $\nexists\ (a^*,u_2^*) \in D$ s.t. $\hat{u}_2 \notin \bar{N}(u_2^*,\frac{a^*+b}{c})$. By the definition of D we have $\forall (a^*,u_2^*) \in D$

(a) $$\bar{N}(u_2^*,\frac{a^*+b}{c}) \supseteq \{u_2 \in U_2 : h(u_2) \leqslant b\}$$

$$\supseteq \Gamma_1(b) \cup \Gamma_2(b)\ ,$$

where $\Gamma_1(b) = \{u_2 \in U_2 : f_1(u_2) \leqslant b\}$

$\Gamma_2(b) = \{u_2 \in U_2 : f_2(u_2) \leqslant b\}$.

Now as $f_3(\hat{u}_2) = b$ and f_3 is Φ_c convex, we have

(b) $\hat{u}_2 \in \text{bdd}\ \Gamma_3(b) = \{u_2 : f_3(u_2) = b\}$.

From (a) we know that $\hat{u}_2$ must be inside any convex set containing $\Gamma_1(b) \cup \Gamma_2(b)$ and we know from (b) that $\hat{u}_2$ is on the boundary of a particular convex set containing $\Gamma_1(b) \cup \Gamma_2(b)$. Hence $\hat{u}_2 \in \text{bdd } M(b)$ the minimal convex set containing $\Gamma_1(b) \cup \Gamma_2(b)$. As M(b) is convex we must have one of the following cases:

(i) $\hat{u}_2 \in \text{bdd } \Gamma_1(b)$;

(ii) $\hat{u}_2 \in \text{bdd } \Gamma_2(b)$;

(iii) $\hat{u}_2 \in$ plane touching bdd $\Gamma_1(b)$ and bdd $\Gamma_2(b)$.

If (i) or (ii) holds, then we have

$$b = f_3(\hat{u}_2) < \min\{f_1(\hat{u}_2),\ f_2(\hat{u}_2)\} \leqslant b$$

which is impossible. If we have (iii) occurring, then, as $\Gamma_3(b)$ is convex and $\hat{u}_2 \in \text{bdd } \Gamma_3(b)$, we must have this particular plane as part of the boundary of $\Gamma_3(b)$ and hence the boundary of $\Gamma_3(b)$ must touch the boundary of both $\Gamma_1(b)$ and $\Gamma_2(b)$. That is, $\exists\ u_2 \in \text{bdd } \Gamma_3(b)$ s.t. $u_2 \in \text{bdd } \Gamma_1(b)$ (say), ie.,

$$b = f_3(u_2) < \min\{f_1(u_2),\ f_2(u_2)\} \leqslant b\,,$$

again a contradiction.

Finally we note that as U_2 is compact, the Φ_c convex functions form a continuous lattice and since

$$f_3(u_2) < \sup_D c\|u_2^* - u_2\| - a^* = g(u_2),\ \ \forall\, u_2 \in U_2\,,$$

where $g(\cdot)$ is a Φ_c convex function, we can apply Proposition 2.2 to deduce the existence of a finite approximation. □

Lemma 2.7 : If $\psi : U \to 2^{\bar{u}}$ then the following are equivalent :

(a) ψ is l.s.c. multi-valued mapping ,

(b) if $u \in U$, $\bar{u} \in \psi(u)$ and V is a neighbourhood of $\bar{u}$ in $\bar{U}$, then $\exists$ a neighbourhood N of u s.t. $\forall\, u' \in N$ $\psi(u') \cap V \neq \phi$.

Proof : Reference [8] Proposition 2.1. □

Lemma 2.8 : Suppose $f : U_2 \to R$ is Φ_c convex $\psi \in \Phi_c$ and $B^* = \{b : \Gamma(b) \neq \phi\}$. Then

(a) $\Gamma(b) = \{u_2 \in U_2 : f(u_2) \leq b\} \subseteq \{u_2 \in U_2 : \psi(u_2) \leq b\}, \forall\, b \in B^*$

iff

(b) $\psi(u_2) \leq f(u_2); \forall\, u_2 \in U_2$.

Proof : The implication (b) $\to$ (a) is obvious. Suppose (a) holds and let $u_2 \in U_2$. Then $f(u_2) = b \in R$.

If $b = f(u_2) < \psi(u_2)$, then $u_2 \notin \{u_2 : \psi(u_2) \leq b\}$, a contradiction. □

Proposition 2.9 : Suppose for $U_1 \subseteq U$ and $U_2 \subseteq \bar{U}$ we have:

(i) U_1 is a compact metric space;

(ii) U_2 is a compact subset of a reflexive Banach space endowed with a norm (not necessarily the norm on $\bar{U}$);

(iii) the multi-functions $\Gamma, \Gamma_\varepsilon : U_1 \to KV(U_2)$, where $\Gamma(\cdot)$ is u.s.c. and $\Gamma_\varepsilon(\cdot)$ is l.s.c. with respect to the corresponding metrics on U_1 and U_2, and

(iv) $\Gamma(\cdot) \subseteq \Gamma_\varepsilon(\cdot)$.

We define

$$\psi(u_1,u_2) = \{x \in R : d(u_2,\Gamma_\varepsilon(u_1)) - 2\varepsilon < x \leq d(u_2,\Gamma(u_1)) - \varepsilon\}$$

and

$$\hat{\psi}(u_1) = \{h(\cdot) \; ; \; h(u_2) = \sup_{i=1,..,n} \|u_2 - u_2^i\| - a_i$$

$$u_2^i \in \bar{U} \; ; \; a \in R \; ; \; n \in Z^+ \text{ and } h(\cdot) \text{ a selection of } \psi(u_1,\cdot)\}.$$

Then $\hat{\psi}(\cdot)$ is a l.s.c. multi-valued mapping from U_1 to the subsets of $C(U_2)$, the space of continuous functions on U_2 endowed with the supremum norm.

Proof : We define for $\varepsilon > 0$ a multi-valued mapping with half-open interval image sets in R;

$$A(\varepsilon,\alpha,u_1,u_1^0,u_2) = \psi(u_1,u_2) \cap N(\alpha(u_1^0,u_2), \varepsilon),$$

where

$$\alpha(u_1^0,\cdot) \in \hat{\psi}(u_1^0),$$

$$u_1^0 \in U_1.$$

As $\{\psi(\cdot,u_2) : u_2 \in U_2\}$ is equi-l.s. continuous family, $\forall\ \varepsilon > 0$ $\exists\ \delta(u_1^0) > 0$ s.t.

$$\psi(u_1,u_2) \cap N(y,\varepsilon) \neq \phi,$$

$$\forall\ y \in \psi(u_1^0,u_2)$$

and

$$\forall\ (u_1,u_2) \in N(u_1^0,\delta) \times U_2.$$

Since

$$\alpha(u_1^0,u_2) \in \psi(u_1^0,u_2);\ \forall\ u_2 \in U_2$$

this implies,

$$A(\varepsilon,\alpha,u_1,u_1^0,u_2) \neq \phi$$

for

$$(u_1,u_2) \in N(u_1^0,\delta) \times U_2 \cdot$$

So if we take $0 < \bar{\delta} < \delta$ we have

$$A(\varepsilon,\alpha,u_1,u_1^0,u_2) \neq \phi$$

on the metric space

$$\bar{N}(u_1^0,\bar{\delta}) \times U_2 \subseteq U_1 \times U_2.$$

As $u_2 \to A(\varepsilon,\alpha,u_1,u_1^0,u_2)$ is half open interval valued in R we have

$$\text{Int } A(\varepsilon,\alpha,u_1,u_1^0,u_2) \neq \phi$$

iff $A(\varepsilon,\alpha,u_1,u_1^0,u_2) \neq \phi$. Hence we have the following;

(i) $\inf A(\varepsilon,\alpha,u_1,u_1^0,u_2) < \sup A(\varepsilon,\alpha,u_1,u_1^0,u_2)$;

(ii) $\inf A(\varepsilon,\alpha,u_1,u_1^0,u_2) = \sup\{d(u_2,\Gamma_\varepsilon(u_1)) - 2\varepsilon,\ \alpha(u_1^0,u_2) - \varepsilon\}$ and since $u_2 \to d(u_2,\Gamma_\varepsilon(u_1))$ is Φ_1 convex and $u_2 \to \alpha(u_1^0,u_2)$ is Φ_1 convex, so is $\inf A(\varepsilon,\alpha,u_1,u_2^0,u_2)$. We also have;

(iii) $\sup A(\varepsilon,\alpha,u_1,u_1^0,u_2)$

$$= \inf \{d(u_2,\Gamma_\varepsilon(u_1)) - 2\varepsilon,\ \alpha(u_1^0,u_2) - \varepsilon\}.$$

As a consequence, (i), (ii), (iii) and Lemma 2.6 allow us to select as follows.

$$\inf A(\varepsilon,\alpha,u_1,u_1^0,u_2) < \sup_{i=1,..,n} \|u_2 - \bar{u}_2^i\| - a_i \leqslant \sup\ A(\varepsilon,\alpha,u_1,u_1^0,u_2)$$

Hence we can say

$$\forall\ u_1 \in \bar{N}(u_1^0,\bar{\delta})\ \exists \text{ a } \Phi_1 \text{ selection.}$$

$$\bar{\alpha}(u_1,\cdot) = \sup_{i=1,..,n} \|u_2 - \bar{u}_2^i\| - a_i$$

of $A(\varepsilon,\alpha,u_1,u_1^0,\cdot)$ (ie. $\bar{\alpha}(u_1,\cdot) \in A(\varepsilon,\alpha,u_1,u_1^0,\cdot) \neq \phi$). Now $\bar{\alpha}(u_1,\cdot)$ is a continuous function of u_2 and as $\bar{\alpha}(u_1,\cdot) \in \psi(u_1,u_2) \cap N(\alpha(u_1^0,u_2),\varepsilon)$ we have

(iv) $\bar{\alpha}(u_1,u_2) \in \psi(u_1,u_2);\ \forall\ (u_1,u_2) \in \bar{N}(u_1^0,\bar{\delta}) \times U_2$

$\alpha(u_1,\cdot) \in \hat{\psi}(u_1);\ \forall\ u_1 \in \bar{N}(u_1^0,\bar{\delta})$

(v) $\bar{\alpha}(u_1,u_2) \in N(\alpha(u_1^0,u_2),\varepsilon)$;

$\forall\ (u_1,u_2) \in \bar{N}(u_1^0,\bar{\delta}) \times U_2$

and as a consequence

$\bar{\alpha}(u_1,\cdot) \in N(\alpha(u_1^0,\cdot),\varepsilon);\ \forall\ u_1 \in \bar{N}(u_1^0,\bar{\delta})$

where $N(\alpha(u_1^0,\cdot),\varepsilon) = \{h(\cdot) \in C(U_2): \sup_{u_2}|h(u_2) - \alpha(u_1^0,u_2)| < \varepsilon\}$.

Hence

$\bar{\alpha}(u_1,\cdot) \in \hat{\psi}(u_1) \cap N(\alpha(u_1^0,\cdot),\varepsilon); \forall\ u_1 \in \bar{N}(u_1^0,\bar{\delta})$.

so that

$\forall\ u_1 \in \bar{N}(u_1^0,\bar{\delta})$

$\hat{\psi}(u_1) \cap N(\alpha(u_1^0,\cdot),\varepsilon) \neq \phi$

for any given $\alpha(u_1^0,\cdot) \in \hat{\psi}(u_1^0)$ which is equivalent to l.s.c. by Lemma 2.7. □

We note that the delta we provide for a given epsilon is obtained directly from the equi-l.s. continuity of $\{\psi(\cdot,u_2): u_2 \in U_2\}$ and hence may depend on u_1^0 but is independent of $\alpha(u_1^0,\cdot)$. We now concentrate on the class $F = \{h(u_2) = \sup_{i=1,..,n}\|u_2 - \bar{u}_2^i\| - a_i\ ;\ \{\bar{u}_2^i\}_{i=1}^n \subseteq \bar{U}$ $\{a_i\}_{i=1}^n \subseteq R;\ n \in Z^+\}$ and define a concept of convexity on this class of functions.

<u>Definition 2.11</u> : For $\lambda \in [0,1]$, u_2, $\bar{u}_2 \in U$, $\delta > 0$, $\bar{\delta} > 0$, we let

$$(1-\lambda) \odot N(u_2,\delta) \oplus \lambda \odot N(\bar{u}_2,\bar{\delta})$$

$$= N((1-\lambda)u_2 + \lambda\bar{u}_2,\ (\delta-\bar{\delta})(1-\lambda) + \bar{\delta})\ .$$

So, if $\lambda = 0$ we get $\bar{N}(u_2,\delta)$ and if $\lambda = 1$ we get $\bar{N}(\bar{u}_2,\bar{\delta})$ as one would wish. Now if f_1, $f_2 \in F$, then as usual we have

$$\Gamma_1(b) = \bigcap_{i=1}^{n} \bar{N}(\bar{u}_2^i,a_i+b)$$

and

$$\Gamma_2(b) = \bigcap_{j=1}^{m} \bar{N}(\hat{u}_2^j,\hat{a}_j+b).$$

Definition 2.12 : For $\Gamma_1(\cdot)$, $\Gamma_2(\cdot)$ as above we let

$$(1-\lambda) \odot \Gamma_1(b) \oplus \lambda \odot \Gamma_2(b)$$

$$= \bigcap_{i,j} (1-\lambda) \odot \bar{N}(\bar{u}_2^i, b+a_j) \oplus \lambda \odot \bar{N}(\hat{u}_2^j, b+a_j)$$

$$= \{u_2 : \sup_{i,j} \| u_2 - [(1-\lambda)\bar{u}_2^i + \lambda\hat{u}_2^j] \| - [(a_i - \hat{a}_j)(1-\lambda) + \hat{a}_j] \leqslant b\}.$$

Definition 2.13 : f_1, $f_2 \in F$; $\lambda \in [0,1]$ we let

$$(1-\lambda) \odot f_1(u_2) \oplus \lambda \odot f_2(u_2)$$

$$= \sup_{i,j} \| u_2 - [(1-\lambda)\bar{u}_2^i + \hat{u}_2^j] \| - [(a_i - a_j)(1-\lambda) + \hat{a}_j] \in F$$

where

$$f_1(u_2) = \sup_{i=1,..,n} \| u_2 - \bar{u}_2^i \| - a_i$$

$$f_2(u_2) = \sup_{j=1,..,n} \| u_2 - \hat{u}_2^j \| - \hat{a}_j .$$

Lemma 2.9 : Suppose $\bar{U}$ is a reflexive Banach space on which we have a smooth, strictly convex norm. Then

(a) $\bar{N}((1-\lambda)u_2 + \lambda\bar{u}_2, (1-\lambda)\delta + \lambda\bar{\delta})$

$\subseteq \overline{co}\,\{\bar{N}(u_2,\delta) \cup \bar{N}(\bar{u}_2,\bar{\delta})\}$,

(b) $\bigcap_{i,j} \overline{co}\,\{\bar{N}(u_2^i,\delta_i) \cup \bar{N}(\bar{u}_2^j,\bar{\delta}_j)\}$

$\equiv \overline{co}\,\{\bigcap_i \bar{N}(u_2^i,\delta_i) \cup \bigcap_j \bar{N}(\bar{u}_2^j,\bar{\delta}_j)\}$,

where $\overline{co}$ denotes the convex closure.

Proof :

(a) As $\bar{N}(u_2,\delta) \cup \bar{N}(\bar{u}_2,\bar{\delta})$ is a bounded set in a reflexive Banach space, we may interpret the $\overline{co}$ operation to be the intersection of all closed balls containing the set. So if

(i) $\bar{N}(u_2,\delta) \cup \bar{N}(\bar{u}_2,\bar{\delta}) \subseteq \bar{N}(\hat{u}_2,\hat{\delta})$ implies

$\bar{N}(1-\lambda)u_2 + \lambda\bar{u}_2, (1-\lambda)\delta + \lambda\bar{\delta}) \subset \bar{N}(\hat{u}_2,\hat{\delta})$, then we have proven (a).

Suppose $\|u_2 - \hat{u}_2\| + \delta \leqslant \hat{\delta}$,

$\|\bar{u}_2 - \hat{u}_2\| + \bar{\delta} \leqslant \hat{\delta}$,

which is equivalent to (i) and let

$$u_2^* \in \bar{N}((1-\lambda)u_2 + \lambda\hat{u}_2, (1-\lambda)\delta + \lambda\bar{\delta}),$$

ie. $\|u_2^* - [(1-\lambda)u_2 + \lambda\bar{u}_2]\| \leqslant (1-\lambda)\delta + \lambda\bar{\delta}$.

Then

$$\begin{aligned}\|u_2^* - \hat{u}_2\| &\leqslant \|u_2^* - [(1-\lambda)u_2 + \lambda\bar{u}_2]\| + \|(1-\lambda)u_2 + \lambda\bar{u}_2 - \hat{u}_2\| \\ &\leqslant (1-\lambda)\delta + \lambda\bar{\delta} + \|(1-\lambda)(u_2-\hat{u}_2) + \lambda(\bar{u}_2-\hat{u}_2)\| \\ &\leqslant (1-\lambda)\delta + \lambda\bar{\delta} + (1-\lambda)\|u_2-\hat{u}_2\| + \lambda\|\bar{u}_2-\hat{u}_2\| \\ &\leqslant (1-\lambda)\delta + \lambda\bar{\delta} + (1-\lambda)(\hat{\delta}-\delta) + \lambda(\hat{\delta}-\bar{\delta}) \equiv \hat{\delta}.\end{aligned}$$

(b) This follows immediately from the observation that

$$u_2 \in \bigcap_{i,j} \overline{co}\,\{\bar{N}(u_2^i,\delta_i) \cup \bar{N}(\bar{u}_2^j,\delta_j)\}$$

is either a vertex,and hence must lie on the boundary of either

$\bar{N}(u_2^i,\delta_i)$ or $\bar{N}(\bar{u}_2^j,\delta_j)$ for some i,j (and inside all others),or must be internal to the convex set

$$\bigcap_{i,j} \overline{co}\,\{\bar{N}(u_2^i,\delta_i) \cup \bar{N}(u_2^j,\delta_j)\}.$$

In the latter case it must lie on the line segment which can be made "parallel" (ie. in the "direction" of $(u_2^i - u_2^j)$) to the axis of the set $\overline{co}\,\{\bar{N}(u_2^i,\delta_i) \cup \bar{N}(\bar{u}_2^j,\delta_j)$ for the i,j which obtains the minimum of $d(u_2, \text{bdd}\ \overline{co}\,\{\bar{N}(u_2^i,\delta_i) \cup \bar{N}(u_2^j,\delta_j)\})$. This line segment may then be extended so that the end points lie in $\bar{N}(u_2^i,\delta_i)$ and $\bar{N}(\bar{u}_2^j,\delta_j)$.

<u>Proposition 2.10</u> : Let $f_1, f_2 \in F$; $f_3 \in SC_1(U_2)$.

(i) If $f_3 \leqslant f_1$ and $f_3 \leqslant f_2$, then $f_3 \leqslant (1-\lambda) \odot f_1 \oplus \lambda \odot f_2$

(ii) If $f_1 \leqslant f_3$ and $f_2 \leqslant f_3$, then $(1-\lambda) \odot f_1 \oplus \lambda \odot f_2 \leqslant f_3$.

<u>Proof</u> :

(i) Let $f_1(u_2) = \sup_{i=1,..,n} \| u_2 - \bar{u}_2^i \| - a_i$,

$$f_2(u_2) = \sup_{j=1,..,n} \| u_2 - \hat{u}_2^j \| - a_j ,$$

$$f_3(u_2) = \sup_D \| u_2 - \bar{u}_2 \| - a.$$

Now as $\forall (a,\bar{u}_2) \in D$

$$\| u_2 - \bar{u}_2 \| - a \leqslant f_3(u_2) \leqslant \sup_{i=1,..,n} \| u_2 - \bar{u}_2^i \| - a_i$$

and

$$\| u_2 - \bar{u}_2 \| - a \leqslant f_3(u_2) \leqslant \sup_{j=1,..,n} \| u_2 - \hat{u}_2^j \| - \hat{a}_j,$$

we have $\forall$ b that

$$\bar{N}(\bar{u}_2, a+b) \supseteq \bigcap_{i=1}^{n} \bar{N}(\bar{u}_2^i, a_i + b),$$

$$\bar{N}(\bar{u}_2, a+b) \supseteq \bigcap_{j=1}^{m} \bar{N}(\hat{u}_2^j, \hat{a}_j + b).$$

From Lemma 2.9 we can deduce

$$\bigcap_{i,j} (1-\lambda) \odot \bar{N}(\bar{u}_2^i, a_i + b) \oplus \lambda \odot \bar{N}(\hat{u}_2^j, \hat{a}_j + b)$$

$$\subseteq \bigcap_{i,j} \overline{co}\{\bar{N}(\bar{u}_2^i, a_i + b) \cup \bar{N}(\hat{u}_2^j, \hat{a}_j + b)\}$$

$$= \overline{co}\{ \bigcap_{i=1}^{n} \bar{N}(\bar{u}_2^i, a_i + b) \cup \bigcap_{j=1}^{m} \bar{N}(\hat{u}_2^j, \hat{a}_j + b)\}$$

$$\subseteq \bar{N}(\bar{u}_2, a+b); \ \forall \ b.$$

Hence by Lemma 2.8 and Definition 2.12

$$(1-\lambda) \odot f_1(u_2) \oplus \lambda \odot f_2(u_2) \geqslant \|u_2 - \bar{u}_2\| - a$$

$$\forall\ (a,\bar{u}_2) \in D$$

and hence

$$(1-\lambda) \odot f_1(u_2) \oplus \lambda \odot f_2(u_2) \geqslant \sup_D \|u_2 - \bar{u}_2\| - a$$

$$= f_3(u_2).$$

We have

(ii) $$(1-\lambda) \odot f_1(u_2) \oplus \lambda \odot f_2(u_2)$$

$$\equiv \sup_{i,j} \|u_2 - \{(1-\lambda)\bar{u}_2^i + \lambda\hat{u}_2^j\}\| - \{(1-\lambda)a_i + \lambda\hat{a}_j\}$$

$$\leqslant \sup_{i,j} (1-\lambda)\|u_2 - \bar{u}_2^i\| - (1-\lambda)a_i + \lambda\|u_2 - \hat{u}_2^j\| - \lambda\hat{a}_j.$$

Hence

$$(1-\lambda) \odot f_1(u_2) \oplus \lambda \odot f_2(u_2)$$

$$\leqslant (1-\lambda)\{\sup_{i=1,..,n} \|u_2 - \bar{u}_2^i\| - a_i\} + \lambda\{\sup_{j=1,..,n} \|u_2 - \hat{u}_2^j\| - \hat{a}_j\}$$

$$\leqslant (1-\lambda)f_3(u_2) + \lambda f_3(u_2) = f_3(u_2).$$ □

Lemma 2.10 : Suppose $\lambda : U_1 \to [0,1]$ is a continuous function. Then $u_1 \to (1-\lambda(u_1)) \odot f_1 \oplus \lambda(u_1) \odot f_2$ for $f_1, f_2 \in F$ is continuous from U_1 to $C(U_2)$.

Proof : Suppose $u_1^n \to u_1$ in U_1 and let

$$(1-\lambda(u_1))\bar{u}_2^i + \lambda(u_1)\hat{u}_2^j = u(\lambda(u_1))$$

$$(1-\lambda(u_1))a_i + \lambda(u_1)\hat{a}_j = a(\lambda(u_1)).$$

Then

$$\sup_{u_2}\Big| \|u_2 - u(\lambda(u_1))\| - a(\lambda(u_1)) - \{\|u_2 - u(\lambda(u_1^n))\| - a(\lambda(u_1^n))\}\Big|$$

$$= \sup_{u_2}\Big| \|u_2 - u(\lambda(u_1))\| - \|u_2 - u(\lambda(u_1^n))\| - a(\lambda(u_1)) + a(\lambda(u_1^n))\Big|$$

$$\leq \sup_{u_2}\Big| \|u_2 - u(\lambda(u_1))\| - \|u_2 - u(\lambda(u_1^n))\|\Big| + |a(\lambda(u_1^n)) - a(\lambda(u_1))|$$

$$\leq \|u(\lambda(u_1)) - u(\lambda(u_1^n))\| + |a(\lambda(u_1^n)) - a(\lambda(u_1))|$$

$$\leq \|-(\lambda(u_1) - \lambda(u_1^n))\bar{u}_2^i + (\lambda(u_1) - \lambda(u_1^n)\hat{u}_2^j\|$$

$$+ |-(\lambda(u_1) - \lambda|u_1^n|)a_j + (\lambda(u_1) - \lambda(u_1^n))\hat{a}_j|$$

$$\leq |\lambda(u_1) - \lambda(u_1^n)|\|\hat{u}_2^j - \bar{u}_2^i\| + |\lambda(u_1) - \lambda(u_1^n)|\cdot|\hat{a}_j - a_i|_{n\to\infty} \to 0 . \quad \square$$

Theorem 2.9 : Suppose $\bar{U}$ is reflexive locally F-normable, the conditions of Proposition 2.9 are satisfied and $\hat{\psi}(u_1)$ is defined as before. Then $\forall\ \varepsilon > 0$, the mapping $u_1 \to \bar{N}(\hat{\psi}(u_1),\varepsilon)$, where the neighbourhood is taken in $C(U_2)$, admits a continuous selection from U_1 to the space F considered as a subset of $C(U_2)$.

Proof : For every $h \in F$ we let

$$V(h) = \{u_1 : h \in N(\hat{\psi}(u_1),\varepsilon)\}$$

$$= \{u_1 : \hat{\psi}(u_1) \cap N(h,\varepsilon) \neq \phi\}.$$

Now, as ψ is l.s.c. from U_1 to $C(U_2)$ and $h \in C(U_2)$, we know $V(h)$ is open in U_1. As $\hat{\psi}(u_1) \subseteq F$; $\forall\ u_1 \in U_1$ we know $\{V(h) : h \in F\}$ is an open cover of U_1 and as U_1 is compact there exists a finite refinement $\{V(h_i) : i=1,..,n\}$ which covers U_1. Let $\{\lambda_i(\cdot) : i=1,..,n\}$ be a partition of unity subordinate to this cover. Then $u_1 \to \lambda_i(u_1)$ is continuous

$$\lambda : U_1 \to [0,1]; \sum_{i=1}^{n} \lambda_i(u_1) = 1$$

and

$$\lambda_i(u_1) \neq 0 \quad \text{iff} \quad h_i \in N(\hat{\psi}(u_1),\varepsilon);$$

$$i=1,..,n \qquad (S).$$

From Lemma 2.10 we know that

$$u_1 \to f(u_1) = \rho_1(u_1) \odot h_1 \oplus \rho_2(u_1) \odot h_2 \ldots \oplus \rho_n(u_1) \odot h_n,$$

$f : U_1 \to C(U_2)$ is a continuous function and $f(u_1) \in F$.

From Proposition 2.10 we can conclude, since

$$d(\cdot,\Gamma_\varepsilon(u_1)) - 3\varepsilon < h_i(\cdot) < d(\cdot,\Gamma(u_1)); \quad \forall\ i=1,..,n$$

$$\text{iff } \lambda_i(u_1) \neq 0$$

and $d(\cdot,\Gamma_\varepsilon(u_1))$, $d(\cdot,\Gamma(u_1)) \in SC_1(U_2)$, that

$$d(\cdot,\Gamma_\varepsilon(u_1)) - 3\varepsilon \leqslant f(u_1) \leqslant d(\cdot,\Gamma(u_1)); \forall\ u_1 \in U_1.$$

That is

$$f(u_1) \in \bar{N}(\hat{\psi}(u_1),\varepsilon); \ \forall\ u_1 \in U_1. \qquad \square$$

Corollary 2.91 : Suppose the condition of Theorem 2.9 hold and $\Gamma(\cdot)$ is a convex valued u.s.c. multi-valued mapping which is being approximated above by a l.s.c. multi-valued mapping

$$\Gamma_\varepsilon(\cdot) \supseteq \Gamma(\cdot) \text{ s.t. } d^*(F_\varepsilon,G) \leqslant \varepsilon.$$

Then $\exists\ f : U_1 \to C(U_2)$ continuous s.t. $f(u_1) \in F$,

$$u_1 \to T_\varepsilon(u_1) = \{u_2 \in U_2 : f(u_1)(u_2) \leqslant 0\}$$

is Hausdorff continuous convex closed valued and

$$N(\Gamma_\varepsilon(u_1),3\varepsilon) \supseteq T_\varepsilon(u_1) \supseteq \Gamma(u_1).$$

Proof : We choose f as in our previous theorem. The last assertion follows immediately from our choice of f and the definition of $\hat{\psi}(\cdot)$. We need only show that $T_\varepsilon(\cdot)$ is Hausdorff continuous, which amounts to showing $T_\varepsilon(\cdot)$ is uniformly u.s.c. on U_1 (see Corollary 1.13).

As $\forall\ \varepsilon > 0\ \exists\ \delta(\bar{u}_1) > 0$ s.t.

$$|f(u_1)(u_2) - f(\bar{u}_1)(u_2)|$$

$$\leq \|f(u_1) - f(\bar{u}_1)\| < \varepsilon \text{ for } u_1 \in N(\bar{u}_1,\delta),$$

$\{f(\cdot,u_2) : u_2 \in U_2\}$ is an equi-continuous class of single valued mappings with respect to u_1. Now as U_1 is compact, $f : U_1 \to C(U_2)$ must be uniformly continuous, and we may choose $\forall\ \varepsilon > 0$ a $\delta(\varepsilon) > 0$ independent of $\bar{u}_1 \in U_1$ and of course $u_2 \in U_2$ (because of the equi-continuity).

Hence $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0$ s.t.

$$f(\bar{u}_1)(u_2) - \varepsilon \leq f(u_1)(u_2);\ \forall\ u_1 \in N(\bar{u}_1,\delta).$$

Thus

$$\begin{aligned} T_\varepsilon(u_1) &= \{u_2 \in U_2 : f(u_1)(u_2) \leq 0\} \\ &\subseteq \{u_2 \in U_2 : f(\bar{u}_1)(u_2) - \varepsilon \leq 0\} \\ &= \{u_2 : f(\bar{u}_1)(u_2) \leq \varepsilon\} \\ &= N(\{u_2 : f(\bar{u}_1)(u_2) \leq 0\},\ \varepsilon) \\ &= N(T_\varepsilon(\bar{u}_1),\varepsilon), \end{aligned}$$

the last equality following from $f \in F$ and corollary 2.7 as the cut sets are metrically increasing with a rate $\eta(x) = x$. □

<u>Corollary 2.92</u> : Suppose all the conditions of Proposition 2.7 are satisfied. In particular, U_2 satisfies condition (ii) and $\Gamma : U_1 \to KV(U_2)$ is u.s.c.. Then $\forall\ \varepsilon > 0;\ \exists\ \Gamma_\varepsilon : U_1 \to KV(U_2)$ Hausdorff continuous s.t. if G is the graph of $\Gamma(\cdot)$

G_ε is the graph of $\Gamma_\varepsilon(\cdot)$

then $d^*(G_\varepsilon,G) \leq 4\varepsilon$.

<u>Proof</u> : This follows immediately from the previous corollary. □

CHAPTER III

Since Kakutani, it has been observed that certain multi-valued mappings admit fixed points. Convexity of the image sets of these mappings has played an essential role in the development of such theorems. Little progress has been made in relaxing convexity requirements. Conversely no totally geometric proof of Kakutani's theorem has been given. One notes that even in going from R to R^n, one loses the property that all continuous multi-valued mappings admit fixed points. This contrasts dramatically with single valued mappings. One needs to restrict the shape of the image set, or how it "changes", to provide an affirmative answer to the fixed point problem.

The other area of mathematics which uses convexity to high degree is the theory of nonlinear optimization. Researchers have been much more successful, in recent years, in weakening (and removing) convexity assumptions in this area. Since in the context of reflexive Banach spaces, one can approximate upper semi-continuous multi-functions, at least as well with continuous multi-functions as one can with lower semi-continuous multi-functions, we are able to view Kakutani's theorem as a consequence of nonlinear optimization. To do this we use the work of Arrigo Cellina.

This approach allows us to reduce the problem of finding a fixed point of a multi-valued mapping, to the problem of finding a fixed point of a single valued mapping. The natural question of, how large is the class of problems amenable to this approach, arises. An attempt is made to identify the essential ingredients required to apply this approach to a general mapping. The lattice theoretic nature of convexity enters in a natural way and continuous lattice theory proves usefuly in analysing such an approach.

Convexity assumptions are not removed but their role redefined, in the context of the abovementioned spaces. Quasi-convexity and strictly quasi-convex functions enter naturally in an attempt to understand the contribution of the "changing shape" of the image set has on the over all "motion" of the set valued mapping. We show that if a quasi-convex function can be written as the pointwise supremum of a collection of strictly quasi-convex functions, then the resultant set valued mappings in fact approximate each other in graph. This implies that the fixed points of the approximating set valued mappings approximate the fixed points of the original.

§3.1 Fixed Points of Multi-Valued Mappings

Arrigo Cellina observed the following.

Proposition 3.1 : Let U be a compact metric space having the fixed point property. Let $\Gamma : U \to 2^{U}$ be a closed multi-valued mapping. Assume for an arbitrary $\varepsilon > 0$ there exists a continuous mapping $f : U \to U$, depending on ε, such that if G_ε and G denote the graphs of f and Γ respectively, we have

$$d^*(G_\varepsilon, G) < \varepsilon.$$

Then Γ has a fixed point in U.

Proof : Reference [14] proposition 1. □

He obtained generalizations of certain fixed point theorems, obtaining his particular f by selecting from a l.s.c. approximation to Γ. In relative 'nice' spaces we can approximate the graph of Γ with the graph of a Hausdorff continuous mapping. Does this tell us anything more?

Definition 3.1 : Let g be a continuous numerical function defined on a topological space U. A family of compact sets $\{K_i : i \in I\}$ is said to be selective with respect to g if there exists one and only one $\bar{u}_i$

for each i s.t. $\bar{u}_i \in K_i$; $g(\bar{u}_i) = \max\{g(u_i) : u_i \in K_i\}$.

In a Banach space the strongly compact convex sets are selective. This follows by choosing $g(u) = -d(0,u)$. In a strictly convex space the sets $\{\{u : h_i(u) \leqslant b\};\ i \in I\}$ for $h_i \in F$ are selective. This follows from the observation that

$$\Gamma_i(b) = \{u : h_i(u) \leqslant b\} = \bigcap_{j=1}^{m} \bar{N}(\bar{u}_j, a_j + b)$$

and that any continuous non-identically zero linear functional takes a minimum on the closed unit ball at only one point. As a consequence any linear functional non-identical zero will do for $g(\cdot)$, since $\Gamma_i(b)$ is the finite intersections of closed unit balls.

Theorem 3.1 : Let $\Gamma(\cdot) : U_1 \to 2^{u_2}$ be a continuous multi-valued mapping. If the family $\{\Gamma(u_1) : u_1 \in U_1\}$ is selective, there is a single valued continuous mapping $\alpha : U_1 \to U_2$ s.t. $\alpha(u_1) \in \Gamma(u_1)$; $\forall\, u_1 \in U_1$.

Proof : Reference [1] theorem 3, page 117. □

It has been known since Schauder that the strongly compact convex subsets of a Banach space have the fixed point property for strongly continuous mappings and the convex weakly compact subsets of a separable Banach space have the fixed point property for weakly continuous mappings. As a consequence we can deduce the following.

Theorem 3.2 : Suppose;

(i) U is a reflexive Banach space,

(ii) $U_1 \subseteq U$ is a convex, weakly compact locally F-normable set in U, and

(iii) $\Gamma : U_1 \to KV(U_1)$ is weakly u.s.c. (in fact, weakly Hausdorff u.s.c. with respect to the F-norm).

Then Γ has a fixed point.

Proof: This follows from corollary 2.91, proposition 3.1 and theorem 3.1 noting that U_1 has the fixed point property as it is weakly compact, convex and separable (since all compact metric spaces are separable).

□

This forms a complementary result to the Kakutani theorem in Banach spaces. In the same fashion we could have deduced the Kakutani theorem.

How far can we extend this approach? If one checks the proof of Theorem 3.1 then one sees that the selection α of Γ was obtained by (A):

$$\alpha(u_1) = \{u_2 : u_2 \in \Gamma(u_1) : g(u_2) = M(u_1)\},$$

$$M(u_1) = \max\{g(u_2) = u_2 \in \Gamma(u_1)\},$$

where $\Gamma(u_1)$ is selective with respect to g.

The continuity of α follows from the fact that in general $\alpha(u_1)$ would be u.s.c. multi-valued, but since it reduces to a single point mapping it is continuous. The scenario of the proof proceeds as follows.

First we need to decide when one can approximate an upper semi-continuous mapping from above by continuous multi-valued mappings. We need to impose some sort of convexity restriction on the image sets of $\Gamma(\cdot)$ for this to happen. We will not answer this question but will reword it to emphasise the role of convexity. We begin by noting that the notions of generalized convexity can be extended from function $f : U_1 \to R^*$ to mappings $\Gamma : U_1 \to L$ where L is a continuous lattice.

Definition 3.2 : Suppose L is a continuous lattice. $\Gamma:U_1 \to L$ is called Φ convex, where Φ is an arbitrary set of mappings $\psi:U_1 \to L$,

if $\exists\ \Phi' \subseteq \Phi$ s.t.

$$\Gamma(u_1) = V_{\psi\in\Phi'}\ \psi(u_2).$$

In this way for a continuous lattice $O(U_2)$ the Scott continuous mappings $[U_1, \Sigma O(U_2)]$ can be considered convex, since by proposition 1.10

$$\cup_{i\in I}\Gamma_i(\cdot) \in [U_1, \Sigma O(U_2)] \text{ if } \forall\ i, \Gamma_i(\cdot) \in [U_1, \Sigma O(U_2)].$$

The question then arises whether there exist a class $\Phi \subseteq [U_1, \Sigma O(U_2)]$ of continuous multi-valued mappings which generates $[U_1, \Sigma O(U_2)]$. In general the answer is no. We need to restrict the lattice $L = O(U_2)$ to have any hope of a positive answer. We do this by using $L = C\Phi_{ops}(U_2)$, the continuous lattice of complements of Φ-convex sets on a compact Hausdorff space U_2. Once again the class $[U_1, \Sigma\ C\Phi_{ops}(U_2)]$ is closed with respect to arbitrary unions if $O(U_1)$ is a continuous lattice, itself. This class can be considered as consisting of convex functions in the sense of definition 3.2 and hopefully by choosing Φ-correctly we may find a generating class $\mathcal{L} \subseteq [U_1, \Sigma C\Phi_{ops}(U_2)]$ which consists of Hausdorff continuous mappings. To achieve a generalization we need $\mathcal{L}$ and Φ to satisfy two more conditions.

First, $C\Phi_{ops}(U_2)$ must admit a generating class Φ which is selective with respect to some continuous mapping $g(\cdot)$ and $C\Phi_{op}(U_2)$ must be compatible with the metric on the space U_2 in the following sense. If $S \subseteq U_2$ is Φ convex then so is $A(S,\varepsilon)$; $\forall\ \varepsilon > 0$.

Secondly, the $\mathcal{L}$ we are seeking must consist of Hausdorff continuous mappings $T : U_1 \to A = \{S=\{u_2 : \psi(u_2) > a\};\ \psi \in \Phi\}$. This amounts in practice to the following problem. Our multi-valued mapping $\Gamma(\cdot) \in [U_1, \Sigma C\Phi_{ops}(U_2)]$ is given by $\Gamma(u_1) = \{u_2 : \sup_{\psi\in\Phi'}\psi(u_1,u_2) > a\}$, where $f(u_1,u_2) = \sup_{\psi\in\Phi'}\psi(u_1,u_2)$ is most probably l.s.c. with respect to $U_1 \times U_2$ (to ensure u.s.c.) and the ψ's are continuous on $U_1\times U_2$. We need to know when the class $\mathcal{L} = \{T(u_1) = \{u_2 : \psi(u_1,u_2) > a\}; \psi \in \Phi'\}$ is Hausdorff continuous.

We also need to be able to shrink the image sets of our multi-valued mappings. The lattice of sets $C\Phi_{ops}(U_2)$ cannot be an arbitrary class of open sets. We define for $A \in C\Phi_{ops}(U_2)$ $S(A,\varepsilon) = [\overline{N}(A^c,\varepsilon)]^c$, the shrinkage of the open set A. If we shrink a set we may not be able to recover the original set by expanding, i.e. $N(S(A,\varepsilon),\varepsilon) \neq A$. For example let A be the union of a collection of disjoint balls $N(u_n,\frac{1}{n})$ i.e.

$$A = \cup_n N(u_n,\tfrac{1}{n}).$$

This set is by definition open, but we cannot shrink it by any $\varepsilon > 0$ without losing some of these disjoint balls.

We need to be able to shrink our $C\Phi_{ops}(U_2)$ set a small amount and be able to recover it again, i.e.

$$N(S(A,\varepsilon),\delta) = S(A,\varepsilon-\delta) \qquad \text{for } 0 < \delta < \varepsilon,$$

for ε sufficiently small. We will call such a set shrinkable if there exists an $\bar{\varepsilon} > 0$ s.t. for $0 < \varepsilon < \bar{\varepsilon}$, the above equality holds for all $0 < \delta < \varepsilon$. If the set A is generated by a "constraint" function $f(\cdot) = (f_1(\cdot),f_2(\cdot),\ldots,f_m(\cdot))$ the above definition of the shrinkage becomes equivalent to that of reference [13].

That is, if $A = \{u;f(u) > \bar{b}\}$ for some continuous function $f(\cdot)$, we have in the case when

$$\text{bdd}A = \{u \in \text{cl}A : f_j(u) = b_j \text{, some } j\}$$

that

$$S(A,\varepsilon) = \{u \in A : d(u,\mathrm{bdd}A) > \varepsilon\}.$$

Lemma 3.1 : Suppose U is a metric space and A is a closed set. Then

(i) $A^c \neq \phi$ implies $S(A^c,\varepsilon) \neq \phi$ for $\varepsilon > 0$ sufficiently small;

(ii) $S(N(A^c,\varepsilon),\delta) = N(A,\varepsilon-\delta)$ for $0 < \delta < \varepsilon$; and

(iii) if, some set B and $\bar{\varepsilon} > 0$ $A^c = N(B,\bar{\varepsilon})$, we have

$$N(S(A^c,\varepsilon),\delta) = S(A^c,\varepsilon-\delta) \text{ for } 0 < \delta < \varepsilon < \bar{\varepsilon}.$$

Proof : We begin by first showing that

$$S(N(u,\varepsilon),\delta) = N(u,\varepsilon-\delta),$$

for $0 < \delta < \varepsilon$.

Since

$$N(u,\varepsilon) = \{\bar{u} : d(u,\bar{u}) < \varepsilon\},$$

we have

$$\bar{N}(N^c(u,\varepsilon),\delta) = \{u' : d(u',\bar{u}) \leqslant \delta \text{ and } d(u,\bar{u}) \geqslant \varepsilon\}.$$

Hence

$$d(u',u) \geqslant d(u,\bar{u}) - d(\bar{u},u') \geqslant \varepsilon - \delta$$

implying

$$\bar{N}(N^c(u,\varepsilon),\delta) \subseteq N^c(u,\varepsilon-\delta),$$

that is

$$S(N(u,\varepsilon),\delta) \supseteq N(u,\varepsilon-\delta).$$

Now suppose $u' \in N(u,\varepsilon-\delta)$.

We must show that there exists a

$$\bar{u} \in N^c(u,\varepsilon) \text{ s.t.}$$

$$d(\bar{u},u') > \delta.$$

Since

$$u' \in N(u,\varepsilon-\delta) \subseteq N(u,\varepsilon)$$

$$u' \notin N^c(u,\varepsilon).$$

Let $\bar{u}$ be the closest point in $N^c(u,\varepsilon)$ to u'. This is unique and $\bar{u} \in \text{bdd } N(u,\varepsilon) = \{u : d(u,\bar{u}) = \varepsilon\}$. This implies that,

$$d(u,u') \geqslant d(\bar{u},u) - d(u,u')$$

$$\geqslant \varepsilon - d(u,u')$$

$$> \varepsilon - (\varepsilon-\delta) = \delta$$

and subsequently,

$$u' \in \overline{N^c}(N^c(u,\varepsilon),\delta)$$

$$= S(N(u,\varepsilon),\delta).$$

We now show that for $0 < \delta < \varepsilon$ we have

$$S(N(A,\varepsilon),\delta) = N(A,\varepsilon-\delta).$$

By writing

$$N(A,\varepsilon) = \cup\{N(u,\varepsilon) : U \in A\},$$

we have

$$\overline{N}(N^c(A,\varepsilon),\delta) = \overline{N}(\cap\{N^c(u,\varepsilon) : u \in A\}, \delta)$$

$$= \cap\{\overline{N}(N^c(u,\varepsilon),\delta) : u \in A\}$$

$$= \cap\{N^c(u,\varepsilon-\delta) : u \in A\}$$

$$= (\cup\{N(u,\varepsilon-\delta) : u \in A\})^c$$

$$= N^c(A,\varepsilon-\delta)$$

implying the above result.

The first result of the ~~anno~~enunciation of the lemma follows by considering

$$u \in A^c.$$

Since A^c is open, $\exists\ \delta > 0$ s.t.

$$N(u,\varepsilon) \subseteq A^c.$$

Hence

$$S(N(u,\delta),\varepsilon) \subseteq S(A^c,\varepsilon)$$

and

$$u \in N(u,\delta-\varepsilon) \subseteq S(A^c,\varepsilon), \quad \text{for } 0 < \varepsilon < \delta.$$

The last part of the lemma follows almost immediately from what has been done. If $A^c = N(B,\bar{\varepsilon})$ then for $0 < \varepsilon < \bar{\varepsilon}$ we have,

$$S(N(B,\bar{\varepsilon}),\varepsilon) = N(B,\bar{\varepsilon}-\varepsilon).$$

Hence we have for $0 < \delta < \varepsilon < \bar{\varepsilon}$,

$$N(S(A^c,\varepsilon),\delta) = N(B,\bar{\varepsilon}-(\varepsilon-\delta))$$
$$= S(N(B,\bar{\varepsilon}),\varepsilon-\delta)$$
$$= S(A^c,\varepsilon-\delta). \qquad \square$$

<u>Lemma 3.2</u> : Suppose U is a reflexive Banach space and $A \subseteq U$ is a weakly compact, convex subset.

Then there exists a set B and $\bar{\varepsilon} > 0$ s.t.

$$A^c = N(B,\bar{\varepsilon}).$$

<u>Proof</u> : Choose $\bar{\varepsilon} > 0$. Since A is weakly compact and convex it can be expressed as the intersection of a collection of closed balls (see Theorem 2.3). For any such ball $N(u',b)$ and $u \in \text{bdd}\, N(u',b)$, there exists a ball

$$N(\bar{u},\bar{\varepsilon}) \subseteq N^c(u',b) \text{ s.t.}$$

$$\overline{N}(\bar{u},\bar{\varepsilon}) \cap \overline{N}(u',b) = \{u\}.$$

As a consequence we can express A^c as the union of a collection of balls of radius $\bar{\varepsilon}$.

Let

$$\{N(\bar{u}_i,\bar{\varepsilon});\ i \in I\}$$

be such a collection. We define

$$B = \{\bar{u}_i : \overline{N}(\bar{u}_i,\bar{\varepsilon}) \cap A \neq \phi\} \cup S(A^c,\bar{\varepsilon}).$$

If $u \in B$ then either,

(i) $u \in S(A^c,\bar{\varepsilon})$ and $d(u,A) > \bar{\varepsilon}$, or

(ii) $u = \bar{u}_i$, for some $i \in I$, in which case

$$d(u,A) = d(\bar{u}_i,A) > \bar{\varepsilon}.$$

Hence

$$N(u,\bar{\varepsilon}) \subseteq A^c \quad \text{and} \quad N(B,\bar{\varepsilon}) \subseteq A^c.$$

Take $u \in A^c$, then either

(i) $u \in S(A^c, \bar{\varepsilon})$ and $u \in B$, or

(ii) $d(u,A) \leqslant \bar{\varepsilon}$. In the latter case

$$u \in N(\bar{u}_i, \bar{\varepsilon}), \text{ for some } i \in I \text{ s.t.}$$

$$\overline{N}(\bar{u}_i, \bar{\varepsilon}) \cap A \neq \phi.$$

That is

$$N(B, \bar{\varepsilon}) = A^c. \qquad \square$$

<u>Proposition 3.2</u> : Suppose U_1 and U_2 are compact and metric and $\Gamma(\cdot) \in [U_1, \Sigma C\Phi_{ops}(U_2)]$, which has a generating class $\mathcal{L}$ derived from

$$\Phi' = \{\psi : U_1 \times U_2 \to R\},$$

Φ being compatible metrically and consisting of l.s. continuous functions. Suppose also that the $C\Phi_{ops}(U_1)$ sets are shrinkable.

Let $\Gamma(u_1) = \{u_2 : f(u_1,u_2) > a\}$ and suppose $T(u_1) = \{u_2 : \psi(u_1,u_2) > a\}$ is Hausdorff continuous.

Then $\exists$ a class of Hausdorff continuous mappings $T_\varepsilon(\cdot)$, $C\Phi_{ops}(U_2)$ - convex s.t.

$$d^*(G_\varepsilon, G) \leqslant \varepsilon;\ \forall\ \varepsilon > 0 \ \text{ and } T_\varepsilon(u_1) << \Gamma(u_1);\ \forall\ u_1 \in U_1,$$

where G_ε is the graph of $T^c_\varepsilon(\cdot)$ and G is the graph of $\Gamma^c(\cdot)$.

<u>Proof</u> : As $\Gamma(\cdot) \in [U_1, \Sigma C\Phi_{ops}(U_2)]$, U_2 a compact Hausdorff space, then by proposition 1.8 Γ is i.s.c. and hence $\Gamma^c(\cdot)$ is a closed valued u.s.c. multi-valued mapping with Φ-convex image sets.

Thus by Theorem 2.7 there is a l.s.c. multi-valued mapping $M_{\varepsilon/2}(\cdot)$ s.t.

$$M_{\varepsilon/2}(\cdot) \supseteq \Gamma^c(\cdot)$$

and

$$d^*(\text{Graph } M_{\varepsilon/2}(\cdot), \text{ Graph } \Gamma^c(\cdot)) \leq \varepsilon/2.$$

We define

$$co\Phi(A) = \cap\{S : A \subseteq; S \ \Phi\text{-convex}\}$$

and show that

$$co\Phi \ M_{\varepsilon/2}(\cdot)$$

is l.s.c. as well.

For any $\bar{\varepsilon} > 0$

$$M_{\varepsilon/2}(u_1) \subseteq coM_{\varepsilon/2}(u_1)$$

implies

$$N(M_{\varepsilon/2}(u_1),\bar{\varepsilon}) \subseteq \bar{N}(co\Phi M_{\varepsilon/2}(u_1),\bar{\varepsilon})$$

a Φ-convex set itself.

For $\forall \ \bar{\varepsilon} > 0$, $\exists \ \delta > 0$ s.t. if $u_1 \in N(\bar{u}_1,\delta)$, then

$$M_{\varepsilon/2}(\bar{u}_1) \subseteq N(M_{\varepsilon/2}(u_1),\bar{\varepsilon})$$

implying

$$co \, \Phi \, M_{\varepsilon/2}(\bar{u}_1) \subseteq co \ \Phi \ N(M_{\varepsilon/2}(u_1),\bar{\varepsilon})$$

$$\subseteq N(co \ \Phi \ M_{\varepsilon/2}(u_1),\bar{\varepsilon}).$$

Similarly $\Gamma_\varepsilon(u_1) = \bar{N}(co \ \Phi \ M_{\varepsilon/2}(u_1), \ \varepsilon/2]$ is Φ-convex and by proposition 1.11 it is also i.s.c. and hence l.s.c..

Also $d^*(\text{Graph } \Gamma_\varepsilon(\cdot), \text{Graph } \Gamma^c(\cdot)) \leq \varepsilon$ as

$$M_{\varepsilon/2}(u_1) \subseteq N(\Gamma^c(\bar{u}_1), \varepsilon/2)$$

implies

$$\text{co } \Phi\ M_{\varepsilon/2}(u_1) \subseteq \text{co } \Phi\ N(\Gamma^c(\bar{u}_1), \varepsilon/2) \subseteq N(\text{co}\Phi\Gamma^c(\bar{u}_1), \varepsilon/2)$$

once again due to Φ's metric compatibility and the fact that

$$\text{co } \Phi\ \Gamma^c(\bar{u}_.) = \Gamma^c(\bar{u}_1).$$

By letting $K_\varepsilon(u_1) = \Gamma^c_\varepsilon(u_1)$ we obtain an u.s.c. multi-valued mapping as $\Gamma_\varepsilon(\cdot)$ is i.s.c..

Now $N(\Gamma^c_\varepsilon(u_1), \varepsilon/2) = [\text{co } \Phi\ M_{\varepsilon/2}(u_1)]^c \subseteq \Gamma(u_1)$ for all u_1. Hence $\Gamma_\varepsilon(u_1) << \Gamma(u_1)$; $u_1 \in U_1$, where $<<$ is the way below relation on $C\phi_{op}(U_2)$. We now argue similarly to Lemma 2.1. As $\Gamma(\cdot)$ is generated by $\mathcal{L}$, $\exists$ a class $\Phi' \subseteq \Phi$ with

$$\Phi' = \{\psi_i(u_1,u_2);\ i \in I\} \text{ s.t.}$$

$$\Gamma(\cdot) = \cup_{i\in I}\{u_2\colon \psi_i(u_1,u_2) > a\}.$$

If we define $S(\Gamma(u_1),\delta) = [\bar{N}(\Gamma^c(u_1),\delta)]^c$

$$\rho(u_1) = \sup\{\delta\colon \Gamma_\varepsilon(u_1) << S(\Gamma(u_1),\delta) \text{ and}$$

$$\bar{N}(S(\Gamma(u_1),\delta),\bar{\varepsilon}) = S(\Gamma(u_1),\delta-\bar{\varepsilon})\ \forall\ 0 < \bar{\varepsilon} < \bar{\delta}\}$$

and note that $\rho(u_1) > 0\ \forall\ u_1 \in U_1$, since $\Gamma(u_1)$ is shrinkable.

By using the compactness of U_1 we can show $\rho(u_1)$ is bounded away from zero on U_1. Suppose not, then $\exists\ u_1^n$ s.t.

$$\rho(u_1^n) < 1/n;\ \forall\ n \in Z^+.$$

By the compactness of U_1 there is a convergent subsequence to $\bar{u}_1$ say. After renumbering $u_1^n \to \bar{u}_1$ and $\rho(u_1^n) \to 0$; $n \to \infty$.

We know that $\forall\ 0 < \delta < \rho(\bar{u}_1)$, we have $\Gamma_\varepsilon(\bar{u}_1) \ll S(\Gamma(u_1),\delta)$. As $\Gamma_\varepsilon(\cdot)$ is u.s.c., if we let $o < \bar{\varepsilon} < \delta < \rho(\bar{u}_1)$, $\exists$ a neighbourhood N_1 of $\bar{u}_1$ s.t.

$$\Gamma_\varepsilon(u_1) \subseteq N(\Gamma_\varepsilon(\bar{u}_1),\bar{\varepsilon});\ \forall\ u_1 \in N_1.$$

Let $\varepsilon' = \frac{1}{2}(\delta-\bar{\varepsilon}) > 0$. Then $\exists\ N_2$ a neighbourhood of $\bar{u}_1$ s.t.

$$\Gamma^c(u_1) \subseteq \bar{N}(\Gamma^c(\bar{u}_1),\varepsilon');\ \forall\ u_1 \in N_2.$$

(Note that we may make ε' as small as we like by letting δ be smaller.)
As

$$\Gamma(u_1) \supseteq [\bar{N}(\Gamma^c(\bar{u}_1),\varepsilon')]^c$$

$$= S(\Gamma(\bar{u}_1),\varepsilon');\ \forall\ u_1 \in N_1 \cap N_2,$$

we have

$$\Gamma_\varepsilon(u_1) \subseteq N(\Gamma_\varepsilon(\bar{u}_1),\bar{\varepsilon}) \ll N(S(\Gamma(\bar{u}_1),\delta),\bar{\varepsilon})$$

$$= S(\Gamma(\bar{u}_1),\delta-\bar{\varepsilon}) = S(\Gamma(\bar{u}_1),2\varepsilon') = S(S(\Gamma(\bar{u}_1),\varepsilon'),\varepsilon')$$

$$\subseteq S(\Gamma(u_1),\varepsilon').$$

For n sufficiently large we have $u_1 \in N_3 \subseteq N_1 \cap N_2$, where N_3 is a neighbourhood of $\bar{u}_1$. Hence $\rho(\bar{u}_1^n) \geqslant \varepsilon'$ for n sufficiently large, a contradiction.

Now as $\Gamma^c(\cdot)$ is u.s.c. we have $\bar{N}(\Gamma^c(\cdot),\delta)$ is u.s.c. and hence $[\bar{N}(\Gamma^c(\cdot),\delta)]^c = S(\Gamma(\cdot),\delta)$ is i.s.c.. Since $\bar{N}(\Gamma^c(u_1),\delta)$ is Φ-convex $\forall\ \delta > 0$, we have $S(\Gamma(\cdot),\delta) \in [U_1, \Sigma C\Phi_{o\,ps}(U_2)]$.

If we choose $0 < \delta < \inf\{\rho(u_1): u_1 \in U_1\}$ then

$$\Gamma_\varepsilon(u_1) \ll S(\Gamma(u_1),\delta);\ \forall\ u_1 \in U_1.$$

By hypothesis there exists a class

$\Phi' = \{\psi_i : U_1 \times U_2 \to R;\ i \in I\}$ s.t.

$\cup_{i\in I}\{u_2 : \psi_i(u_1,u_2) > a\} = S(\Gamma(u_1),\delta)$

with

$\{u_2 : \psi_i(u_1,u_2) > a\} \in [U_1,\ \Sigma C\Phi_{ops}(U_2)]$

Hausdorff continuous.

Since for each $\bar{u}_1 \in U_1$ these sets are in the lattice $C\Phi_{ops}(U_2)$, $\exists$ a finite number, $i = 1, .., N(\varepsilon)$, s.t.

$\cup_{i=1}^{N(\epsilon)} \{u_2 : \psi_i(\bar{u}_1,u_2) > a\} >> \Gamma_\varepsilon(\bar{u}_1)$

and

$S(\Gamma(u_1),\delta) \supseteq \cup_{i=1}^{N(\epsilon)} \{u_2 : \psi_i(\bar{u}_1,u_2) > a\}.$

We let

$\wedge_\varepsilon(\bar{u}_1) = \cup_{i=1}^{N(\epsilon)} \{u_2 : \psi_i(\bar{u}_1,u_2) > a\}$

and note that since

$\wedge_\varepsilon(\bar{u}_1) >> \Gamma_\varepsilon(\bar{u}_1)$, $\quad \exists\ \hat{\delta},\ \bar{\delta};\ 0 < \hat{\delta},\bar{\delta} < \delta$ s.t.

$S(\wedge_\varepsilon(\bar{u}_1),\hat{\delta}) \supseteq N(\Gamma_\varepsilon(\bar{u}_1),\bar{\delta}).$

As $\Gamma_\varepsilon(\cdot)$ is u.s.c. at $\bar{u}_1$, $\exists$ a neighbourhood N_4 of $\bar{u}_1$ s.t.

$N(\Gamma_\varepsilon(\bar{u}_1),\bar{\delta}) \supseteq \Gamma(u_1);\ \forall\ u_1 \in N_4.$

As $\wedge^c_\varepsilon(\cdot)$ is u.s.c. at $\bar{u}_1$, $\exists$ a neighbourhood N_5 of $\bar{u}_1$ s.t.

$N(\wedge^c_\varepsilon(\bar{u}_1),\hat{\delta}) \supseteq \wedge^c_\varepsilon(u_1);\ \forall\ u_1 \in N_5$.

Hence $\forall\ u_1 \in N_6 \subseteq N_4 \cap N_5$, a neighbourhood of $\bar{u}_1$, we have

$$\Gamma_\varepsilon(u_1) \subseteq N(\Gamma_\varepsilon(\bar{u}_1),\bar{\delta})$$

$$\subseteq S(\Lambda_\varepsilon(\bar{u}_1),\hat{\delta})$$

$$\subseteq \Lambda_\varepsilon(u_1)$$

$$\subseteq S(\Gamma(u_1),\delta) \subseteq \Gamma(u_1).$$

This implies

$$\Gamma_\varepsilon(u_1) << \Lambda_\varepsilon(u_1) << \Gamma(u_1);\ \forall\ u_1 \in N_6,$$

and since $\bar{u}_1 \in U_1$ is arbitrary the collection of all such neighbourhoods forms an open-cover of U_1. Since U_1 is compact there exists a finite sub-cover $\{N(u_1^i);\ i=1,..,M\}$, say. For each i we have a $\Lambda_\varepsilon^i(u_1)$ s.t.

$$\Gamma_\varepsilon(u_1) << \Lambda_\varepsilon^i(u_1) << \Gamma(u_1);\ \forall\ u_1 \in N(u_1^i).$$

We define

$$T_\varepsilon(u_1) = \bigcup_{i=1}^{M} \Lambda_\varepsilon^i$$

and note that

$$\Gamma_\varepsilon(u_1) << T_\varepsilon(u_1) << \Gamma(u_1);\ \forall\ u_1 \in U_1,$$

since each Λ_ε^i is defined by a sub-collection the mappings

$$\{\psi_j\,(u_1,u_2) : j \in I\},\text{where}$$

$$\cup_{j\in I}\{u_2 : \psi_j\,(u_1,u_2) > a\} \subseteq S(\Gamma(u_1),\delta) << \Gamma(u_1).$$ □

For the problem, alluded to above, of finding fixed points of multi-valued mappings, we can approximate the fixed points of the original mapping by the fixed points of a mapping $T_\varepsilon^c(u_1) = \bigcap_{i=1}^{N} \{u_2 : \psi_j\,(u_1,u_2) \leqslant a\}$ for an appropriate choice of ψ_i 's.

Since these sets are Φ-convex and the sets $\{u_2 : \psi(u_1,u_2) \leqslant a\}$ are selective with respect to a given continuous function $g(\cdot)$, the image sets of $T^c_\varepsilon(\cdot)$ are selective with respect to $g(\cdot)$ as well, since

$$\max\{g(u_2) : u_2 \in T^c_\varepsilon(u_1)\}$$

$$= \min_{i=1,..,N} \max\{g(u_2) : \psi_i(u_1,u_2) \leqslant a\}.$$

If we suppose this max is achieved at more than one point, at $\hat{u}_2$ and $\bar{u}_2$ say, then $g(\hat{u}_2) = g(\bar{u}_2)$.

As $T^c_\varepsilon(u_1) \subseteq \{u_2 : \psi_i(u_1,u_2) \leqslant a\}; \forall\ i$ and as we can see from above $\hat{u}_2$ must be the unique max of g on one of the generating sets, on set i, say. We have

$$\hat{u}_2,\ \bar{u}_2 \in \{u_2 : \psi_i(u_1,u_2) \leqslant a\}$$

and hence $g(\hat{u}_2) < g(\bar{u}_2)$ a contradiction.

A continuous selection of $T^c_\varepsilon(u_1)$ where $M_\varepsilon(u_1) = \max\{g(u_2) : u_2 \in T^c_\varepsilon(u_1)\}$, is

$$\alpha_\varepsilon(u_1) = \{u_2 : M_\varepsilon(u_1) = g(u_2) : \psi_i(u_1,u_2) \leqslant a;\ \forall\ i=1,..,N\}$$

and its fixed points can be used to approximate those of the original problem. The problem of finding $\alpha_\varepsilon(u_1)$ for each u_1 is a constrained non-linear optimization problem. Much work has been done on this problem for $R^n = U_1 = U_2$. Recently the constrained optimization problem has been investigated in more general spaces (see reference [5], [6], [11]). We will not deliberate on the Banach space fixed point problem any longer in this discussion, but turn to the problem in R^n.

Even in the case R^n the question of what continuous multi-valued mappings admit fixed points has not been fully explored. We know that in [a,b] all continuous multi-valued mappings admit fixed points but in even going over to $[a,b] \times [a,b]$ we lose this property.

The question of selectivity of sets in R^n has not been investigated except for convex sets of course. The other question of what condition ensure Hausdorff continuity has been investigated and deserves a mention.

Theorem 3.3 : Given a continuous function $f : R^n \to R^n$, suppose we define

$$\Gamma(b) = \{u \in R^n \ ; \ f(u) \leqslant b\} \text{ for } b \in R^n.$$

(a) Then the mapping Γ is u.s.c. at $\bar{b}$ iff $\exists \ \hat{b} > \bar{b}$ s.t. $\Gamma(\hat{b})$ is compact

(b) If $\Gamma(\bar{b})$ is compact $I(\bar{b}) \neq \phi$ (ie. $\bar{b} \in \text{int } B$), then the mapping Γ is l.s.c. at $\bar{b}$ iff $\text{cl } I(\bar{b}) = \Gamma(\bar{b})$,

$$I(b) = \{u \in R^n \ ; \ f(u) < b\}.$$

Proof : See reference [13]. □

We let

$$G(\bar{b},\bar{g}) = \{g : g \text{ cont.}, \{u \in R^n : g(u) \leqslant \bar{b}\} \neq \phi,$$

$$\max_{j=1,..,n} \sup_u |g_j(u) - \bar{g}_j(u)| < \infty\}$$

and define a metric on $G(\bar{b},\bar{g})$ using

$$d(f,g) = \max_{j=1,..,n} \sup_u |g_j(u) - f_j(u)|$$

and

$$\sigma(g) = \{u \in R^n : g(u) \leqslant \bar{b}\} \text{ for } g \in G(\bar{b},\bar{g}).$$

We can discuss upper and lower semi continuity of $\sigma(\cdot)$ with respect to the metric space $G(\bar{b},\bar{g})$ and R^n. As usual $\Gamma(b) = \{u : \bar{g}(u) \leqslant b\}$.

Theorem 3.4 :

(a) σ is u.s.c. at $\bar{g}$ iff Γ is u.s.c. at $\bar{b}$.

(b) Let $I(\bar{b}) \neq \phi$ (ie. $\bar{b} \in \text{int } B(\bar{g})$). Then σ is l.s.c. at $\bar{g}$ iff Γ is l.s.c. at $\bar{b}$.

Proof : See reference [20]. □

Theorem 3.5 : Suppose g is l.s. continuous.

(a) If g is strictly quasi convex and $I(\bar{b}) \neq \phi$, then cl $I(\bar{b}) = \Gamma(\bar{b})$.

(b) If $g(\cdot)$ is quasi convex and $\Gamma(\bar{b})$ is compact, then $\exists\ \tilde{b} > \bar{b}$ s.t. $\Gamma(\tilde{b})$ is compact.

Proof : Direct modification of those in reference [13], which assume g is continuous instead of l.s.c..

In (a) we note that the l.s.c. of $g(\cdot)$ suffices for $\Gamma(\bar{b})$ to be a closed set.

In (b) we note that given $b_{n_j} \to \bar{b}, u_{n_j} \to u_0$ and $g(u_{n_j}) \leq b_{n_j}$, then $\forall\ \varepsilon > 0$; n_j sufficiently large,

$$g(u_0) - \varepsilon \leq g(u_{n_j}) \leq b_{n_j} \leq \bar{b} + \varepsilon.$$

Hence $g(u_0) \leq \bar{b} + 2\varepsilon$ and ε arbitrary implies $g(u_0) \leq \bar{b}$. □

Corollary 3.5 : A l.s.c. function $g : U \to R^n$, for $U \subseteq R^n$ convex, is strictly quasi convex iff

(i) $\Gamma(b) = \{u : g(u) \leq b\}$

is closed convex $\forall$ b, and

(ii) for b s.t.

$I(b) = \{u : f(u) < b\} \neq \phi$

we have cl $I(b) = \Gamma(b)$.

Proof : Because of the previous theorem we need only to show the conditions are sufficient.

Obviously g is quasi convex. To show strict quasi convexity, we need to consider $u, \bar{u} \in U$ where $g(u) < g(\bar{u})$.

We have $u \in I(\bar{b})$ where $\bar{b} = g(\bar{u})$. It follows that

$$u \in \text{Int}\ \Gamma(\bar{b}) = \text{reint}\ \Gamma(\bar{b})$$

since

$$\text{cl}\ I(\bar{b}) = \Gamma(\bar{b}),$$

where re-int stands for the relative interior of $\Gamma(\bar{b})$ (see reference [23] pages 44, theorem 6.1).

As a consequence $\lambda u + (1-\lambda)\bar{u} \in \text{re int}\ \Gamma(\bar{b}); \lambda \in (0,1)$, that is,

$$\lambda u + (1-\lambda)\bar{u} \in \text{Int}\ \Gamma(\bar{b})$$

$$\equiv I(\bar{b}).$$

Hence $g(\lambda u + (1-\lambda)\bar{u}) < \bar{b} = g(\bar{u})$ and g is strictly quasi convex. □

Theorem 3.6 : Suppose f is l.s.c. on $U \subseteq R^n$ and quasi convex. If $\Gamma(\cdot)$ is l.s.c. at b $\forall$ b $\in$ B, then f is strictly quasi convex.

Proof : Once again this is a direct adaptation of that in [20], which assumes that f is continuous. We note that in fact the author usès only a one sided inequality in his proof which is associated with the l.s.c. of f. □

We wish to conjecture at this point that all l.s.c. quasi convex functions can be obtained as the supremum of l.s.c. strictly quasi convex functions.

Theorem 3.7 : Suppose f is lower semi continuous and defined on a convex subset $U \subseteq R^n$. If f is strictly quasi convex on U, then f is quasi convex on U but not conversely.

Proof : See reference [19] page 139. □

The above theorem supports our conjecture in that the class of l.s.c. strictly quasi convex functions is a subclass of the quasi convex l.s.c. functions.

It is easily seen that the supremum of l.s.c. quasi convex functions is once again quasi convex l.s.c., since

$$\Gamma(\bar{b}) = \{u : \sup_{i \in I} f_i(u) \leqslant \bar{b}\}$$

$$= \cap_{i \in I} \Gamma_i(\bar{b})$$

$$= \cap_{i \in I} \{u : f_i(u) \leqslant \bar{b}\}$$

is closed convex iff all $\Gamma_i(\bar{b})$ are closed convex.

From corollary 2.2 we can observe that if our conjecture is correct then for closed convex bounded sets $U \subseteq R^n$ the class

$$QC(U) = \{f : U \to R^n \; ; \; U \subseteq R^n \quad \text{l.s.c. quasi convex}\}$$

is a continuous lattice generated by

$$SQC(U) = \{f : f : U \to R^n \; ; \; U \subseteq R^n \text{ l.s.c. strictly quasi convex}\}.$$

As usual we would use the lattice ordering of R^n ie.
ie. $u = (u_1,..,u_n) \leqslant (\bar{u}_1,...,\bar{u}_n) = \bar{u}$ iff $u_i \leqslant \bar{u}_i \; \forall \; i=1,..,n$.

We will justify this assumption in the last chapter. For now we will investigate the method of choosing a continuous selection to approximate the points of the original multi-valued mapping. We are dealing

with the minimization (or max.) problem
(MP);

$f_i : U_1 \times U_2 \to R$ jointly continuous for all $i=1,..,m$,

$$M(u_1) = \sup\{g(u_1) : f_i(u_1,u_2) \leq \bar{b};\ i=1,..,m\},$$

$$\alpha(u_1) = \{u_2 : g(u_2) = M(u_1);\ f_i(u_1,u_2) \leq \bar{b};\ i=1,..,m\}.$$

In order to find $\alpha(u_1)$ we use a selecting function $g(u_2) = -d(0,u_2)$ or $g(\cdot)$ any strictly concave function, as the following indicates.

Theorem 3.8$'$: Suppose

$$\Gamma_m(b) = \{u_2 : \sup_{i=1,..,m} f_i(u_1,u_2) \leq b\}$$

is a convex set and g strictly concave. If $\bar{u}$ is a solution to (MP) then $\bar{u}$ is the unique solution of (MP).

Proof : See reference [19], page 73. □

Theorem 3.9 : Suppose $f(u_1,\cdot)$ is quasi-convex and $\exists\ \{f_i\}_{i=1}^{\infty}$, $f_i : U_1 \times U_2 \to R^n$ is continuous on $U_1 \times U_2 \subseteq R^n$, where U_2 is compact and both U_1 and U_2 are convex.

Suppose

(a) $h_m(u_1,u_2) = \sup_{i=1,..,m} f_i(u_1,u_2) < f(u_1,u_2)$

where the $f_i(u_1,u_2)$ are strictly quasi convex ,

(b) $T_m(u_1) = \{u_2 : h_m(u_1,u_2) \leq \bar{b}\}$

where

$$T(u_1) = \{u_2 : f(u_1,u_2) \leq \bar{b}\} \neq \phi$$

and

(c) $h_m(u_1,u_2) \uparrow f(u_1,u_2)$ pointwise.

Then $T_m(u_1)$ is Hausdorff continuous $\forall$ m and

$$\cap_m T_m(u_1) = T(u_1).$$

<u>Proof</u> : First, as the f_i's are strictly quasi continuous and

$$S_i(u_1) = \{u_2 : f_i(u_1,u_2) < \bar{b}\} \supseteq T(u_1) \neq \phi$$

is open, $S_i^c(u_1) = \{u_2 : f_i(u_1,u_2) \geq \bar{b}\}$ is u.s.c. (has a closed graph and U_2 is compact). From proposition 1.8 we can conclude $S_i(u_1)$ is Scott continuous. As a consequence so is $\cap_{i=1}^m S_i(u_1)$.

As Scott continuous mappings are l.s.c. multi-valued, we have

$$\text{cl } \cap_{i=1}^m S_i(u_1)$$

is l.s.c. multi-valued and

$$\text{cl } \cap_{i=1}^m S_i(u_1) = \cap_{i=1}^m \text{cl } S_i(u_1) = \cap_{i=1}^m \{u_2 : f_i(u_1,u_2) \leq \bar{b}\} = T_m(u_1)$$

(since the f_i's are strictly quasi-convex).

As U_2 is compact and the graph of $T_m(\cdot)$ is closed, $T_m(u_1)$ must also be u.s.c. and hence Hausdorff continuous.

The last statement follows from

$$\cap_m T_m(\;) = \{u_2 : \sup_i f_i(u_1,u_2) \leq \bar{b}\}$$

$$= \{u_2 : f(u_1,u_2) \leq \bar{b}\} = T(u_1). \qquad \square$$

This demonstrates the generalized convexity nature of the problem. As with what we have seen, we are most interested in the convexity generating class $\mathcal{L} = \{f : f : U_1 \to R^n$ continuous; cl $I(b) = \Gamma(b)$ $\forall\, b \in \text{int } B\}$ for $U_1 \subseteq R^n$ convex and compact.

Corollary 3.9 : If we make the assumptions of Theorem 3.9 and also assume U_1 to be compact, then $\exists$ M s.t. for m > M we have

$$d^*(G_m, G) \leq \varepsilon$$

where G_m is the graph of $T_m(\cdot)$
and G is the graph of $T(\cdot)$.

Proof : Let the generating class of Φ be

$$\mathcal{L} = \{\psi : U_1 \times U_2 \to R \text{ continuous}$$
$$\psi(u_1, \cdot) \text{ strictly quasi convex } \forall\, u_1 \in U_1\} .$$

Then all the assumptions of proposition 3.2 are satisfied and we are assured of the existence of a $T_\varepsilon(\cdot) \in [U_1, \Sigma C\Phi_{ops}(U_2)]$ s.t. $d^*(G_\varepsilon, G) < \varepsilon$ where G_ε is the graph of $T^c_\varepsilon(\cdot)$
and G is the graph of $T(\cdot)$.

Since we have also

$$T_\varepsilon(u_1) << T^c(u_1);\ \forall\, u_1 \in U_1,$$

where $T_\varepsilon(\cdot)$ is Hausdorff continuous ,

all that remains to be shown is that for m sufficiently large

$$T_\varepsilon(u_1) \subseteq T_m(u_1) \subseteq T^c(u_1);\ \forall\, u_1 \in U_1$$

where $T_m(u_1)$ is defined as in Theorem 3.9.

Since $T^c_m(\bar{u}_1) \in \Sigma C\Phi_{ops}(U_2)$ and $\cup_m T^c_m(\bar{u}_1) = T^c(\bar{u}_1)$ we can define a directed set D in

$$\Sigma C\Phi_{ops}(U_2) \quad \text{by} \quad \{\cup^k_{m=1} T^c_m(\bar{u}_1);\ k=1,2,..\}$$

for which

$$\cup\{A \in D\} = T^c(\bar{u}_1).$$

As

$$T_\varepsilon(\bar{u}_1) << T^c(\bar{u}_1),$$

there exists a finite k s.t.

$$T^c(\bar{u}_1) >> T^c_k(\bar{u}_1) = \cup^k_{m=1} T^c_m(\bar{u}_1) >> T_\varepsilon(\bar{u}_1).$$

As a consequence $\exists\ \delta > 0$ s.t.

$$S(T^c_k(\bar{u}_1),\delta) \supseteq N(T_\varepsilon(\bar{u}_1),\delta),$$

where

$$S(T^c_k(\bar{u}_1),\delta) = [\bar{N}(T_k(\bar{u}_1),\delta)]^c$$

and

$$N(S(T^c_k(\bar{u}_1),\delta),\delta) = T^c_k(\bar{u}_1),$$

due to the openess of $T^c_k(\bar{u}_1)$.

Since $T_\varepsilon(\cdot)$ is u.s.c. at $\bar{u}_1$ there must exist a neighbourhood of $\bar{u}_1$, $N(\bar{u}_1)$ say, for which

$$T_\varepsilon(u_1) \subseteq N(T_\varepsilon(\bar{u}_1),\delta);\ \forall\ u_1 \in N_1(\bar{u}_1)\ ,$$

in which case we have

$$T_\varepsilon(u_1) \subseteq S(T^c_k(\bar{u}_1),\delta);\ \forall\ u_1 \in N_1(\bar{u}_1).$$

Since $T_k(\cdot)$ is u.s.c. at $\bar{u}_1$, $\bar{N}(T_k(u_1),\delta)$ is u.s.c. at $\bar{u}_1$ and as a consequence

$$S(T^c_k(u_1),\delta) = [\bar{N}(T_k(u_1),\delta)]^c$$

is i.s.c..

Since $T_k^c(\bar{u}_1) \gg S(T_k^c(\bar{u}_1),\delta)$, $\exists\ \bar{\delta} > 0$ s.t. $(0 < \bar{\delta} < \delta)$

$$T_k^c(\bar{u}_1) \gg N(S(T_k^c(\bar{u}_1),\delta),\bar{\delta})$$

that is

$$T_k^c(\bar{u}_1) \supseteq \bar{N}(S(T_k^c(\bar{u}_1),\delta),\bar{\delta}).$$

By the definition of i.s.c., $\exists$ a neighbour $N_2(\bar{u}_1)$ s.t. $\forall\ u_1 \in N_2(\bar{u}_1)$

$$T_k^c(u_1) \supseteq \bar{N}(S(T_k^c(\bar{u}_1),\delta),\bar{\delta})$$

$$\supseteq N(S(T_k^c(\bar{u}_1),\delta),\bar{\delta}).$$

If we let $N_3(\bar{u}_1) \subseteq N_1(\bar{u}_1) \cap N_2(\bar{u}_1)$ be a neighbourhood of $\bar{u}_1$, we have

$$T_k^c(u_1) \supseteq S(T_k^c(\bar{u}_1),\delta) \supseteq T_\varepsilon(u_1); \quad \forall\ u_1 \in N(\bar{u}_1).$$

Now k depends on $\bar{u}_1$ at this point, but since U_1 is compact there exists a finite sub cover to the cover

$$\{N(\bar{u}_1) : T_\varepsilon(u_1) \subseteq T_k^c(u_1);\ \text{for some } k(\bar{u}_1);\ \bar{u}_1 \in U_1\},$$

$$\{N(u_1^e) : e=1,..,q\},\ \text{say}.$$

For each u_1^e there is a $k(u_1^e) \equiv k_e$ s.t.

$$T_\varepsilon(u_1) \subseteq T_{k_e}^c(u_1);\ \forall\ u_1 \in N(u_1^e).$$

We let $m = \max\{k_e : e=1,..,q\}$ and note that

$$\cup\{T_{k_e}^c(u_1) : e=1,..,q\}$$

$$= \{u_2 : \sup_{e=1,..,q} h_{k_e}(u_1,u_2) > \bar{b}\}$$

$$= \{u_2 : \sup_{i=1,..,m} f_i(u_1,u_2) > \bar{b}\}$$

$$= T_m^c(u_1).$$

If we let $u_1 \in U_1$ be arbitrary there must exist an e s.t. $u_1 \in N(u_1^e)$. Hence

$$T_\varepsilon(u_1) \subseteq T^c_{ke}(u_1) \subseteq T^c_m(u_1) \subseteq T^c(u_1)$$

and our result is proven. □

In the quasi convex case one could also conjecture that the pseudo-convex functions are in fact good enough to approximate the l.s.c. quasi convex functions (see definition 2.7). If so, this is advantageous because of their simple differentiable characterization. We note.

Theorem 3.10 : Let $U \subseteq R^n$ be a convex set and g a numerical function defined on an open set containing U. If g is pseudo-convex on U then g is strictly quasi convex on U and hence quasi convex.

The converse is not true.

Proof : Reference [19], page 143. □

CHAPTER IV

We have considered the approximation, in graph, of upper semi-continuous, convex-imaged multi-functions with continuous, convex-imaged multi-functions. This enables what is usually a fixed point matter to be placed in the context of non-linear optimization. The continuity of the "constraint set" or multi-function is essential in order to produce a sufficiently smooth problem for this to be implemented. Thus the conditions under which a constraint set, depending on a particular parameter, can be considered to be a continuous multi-function, is of interest.

In this Chapter, we begin by reviewing the work of M.H. Stern and D.M. Topkis on rates of continuity of such multi-valued mappings, as arise in non-linear optimization. We go on to extend these results to a broader class depending on a more general parametrization. In their work in reference [24] the above authors consider a multi-valued mapping

$\Gamma(b) = \{u_2 : g_j(u_2) \leqq b_j ; j=1,..,m\}$ and show that

the Cottle constraint qualification plays an important role in producing, not only continuity, but in fact, local linear continuity. We show that under very similar assumptions the multi-valued mapping

$$g \to \Gamma(g,\bar{b}) = \{u_2 : g(u_2) \leqslant \bar{b}\}$$

can be considered to be a locally, linearly continuous multi-function, mapping, from the Banach space of continuously differentiable functions into to $C(U_2)$. These properties flow on to produce locally Lipschitz marginal mappings

$$g \to M(g,b) = \max \{f(u_2) : u_2 \in \Gamma(g,\bar{b})\}$$

and ε-optimal set mappings

$$g \to \alpha(g,\bar{b},\varepsilon) = \{u_2 : g_j(u_2) \leqslant \bar{b};\ j=1,\ldots,m;\ f(u) \geqslant M(g,\bar{b}) - \varepsilon\}$$

It turns out to be much harder to establish local linear continuity of the mapping $b \to \alpha(\bar{g},b,0)$. We are not assured of local linear upper semi-continuity even when $f(\cdot)$ is linear and the Slater constraint qualification holds. Local linear lower semi-continuity may exist in this case but remains an open question. We show that local linear upper semi-continuity, plus the usual constraint qualification assumptions used to produce the local linear continuity of $\Gamma(g,b)$, imply the local linear lower semi-continuity of $\alpha(\bar{g},b,0)$. In fact the rate of local uniform upper semi-continuity is related to the rate of local uniform lower semi-continuity (as was indicated in Chapter One).

Despite the difficulty in establishing the lower semi-continuity of $b \to \alpha(\bar{g},b,0)$ we are able to show that, when $\alpha(\bar{g},b,0)$ is uniformly compact near $\bar{b}$ and $\alpha(\bar{g},\bar{b},0)$ consists only of isolated local minima, then we have the lower semi-continuity of $b \to \alpha(\bar{g},b,0)$ at $\bar{b}$. Lower semi-continuity turns out to be crucial in showing the equivalence of the marginal mapping $b \to M(\bar{g},b)$ and the localized version

$$b \to \hat{M}(\bar{g},b) = \max\{f(u_2) : u_2 \in \Gamma(\bar{g},b) \cap \overline{N}(\bar{u}_2,\delta)\},$$

in some neighbourhood of $\bar{b}$, when $\bar{u}_2$ is a local optimum.

This property is used when showing a Lagrange multiplier is, in fact, a solution to the dual problem of an augmented Lagrangian. The augmented Lagrangian we deal with is that investigated by R.T. Rockafellar and D.P. Bertsekas in references [22], [28] and [32]. This Lagrangian is useful in shedding light on the "generalized differentiability" properties of the non-linear optimization problem we have described above.

As has been shown by various authors, the local Lipschitzness of single (and multi-valued) mappings implies a very general type of differentiability. J. Gauvin showed in references [27] and [29] that the Clarke derivative of the marginal mapping exists under certain conditions, which include the Cottle constraint qualification. He goes on to show that the Clarke derivative can be contained in the convex hull of elements, produced by evaluating the gradient of the usual Lagrangian at all optimal solutions and associated Lagrange multipliers. These theorems can be viewed as a first step towards producing techniques to solve problems such as

$$m(\bar{u}_1) = \min\{\|u_1 - u_2\|^2 : u_2 \in \Gamma(u_1)\}$$

where

$$\Gamma(u_1) = \{u_2 : g_j(u_1, u_2) \leqslant \bar{b}_j \,;\ j=1,..,m\}.$$

Of course when $m(\bar{u}_1) = 0$ we have found a fixed point of the multi-function $\Gamma(u_1)$. For this reason the characterization of the Clarke derivative is of interest.

When we deal with the simpler problem $b \to \bar{m}(b)$, where

$$\bar{m}(b) = \min\{f(u_2) : u_2 \in \Gamma(\bar{g}, b)\}$$

the theorem of J. Gauvin can be stated as

$$\partial\bar{m}(\bar{b}) \subseteq \overline{co}\{-\bar{y} : \exists\ \bar{u}_2 \text{ satisfying with } \bar{y} \text{ the Kuhn-Tucker conditions}\}.$$

We do not attempt to show equivalence of the Clark derivative $\partial\bar{m}(\bar{b})$ to this set but deal with the alternative set of optimal dual solutions to our augmented Lagrangian. That is, we look at the solutions $(\bar{y},\bar{c})$ to the dual problem, for the Lagrangian

$$L(u_2,y,c) = f(u_2) + \sum_{j=1}^{m} y^j \max\{\bar{g}_j(u_2) - \bar{b}_j , \frac{-y^j}{c}\}$$

$$+ (\frac{c}{2}) \sum_{j=1}^{m} \max^2\{\bar{g}_j(u_2) - \bar{b}_j , \frac{-y^j}{c}\}$$

$$= f(u_2) + (\frac{1}{2C}) \sum_{j=1}^{m} \psi(\bar{g}_j(u_2) - \bar{b}_j , y^j),$$

where

$$\psi(\alpha,B) = [\max^2\{0,\beta+c\alpha\} - \beta^2].$$

We show that under some very general conditions, which include local order two Lipschitzness of $b \to \bar{m}(b)$, we have that

$$\partial\bar{m}(\bar{b}) = \{-\bar{y} : (\bar{y},\bar{c}) \text{ is a solution of the dual problem for some } \bar{c} > 0\}.$$

Since the dual variable $\bar{y}$, associated with some optimal solution $\bar{u}_2$, always satisfies the Kuhn-Tucker conditions, we have tightened the previous inclusion by removing the convex closure. There is no guarantee that equivalence can be forced in the former relation and as a consequence, the dual solutions can be thought of as a more "refined" set of Lagrange multipliers.

§4.1 Rates of Continuity in Nonlinear Programming

Definition 4.1 : Let $g_j : U \to R$; $U \subseteq R^n$; $j=1,..,m$ be m functions. For $b \in R^n$ we can define

$$\Gamma(b) = \{u \in U \ ; \ g(u) \leqslant b\} \text{ where}$$

$$g(u) = (g_1(u), \ldots, g_m(u))$$

and for $u \in \Gamma(b)$ we let $b = (b_1, .., b_m)$ and $J(u,b) = \{j : g_j(u) = b_j\}$.

We say the Cottle constraint qualification is satisfied at $\bar{u} \in \Gamma(\bar{b})$ for differentiable $g_j : j=1,..,m$ iff

$$\sum_{j \in J(\bar{u},\bar{b})} \lambda_j \nabla g_j(u) = 0$$

has no semi-positive (ie. non zero, non negative) solutions in the λ's. It is said to hold at $\bar{b}$ if it holds for each $\bar{u} \in \Gamma(\bar{b})$.

Definition 4.2 : The Slater constraint qualification holds at $\bar{u} \in \Gamma(\bar{b})$ if $g_j(u)$ is pseudo-convex for each $j \in J(\bar{u},\bar{b})$ and there exists a $\hat{u}$ s.t. $g_j(\hat{u}) < \bar{b}$ for each $j \in J(\bar{u},b)$.

It is well known that if the Slater constraint qualification holds then the Cottle constraint qualification holds.

The Cottle constraint qualification is known to be equivalent to the existence of a vector e such that

$$<\nabla g_j(\bar{u}), e> < 0 \quad \text{for all } j \in J(\bar{u},\bar{b}).$$

This was used as the constraint qualification in reference [27]. Since these are equivalent we will quote, when referencing J. Gauvin's results, the Cottle constraint qualification. The next result follows from this equivalence.

Theorem 4.1 : If the Cottle constraint qualification holds for $\bar{b}$, then $\mathrm{cl}\, I(\bar{b}) = \Gamma(\bar{b})$.

Proof : Reference [24], Theorem 1.3. □

For a particular $\bar{g} = (\bar{g}_1, \ldots, \bar{g}_m)$ we let $B(\bar{g}) = \{\bar{b} : \{u : \bar{g}(u) \leqslant \bar{b}\} \neq \phi\}$. To obtain results on the uniform linear continuity of $\Gamma(\cdot)$ we look to the work of Stern and Topkis (reference [24]). To obtain such results they first investigated a lower bound on

$$\left| \sum_{j \in J(u,b)} \lambda_j \nabla g_j(u) \right|$$

in terms of $|\lambda|$, where $\lambda_j \geqslant 0$; $\lambda = 0$ for $j \notin J(u,b)$; $u \in \Gamma(b)$ and b is in a prescribed set. We let $(-\infty, \hat{b}] = \{b \in R^m : b \leqslant \hat{b}\}$.

Lemma 4.1 : If $\Gamma(\hat{b})$ is bounded, then $B(\bar{g}) \cap (-\infty, \hat{b}]$ is compact.

Proof : Reference [24] Lemma 2.1. □

We let $D_p(\Omega)$ be the space of functions with p bounded and uniformly continuous derivatives. It can be viewed as a Banach space with the norm

$$\|g\|_p = \max_{0 \leqslant |\alpha| \leqslant p} \sup_{u \in \Omega} |\nabla^{\alpha} g(u)|$$

where $|x|$ denotes the Euclidean norm on R^m, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $\nabla^{\alpha} g(u) = \dfrac{\partial^{\alpha} g(u)}{\partial^{\alpha_1} u_1 \ldots \partial^{\alpha_n} u_n}$.

We shall discuss the continuity of a lower bound on

$$\left| \sum_{j \in J(u,b)} \lambda_j \nabla g_j(u) \right|$$

with respect to the functions g.

Theorem 4.2 : Suppose $g \in D_1(\Gamma(\hat{b}))$ B is a closed subset of B(g), there exists $\hat{b}$ such that $\Gamma(\hat{b})$ is bounded and the Cottle constraint qualification holds for b in $\hat{B} \cap (-\infty,\hat{b}]$. Then these exists $K > 0$ such that

$$\left| \sum_{j\in J(u,b)} \lambda_j \nabla g_j(u) \right| \geq K|\lambda|$$

for all $u \in \Gamma(b)$; $b \in \hat{B} \cap (-\infty,\hat{b}]$, and $|\lambda|=1, \lambda \geq 0$; $\lambda_j = 0$ for all $j \in J(u,b)$.

Proof : Theorem 2.1 of reference [24]. □

The lower bound on K(g) is obtained in the following way

$$K(g) = \min\{K_J(g) : J \subseteq \{1,..,m\};\ J \neq \phi\},$$

$$K(g) = \begin{cases} \inf\{|\sum_{j\in J} \lambda_j \nabla g_j(u)|;\ (u,b,\lambda) \in T_J(g)\}, \\ +\infty \quad \text{otherwise} \end{cases}$$

where

$$T_J(g) = \{(u,b,\lambda) : g(u) \leq b,\ b \in \hat{B} \cap (-\infty,\hat{b}],$$

$$g_j(u) = b \text{ for all } j \in J,\ |\lambda| = 1,\ \lambda \geq 0 \text{ and } \lambda_j = 0 \text{ if } j \notin J\}.$$

One could consider $T_J(\cdot)$ being a function of g and hence $K_J(\cdot)$ and $K(\cdot)$ functions of g. In passing we note that $K(\cdot)$, considered as a function of b, for fixed g, (in a similar way) is monotonically decreasing.

Quite often it is easy to deduce that a multi-valued mapping is closed but much harder to deduce upper-semi-continuity. If the image sets are contained in a compact space then closedness immediately implies upper semi-continuity. A weaker assumption which replaces upper semi-continuity at a point is that the mapping is closed and "uniformly compact" at that point. That is; given $u \to \Omega(u)$, then $\Omega(\cdot)$ is uniformly compact near $\bar{u}$ if there is a neighbourhood N of $\bar{u}$ such that the closure of the set $\cup\{\Omega(u) : u \in N\}$ is compact.

Lemma 4.2 :

Suppose the condition of Theorem 4.2 hold for $\hat{B} \subseteq \text{int } B(\bar{g})$, $\bar{g} \in \mathcal{D}_1(R^n)$ and that $\Gamma(\bar{g},\bar{b})$ is bounded for $\bar{b} > \hat{b}$. Then $K(\cdot)$ is lower semi-continuous at $\bar{g}$ in the space $\mathcal{D}_1(R^n)$.

Proof : In view of Theorem 1.18 and the fact that $|\sum_{j\in J} \lambda_j \nabla g_j(u)|$ is continuous in (u,b,λ,g) jointly, we only need to show that the multi-valued mapping $T_J(\cdot)$ is non-empty in a $\mathcal{D}_1(R^n)$ neighbourhood of $\bar{g}$, closed and uniformly compact near $\bar{g}$.

First of all we need to show $T_J(g)$ is non-empty in a neighbourhood of $\bar{g}$. If int $B(g)$ can be shown to be i.s.c., then $\hat{B} \subseteq \text{int } B(\bar{g})$ will imply $\hat{B} \subseteq B(g)$ for $g \in N(\bar{g},\delta)$ (some $\delta > 0$), in which case $T_j(g) \neq \phi$.

We have

$$(\text{int } B(g))^c = (\text{int } \{b : \Gamma(g,b) \neq \phi\})^c$$

$$= \text{cl } \{b : \Gamma(g,b) \neq \phi\}^c$$

$$= \text{cl } \{b : \Gamma(g,b) = \phi\}.$$

Now $\Gamma(g,b) = \phi$ iff

$$\inf\{g_j(u) : u \in U\} > b_j \text{ for some } j.$$

Hence $b \in \text{cl } \{b : \Gamma(g,b) = \phi\}$ iff

$$F_j(g) = \inf\{g_j(u) ; u \in U\} \geqslant b_j \text{ for some } j,$$

that is,

$$(\text{int } B(g))^c = \bigcup_{j=1}^{m} \{b : F_j(g) \geq b_j\}.$$

For a fixed $u \in U$;$g \to g_j(u)$ is continuous in $\mathcal{D}_1(R^n)$. As a consequence

$$g \to \inf\{g_j(u) : u \in U\}$$

is u.s.c. in $\mathcal{D}_1(R^n)$, being an infimum of a class of continuous mappings.

The mapping $g \to \{b : F_j(g) \geq b_j\}$ is clearly u.s.c. multi-valued and so is $\bigcup_{j=1}^{m} \{b : F_j(g) \geq b_j\}$, being a finite union of u.s.c. multi-valued mappings. This establishes the non-emptiness of $T_J(\cdot)$ in a neighbourhood of $\bar{g}$.

In order to establish the uniform compactness we note that $\hat{B} \cap (-\infty,\hat{b}] \subseteq B(\bar{g}) \cap (-\infty,\hat{b}]$ is compact, as $\Gamma(\bar{g},\hat{b})$ is bounded and that the λ's are always contained in a compact set. If we can establish that $\Gamma(g,\hat{b})$ is contained in a compact set for all g in a neighbourhood of $\bar{g}$ then so will be $T_j(\cdot)$.

Since $\Gamma(\bar{g},\bar{b})$ is bounded, $\bar{b} > \hat{b}$, we have, using Theorem 3.3., shown $\Gamma(\bar{g},\hat{b})$ to be upper semi-continuous at $\hat{b}$. Using Theorem 3.4 we can deduce the upper semi-continuity of $g \to \Gamma(g,\hat{b})$ at $\bar{g}$. Let $N(\bar{g},\delta)$ be a neighbourhood of $\bar{g}$ for which $\hat{B} \subseteq B(g)$. Since $(\bar{g},\hat{b})$ is bounded so is $N(\Gamma(\bar{g},\hat{b}),\varepsilon)$. By the u.s.c. of $\Gamma(g,\hat{b})$ at $\bar{g}$ we have $\Gamma(g,\hat{b}) \subseteq N(\Gamma(\bar{g},\hat{b}),\varepsilon)$ for $\forall\, g \in N(\bar{g},\delta)$, for some δ sufficiently small.

We let

$$S = \{\lambda : \lambda \geq 0;\ |\lambda|^2 = \sum_{i=1}^{m} \lambda_i^2 = 1\}$$

and note that

$$\cup\{T_J(g) : g \in N(\bar{g},\delta)\}$$

$$\subseteq Z \times (\hat{B} \cap (-\infty,\hat{b}]) \times S,$$

which is compact.

One can easily verify that $T_J(\cdot)$ is closed to complete the proof. □

<u>Theorem 4.3</u> : Suppose the Cottle constraint qualification holds at $\bar{b} \in B$, there exists $\hat{b} > \bar{b}$ such that $\Gamma(\hat{b})$ is bounded, and each $g_j(u)$ has continuous second derivatives on R^n. Then there exists $\delta > 0$ such that $\Gamma(b)$ is uniformly linearly continuous on $B(\bar{g}) \cap N(\bar{b},\delta)$ with a constant $2/K(\bar{g})$, ie.

$$d(\Gamma(b), \Gamma(b'))\ (\tfrac{1}{2}K(\bar{g})) \leqslant |b-b'|$$

for all $b, b' \in B(\bar{g}) \cap N(\bar{b},\delta)$.

<u>Proof</u> : Reference [24], theorem 3.2. □

<u>Corollary 4.3</u> : Suppose each $\bar{g}_j$; $j=1,..,m$ are pseudo-convex and have continuous second derivatives on R^n. Then $\Gamma(b)$ is locally uniformly linearly continuous on

$$\text{int } B(\bar{g}) \cap \{\bar{b} \in R^n : \Gamma(\bar{b}) \text{ is bounded}\}$$

$$= \bar{B} \text{ (say)}.$$

<u>Proof</u> : The Cottle constraint qualification holds for all $b \in \text{int } B(\bar{g})$ as the Slater condition holds (i.e. $\bar{g}_j$ are pseudo-convex). As the g_j are strictly quasi-convex, Theorems 3.5(b) and 4.3 establish the result. □

The function $f = R^n \to R$ is said to satisy a Lipschitz condition order $\beta > 0$ if there exists some $L > 0$ such that

$$|f(u) - f(\hat{u})| \leq L\|u-\hat{u}\|^{\beta}.$$

In the following we investigate the properties of $M(b) = \max\{f(u):u\in\Gamma(b)\}$.

Theorem 4.4 : Suppose

(i) the Cottle constraint qualification holds at $\bar{b} \in B(\bar{g})$,

(ii) there exists $\hat{b} > \bar{b}$ such that $\Gamma(\hat{b})$ is bounded,

(iii) each $\bar{g}_j(\cdot)$ has continuous second derivatives on R^n and

(iv) $f(\cdot)$ satisfies a Lipschitz condition order $\beta > 0$ on $\Gamma(\hat{b})$.

Then there exists $\delta > 0$ such that $M(b)$ satisfies a Lipschitz condition order $\beta > 0$ on $B(\bar{g}) \cap N(\bar{b},\delta)$.

Proof : A direct adaptation of Corollary 4.2 and Theorem 4.1 of reference [24] with the obvious modifications. □

We let for $\hat{b}$, $b \in R^m$

$$(-\infty,\hat{b}] = \{x \in R^m : x \leq \hat{b}\}$$

and

$$[b,\hat{b}] = \{x \in R^m : b \leq x \text{ and } x \leq \hat{b}\}.$$

Corollary 4.4 : Suppose $\Gamma(\hat{b})$ is bounded for $\hat{b} \in R^m$, each g_j ; $j=1,..,m$ is pseudo-convex and has continuous second derivatives on R^m. Suppose also that $f(\cdot)$ satisfies a Lipschitz condition order $\beta > 1$. Then $M(\cdot)$ satisfying a Lipschitz condition order β on

$$B(\bar{g}) \cap (-\infty,\hat{b}].$$

Proof : From Lemma 4.1 we know $B(\bar{g}) \cap (-\infty,\hat{b}]$ is compact. Now

$$b \in B(\bar{g}) = \{b : \Gamma(\bar{g},b) \neq \phi\}$$

if $\exists$ u s.t. $\bar{g}(u) \leq b$.

If we restrict $b \leq \hat{b}$, then $u \in \Gamma(\bar{g},\hat{b})$, a compact set. In this case $b \in B(\bar{g})$ iff

$$\bar{b} = \min\{\bar{g}(u) : u \in \Gamma(\bar{g},\hat{b})\}$$

$$= \inf\{\bar{g}(u) : u \in R^n\} \leqslant b.$$

We have

$$B(\bar{g}) = [\bar{b},+\infty)$$

a convex set and

$$B(\bar{g}) \cap (-\infty,\hat{b}]$$

$$= [b.+\infty) \cap (-\infty,\hat{b}] \text{ a convex set.}$$

Obviously for $b > \bar{b}$ we have $b \in \text{int } B(\bar{g})$. For $[b,\hat{b}]$ there exists a finite sub-cover of the cover $S = \{N(b,\delta) : b \in \text{int } B(\bar{g})$; $M(\cdot)$ is locally Lipschitz order β on $B(\bar{g}) \cap N(b,\delta)$; $\delta > 0$ and $b \in B(\bar{g}) \cap (-\infty,\hat{b}]\}$ of $B(\bar{g}) \cap (-\infty,\hat{b}]$. Suppose

$$S' = \{N(b_i,\delta_i),\ i=1,..,\ell\}$$

is the sub-cover. For b, $\bar{b} \in [b,\hat{b}]$ we let

$$P = \{b' \in R^n : b' = \lambda b + (1-\lambda)\bar{b}, \lambda \in [0,1]\}.$$

Then $\exists\ b = b_0, b_1, \ldots, b_k = \bar{b} \in P$ s.t. $b_j = \lambda_j b + (1-\lambda_j)\bar{b}$; $\lambda_j < \lambda_{j+1}$ for $j=0,1,..,k-1$ and b_j, $b_{j+1} \in N(b_i,\delta_i)$ for some $b_i \in \{1,.,\ell\}$ $\forall$ j.

This follows from the fact that P is compact, connected and hence chainable, using also the openess of the balls $N(b_i,\delta_i)$. We note that

$$\sum_{j=0}^{k-1} |\lambda_{j+1} - \lambda_j| = 1.$$

If

$$Q = \max\{\bar{K}_i \ ; \text{ the Lipschitz constant on } N(b_i,\delta_i)\}$$

then

$$
\begin{aligned}
|M(b) - M(\bar{b})| &= \Big|\sum_{j=0}^{k-1} M(b_{j+1}) - M(b_j)\Big| \\
&\leq \sum_{j=0}^{k-1} |M(b_{j+1}) - M(b_j)| \\
&\leq Q \sum_{j=0}^{k-1} \| b_{i+1} - b_i \|^{\beta} \\
&= Q \sum_{j=0}^{k-1} |\lambda_{j+1} - \lambda_j|^{\beta} \| b - \bar{b} \|^{\beta} \\
&\leq Q \| b - \bar{b} \|^{\beta} \Big(\sum_{j=0}^{k-1} |\lambda_{j+1} - \lambda_j| \Big) \\
&= Q \| b - \bar{b} \|^{\beta}
\end{aligned}
$$

using $1 \geq |\lambda_{j+1} - \lambda_j| \geq |\lambda_{j+1} - \lambda_j|^{\beta}$, as $\beta \geq 1$. □

We could have equivalently assumed that the Cottle constraint qualification holds at all $b \in \text{int } B(\bar{g})$.

Obviously we have trouble at $\bar{b}$ since

$$
\begin{aligned}
\bar{b} &= \min \{\bar{g}(u) : u \in R^n\} \\
&= \min \{\bar{g}(u) : u \in \Gamma(\bar{g},\hat{b})\}
\end{aligned}
$$

implying the minimum is attained at the points $S = \{u : g(u) = \bar{b};$ $u \in \Gamma(\bar{g},\hat{b})\}$. That is for $u \in S$ all the constraints are active since $g(u) = \bar{b}$ and since $\nabla g_j (u) = 0$, $\forall\, j = 1,\ldots,m$ the Cottle constraint qualification could not possibly hold at $\bar{b}$.

<u>Lemma 4.3</u> : Suppose the Cottle constraint qualification holds at $\bar{b} \in B$ and there exists a $\hat{b} > \bar{b}$ such that $\Gamma(\bar{g},\hat{b})$ is bounded. Then $\exists\, \delta > 0$ s.t. the Cottle constraint qualification holds at $\bar{b}$ for all $g \in G_2(\bar{b},\bar{g},\delta)$ where

$$G_2(\bar{b},\bar{g},\delta) = \{g \in D_2(\Gamma(\bar{g},\hat{b})) : \Gamma(g,\bar{b}) \neq \phi;\ \| g-\bar{g} \|_2 < \delta\}$$

and

$$\Gamma(g,\bar{b}) = \{u : g(u) \leq \bar{b}\}.$$

Proof : First we show that $\Gamma(g,\bar{b})$ is bounded for δ sufficiently small. We let

$$\Delta(g,\bar{g}) = (\sup_u |\bar{g}_1(u) - g_1(u)|, \ldots, \sup_u |\bar{g}_m(u) - g_m(u)|)$$

and show

$$\Gamma(\bar{g},\bar{b} - \Delta(g,\bar{g})) \subseteq \Gamma(g,\bar{b}) \subseteq \Gamma(\bar{g},\bar{b} + \Delta(g,\bar{g})).$$

Let $u \in \Gamma(\bar{g},\bar{b} - \Delta(g,\bar{g}))$. Then since $\bar{g}(u) \leqslant \bar{b} - \Delta(g,\bar{g})$ we have

$$g(u) \leqslant \bar{g}(u) + \Delta(g,\bar{g}) \leqslant \bar{b} \text{ implying}$$

$$g(u) < \bar{b} \quad \text{and} \quad u \in \Gamma(g,\bar{b}).$$

Similarly if $u \in \Gamma(g,\bar{b})$ then $g(u) < \bar{b}$. Since $\bar{g}(u) - \Delta(g,\bar{g}) \leqslant g(u)$ we have $\bar{g}(u) \leqslant \bar{b} + \Delta(g,\bar{g})$ and $u \in \Gamma(\bar{g},\bar{b} + \Delta(g,\bar{g}))$.

As $\Gamma(\bar{g},b)$ is bounded for $\hat{b} > \bar{b}$ we choose $0 < \delta < \hat{b} - \bar{b}$ and $\forall\, g \in G_2(\bar{b},\bar{g},\delta)$ we have $\Gamma(g,\bar{b})$ bounded.

We now argue in a similar manner to Lemma 2.2 of Reference [24]. Suppose the contrary is true; that is there exists a sequence $g^k \in G(\bar{b},\bar{g},\delta)$ with $\lim_{k\to\infty} g^k = \bar{g}$ in $\mathcal{D}_2(R^n)$ such that the Cottle constraint qualification doesn't hold for any g^k at $\bar{b}$. Thus $\exists\, u_k$ such that $g^k(u_k) \leqslant \bar{b}$ $J(u_k,\bar{b})$ is non-empty, $\exists\, \lambda^k > 0$, $|\lambda^k| = 1$; $\lambda_j^k = 0$, $\forall\, j \notin J(u_k,\bar{b})$ and $\sum_{j\in J(u_k,\bar{b})} \lambda_j^k \nabla g_j^k(u_k) = 0$. As $\Gamma(g_k,\bar{b})$ is bounded for k large, $u_k \in \Gamma(g^k,\bar{b})$ and $|\lambda^k| = 1$ for all , there exists a convergent subsequence of $(\bar{b},u_k,\lambda^k)$ with limit $(\bar{b},\bar{u},\bar{\lambda})$ such that $J(u_k,\bar{b}) = J$ and $\lambda_j^k = 0$ for $j \notin J$ for all k in the subsequence. Then $\bar{\lambda} \geqslant 0$, $|\bar{\lambda}| = 1$ and $\bar{\lambda}_j = 0$ for $j \notin J$.

By continuity $J \subseteq J(\bar{u},\bar{b})$. For if we suppose $j \notin J(\bar{u},\bar{b})$ then $\bar{g}_j(\bar{u}) < \bar{b}$, which implies for k large $g_j^k(u_k) < \bar{b}$. Since $\sum_{j\in J} \bar{\lambda}_j \nabla\bar{g}_j(\bar{u}) = 0$ we have a contradiction. □

<u>Theorem 4.5</u>* : Suppose

(i) The Cottle constraint qualification holds at $\bar{b} \in B(\bar{g})$,

(ii) there exists $\hat{b} > \bar{b}$ such that $\Gamma(\hat{b})$ is bounded, and

(iii) each $\bar{g}_j(u)$ is twice continuously differentiable on R^n.

Then the multi-valued mapping $g \to \Gamma(g,\bar{b}) \subseteq R^n$ is uniformly linear continuous for some $\bar{\delta} > 0$ on $G_2(\bar{g},\bar{b},\bar{\delta})$, i.e., if we choose $\bar{K}(\bar{g}) > 2/K(\bar{g})$, $\exists\ \delta > 0$ s.t.

$$d(\Gamma(g,\bar{b}),\ \Gamma(\hat{g},\bar{b})) \leqslant \bar{K}(\bar{g})\| g - \hat{g}\|_2 ; \forall\ g,\hat{g} \in G_2(\bar{g},\bar{b},\delta).$$

<u>Proof</u> : First we show that for $\hat{K} < \frac{1}{2}K(\bar{g}), \exists\ \bar{\delta} > 0$ s.t.

$$\forall\ g \in G(\bar{g},\bar{b},\bar{\delta})$$
$$b \in N(\bar{b},\bar{\delta})$$
$$d(\Gamma(g,b),\ \Gamma(g,\bar{b}))\hat{K} \leqslant |b - \bar{b}|$$

and then let $\bar{K}(\bar{g}) = 1/\hat{K}$. Suppose not. Then for $\hat{K} < \frac{1}{2}K(\bar{g})$

$$\bar{\delta} = 1/k \to 0, \exists\ g^k \in G\ (\bar{g},\bar{b},1/k)$$

s.t.

$$d(\Gamma(g^k,b),\ \Gamma(g^k,\bar{b}))\hat{K} \geqslant |b_k - \bar{b}| \text{ for some} \qquad \ldots(1)$$

$b_k \in N(\bar{b}, 1/k)$.

As $K(g)$ is l.s.c. at $\bar{g}$ (Lemma 3.2), by letting $0 < \varepsilon < \frac{1}{2}K(\bar{g}) - \hat{K}$, we have $\frac{1}{2}K(g^k) > \frac{1}{2}K(\bar{g}) - \varepsilon > \hat{K}$ for k sufficiently large.

Hence (1) implies

$$d(\Gamma(g^k,b),\ \Gamma(g^k,\bar{b}))\tfrac{1}{2}K(g^k) \geqslant |b_k - \bar{b}| \text{ for some} \qquad \ldots(2)$$

$b_k \in N(\bar{b}, 1/k)$ and k sufficiently large.

* It has been brought to the author's attention that Lemma 4.3 and Theorem 4.5 are related to Theorems of references [38] and [39].

As $g^k \in \mathcal{D}_2(R^n)$ it has continuous second derivatives and by Lemma 4.3 the Cottle constraint qualification holds for g^k at $\bar{b}$ for k sufficiently large. If we let $\bar{b} < \tilde{b} < \hat{b}$, then as in Lemma 4.3 we have

$$\Gamma(g^k,\tilde{b}) \subseteq \Gamma(\bar{g}, \tilde{b} + \Delta(g^k,\bar{g})).$$

As $g^k \in G_2(\bar{g},\bar{b}, 1/k)$ for k sufficiently large, we have

$$\tilde{b} + \Delta(g^k,\bar{g}) < \hat{b}$$

and hence $\Gamma(g^k,\tilde{b})$ is bounded for $\bar{b} < \tilde{b}$ and hence bounded at $\bar{b}$. All conditions of Theorem 4.3 hold for g^k for k sufficiently large hence (2) constitutes a contradiction and the result is established.

We note the following. Let

$$\Delta(g,\hat{g}) = (\sup_u |g_1(u) - \hat{g}_1(u)|, \ldots, \sup_u |g_m(u) - \hat{g}_m(u)|) .$$

Then if $b^0 = \bar{b} + \Delta(g,\hat{g})$, we have

$$|b^0-b| = |\Delta(g,\hat{g})| = \sup_u |g-\hat{g}|$$

$$\leq \max_{0\leq|\alpha|\leq 2} \sup_u |\nabla^\alpha g(u)-\nabla\hat{g}(u)|$$

$$\equiv \| g_j - \hat{g}_j \|_2 .$$

Hence if $g,g \in G_2(\bar{g},\bar{b},\delta)$ we have

$$\Gamma(g,\bar{b}) \subseteq \Gamma(\hat{g},b + \Delta(\hat{g},g))$$

$$= \Gamma(\hat{g},b^0);\ b^0 \text{ as above}$$

$$\subseteq N(\Gamma(\hat{g},\bar{b});\ 1/\hat{K}\ |b^0 - \bar{b}|)$$

$$\subseteq N(\Gamma(\hat{g},\bar{b}),1/\hat{K}\ \| g_j - \hat{g}_j \|_2).$$

Due to the symmetry between $g,\hat{g}$ we may interchange $g,\hat{g}$ to obtain the result. □

Corollary 4.5 : Suppose each $\bar{g}_j$; j=1,..,m arepseudo-convex and have continuous second derivatives on R^n, $\Gamma(\bar{g},\bar{b})$ bounded. Then $\Gamma(g,\bar{b})$ is uniformly linearly continuous on some $G_2(\bar{g};\bar{b},\delta)$ whenever $\bar{b} \in \text{int } B(\bar{g})$.

Proof : The Cottle constraint qualification holds at $\bar{b} \in \text{int } B(\bar{g})$ as the Slater condition holds. As $\bar{g}_j$ is quasi convex, Theorems 3.5(b) and 4.4 establish the result. □

Theorem 4.6 : Suppose

(i) The Cottle constraint qualification holds at $\bar{b} \in B$,

(ii) there exists $\hat{b} > \bar{b}$ such that $\Gamma(\bar{g},\hat{b})$ is bounded

(iii) each $\bar{g}_j(\cdot)$ is twice continuously differentiable, and

(iv) the function $f(\cdot)$ satisfies a Lipschitz condition order $\beta > 0$ on $\Gamma(\bar{g},\hat{b})$.

Then $M(g) = \max\{f(u) : u \in \Gamma(g,\bar{b})\}$ satisfies a Lipschitz condition order $\beta > 0$ on $G_2(\bar{g},\bar{b},\delta)$ for some $\delta > 0$.

Proof : From our previous theorem we have $\delta > 0$ s.t.

$\forall\, g,\hat{g} \in G_2(\bar{g},\bar{b},\delta)$

$$d(u,\Gamma(\hat{g},\bar{b})) \leqslant \bar{K}(\bar{g})\|g - \hat{g}\|_2,\ \forall\, u \in \Gamma(g,\bar{b}).$$

As $\Gamma(g,\bar{b}) \subseteq \Gamma(\bar{g},\bar{b} + \Delta(g,\bar{g}))$ if we take $\delta < \Delta(g,\bar{g}) < \hat{b} - \bar{b}$ then f is Lipschitz on all $\Gamma(g,\bar{b})$ for $g \in G_2(\bar{g},\bar{b},\delta)$ with constant L.

Pick $g,\hat{g} \in G_2(\bar{g},\bar{b},\delta)$. Without loss of generality we may assume $M(\hat{g}) \leqslant M(g)$. Pick $u \in \Gamma(g,\bar{b})$ such that $M(g) = f(u)$. Then pick $\hat{u} \in \Gamma(\hat{g},\bar{b})$ so that

$$d(u,\Gamma(\hat{g},\bar{b})) = \|u - \hat{u}\|.$$

Hence $|M(g) - M(\hat{g})| = M(g) - M(\hat{g}) = f(u) - M(\hat{g}) \leqslant f(u) - f(\hat{u})$ $\leqslant L\|u - \hat{u}\|^{\beta}$, so

$|M(g) - M(\hat{g})|$

$\leqslant L\|u - \hat{u}\|^{\beta} = L[d(u,\Gamma(\hat{g},\bar{b}))]^{\beta}$

$\leqslant L\bar{K}(\bar{g})^{\beta}\|g - \hat{g}\|_2^{\beta}.$ □

Theorem 4.7 : Suppose each $\bar{g}_j$; j=1,..,m are pseudo-convex and have continuous second derivatives on R^n, $\Gamma(\bar{g},\bar{b})$ bounded for $\bar{b} \in \text{int } B(\bar{g})$ and $-f(\cdot)$ pseudo-convex and also twice continuously differentiable. Then $\forall\ \varepsilon > 0$

$$\alpha(g,\varepsilon) = \{u : g_j(u) \leqslant \bar{b};\ j=1,..,m; f(u) \geqslant M(g) - \varepsilon\}$$

is uniformly linearly continuous on $G_2(\bar{g},\bar{b},\delta)$ for some $\delta > 0$.

Proof : First we choose δ_1 sufficiently small so that

$\forall\ g \in G_2(\bar{g},\bar{b},\delta_1)$ we have

$\Gamma((g,-f), (\bar{b},-M(\bar{g})+\varepsilon)) \neq \phi,$

which is possible since the mapping $g \to M(g)$ is continuous.

Let $M(g) = \max\{f(u) : g_j(u) \leqslant \bar{b}_j ;\ j=1,..,m\}$ and

$$F(g_1,..,g_m,g_{m+1}) = (g_1,...,g_m,g_{m+1}+M(g_1,..,g_m) - M(\bar{g}_1,..,\bar{g}_m)).$$

Then $g \to F(g)$ is Lipschitz continuous, from Theorem 4.5, as the Cottle constraint qualification holds and $\Gamma(\bar{g},\hat{b})$ is bounded for some $\hat{b} > \bar{b}$, since g is quasi convex.

As f is continously differentiable and $\Gamma(\bar{g},\hat{b})$ is bounded, f is Lipschitz on $\Gamma(\bar{g},\hat{b})$ implying $M(\cdot)$ is Lipschitz locally with some constant $L > 0$.

We now apply Theorem 4.5 to the multi-valued mapping (we actually hold f constant in the final analysis)

$$\Gamma((g_1,..,g_m,g_{m+1}),(\bar{b}_1,..,\bar{b}_m,-M(\bar{g})+\varepsilon))$$

to deduce its local linear continuity (as a function of $(g_1,..,g_m,g_{m+1})$). Hence

$$d(\Gamma((g,g_{m+1}),(\bar{b},-M(\bar{g})+\varepsilon)),\ \Gamma((\hat{g},\hat{g}_{m+1}),(\bar{b},-M(\bar{g})+\varepsilon)))$$

$$\leq \bar{K}_\epsilon(\bar{g},\bar{g}_{m+1})\|(g,g_{m+1})-(\hat{g},\hat{g}_{m+1})\|_2$$

$$\forall\ (g,g_{m+1}),(\hat{g},\hat{g}_{m+1}) \in G_2((\bar{g},-f),(\bar{b},-M(\bar{g})+\varepsilon),\delta_2).$$

Since $\alpha(g,\varepsilon) = \Gamma(F((g,-f)),\ (\bar{b},-M(\bar{g})+\varepsilon))$, this implies

$$d(\alpha(g,\varepsilon),\ \alpha(\hat{g},\varepsilon))$$

$$\leq \bar{K}_\epsilon(\bar{g},-f)\|F(g,-f)\ -\ F(\hat{g},-f)\|_2$$

$$= \bar{K}_\epsilon(\bar{g},-f)(\|g\ -\ \hat{g}\|_2 + |M(g)-M(\hat{g})|)$$

$$\leq \bar{K}_\epsilon(\bar{g},-f)(\|g\ -\ \hat{g}\|_2 + L\|g\ -\ \hat{g}\|_2)$$

$$= \bar{K}_\epsilon(\bar{g},-f)(L+1)\|g\ -\ \hat{g}\|_2$$

$$\forall\ g,\hat{g} \in G_2(\bar{g},\bar{b},\bar{\delta}) \text{ for } 0 < \bar{\delta} < \delta_1$$

sufficiently small so that M( ) is locally Lipschitz and

$$F(g,-f),\ F(\hat{g},-f) \in G_2((\bar{g},-f),\ (-M(\bar{g})+\varepsilon),\delta_2).$$ □

It was demonstrated in reference [24] via numerous counter examples that there is little hope of proving a similar result to this replacing pseudo-convexity by any weaker a notion. It has remained an open question whether the simpler problem $b \to \alpha(b,0)$, involving the variation in b, would be linearly lower semi-continuous if $g(\cdot)$ satisfies the Slater condition and $-f(\cdot)$ is linear or convex. It appears to be much harder to establish results when $\varepsilon = 0$, especially lower semi-continuity. It is still possible that $b \to \alpha(b,\varepsilon)$ may be, in some circumstances, locally upper semi-continuous at some uniform rate $g(\cdot)$ which is strictly increasing and continuous. Possibly the conditions of Theorem 4.6 would imply this.

In the following we call;

$$\alpha(b,0) = \{u : -f(u) \leqslant -M(b);\ u \in \Gamma(b)\}$$

$$M(b) = \max \{f(u) : u \in \Gamma(b)\}$$

and

$$\Gamma(b) = \{u : g_j(u) \leqslant b_j ;\ j=1,..,m\}.$$

Theorem 4.8 : Suppose

(i) $\Gamma(b)$ is uniformly linear continuous with a constant K for $b \in \hat{B}$, and

(ii) $f(\cdot)$ satisfies a Lipschitz condition with a constant M on $\cup\{\Gamma(b);\ b \in \hat{B}\}$.

Then

$$d(\bar{u}, \alpha(b,2KM|b-b'|)) \leqslant K|b-b'|$$

for each $b, b' \in \hat{B}$, $\varepsilon \geqslant 0$ and $\bar{u} \in \alpha(b',0)$.

Proof : Reference [24] Theorem 5.4. □

Combining this with previous results we have.

Theorem 4.9 : Suppose

(i) The Cottle constraint qualification holds at $\bar{b} \in B(\bar{g})$,

(ii) there exists $\hat{b} > \bar{b}$ such that $\Gamma(\hat{b})$ is bounded,

(iii) each $\bar{g}_j(\cdot)$ has continuous second derivatives on R^n, and

(iv) $f(\cdot)$ satisfies a Lipschitz condition on $\Gamma(\hat{b})$.

Then $\exists\ K > 0$, $M > 0$ and $\delta > 0$ such that

$$d(\bar{u}, \alpha(b,K|b-b'|)) \leqslant M|b-b'|$$

for each b', $b \in B(\bar{g}) \cap \bar{N}(\bar{b},\delta)$ and $\bar{u} \in \alpha(b',0)$.

Proof : Reference [24] Corollary 5.2 with $\varepsilon = 0$. □

This looks very much like a sort of lower semi-continuity at $b = \bar{b}$. Unfortunately, we require more to achieve this.

Corollary 4.9 : Suppose the conclusion of Theorem 4.9 holds and suppose also that the multi-valued mapping $\varepsilon \to \alpha(b,\varepsilon)$ is uniformly linearly upper semi-continuous at $\varepsilon = 0$, locally around $\bar{b}$ (i.e., for $b \in B(\bar{g}) \cap \bar{N}(\bar{b},\delta)$). Then the multi-valued mapping

$$b \to \alpha(b,0)$$

is locally, linearly lower semi-continuous around $\bar{b}$ and hence locally, linearly continuous there.

Proof : The conclusion of Theorem 4.8 can be written as

$$d^*(\alpha(b',0), \alpha(b,K|b-b'|)) \leqslant M|b-b'|$$

for b', $b \in B(\bar{g}) \cap \bar{N}(\bar{b},\delta)$.

The assumption of linear upper semi-continuity, locally around $\bar{b}$, implies that for any $r > 0$ there exists $\delta' > 0$ and $L > 0$ s.t.

for $b \in B(\bar{g}) \cap \bar{N}(\bar{b},\delta')$, we have

$$\alpha(b,\bar{N}(0,Lr)) = \alpha(b,Lr)$$

we have, after letting $r = (\frac{K}{L})|b-b'|$,

$$\alpha(b,K|b-b'|) \subseteq \bar{N}(\alpha(b,0), (\frac{K}{L})|b-b'|).$$

That is,

$$d^*(\alpha(b,K|b-b'|), \alpha(b,0)) \leq (\frac{K}{L})|b-b'|.$$

Finally for b, b' $\in \bar{N}(\bar{b},\delta'')$, where $\delta'' = \min(\delta,\delta')$, we have

$$d^*(\alpha(b',0), \alpha(b,0))$$

$$\leq d^*(\alpha(b',0), \alpha(b,K|b-b'|))$$

$$+ d^*(\alpha(b,K|b-b'|), \alpha(b,0))$$

$$\leq M|b-b'| + (\frac{K}{L})|b-b'|$$

$$= (M + \frac{K}{L})|b-b'|,$$

implying the required result. □

The local nature of the upper semi-continuity would follow naturally if conditions for the linear upper semi-continuity of $\varepsilon \to \alpha(\bar{b},\varepsilon)$, at $\varepsilon = 0$, could be established. Our previous results are unfortunately useless when addressing this problem. The strict interior of $\alpha(\bar{b},0)$ is empty. The Cottle constraint qualification cannot possibly hold at $(\bar{b},0)$ for the problem (AP) involving constraints

$$g_j(\bar{u}) \leq \bar{b}_j \text{ ; } j=1,..,m$$

and

$$-f(\bar{u}) + M(\bar{b}) = 0.$$

For the problem (P) we demand the Cottle constraint qualification to hold at $\bar{b}$. Hence there exists a Lagrange multiplier and $\sum_{j\in J(\bar{u},\bar{b})} \lambda_j \nabla g_j(\bar{u}) = \nabla f(\bar{u})$. For any vector e s.t. $<\nabla g_j(\bar{u}),e> < 0$ for $j \in J(\bar{u},\bar{b})$, we necessarily have $<\nabla f(u),e> = \sum_{j\in J(\bar{u},\bar{b})} \lambda_j <\nabla g_j(\bar{u}),e>$ implying $<-\nabla f(u),e> > 0$. Hence (AP)'s Cottle constraint qualification cannot hold.

Many of the counter examples exploit the disconnectedness of the images of $\alpha(\bar{b},0)$. Convexity requirements or uniqueness may avoid these problems.

Weaker requirements for the uniform upper semi-continuity are required in order to obtain simple lower semi-continuity.

Theorem 4.10 : Suppose

(i) the Cottle constraint qualification holds at $\bar{b} \in B(\bar{g})$,

(ii) there exists $\hat{b} > \bar{b}$ such that $\Gamma(\hat{b})$ is bounded,

(iii) each $\bar{g}_j(\cdot)$ has continuous second derivatives on R^n,

(iv) $f(\cdot)$ satisfies a Lipschitz condition on $\Gamma(\hat{b})$,

(v) $\varepsilon \to \alpha(b,\varepsilon)$ is upper semi-continuous at $\varepsilon = 0$, locally around $\bar{b}$, at a uniform rate $q(\cdot) : (0,r_0) \to R_+$, and

(vi) $q(\cdot)$ has a continuous inverse.

Then $b \to \alpha(b,0)$ is lower semi-continuous, locally around $\bar{b}$.

Proof : Arguing as in Corollary 4.8, we have the existence of $\delta' > 0$ s.t. for $b \in B(\bar{g}) \cap N(\bar{b},\delta'), \alpha(b,q(r)) \subseteq \bar{N}(\alpha(b,0),r)$.

By letting $r = q^{-1}(K|b-b'|)$ we obtain

$$d^*(\alpha(b,K|b-b'|), \alpha(b,0)) \leq q^{-1}(K|b-b'|).$$

This implies for

$b, b' \in N(\bar{b}, \delta)$ (some $\delta > 0$)

$d^*(\alpha(b',0), \alpha(b,0)) \leq M|b-b'| + q^{-1}(K|b-b'|) \to 0$ as $b \to b'$. □

Establishing a uniform local upper semi-continuity is not an easy task either. We certainly cannot be guaranteed of linear upper semi-continuity, even when the Slater condition holds and $f(\cdot)$ is linear!

In Chapter one, we established a close relationship between δ-u.H.s.c. and local, uniform u.H.s.c.. Proposition 1.5 states that in the case of compact image sets, the uniform δ-u.H.s.c. of $\Gamma(\cdot)$ at every $(\bar{u}_1, \bar{u}_2)$, for all $\bar{u}_2 \in \Gamma(\bar{u}_1)$, is equivalent to the local, uniform u.H.s.c. of $\Gamma(\cdot)$ around $\bar{u}_1$. It is not hard to see that if we could establish linear δ-u.H.s.c. at every $(\bar{u}_1, \bar{u}_2)$ such that $\bar{u}_2 \in \Gamma(\bar{u}_1)$ then local, linear u.H.s.c. of $\Gamma(\cdot)$ would follow. S. Dolecki and S. Rolewicz derive various conditions which imply the linear δ-u.H.s.c. of $\Gamma(\cdot)$ at a point $(\bar{u}_1, \bar{u}_2)$, in reference [9].

<u>Proposition 4.1</u> : Suppose $\Gamma(b)$ is continuous at $\bar{b}$ and $f(\cdot)$ is continuous on $\bar{b} \times \Gamma(\bar{b})$. Suppose also that $\alpha(b,0)$ is uniformly compact near $\bar{b}$ and $\alpha(\bar{b},0)$ consists of a collection of isolated local maxima for the problem.

Then $b \to \alpha(b,0)$ is lower semi-continuous at $\bar{b}$.

<u>Proof</u> : We know from Theorem 1.22 that $b \to \alpha(b,)$ is closed at $\bar{b}$. The uniform boundedness establishes the u.s.c. of the multi-function at $\bar{b}$. We establish the lower semi-continuity by first noting that for any open set Q, $b \to \alpha(b,0) \cap \bar{Q}$ is u.H.s.c. at $\bar{b}$. In fact Theorem 1.7 indicates that this property "characterises" u.s.c. as distinct from u.H.s.c.

Since $\alpha(\bar{b},0)$ consists of a collection of isolated local maxima, for each $\bar{u}_2 \in \alpha(\bar{b},0)$ there exists a neighbourhood Q such that $\alpha(\bar{b},0) \cap \bar{Q} = \{\bar{u}_2\}$. We can deduce the l.s.c. at $\bar{b}$ as follows. Let $b^n \to \bar{b}$ where

$$u_2^n \in \alpha(b^n,0) \cap \bar{Q}.$$

Necessarily $u_2^n \to \bar{u}_2$ and hence for n sufficiently large $\bar{u}_2 \in N(u_2^n,\varepsilon)$.

That is, $\alpha(\bar{b},0) \cap \bar{Q} = \{\bar{u}_2\} \subseteq N(\alpha(b^n,0) \cap \bar{Q},\varepsilon)$, which is the definition of l.H.s.c.. Since l.H.s.c. implies l.s.c. we have a localised l.s.c. By Theorem 1.12 part (ii) we know that

$$\begin{aligned} b \to \bar{U}\{\alpha(b,0) \cap \bar{Q} : \{\bar{u}_2\} = \alpha(\bar{b},0) \cap \bar{Q} \text{ for a nbhd } Q \text{ of } \\ \bar{u}_2 \in \alpha(\bar{b},0)\} \\ \subseteq \text{cl } \alpha(b,0) = \bar{\alpha}(b,0) \end{aligned}$$

is lower semi-continuous at $\bar{b}$. This multi-function is of course equal to $\alpha(\bar{b},0)$ at $\bar{b}$. It is in fact equal to $\bar{\alpha}(b,0)$ for b sufficiently close to $\bar{b}$.

Since $\alpha(b,0)$ is u.s.c. at $\bar{b}$ and $W = U\{Q : \{\bar{u}_2\} = \alpha(\bar{b},0) \cap \bar{Q}$ for a nbhd of $\bar{u}_2 \in \alpha(\bar{b},0)\}$ is a neighbourhood of $\alpha(\bar{b},0)$, we must have $\alpha(b,0) \subseteq W$ for b sufficiently close to $\bar{b}$. That is

$$\begin{aligned} \alpha(b,0) &= \alpha(b,0) \cap W \\ &= U\{\alpha(b,0) \cap Q : \{\bar{u}_2\} = \alpha(\bar{b},0) \cap \bar{Q} \\ &\qquad \text{for a nbhd of } \bar{u}_2 \in \alpha(\bar{b},0)\} \end{aligned}$$

imply the l.s.c. of $b \to \text{cl } \alpha(b,0)$ at $\bar{b}$. Using Theorem 1.10 (i) we can deduce the l.s.c. of $b \to \alpha(b,0)$ at $\bar{b}$. □

The conditions which ensure the local continuity of $\Gamma(b)$ around $\bar{b}$ involve the boundedness of $\Gamma(\hat{b})$ for some $\hat{b} > \bar{b}$. This in itself would imply the uniform compactness of $\alpha(b,0)$ near b in R^n.

It seems unlikely that linear lower semi-continuity will be a very common a property for $\alpha(b,0)$ to possess. Simple lower semi-continuity is most probably a much more common phenomena. The Slater condition plus some sort of assumption about the behaviour of the function $f(\cdot)$ near the critical set, would probably suffice as well.

§4.2 The Differentiability Properties of Locally Lipschitz Mappings

Ever since F.H. Clarke published his papers on the theory of generalized gradients (see reference [29]), much interest has surrounded the development of these theories. Locally Lipschitz functions play an important role due to their equivalience to a type of differentiability. We begin by reviewing some aspects of the theory's present state.

We let for $f(\cdot) : R^n \to R^m$

$$A \in L(R^n, R^m)$$

and

$$u,h \in R^n;\ t > 0$$

$$U_f(u;h,t) = (f(u+th) - f(u))/t,$$

$$O_{f,A}(u;h) = \|f(u+h) - f(u) - A.h\|.$$

<u>Definition 4.3</u> : We call

(i) $f'(u,h) \in R^m$ the one sided directional derivative of $f(\cdot) : R^n \to R^m$ if

$$f'(u,h) = \lim_{t\to 0-} U_f(u;h,t)$$

(ii) the linear mapping $A \in L(R^n, R^m)$ the Gâteaux derivative of $f(\cdot) : R^n \to R^m$ at $u \in R^n$ if

$$A.h = f'(x,h) \text{ for any } h \in R^n$$

(iii) the linear mapping $A \in L(R^n, R^m)$ the Fréchet derivative of $f(\cdot) : R^n \to R^m$ at $u \in R^n$ if

$$\lim_{h \to 0} O_{f,A}(u;h)/\|h\| = 0$$

and the strict Fréchet derivative at $u \in R^n$ if

$$\lim_{(\bar{u},h) \to (0,0)} O_{f,A}(\bar{u};h)/\|h\| = 0.$$

Definition 4.4 : For the set valued mapping $(u,h,t) \to \{U_f(u;h,t)\}$ or any other set valued mapping

$$F : Y \to \mathcal{P}(R^m)$$

we use

$$\liminf_{\bar{y} \to y} F(\bar{y}) = \{u \in R^m : \forall\, y_k \to y;\ \exists\, u_k \in F(y_k) : u_k \to u\}$$

$$\limsup_{\bar{y} \to y} F(\bar{y}) = \{u \in R^m : \exists\, y_k \to y;\ \forall\, u_k \in F(y_k) : u_k \to u\}.$$

We call

(i) the set valued mapping $\bar{K}f(u;h)$ $\bar{K}f : R^n \times R^n \to \mathcal{P}(R^m)$ defined by $\bar{K}f(u;h) = \limsup_{(g,t) \to (h,0+)} \{U_f(u;g,t)\}$ the contingent of f and the mapping $\bar{P}f : R^n \times R^n \to \mathcal{P}(R^m)$ defined by

$$\bar{P}f(u;h) = \limsup_{(\bar{u},g,t) \to (u,h,0^+)} \{Uf(\bar{u};g,t)\}$$

the paratingent.

We call

$$Kf(u;h) = \text{co } \bar{K}f(u;h)$$

$$Pf(u;h) = \text{co } \bar{P}f(u;h)$$

the convex contingent and convex paratingent respectively.

(ii) The upper paratingential derivative at $u \in R^n$ in the direction $h \in R^n$ is

$$f_p^+(u;h) = \lim_{(y,g,t)\to(u,h,0+)} \sup U_f(y;g,t).$$

(iii) The Clarke directional derivative at $u \in R^n$ in the direction $h \in R^n$ is

$$f_c^+(u,h) = \lim_{(y,t)\to(u,0+)} \sup U_f(y;h,t).$$

As it turns out the local Lipschitzness of f is crucial for many of these to be well defined.

<u>Proposition 4.2</u>: Suppose $f(\cdot): R^n \to R^m$ is continuous in a neighbourhood of $u \in R^n$. Then the paratingent $\bar{P}f(u;h)$ is a non-empty bounded set for any $h \in R$ if and only if $f(\cdot)$ is locally Lipschitz at u. In this case $\bar{P}f(u,\cdot) : R^n \to V(R^m)$ is Lipschitz sub linear symmetric multi-function

(ie. $\bar{P}f(u,-h) = -\bar{P}f(u,h)$ (symmetric) and
$\bar{P}f(u; h) = t\bar{P}(u;h)$
$\bar{P}f(u;h_1+h_2) \subseteq \bar{P}f(u;h_1) + \bar{P}f(u;h_2)$ (sub-linear))
and is given by

$$\bar{P}f(u;h) = \lim_{(y,t)\to(u,0+)} \sup \{U_f(y,h,t)\}.$$

<u>Proof</u> : Reference [25] page 1348, Prop. 3.11. □

<u>Proposition 4.3</u> : Let $f : R^n \to R$, $u \in R^n$ and let $f_p^+(u;h)$ be defined as in Definition 4.4 for any $h \in R^n$. Then $f_p^+(u;h)$ is finite for any $h \in R^n$ iff $f(\cdot)$ is locally Lipschitz at u. In this event $f_p^+(u;h)$ coincides with the Clark directional derivative, $f_c^+(u;h)$. Moreover $f_p^+(u;\cdot) : R^n \to R$ is the support function of the convex paratingent $Pf(u;\cdot)$. Hence we have

$$Pf(u;h) = [-f_p^+(u,-h),\ f_p^+(u;h)] \text{ for any } h \in R^n.$$

Proof : Reference [25] page 1348, Prop. 3.12. □

It has been shown (see Reference [25]) that whenever the paratingent is a non-empty bounded set (ie. $f(\cdot)$ locally Lipschitz) the convex paratingent $Pf(u;\cdot)$ is generated by a set of linear mappings. Moreover, if $\min(n,m) > 1$, then $Pf(u;\cdot)$ may be generated by different sets of linear mappings.

These results are based on Rademacher's theorem stating that if $f(\cdot)\ R^n \to R^m$ is locally Lipschitz in an open neighbourhood of $u \in R^n$ G (say), then $f(\cdot)$ is a.e. Fréchet differentiable on G and moreover, its derivative, $f'(\cdot) : G \to L(R^n,R^m)$ is a measurable and bounded mapping

Two such sets of linear mappings are

(i) $Jf(u) = co\{A \in L(R^n,R^m) : \exists\ x_k \to u;\ \exists\ f'(x_k) \to A\}$

the generalized Jacobian of Clarke and

(ii) Pourciau's generalized derivative defined by

$$J^P f(u) = co\{A \quad L(R^n,R^m) : \exists\ x_k \in L(f'(\cdot)),\ x_k \to u,\ f'(x_k) \to A\}$$

where $L(f'(\cdot))$ is the set of Lebesque points of $f'(\cdot)$.

Obviously $J^P f(u) \subset Jf(u)$ and in both cases

$$Pf(u;h) = \sup\{A.h : A \in Jf(u)\}$$

$$= \sup\{A.h : A \in J^P f(u)\}.$$

F.H. Clarke (reference [29]) defined, in the case m=1, the Clarke directional derivative $f_c^+(u;h)$ using the above technique.

Definition 4.5 : The generalized gradient of f at u, denoted $\partial f(u)$, is the convex hull of the set of limits of the form

$$\lim \nabla f(u+h_i), \text{ where } h_i \to 0 \text{ as } i \to \infty.$$

It follows that $\nabla f(u)$ is convex compact and non-empty if $f(\cdot)$ is locally Lipschitz. As in the convex case the mapping $u \to \partial f(u)$ is upper semi-continuous. Also $\partial f(u)$ is a Singleton for all $u \in \Omega$ if and only if $f \in \mathcal{D}_1(\Omega)$. If $\partial f(u) = \{x\}$ then $\nabla f(u) = x$.

He proceeds to show that

$$f_c^+(u;h) = \lim_{(y,t)\to(u,0^+)} \frac{f(y+th)-f(y)}{t}$$

$$= \max\{<x;h> : x \in \partial f(u)\}$$

and that in fact if

$$<x;h> \leqslant \limsup_{t\to 0_+} \frac{f(u+th)-f(u)}{t}$$

for all $h \in R^n$ then $x \in \partial f(u)$.

A function $f(\cdot)$ is said to be Clarke regular if $f'(u;h)$ (the directional derivative) exists and equals $f_c^+(u;h)$ for every $h \in R^n$.

F.H. Clarke proves also that $\partial(f_1+f_2)(u) \subset \partial f_1(u) + \partial f_2(u)$, for suitable functions f_1 and f_2. R.S. Womersley proved the following in reference [31].

Lemma 4.4 : Let $f : R^n \to R$ be a smooth function and let $h : R^n \to R$ be locally Lipschitz. Then the function $F(x) = f(x) + h(x)$ is locally Lipschitz and

$$\partial F(x) = \{\nabla f(x) + u : u \in \partial h(x)\}$$

$$= \partial h(x) + \nabla f(x).$$

Proof : Reference [31], pp.62. □

This approach to generalized derivatives is inspired by the following theorem on the sub-derivative of a convex function $h(\cdot)$,

$$\partial h(u) = \{u^* : h(x) - h(u) \geq \langle u^*, x-u\rangle \text{ for all } x\}.$$

Theorem 4.11 : Let $h(\cdot)$ be lower semi-continuous, bounded below and not identically $+\infty$. Suppose also that $u \in \text{int(dom}h)$.

Then

$$\partial h(u) = \overline{\text{co}}\, S(u)$$

where $S(u)$ is the set of all limits of sequences $\{\nabla h(u_i)\}_{i=1}^{\infty}$ such that h is differentiable at u_i and u_i tends to u.

Proof : Reference [23], Theorem 25.6. □

For a convex function the condition $0 \in \partial h(u)$ implies $h(\cdot)$ achieves its global minimum at u.

In the case $f(\cdot) : R^n \to R$, if f is convex $\partial f(u) = Jf(u) = J^P f(u)$ the convex sub-differential with respect to the class of affine mappings and in the case f Gâteaux differentiable

$$\partial f(u) = Jf(u) = J^P f(u) = \{A\} \text{ where } A.h = f'(u;h).$$

This is the real power of the theory at present. Whenever stronger forms of differentiability exist then the weaker form reduces to the stronger.

One can also define derivatives of set valued mappings.

Definition 4.6 : Suppose we let for $F(\cdot), \phi(\cdot) : R^n \to C(R^m)$

$$0_{F,\phi}(u,h) = d(F(u+h), F(u)+\phi(h))$$

and

$$O^*_{F,\phi}(u;h) = \max\{d^*(F(u+h),F(u)+\phi(h)),\ d^*(F(u).F(u+h)-\phi(h))\}$$

(obviously $O^*_{F,\phi}(u;h) \leqslant O_{F,\phi}(u;h)$). Then $\phi(\cdot) : R^n \to CV(R^m)$ a positively homogeneous u.s.c. multi-function is said to be

(i) an upper strict prederivative of $F(\cdot)$ at u if

$$\lim_{(y,h)\to(u,0)} O^*_{F,\phi}(y,h)/\|h\| = 0,$$

(ii) a strict prederivative of $F(\cdot)$ at u if

$$\lim_{(y,h)\to(u,0)} O_{F,\phi}(y,h)/\|h\| = 0.$$

The prederivatives are not unique but one may define the infimum and minimal (when it exists) prederivatives with respect to the lattice induced by set inclusion on $CV(R^m)$.

Proposition 4.4 : The set valued mapping $F(\cdot) : R^n \to C(R^m)$ has a upper strict prederivative iff it is locally Lipschitz at the relevant point $u \in R^n$.

Proof : Reference [25] page 1354, Prop. 4.15. □

Proposition 4.5 : Suppose that $f(\cdot):R^n\to R^m$ defines a set valued mapping $F(\cdot) = \{f(\cdot)\}$. Then $F(\cdot)$ has a strict prederivative $\phi(\cdot) : R^n \to CV(R^m)$ at $u \in R^n$ iff f is strictly Fréchet differentiable. In this case, $f'(u) \in L(R^n,R^m)$ satisfies $\phi(h) = \{f'(u).h\} = \{f'_p(u;h)\}$ for any $h \in R^n$ where $f'_p(u;h)$ is the paratingential derivative

$$f'_p(u;h) = \lim_{(y,g,t)\to(u,h,0^+)} U_f(y;g,t).$$

Proof : Reference [25] page 1363, Prop. 6.2. □

Particularly regular is the behaviour of the directional derivatives of the convex functions $f(\cdot) : R^n \to R$ for which

$f'(u;h) = f_p^+(u;h) = f_p'(u;h) = f_c^+(u;h)$. We also have that if f is strictly Fréchet differentiable $f_p'(u;h) = f_c^+(u;h)$.

Proposition 4.6 : If $f(\cdot) : R^n \to R^m$ is locally Lipschitz at u then its convex paratingent $Pf(u;\cdot)$ is continuous and is the minimal (unique) upper strict prederivative of $F(u) = \{f(u)\}$.

Proof : Reference [25] page 1354. □

Of course we are not always assured of a strict prederivative in the case of local Lipschitzness but this indicates how regular the problem is in Theorem 4.6.

One needs only to introduce stronger convexity assumptions to obtain conditions for the existence of the one sided directional derivative for the problem;

$$M(u_1) = \sup \{f(u_1,u_2);\ g_i\,(u_1,u_2) \leq b;\ i=1,..,m\}$$

$$\alpha(u_1) = \{u_2 : M(u_1) \leq f(u_1,u_2);\ g_i\,(u_1,u_2) \leq b;\ i=1,..,m\}$$

We let

$$Y(u_1) = \{\bar{y} \geq 0: L(u_1,\bar{y}) = \inf_{y\geq 0} L(u_1,y)\}$$

$$L(u_1,y) = \sup_{u_2\in U_2} \{f(u_1,u_2) - \langle y.g(u_1,u_2)\rangle + \langle y,b\rangle\}.$$

Theorem 4.12 : Suppose

(i) U_2 is a closed convex set,

(ii) $-f(\bar{u}_1,\cdot)$ and $g_j\,(\bar{u}_1,\cdot)$; $j \in \{1,..,m\}$ are convex on U_2 for $\bar{u}_1 \in U_1$, continuously differentiable on $U_2 \times N(\bar{u}_1)$ where $N(\bar{u}_1)$ is some neighbourhood of $\bar{u}_1$,

(iii) $\alpha(\bar{u}_1)$ is non-empty and bounded,

(iv) $M(\bar{u}_1)$ is finite and

(v) there is a point $\hat{u}_2 \in U_2$ such that $g(\bar{u}_1,\hat{u}_2) < 0$.

Then

(a) $M'(\bar{u}_1;h)$ exists and is finite for all $h \in R^n$ and

$$M'(\bar{u}_1;h) = \max_{u_2 \in \alpha(\bar{u}_1)} \min_{y \in Y(\bar{u}_1)} \{<\nabla_1 f(\bar{u}_1,u_2)h> - y'\nabla_1 g(\bar{u}_1,u_2)h\}$$

where $f : U_1 \times U_2 \to R$; $g = (g_1,\ldots,g_m)$.

(b) If $\alpha(\bar{u}_1) \subseteq \text{int } U_2$ then

$$M'(\bar{u}_1;h) = \max_{\substack{u_2 \in \alpha(\bar{u}_1) \\ w,z \in R}} \{<\nabla_1 f(\bar{u}_1,u_2),h> + <\nabla_2 f(\bar{u}_1,u_2),w>\}$$

subject to

$$zg(\bar{u}_1,u_2) + <\nabla_2 g(\bar{u}_1,u_2),w> \leqslant -<\nabla_1 g(\bar{u}_1,u_2),h>.$$

Proof : See Reference [26], Theorem 2. □

J. Gauvin and F. Dubeau extended this result in reference [29].

Theorem 4.13 : Suppose

(i) $\alpha(\bar{u}_1)$ is non-empty,

(ii) $\alpha(u_1)$ is uniformly compact near $\bar{u}$, and

(iii) the Cottle constraint qualification holds at $\bar{b}$.

Then

$$\partial M(\bar{u}_1) \subseteq \text{co}\{ \nabla_1 f(\bar{u}_1,\bar{u}_2) - y'\nabla_1 g(\bar{u}_1,\bar{u}_2) : \bar{u}_2 \in \alpha(\bar{u}_1) \text{ and } y \in K(\bar{u}_1,\bar{u}_2)\}$$

where $K(\bar{u}_1,\bar{u}_2)$ is the compact convex set of Lagrange multipliers associated with the optimal solution $\bar{u}_2$ at $\bar{u}_1$.

Proof : Theorem 5.3 of reference [29]. □

Equality constraints are actually explicitly treated in this paper. We have and will continue to state such results, referring only to the inequality constraint problem we have been dealing with in this

Chapter. J. Dauvin and F. Dubeau go on to deduce the following corollary.

Corollary 4.13 : If in Theorem 4.13, the assumption (iii) is replaced by the assumption of linear independence of the gradients $\{\nabla_2 g_j(\bar{u}_1,\bar{u}_2);\ j=1,..,m\}$ for every $\bar{u}_2 \in \alpha(\bar{u}_1)$, then

$$\partial M(\bar{u}_1) = \mathrm{co}\{\ \nabla_1 f(\bar{u}_1,\bar{u}_2) - \bar{y}'\nabla_1 g(\bar{u}_1,\bar{u}_2)\ :\ \bar{u}_2 \in \alpha(\bar{u}_1)\}$$

where $\bar{y}$ is the unique Lagrange multiplier associated with $\bar{u}_2$. Furthermore, $M(u_1)$ is Clarke regular at $\bar{u}_1$.

Proof : Corollary 5.4 reference [29]. □

Such conditions are a first step towards finding techniques to solve problems like the following

$$m(u_1) = \min\{\|u_1-u_2\|^2\ :\ u_2 \in \Gamma(u_1)\}$$

where

$$\Gamma(u_1) = \{u_2\ :\ g_j(u_1,u_2) \leqslant \bar{b}_j\ ;\ j=1,..,n\}.$$

Naturally we are assuming m=n and $u_1,\ u_2 \in R^n$. This will have a solution $\bar{u}_2$ even if there exists no fixed point for the multi-valued mapping $\Gamma(\cdot)$ but of course, whenever $m(\bar{u}_2) = 0$ our solution is a fixed point. For this reason the criteria which imply an equivalence are of interest.

Convexity plays its role in reformulating the constrained optimization problem as an unconstrained Lagrangian problem. Many Lagrangian methods exist for non-convex problems now.

We will be using the one characterized in references [11], [21], [22], [28] and [32].

In the work on alternate Lagrangians, researchers first looked at the problem:

(PL) : $\min f(u_2)$

subject to $h_i(u_2) = 0$; $i=1,..,m$; using the Lagrangian

$$L(u_2,y_k,c_k) = f(u_2) + \sum_{i=1}^{m} y_k^i h_i(u_2) + \frac{c_k}{2}[h_i(u_2)]^2.$$

The Lagrangian L is minimized over u_2 for a sequence (y_k,c_k); $c_k \geqslant 0$, which is updated via $y_{k+1} = y_k + c_k h(u_2^k)$, where u_2^k is the result of the k^{th} minimization of L (c_k monotonically increases). On supposing $\bar{u}_2$ is an optimal solution of (PL) in order to get a complete theory, one makes the following assumptions concerning the nature of f and h_j in an open ball around $\bar{u}_2$.

(A) The point $\bar{u}_2$ together with a unique Lagrange multiplier vector $\bar{y}$ satisfies the standard second order sufficiency conditions for $\bar{u}_2$ to be a local minimum.

To elucidate the meaning of this statement, we reiterate some well-known propositions. For the moment we assume $m < n$.

Proposition 4.7 : Suppose $\bar{u}_2$ is a local minimum of (PL) and f and h are continuously differentiable locally around $\bar{u}_2$. We let

$$L_0(u_2,y) = L(u_2,y,0)$$

and suppose $\nabla h_1(\bar{u}_2), \ldots, \nabla h_m(\bar{u}_2)$ are linear independent. Then there exists a unique vector $\bar{y}$ such that

$$\nabla_2 L_0(\bar{u}_2,\bar{y}) = 0$$

and if in addition f and h are twice continuously differentiable around $\bar{u}_2$ we have

$$w'\nabla_2^2 L_0(\bar{u}_2,\bar{y})w \geq 0,$$

$\forall\ w \in R^m$ with $\nabla h(\bar{u}_2)'w = 0$.

<u>Proof</u> : Reference [32] Proposition 1.23. □

<u>Proposition 4.8</u> : Let $\bar{u}_2$ be such that $h(\bar{u}_2) = 0$ and suppose f and h are twice continuously differentiable. Assume there exists a vector $\bar{y} \in R^n$ such that

$$\nabla_2 L_0(\bar{u}_2,\bar{y}) = 0$$

and

$$w'\nabla_2^2 L_0(\bar{u}_2,\bar{y})w > 0;\ \forall\ w \neq 0$$

with $\nabla h(\bar{u}_2)'w = 0$. Then $\exists\ \varepsilon > 0$ s.t.

$$f(\bar{u}_2) < f(u_2);\ \forall\ u_2 \in N(\bar{u}_2,\varepsilon) \quad u_2 \neq \bar{u}_2.$$

<u>Proof</u> : Reference [32] Proposition 1.24. □

In other words we can restate (A) as follows:

(A1) The functions f, h_i , i=1,..,m are twice continuously differentiable within a ball around $\bar{u}_2$.

(A2) The gradients $\nabla h_i\ (\bar{u}_2)$; i=1,..,m are linear independent and there exists a unique Lagrange multiplier $\bar{y}$ such that

$$\nabla_2 f(\bar{u}_2) + \sum_{i=1}^{m} \bar{y}^i\, \nabla_2 h_i\ (\bar{u}_2) = 0$$

(A3) The Hessian matrix of the Lagrangian $L_0(u_2,y)$ satisfies $w'\nabla_2^2 L_0(\bar{u}_2,\bar{y})w > 0$ for all $w \in R^m$ $w \neq 0$ with $w'\nabla_2 h_i\ (\bar{u}_2) = 0$.

To get a complete theory we also assume:

(B) The Hessian matrices $\nabla^2 f$ and $\nabla^2 h$ are Lipschitz continuous in an open ball of $\bar{u}_2$.

It can be shown that if Y (usually assumed bounded) contains $\bar{y}$ in its interior, the generated sequence $\{y_k\}$ remains in the interior of Y (or at least can be arranged to by leaving y_k unchanged if $y_{k+1} \notin Y$). If the penalty parameter is sufficiently large (ie. $c_k \geq c^*$) and u_2^k is the minimum of $L(\cdot, y_k, c_k)$ closest to $\bar{u}_2$, then $u_2^k \to \bar{u}_2$ and $y^k \to \bar{y}$. If $c_k \to \bar{c} < \infty$ then convergence is linear (see Reference [28] and [32]).

Inequality constraints can be treated in a simple way by introducing slack variables, as the problem;

(P) $\min\{f(u_2) : g_i(u_2) \leq \bar{b}_i ; \; i=1,..,m\}$

is equivalent to

$$\min\{f(u_2) : g_i(u_2) + z_i^2 = \bar{b}_i ; \; i=1,...,m\}$$

where z_i are additional variables. It is easily shown that

$$(\bar{u}_2, \sqrt{-g_1(\bar{u}_2)+\bar{b}_1}, \; ..., \; \sqrt{-g_m(\bar{u}_2)+\bar{b}_m})$$

is an optimal solution to this problem (together with $\bar{y}$) satisfying A and B, if we demand the inequality constraint problem to satisfy instead of (A) the assumptions

(A') The function f, g_i ; $i=1,..,m$ are twice continuously differentiable around $\bar{u}_2$. The gradients $\{\nabla g_i(\bar{u}_2) ; \; j \in J(\bar{u}_2)\}$ where

$$J(\bar{u}_2) = \{j : g_j(\bar{u}_2) = \bar{b}_j\},$$

are linear independent. We have a Lagrange multiplier s.t.

$$\bar{y}^j [g_j(\bar{u}_2) - \bar{b}_j] = 0,$$

$$\nabla f(\bar{u}_2) + \sum_{j=1}^{m} \bar{y}^j \nabla g_j(\bar{u}_2) = 0 \text{ and } \bar{y}^j \geq 0$$

with

$$\bar{y}^j > 0 \text{ iff } j \in J(\bar{u}_2).$$

Furthermore, we require

$$w'[\nabla_2^2 f(\bar{u}_2) + \sum_{j=1}^{m} \bar{y}^j \nabla_2^2 g_j(\bar{u}_2)]w > 0$$

for all $w \neq 0$ such that $w' \nabla g_j(\bar{u}_2) = 0$ for all $j \in J(\bar{u}_2)$.

If we carry out first the minimization of the inequality constraint Lagrangian with respect to $z_1, \ldots, z_m$ namely,

$$\hat{L}(u,z,y,c) = f(u) + \sum_{j=1}^{m} y^j [g_j(u) - \bar{b}_j + z_j^2]$$
$$+ c/2 \sum_{j=1}^{m} [g_j(u) - \bar{b}_j + z_j^2]^2$$

we get $L(u,y,c) = \min_z \hat{L}(u,z,y,c)$ where

$$\min_z \hat{L}(u,z,y,c) = f(u) + \frac{1}{2c} \sum_{j=1}^{m} \psi(g_j(u) - \bar{b}_j, y^j) \quad \text{and}$$

$$\psi(\alpha,\beta) = \max(0,\beta+c\alpha)^2 - \beta^2.$$

The optimal value of the z_j are given in terms of (u,y,c) by

$$z_j^2(u,y,c) = \max[0, -y^j/c - g_j(u) + \bar{b}_j]; \quad j=1\ldots,m.$$

Minimization of $L(u,y,c)$ with respect to u yields $u(y,c)$, and the multiplier method iteration takes the form

$$y_{k+1}^j = y_k^j + c[g_j(u(y_k,c) - \bar{b}_j] + z_j^2[u(y_k,c),y_k,c]$$

$$= \max[0,y_k+cg_j(u(y_k,c)) - \bar{b}_j c]; \quad j=1,..,m.$$

<u>Proposition 4.9</u> : Suppose $\bar{u}_2$ satisfies $g(\bar{u}_2) \leqslant \bar{b}$ and that (A') holds then $\bar{u}_2$ is a strict local minimum of the problem (P) ie. $\exists\ \varepsilon > 0$ s.t.

$f(\bar{u}_2) < f(u_2); \ \forall \ u_2 \in N(\bar{u}_2,\varepsilon) \quad \bar{u}_2 \neq u_2.$

Proof : Reference [32] Proposition 1.31 . □

If the assumption (A') is satisfied by (P), then the condition (A) is satisfied by the problem above. As a consequence there exists a unique Lagrange multiplier $(\bar{y}_1,..,\bar{y}_m)$ which is the solution to the system of equations, given in the first part of the following (these equations are known as the Kuhn Tucker conditions).

Proposition 4.10 : Let $\bar{u}_2$ be a local minimum of (P) and assume that f and g_i ; i=1,..,m are continuously differentiable in a neighbourhood of $\bar{u}_2$ and that the gradients $\nabla g_j(\bar{u}_2)$; $j \in J(\bar{u}_2)$ are linearly independent. Then there exists a unique vector $\bar{y}$ such that

$$\nabla_2 L(\bar{u}_2,\bar{y}) = 0$$

$$\bar{y}^j \geq 0; \ \bar{y}^j [g_j(\bar{u}_2) - \bar{b}_j] = 0; \ \forall \ j=1,..,m.$$

If in addition f and g_j , j=1,..,m are twice continuously differentiable in a neighbourhood of $\bar{u}_2$,then for all $w \in R^m$ satisfying

$$\nabla g_j(\bar{u}_2)'w = 0; \ j \in J(\bar{u}_2),$$

we have

$$w'\nabla_2^2 L(\bar{u}_2,\bar{y})w \geq 0.$$

Proof : Reference [32] Proposition 1.29. □

There always exists a Lagrange multiplier, satisfying the first equations of Proposition 4.10, when $\bar{u}_2$ is a regular point. Any suitable constraint qualification, such as the Cottle constraint qualification, implies regularity. In this situation though, we do not necessarily have uniqueness.

Of course all theorems for the equality constraint problem are applicable to the inequality problem satisfying sufficiency assumptions (A').

We in fact can replace the assumption that the gradients $\nabla g_j(\bar{u}_2)$; $j \in J(\bar{u}_2)$ are linearly independent by the assumption that $\bar{u}_2$ is strict local minimum and a regular point. In doing so we still retain this equivalence (see reference [32]).

If we assume the gradients $\nabla g_j(\bar{u}_2)$; $j \in J(\bar{u}_2)$ are linearly independent then the Cottle constraint qualification must hold at $\bar{u}_2$. That is there exists no multipliers, not all zero, such that

$$\sum_{j \in J(\bar{u}_2)} y_j \ g_j(\bar{u}_2) = 0 \qquad (y_j \geqslant 0 \text{ or not}).$$

If we assume the Cottle constraint qualification holds then we immediately have the regularity of $\bar{u}_2$ for the problem (P).

Suppose we let

$$\begin{aligned} \hat{m}(b) &= \min\{f(u_2) : g_j(u_2) + z_j^2 - \bar{b}_j = b_j \ ; \ j=1,..,m; \\ &\qquad\qquad (u_2,z) \in N((\bar{u}_2,\bar{z}),\bar{\delta})\} \\ &= m(b+\bar{b}). \end{aligned}$$

Under condition (A') we can use the implicit function theorem to get $\nabla\hat{m}(0) = \nabla m(\bar{b}) = -\bar{y}$, the unique Lagrange multiplier associated with the strict local minimum $\bar{u}_2$. In fact $\hat{m}(\cdot)$ is twice continuously differentiable in a neighbourhood of zero (see reference [21]).

The localization of the minimization allows us to do this. It is somewhat instructive to see how this may be done, but first we investigate the role of the multi-valued mapping

$$\alpha(b) = \{u_2 : u_2 \in \Gamma(b); \ f(u_2) \leqslant \bar{m}(b)\}$$

where

$$\Gamma(b) = \{u_2 : \bar{g}_j(u_2) \leq \bar{b}_j + b_j \; ; \; j=1,..,m\}$$

and $m(b) = \inf\{f(u_2); \; u_2 \in \Gamma(b)\}$.

When $\alpha(0)$ consists of a collection of isolated minima (which is the case for strict local minima) we know that $b \to \alpha(b)$ is lower semi-continuous at $\bar{b}$ (see Proposition 4.1).

Proposition 4.11 : Suppose $f(\cdot)$ and $g_j(\cdot)$; $j=1,..,m$ are continuous functions. Suppose also that the multi-valued mapping $b \to \alpha(b)$ is lower semi-continuous at $b = 0$.

Then $\bar{m}(b) = \hat{m}(b)$ for $b \in N(0,\delta)$, for some $\delta > 0$.

Proof : First we note that

$$\hat{m}(b) = \min\{f(u_2) : g_j(u_2) + z_j^2 = \bar{b}_j + b_j; \quad j=1,..,m; \; (u_2,z) \in N((\bar{u}_2,\bar{z}),\bar{\delta})\}$$

$$\equiv \min\{f(u_2) : g_j(u_2) + z_j^2 = \bar{b}_j + b_j \; ; \quad j=1,..,m; \; (u_2,z_1^2,..,z_m^2) \in N((\bar{u}_2,\bar{z}_1^2,..,\bar{z}_m^2),(\delta,\varepsilon)\}$$

for a suitable δ, $\varepsilon > 0$. Hence

$$\hat{m}(b) \geq \min\{f(u_2) : g_j(u_2) + z_j^2 = \bar{b}_j + b_j \; ; \; j=1,..,m; \; u_2 \in N(\bar{u}_2,\delta)\}.$$

If we let (u_2,z_j^2) be s.t.

$$g_j(u_2) + z_j^2 = \bar{b}_j + b_j$$

then

$$z_j^2 = (\bar{b}_j + b_j) - g_j(u_2).$$

For $\varepsilon > 0$ as above we can choose $\hat{\delta} > 0$ s.t. for $u_2 \in N(\bar{u}_2,\hat{\delta})$; $\forall \; j=1,..,m$

$$g_j(u_2) - \varepsilon_j \leqslant g_j(\bar{u}_2) \leqslant g_j(u_2) + \varepsilon_j .$$

Hence

$$(\bar{b}_j + b_j) - g(u_2) - \varepsilon_j$$

$$\leqslant (\bar{b}_j + b_j) - g(\bar{u}_2) \leqslant (\bar{b}_j + b_j) - g(u_2) + \varepsilon_j,$$

ie., $z_j^2 - \varepsilon_j \leqslant \bar{z}_j^2 \leqslant z_j^2 + \varepsilon_j$,

ie., $(z_1^2,..,z_m^2) \in N((\bar{z}_1^2,..,\bar{z}_m^2),\varepsilon)$.

So for $\hat{\delta}$ sufficiently small $u_2 \in N(\bar{u}_2,\hat{\delta})$, and z_j^2 s.t.

$$g_j(u^2) + z_j^2 = \bar{b}_j + b_j ; \; j=1,..,m ,$$

we have $(u_2,z_j^2) \in N(\bar{u}_2,\bar{z}_1^2,..,\bar{z}_m^2),(\delta,\varepsilon))$. Hence

$$\hat{m}(b) \leqslant \min\{f(u_2) : g_j(u_2) + z_j^2 = \bar{b}_j + b_j ; \; j=1,...,m; \; u_2 \in N(\bar{u}_2,\hat{\delta})\}$$

for $\hat{\delta}$ sufficiently small, ie.

$$\hat{m}(b) \equiv \min\{f(u_2) : g_j(u_2) \leqslant \bar{b}_j + b_j ; \; u_2 \in N(\bar{u}_2,\hat{\delta})\}.$$

By assumption $b \to \alpha(b,0)$ is lower semi-contuous at $b = 0$.

From the definition of l.s.c. at $b = 0$ we have $\exists \; \delta^* > 0$ s.t.

$$N(0,\delta^*) \subseteq \{b : \alpha(b) \cap N(\bar{u}_2,\hat{\delta}) \neq \phi\}.$$

Hence for $b \in N(o,\delta^*)$, $\exists \; u_2 \in N(\bar{u}_2,\hat{\delta})$ s.t.

$$f(u_2) \equiv \bar{m}(b),$$

that is, $\bar{m}(b) \equiv \hat{m}(b)$. □

In reference [22] Rockafellar studies

$$p(b) = \inf\{F(u_2,b);\ u_2 \in U_2\}$$

where for each $(u_2,b) \in U_2 \times R^m$;

$$F(u_2,b) = \begin{cases} f(u_2) \text{ if } \bar{g}_j(u_2) \leq \bar{b}_j + b_j\ ;\ j=1,..,m; \\ +\infty \quad \text{otherwise.} \end{cases}$$

Of course we always have $p(b) = \bar{m}(b)$ whenever $\bar{b} + b \in B(\bar{g})$. In fact if $\bar{b} \in \text{int } B(\bar{g})$ then $p(b) = \bar{m}(b)$ for $b \in N(0,\delta)$, for δ sufficiently small. Rockafellar goes on to define the concept of stability degree 2.

Definition 4.7 : If there exists a twice continuously differentiable function $\psi(\cdot) : N(0,\delta) \to R$, for some $\delta > 0$, s.t.

(a) $\bar{m}(b) \geq \psi(b);\ \forall\ b \in N(0,\delta)$

(b) $\bar{m}(0) = \psi(0)$

then $\bar{m}(\cdot)$ is said to be stable degree 2 or alternatively the problem (P) is said to be stable degree 2.

If $\bar{m}(b)$ is sub-differentiable at zero with respect to the class

$$\Phi_2 = \{\psi(b) = q - r\|b-\bar{b}\|^2;\ q \in R,\ r \in R_+,\ \bar{b} \in R^m\}$$

then obviously it is stable degree 2. It is not hard to see that if $p(\cdot)$ is Φ_2 bounded (i.e. minorized by an element of Φ_2) and stable degree 2 at zero then it is sub-differentiable there with respect to the above class (see Proposition 5.6 of reference [22]).

In reference [22] this was exploited to high degree. We can combine a number of very general results from this reference to obtain the following;

Proposition 4.12 : Suppose $p(\cdot) : R^m \to R$ is lower semi-continuous and Φ_2-bounded then it is in fact Φ_2-convex.

Proof : Reference [22] Theorem 4.2, Proposition 4.13 and example 4.15. □

The class

$$\Phi_{2c} = \{\psi(b) = a - c\|b-\hat{b}\|^2 : a \in R \text{ and } b \in R^m\}$$
$$= Q^c \text{ is of interest to us.}$$

If $p(\cdot)$ is Q^c convex then

$$p(b) = \sup\{-c\|b-\hat{b}\|^2 + a;\ (\hat{b},a) \in S,\ S \subseteq R^m \times R\}.$$

Since

$$\|b-\hat{b}\|^2 = \|b\|^2 - 2\langle b,\hat{b}\rangle + \|\hat{b}\|^2$$

we have

$$p(b) + c\|b\|^2 = \sup\{\langle\bar{b},b\rangle + \bar{a}:\ (\bar{b},\bar{a}) \in S' \subseteq R^m \times R\}$$

which is the supremum of a class of affine mappings.

Thus $p(\cdot)$ is Q^c-convex iff $p(\cdot) + c\|\cdot\|^2$ is convex in the ordinary sense. In this situation we know that $p(\cdot)$ is Q^c sub-differentiable at any point in int(dom f) (reference [11] Theorem 5.11). We have also when $c > 0$;

Proposition 4.13 : Suppose $h(\cdot)$ is Fréchet differentiable and $h'(\cdot)$ Lipschitz continuous on an open convex set B.

Then there exists a $\bar{c} > 0$ s.t. $h(\cdot) + \bar{c}\|\cdot\|^2$ is a convex function on B and hence $h(\cdot)$ is $Q^{\bar{c}}$ convex on B.

Proof : Reference [11] Corollary 5.14. □

In reference [22] Rockafellar uses the following:

$$L(u_2,y,c) = \inf\{F(u_2,b) + \langle y.b\rangle + \frac{c}{2}\|b\|^2;\ b \in R^m\}$$

$$W(y,c) = \inf\{p(b) + \langle y.b\rangle + \frac{c}{2}\|b\|^2;\ b \in R^m\}.$$

In the case when $p(\cdot)$ is twice continuously differentiable in a neighbourhood of zero then $p(b) + \langle y.b\rangle$ is $Q^{\bar{c}}$ convex on the interior of a quasi-compact neighbourhood. This is unlikely in general, but we will be interested in whether

$$b \to P(b) + \langle y.b\rangle + \frac{\bar{c}}{2}\|b\|^2$$

can be made convex on a quasi-compact neighbourhood of zero.

We need the following in order to investigate this question later.

Theorem 4.14 : Suppose $h(\cdot) : R^n \to R$ is lower semi-continuous and Φ_2 bounded. Then $h(\cdot)$ is sub-differentiable on a dense sub-set of its domain.

Proof : Reference [11], Theorem 6.2 with $\alpha = 2$ and $X \equiv B^n$ (obviously uniformly convex). □

Theorem 4.15 : Suppose $p(b)$ is Φ_2 bounded and is stable degree 2. In order that $\bar{u}_2 \in U_2$ is an optimal solution to the problem (P), it is necessary and sufficient that there exists $(\bar{y},\bar{c}) \in T = R^m \times (0,+\infty)$ s.t.

$$L(u_2,\bar{y},\bar{c}) \geq L(\bar{u}_2,\bar{y},\bar{c}) \geq L(\bar{u}_2,y,c)$$

for all $u_2 \in U_2$; $(y,c) \in T$. Moreover, this condition is satisfied by $(\bar{y},\bar{c})$ iff $(\bar{y},\bar{c})$ is optimal for the dual problem

$$(D): \sup_T [\inf_{u_2} L(u_2,y,c)] = \sup_T W(y,c) \text{ where } W(y,c) = \inf_{u_2} L(u_2,y,c).$$

In other words,

$$\bar{m}(0) = \inf_{u_2} \sup_T L(u_2,y,c)$$

$$\equiv \max_T \inf_{u_2} L(u_2,y,c).$$

Indeed $(\bar{y},\bar{c})$ is an optimal solution of (D) for some $\bar{c} > 0$ iff $\bar{y} = -\nabla\psi(0)$ for some function ψ as in the definition of stability degree 2 and in fact $(\bar{y},c)$ is optimal for (D) when $c > \bar{c}$.

<u>Proof</u> : Reference [27] Theorem 5 and Corollary 5.2. □

Let us suppose $b \in \text{int } B(\bar{g})$ and $\bar{m}(\cdot)$ is differentiable twice continuously around zero. Then for any $\psi(\cdot)$ satisfying the definition of stability degree 2 we have the function $\ell(b) = \bar{m}(b) - \psi(b) \geq 0$ taking a local minimum at $b = 0$. This implies

$$\nabla\ell(0) = 0 = \nabla\bar{m}(0) - \nabla\psi(0)$$

and hence

$$-\nabla\bar{m}(0) = -\nabla\psi(0) = -\bar{y}.$$

<u>Corollary 4.15</u> : Suppose

(i) the Cottle constraint qualification holds at $b \in \text{int } B(\bar{g})$;

(ii) there exists a $\hat{b} > 0$ such that $\Gamma(\hat{b})$ is bounded;

(iii) the optimal set $\alpha(\bar{b})$ consists of isolated local minima, and

(iv) the condition (A') is satisfied by a particular $\bar{u}_2 \in \alpha(\bar{b})$.

Then $(\bar{y},\bar{c})$ are the only solutions of the dual problem, where $\bar{c} > 0$ is sufficiently large and $\bar{y}$ is the unique Lagrange multiplier associated with $\bar{u}_2$. In fact $\nabla\bar{m}(0) = -\bar{y}$.

<u>Proof</u> : For our particular optimal solution $\bar{u}_2$ we have $\bar{m}(b) = \hat{m}(b) = \min\{f(u_2) : u_2 \in \Gamma(b) \cap \bar{N}(\bar{u}_2,\delta)\}$ locally around $b = 0$. This follows from Theorem 4.1, Propositions 4.1, 4.9 and 4.11.

Since (A') holds we have a unique Lagrange multiplier $\bar{y}$ associated with $\bar{u}_2$. The implicit function theorem implies under the conditions (A') that $\hat{m}(b)$ is twice continuously differentiable around $b = 0$ and $\nabla\bar{m}(0) = \hat{m}(0) = -\bar{y}$. Theorem 4.15 allows us to deduce that $(\bar{y},\bar{c})$ is a solution of the dual and the above comment allows us to deduce that $(\bar{y},\bar{c})$ are the only solutions when $\bar{c}$ is sufficiently large. □

We note that the following conditions (i), (ii), and (v) are sufficient (and "almost necessary") for $\bar{u}_2$ to be an isolated local optimal solution of (P).

Theorem 4.16 : Suppose the following assumptions are satisfied:

(i) the functions f and $\bar{g}_j$; $j=1,..,m$ are twice continuously differentiable;

(ii) the Cottle constraint qualification holds for $\bar{g}_j(\cdot)$; $j=1,..,m$ at $\bar{b} \in \text{int } B(\bar{g})$;

(iii) $\Gamma(\hat{b}) = \{u_2 : \bar{g}_j(u) - \bar{b}_j \leq \hat{b}_j ; j=1,..,m\}$ is bounded for $\hat{b} > 0$;

(iv) the function $p(\cdot)$ is Φ_2 bounded. For each optimal solution $\bar{u}_2$ there exists a Lagrange multiplier $\bar{y}$ (satisfying the Kuhn-Tucker conditions) for which we have;

(v) the Hessian matrix of $L(\bar{u}_2,\bar{y},0)$

$$\nabla_2^2 L(\bar{u}_2,\bar{y},0) = \nabla_2^2 f(\bar{u}_2) + \sum_{j=1}^{m} \bar{y}_j \nabla_2^2 g_j(\bar{u}_2)$$

verifies the inequality

$$w'\nabla_2^2 L(\bar{u}_2,\bar{y},0)w > 0$$

for all $w \neq 0$ s.t.

(a) $w'\nabla_2\bar{g}_j(\bar{u}_2) = 0$ for

$j \in J_0(\bar{u}_2) = \{j : \bar{g}_j(\bar{u}_2) = 0, \bar{y}_j > 0\}$ and

(b) $w'\nabla_2 g_j(\bar{u}_2) \leq 0$ for

$j \in J_1(\bar{u}_2) = \{j : \bar{g}_j(\bar{u}_2) = 0, \bar{y}_j = 0\}$.

Then (P) is stable degree 2 and for $\bar{c}$ sufficiently large the pair $(\bar{y},\bar{c})$ is an optimal solution of the dual (D).

Proof : The conditions (v) are sufficient for $\bar{u}_2$ to be an isolated locally optimal solution. Assumption (ii) implies the existence of a Lagrange multiplier. Assumptions (ii) and (iii) ensure that $b \to \Gamma(b)$ is continuous locally around $\bar{b}$ and uniformly compact near $\bar{b}$. The conditions of Proposition 4.1 are met and we can deduce the lower semi-continuity of $b \to \alpha(b)$ at zero. The conditions of Proposition 4.11 are satisfied and we have locally around zero

$$p(b) = \bar{m}(b) = \hat{m}(b)$$

$$= \inf\{f(u_2) : u_2 \in N(\bar{u}_2,\delta);\ \bar{g}_j(u_2) \leq \bar{b}_j + b_j \text{ for } j=1,..,m\},$$

for some $\delta > 0$, where $\bar{u}_2$ is any isolated optimal solution of (P).

We can now, in an identical fashion to R.T. Rockafellar, construct a function $\pi(\cdot)$ twice continuous differentiable in a neighbourhood of zero s.t.

(a) $\hat{m}(b) \geq \pi(b)$ in a neighbourhood of zero,

(b) $\hat{m}(0) = \pi(0)$, and

(c) $\nabla\pi(0) = -\bar{y}$.

We refer the reader to Theorem 6 of reference [22] for the details of this construction. The conditions of Theorem 4.15 are now satisfied and our result is established. □

Much interest has been directed towards interpreting the Clarke derivative of the marginal mapping m(b) at b = 0. The generalization of the relation, which holds in the convex case, namely

$$\partial m(0) = \{-\bar{y} : \exists\ \bar{u}_2 \text{ satisfying with } \bar{y} \text{ the Kuhn-Tucker conditions}\},$$

is hoped to hold more generally. J. Gauvin in reference [27] has proved a weaker result.

Theorem 4.17 : Suppose $\alpha(0)$ is non-empty, $\alpha(b)$ is uniformly compact near zero, and the Cottle constraint qualification holds at $\bar{b}$.

Then

$$\partial\bar{m}(0) \subseteq \overline{co}\{-\bar{y} : \exists\ \bar{u}_2 \text{ satisfiying with } \bar{y} \text{ the Kuhn-Tucker conditions}\}.$$

Proof : Reference [27], Theorem 3. □

We will not pursue this particular relation but prove the following equivalence:

$$\partial\bar{m}(0) = \{-\bar{y} : (\bar{y},\bar{c}) \text{ is a solution of the dual of problem (P) for some } \bar{c} \in R_+\}.$$

The above set of dual variables is always a convex set. This can be deduced using the following theorem. If we can show this equivalence then we have shown the compactness of the set.

Theorem 4.18 : The functions $L(u_2,y,c)$ and $w(y,c)$ are concave and upper semi-continuous in $(y,c) \in R^m \times R_+$ and non-decreasing in $c \in R_+$, nowhere $+\infty$. Furthermore, whenever $c > s \geq 0$ one has

$$W(y,c) \geq \max\{W(z,s) - \|y-z\|^2/2\ (c-s);\ z \in R^m\}.$$

Proof : Reference [22], Theorem 1. □

One can also deduce from this that if $(\bar{y},\bar{c})$ is a solution of the dual then $(\bar{y},c)$ will be a solution if $c > \bar{c}$. We always have $W(y,c) \leq p(0)$.

Proposition 4.15 : Suppose

(i) $p(\cdot)$ is Φ_2 bounded,

(ii) $\bar{m}(\cdot)$ is locally Lipschitz order 2 around $b = 0$, and

(iii) $\bar{b} \in \text{int } B(\bar{g})$.

Then the following are equivalent:

(i) zero is a local minimum of $b \to p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2$ for all c sufficiently large;

(ii) $(\bar{y},\bar{c})$ is a solution of the dual for some $c > 0$.

Proof : We know that

$$W(\bar{y},c) = \inf\{p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2;\ b \in R^m\}$$

so we restrict our attention to

$$\{b \in R^m : p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2 = W(\bar{y},c)\}.$$

This is in fact the set of global minima of $p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2$. Since $\bar{m}(\cdot)$ is locally Lipschitz order 2 at $b = 0$ and $\bar{b} \in \text{int } B(\bar{g})$ we have,

(a) $p(0) = \bar{m}(0)$, and

(b) $p(b) = \bar{m}(b) \geq \bar{m}(0) - M\|b\|^2$, locally around $b = 0$ (for some Lipschitz constant $M > 0$).

Thus we establish stability degree 2 by letting $\psi(b) = \bar{m}(b) - M\|b\|^2$. Since $p(\cdot)$ is Φ_2-bounded Theorem 4.15 applies. Hence for $\bar{c} > 0$, sufficiently large, we have

$$p(0) = \bar{m}(0) = \max_T W(y,c) = W(\bar{y},\bar{c})$$

iff

$$b \to p(b) + \langle\bar{y},b\rangle + \frac{\bar{c}}{2}\|b\|^2$$

is minimized at $b = 0$.

The implication (ii) $\to$ (i) is immediate. To show (i) $\to$ (ii) we need to show

$$0 \in \{b \in R^m : p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2 = W(\bar{y},\bar{c})\}$$

for c sufficiently large.

Now

$$\{b \in R^m : p(b) + \langle\bar{y},b\rangle + \tfrac{c}{2}\|b\|^2 = W(\bar{y},c)\}$$

$$\subseteq \{b \in B(\bar{g}) : a - \tfrac{r}{2}\|b\|^2 + \langle\bar{y},b\rangle + \tfrac{c}{2}\|b\|^2 \leqslant \bar{m}(0)\}$$

$$= S(c),$$

where $p(b) \geqslant a - \frac{r}{2}\|b\|^2$ for all $b \in R^m$ (because of $p(\cdot)$'s Φ_2 boundedness).

We can express

$$S(c) = \{b \in B(\bar{g}) : \frac{2}{(c-r)}(\bar{m}(0)-a) + \frac{\|\bar{y}\|^2}{(c-r)^2} \geqslant \|b - \frac{\bar{y}}{(c-r)}\|^2\}.$$

Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. We can choose c to be sufficiently large ($c > \bar{c}$, say) as to ensure that $0 < \frac{2}{(c-r)}(\bar{m}(0)-a) < \varepsilon_1$ and $\frac{\|\bar{y}\|}{(c-r)} = \frac{1}{(c-r)}\|\bar{y}\| < \varepsilon_2$.

We have

$$S(c) \subseteq \{b \in B(\bar{g}) : \varepsilon_1 + \varepsilon_2^2 \geqslant \|b - \frac{\bar{y}}{(c-r)}\|^2\}.$$

That is, if $b \in S(c)$, we have

$$\|b\| \leqslant \|b - \frac{\bar{y}}{(c-r)}\| + \|\frac{\bar{y}}{(c-r)}\| \leqslant (\varepsilon_1+\varepsilon_2^2)^{1/2} + \varepsilon_2$$

for $c > \bar{c}$.

Now since zero is a local minimum we have

$$p(0) = \bar{m}(0) \leqslant p(b) + \langle\bar{y},b\rangle + \tfrac{c}{2}\|b\|^2$$

for all $b \in N(0,\varepsilon)$, for some $\varepsilon > 0$. By ensuring that $\bar{c}$ is sufficiently large that

$$0 < (\varepsilon_1+\varepsilon_2^2)^{1/2} + \varepsilon_2 < \varepsilon,$$

we have

$$W(\bar{y},\bar{c}) = p(b) + \langle\bar{y},b\rangle + \frac{\bar{c}}{2}\|b\|^2$$

for some $b \in N(0,\varepsilon)$ and

$$p(0) = \bar{m}(0) \leq W(\bar{y},\bar{c}).$$

This implies zero is a global minimum and $(\bar{y},\bar{c})$ is a solution of the dual. □

Theorem 4.19 : Suppose

(i) $\bar{b} \in \text{int } B(\bar{g})$

(ii) $b \to \bar{m}(b)$ is locally Lipschitz order 2 around $b = 0$, and

(iii) $p(\cdot)$ is Φ_2 bounded.

Then

$$\partial\bar{m}(0) = \partial p(0)$$

$$= \{-\bar{y} : (\bar{y},\bar{c}) \text{ is a solution of the dual problem (D) for some } \bar{c} > 0\}.$$

Proof : We first show that

$$b \to p(b) + \langle\bar{y},b\rangle + \frac{\bar{c}}{2}\|b\|^2$$

convex on the interior of a quasi-compact neighbourhood of zero for $\bar{c}$ sufficiently large. Suppose $N(0,\delta)$ is a quasi-compact neighbourhood of zero on which $b \to \bar{m}(b)$ is locally order 2 Lipschitz. From Theorem 4.12 we know that $b \to p(b) = \bar{m}(b)$ is Φ_2 sub-differentiable on a dense sub-set of $\overline{N}(0,\delta)$ and from Proposition 4.12 Φ_2-convex.

Let G be a dense sub-set of $\overline{N}(0,\delta)$ s.t. for $\hat{b} \in G$ we have some $r > 0$

$$\bar{m}(b) \geq \bar{m}(\hat{b}) - \frac{r}{2}\|b-\hat{b}\|^2 \quad \text{for all } b \in \overline{N}(0,\delta).$$

We let

$$H(\hat{b}) = \{r > 0 : \frac{r}{2}\|b-\hat{b}\|^2 \geq \bar{m}(\hat{b}) - \bar{m}(b)$$
$$\text{for all } b \in \bar{N}(0,\delta)\}$$

and

$$\bar{r}(\hat{b}) = \inf\{r : r \geq 2M \text{ and } r \in H(\hat{b})\}.$$

We proceed to show that for $\hat{b} \in G$, $b \to \bar{r}(b)$ is upper semi-continuous at $\hat{b}$ if M is the Lipschitz constant of $b \to \bar{m}(b)$. Due to Theorem 1.21 this amounts to showing that

$$b \to \{r : r \geq 2M \text{ and } r \in H(b)\}$$

$$= H(b) \cap [2M,+\infty)$$

is open at any given $\hat{b} \in G$. That is, given

(i) $r \in H(\hat{b}) \cap [2M,+\infty)$,

(ii) $b^n \in G$ s.t. $b^n \to \hat{b}$,

we must show there exists

$$r^n \in H(b^n) \cap [2M,+\infty) \text{ s.t.}$$
$$r^n \to r.$$

We let

$$r^n = \sup\{r\frac{(\|\hat{b}-b\| + \|b^n-\hat{b}\|)^2}{\|b^n-b\|^2};\ b \in \bar{N}(0,\delta)\}$$

and note that;

(iii) $\|b^n-b\|^2 \leq (\|\hat{b}-b\| + \|b^1-\hat{b}\|)^2$

which implies $r^n \geq r \geq 2M$;

(iv) for $b \neq \hat{b}$

$$r^n = \frac{r(\|\hat{b}-b\| + \|b^n-\hat{b}\|)^2}{\|b^n-b\|^2} \to r, \text{ as } n \to \infty \quad \text{and}$$

(v) for $b = \hat{b}$

$$r^n = r \quad \text{for all } n.$$

We are given that

$$\frac{r}{2}\|b-\hat{b}\|^2 \geq \bar{m}(\hat{b}) - \bar{m}(b) \quad \text{for all } b \in \bar{N}(0,\delta).$$

Hence

$$\begin{aligned}
&\bar{m}(b^n) - \bar{m}(b) \\
&\qquad \leq \bar{m}(\hat{b}) - \bar{m}(b) + M\|b^n-\hat{b}\|^2 \\
&\qquad \leq \bar{m}(\hat{b}) - \bar{m}(b) + \frac{r}{2}\|b^n-\hat{b}\|^2 \\
&\qquad \leq \frac{r}{2}(\|b-\hat{b}\|^2 + \|b^n-\hat{b}\|^2) \\
&\qquad \leq \frac{1}{2}\left[\frac{r(\|b-\hat{b}\| + \|b^n-\hat{b}\|)^2}{\|b^n-b\|}\right]\|b^n-b\|^2 \\
&\qquad \leq \frac{1}{2}\, r^n\|b^n-b\|^2 \quad \text{for all } b \in \bar{N}(0,\delta).
\end{aligned}$$

Since $r^n \in H(b^n) \cap [2M,+\infty)$, we have established the u.s.c. of $b \to \bar{r}(b)$ at $\hat{b}$.

Now for each $\hat{b} \in G$ we have $\bar{r}(\hat{b}) < +\infty$ and in fact $\bar{r}(\hat{b})$ is bounded on G since an upper semi-continuous function attains its supremum on the compact set G. Hence

$$\bar{m}(\cdot)$$

is sub-differentiable with respect to $Q^{\bar{c}/2}$ on G for any $\bar{c}$ sufficiently large so that

$$\bar{r}(\hat{b}) \leq \bar{c};\ \forall\, \hat{b} \in G.$$

The continuity of $\bar{m}(\cdot)$ extends this to all of $\bar{N}(0,\delta)$. As a consequence $\bar{m}(\cdot)$ is $Q^{\bar{c}/2}$ convex on $\bar{N}(0,\delta)$ and

$$\bar{m}(b) = \sup\{\psi(b) = \bar{m}(\hat{b}) - \frac{\bar{c}}{2}\|b-\hat{b}\|^2;\ \hat{b} \in \bar{N}(0,\delta)\}.$$

Hence for all $b \in \bar{N}(0,\delta)$

$$p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2$$

$$= \sup\{(\bar{m}(\hat{b}) - \frac{\bar{c}}{2}\|\hat{b}\|^2) + \langle\bar{y}+\bar{c}\hat{b},b\rangle + \frac{(c-\bar{c})}{2}\|b\|^2 :$$

$$\hat{b} \in \bar{N}(0,\delta)\}$$

the supremum of a collection of convex functions in b for $c > \bar{c}$. We can define a proper, lower semi-continuous, convex function on R^m by letting

$$h(b) = \begin{cases} \bar{m}(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2; & \text{if } b \in \bar{N}(0,\delta) \\ +\infty & \end{cases}$$

for $c > \bar{c}$.

Theorem 4.11 is applicable and h(b) achieves its global minimum when

$$0 \in \partial h(b),$$

where $\partial h(b)$, the convex function's sub-derivative, coincides with the Clarke generalized derivative if

$$b \in N(0,\delta).$$

We now apply Lemma 4.4, after first noting that $\bar{b} \in \text{int } B(\bar{g})$ and hence that $p(b) = \bar{m}(b)$ locally around zero. We have, due to its local Lipschitzness (i.e. local Lipschitz order 2 implies local Lipschitz order 1), the existence of $\bar{y}$ s.t.

$$-\bar{y} \in \partial\bar{m}(0).$$

For any such $\bar{y}$

$$0 \in \partial\bar{m}(0) + \bar{y} = \partial(\bar{m}(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2)|_{b=0}$$

$$= \partial(p(b) + \langle\bar{y},b\rangle + \frac{c}{2}\|b\|^2)|_{b=0}$$

$$= \partial h(b)|_{b=0}$$

$$= \partial h(0).$$

For for $c > \bar{c}$ this implies $h(\cdot)$ attains its global minimum at $b = 0$, that is

$$b \to p(b) + \langle \bar{y}, b \rangle + \frac{c}{2}\|b\|^2$$

attains its local minimum at $b = 0$ for any $c > \bar{c}$.

Proposition 4.15 applies and $(\bar{y}, \bar{c})$ is a solution of the dual problem.

To obtain the reversed inclusion we note that since Theorem 4.15 is applicable we have for $\bar{c}$ sufficiently large

$$p(0) = \bar{m}(0) = \max_T W(y,c) = W(\bar{y}, \bar{c})$$

for any $\bar{y} = -\nabla\psi(0)$, for any function satisfying the definition of stability degree 2.

Suppose we have such a function $\psi(\cdot) : R^m \to R$ satisfying

(vi) $p(0) = \bar{m}(0) = \psi(0)$,

(vii) $p(b) = \bar{m}(b) \geqslant \psi(b)$ for $b \in N(0,\delta)$ (for some $\delta > 0$).

This implies that

$$\limsup_{t \to 0_+} \frac{p(te)-p(0)}{t}$$

$$= \limsup_{t \to 0_+} \frac{p(te)-\psi(0)}{t}$$

$$\geqslant \limsup_{t \to 0_+} \frac{\psi(te)-\psi(0)}{t}$$

$$= \lim_{t \to 0_+} \frac{\psi(te)-\psi(0)}{t}$$

$$= \langle \nabla\psi(0), e \rangle.$$

Since this holds for all $e \in R^m$ we have

$$\nabla\psi(0) \in \partial p(0) = \partial\bar{m}(0). \qquad \square$$

Theorem 4.4 gives conditions under which $\bar{m}(\cdot)$ will be locally Lipschitz order 2 around $b = 0$. The role of Φ_2-boundedness is obviously crucial to the above proof and as a consequence needs further exploration. It would be of interest to know what conditions on the functions $f(\cdot)$, $g_j(\cdot)$; $j=1,..,m$ would imply Φ_2-boundedness. R.T. Rockafellar referred to this boundedness as the quadratic growth condition. He gives in reference [22] the following condition

$$\lim_{\|b\|\to+\infty} p(b)/\|b\|^2 > -\infty$$

which is obviously equivalent to Φ_2-boundedness. He goes on to note that this condition holds if and only if $W(y,c)$ is not identically $-\infty$ on T, or, in other words, if and only if (D) has "feasible solutions". The quadratic growth condition is also equivalent to the condition that for some $\bar{y} \in R^m$ (not necessarily $y = 0$) and some $\bar{c} \geq 0$, the infimum of $L(u_2,\bar{y},\bar{c})$ over all $u_2 \in U_2$ is not $-\infty$.

The interesting thing about this equivalence is that even though $\partial\bar{m}(0)$ is, under very generaly conditions, contained in the convex closure of the Lagrange multipliers (see Theorem 4.17), it is not necessarily equivalent to this set. Theorem 4.16 gives conditions under which a Lagrange multiplier associated with an optimal solution would be contained in $\partial m(0)$.

Interestingly enough, the inclusion of $\partial m(0)$ in the set of Lagrange multipliers follows under the conditions of Theorem 4.19 if we assume U_2 is open and the functions $f(\cdot)$ and $\bar{g}_j(\cdot)$; $j=1,..,m$ are continuously differentiable. That is, if $\bar{u}_2 \in U_2$ and $(\bar{y},\bar{c}) \in T$ satisfy the saddle point relation of Theorem 4.15, we have;

$$0 = \frac{\partial L}{\partial y_j}(\bar{u}_2,\bar{y},\bar{c}) = \max\{\bar{g}_j(\bar{u}_2), -\bar{y}_j/\bar{c}\}, \quad \text{for } j=1,..,m,$$

$$0 = \nabla_2 L(\bar{u}_2,\bar{y},\bar{c})$$

$$= \nabla_2 f(\bar{u}_2) + \sum_{j=1}^{m} \max\{0,\bar{y} + \bar{c}\bar{g}(\bar{u}_2)\}\nabla_2\bar{g}_j(\bar{u}_2)$$

$$= \nabla_2 f(u_2) + \sum_{j=1}^{m} [\bar{y}_j + \bar{c}\max\{\bar{g}_j(\bar{u}_2), -\bar{y}_j/\bar{c}\}]\nabla_2\bar{g}_j(\bar{u}_2)$$

implying

$$g_j(\bar{u}_2) \leq 0;\ \bar{y}_j \geq 0;\ \bar{y}_j\bar{g}_j(\bar{u}_2) = 0, \text{ for } j=1,..,m$$

and

$$\nabla_2 f(\bar{u}_2) + \sum_{j=1}^{m} \bar{y}_j \nabla_2\bar{g}_j(\bar{u}_2) = 0.$$

R.T. Rockafellar also notes that if the functions $f(\cdot)$, $\bar{g}_j(\cdot)$; $j=1,..,m$ are twice continuously differentiable one has the condition (v) of Theorem 4.16 almost satisfied in the sense that the inequality

$$w'\nabla_2^2 L(\bar{u}\ ,\bar{y},\bar{c})w > 0,$$

is weakened to

$$w'\nabla_2^2 L(\bar{u}_2,\bar{y},\bar{c})w \geq 0.$$

Corollary 4.19 : Suppose;

(i) $\bar{b} \in \text{int } B(\bar{g})$;

(ii) $b \to \bar{m}(b)$ is locally Lipschitz order 2 around $b = 0$;

(iii) $p(\cdot)$ is Φ_2 bounded;

(iv) the functions $f(\cdot)$, $\bar{g}_j(\cdot)$; $j=1,..,m$, are continuously differentiable on U_2 and

(v) U_2 is an open set.

Then $\partial\bar{m}(0) \subseteq \{-\bar{y} : \exists\ \bar{u}_2$ satisfying with $\bar{y}$ the Kuhn-Tucker conditions$\}$.

Proof : Theorem 4.19 and the above comments. □

In a way the dual solutions can be thought of as a more "refined" set of Lagrange multipliers.

CHAPTER V

Fuzzy sets have been around for a number of years. They were developed to model the concept of "impression". For instance what do we mean by the set of "tall" people? How do we qualify degree of closeness? Initially an extension of ordinary set theory was achieved by extending the idea of the characteristic function $I(A)(\cdot)$ of a set $A \subseteq U$. The characteristic function takes U onto {0,1} and is interpreted as assigning a degree of membership. L.A. Zadeh replaced {0,1} with the unit interval [0,1] giving a continuum of degree of membership. Other authors later replaced [0,1] with much more general lattices. More precisely, $(L,\leqslant,^{c})$ a complete distributive lattice with order reversing involution.

Bruce Hutton (see references [36] and [37]) discussed various separation axiom of the fuzzy topological spaces induced by this "extended" set theory. Normality being one of the few separation axioms which can be defined purely in terms of the properties of open and closed sets (i.e. no mention of points) is of some interest. Bruce Hutton characterised normality in terms of a "Urysohn" type lemma and introduced the fuzzy until interval, which plays the role of the ordinary unit interval in this context.

As we have noted, the original Urysohn lemma is related to the problem of extension of continuous functions (see the comments before Theorem 2.4). Theorems on continuous selection deal with spaces which necessarily are extension spaces with respect to each other, namely if $A \subseteq U_1$ and $g : A \to U_2$ we say U_2 is an extension space with respect to U_1 if we can extend any continuous function $g(\cdot)$ to a continuous $f(\cdot) : U_1 \to U_2$. As we will see the multi-valued mappings we have dealt with in previous chapters can be considered to be members of a particular fuzzy topology. It seems very natural that the concept of fuzzy normality should shed light on the selection problem, involving multi-valued mappings, to which we have devoted much time to in previous chapters. It turns out to be also natural to deal with less general lattices for L and restrict ourselves to continuous lattices which reflect the continuum properties of the unit interval more closely.

In the first part of chapter five we extend slightly some of the representational theorems of continuous lattice theory in the sense that we deal with continuous lattices of sets which are not necessarily topologies. We go on to establish a dual isomorphism of continuous lattices which is closely related to the L-flow theory of C.V. Negoita and D.A. Rolesca (see reference [34]). Using this we can show that every quasi-convex function, taking a compact set U into R^n, can be expressed as the point wise limit of a class of strictly quasi-convex functions. More specifically, the strictly quasi-convex functions are "lower dense" in the lattice of quasi-convex functions.

We go onto consider the following problem. Given a class of Φ-convex sets which are closed under finite infimums, when will the resulting fuzzy topology $\mathcal{L}' = [U_1, \Sigma \subset \Phi_{o\,ps}(U_2)]$, admit the following? The existence of an open-closed set $T(\cdot)$, for any closed set $K(\cdot)$ and open set $U(\cdot)$ where $K(\cdot) \subseteq U(\cdot)$, such that

$$K(\cdot) \subseteq T(\cdot) \subseteq U(\cdot).$$

This of course implies normality of the corresponding fuzzy topology $\mathcal{L}'$. We conclude by showing that the normality of $\mathcal{L}'$ implies the ability to achieve the above for some set $T(\cdot)$, corresponding to a continuous multi-valued mapping.

This particular situation is closely related to the content of Proposition 3.2. It is of interest because perfect normality is equivalent to the existence of a generating class of $[U_1, \Sigma\, O(U_2)]$, when U_2 is a compact Hausdorf space. Since $[U_1, \Sigma\, O(U_2)]$ will consist of i.s. continuous functions, the complements of u.s. continuous functions, the closed fuzzy set $K(\cdot)$ is an upper semi-continuous multi-function and $U(\cdot)$ will be a lower semi-continuous multi-function. The mapping $T(\cdot)$ is continuous and perfect normality will imply an arbitrarily close graph approximation by continuous multi-functions. This demostrates that such a property may be possessed by a large class of problems.

IS

§5.1 Representation of L-fuzzy sets

The usual power set $\mathcal{P}(U)$ can be identified with the indicator functions $\{I(A) : U \to \{0,1\};\ A \in \mathcal{P}(U)\}$. If this is done then a "natural" extension is to replace $\{0,1\}$ by a more general lattice reflecting the degree of membership. More precisely, we use $(L,\leq,c)$ a complete lattice with order reversing involution. The L-fuzzy sets are then the mappings $\{I(A) : U \to L\} \equiv \mathcal{L}_L(U)$.

The usual practice with fuzzy sets is to identify unions, intersections and complements as follows

$$I(\cup_i A_i)(u) = \vee_i I(A_i)(u) \equiv \cup_i \tilde{A}_i$$

$$I(\cap_i A_i)(u) = \wedge_i I(A_i)(u) \equiv \cap_i \tilde{A}_i$$

$$I(A^c)(u) = I(A)(u)^c \equiv \tilde{A}^c .$$

One can, under certain situations, show that there exists a lattice isomorphism between the L-fuzzy sets, lattice and the lattice of L-flow sets.

Definition 5.1 : An L-flow subset of a set U is a family

$$\mathcal{F}_L = (E(\alpha))_{\alpha\in L};\ E(\alpha) \subseteq U;\ \forall\, \alpha \in L$$

s.t.

$$E(\vee_i \alpha_i) = \cup_i E(\alpha_i);\ \forall\ \{\alpha_i\ ;\ i \in I\} \subseteq L.$$

If we consider $f : L \to \mathcal{P}(U)$ with the property $f(\vee_i \alpha_i) = \cup_i f(\alpha_i)$ for all $\{\alpha_i\}_{i \in I} \subseteq L$, we can generate flow subsets via these functions. This was used in reference [34] to obtain an equivalent representation for fuzzy sets. We will derive a variant of this by using a slightly different concept.

Definition 5.2 : An L-deflow subset of a set U is a family

$$\mathcal{F} = (E(\alpha))_{\alpha \in L};\ E(\alpha) \subseteq U;\ \forall\ \alpha \in L$$

s.t.

$$E(\vee D) = \cup\{E(d);\ d \in D\}$$

for all directed subsets $D \subseteq L$.

If we let τ and L be continuous and complete lattices then complete we know (Definition 1.17)

$$[L \to \tau] = \{f : L \to \tau;\ f(\vee D) = \cup\{f(d) : d \in D\} \text{ for all directed sets } D \subseteq L\}$$

is the complete continuous lattice of Scott continuous functions.

We can restrict the class $[L \to \tau]$ as follows

$$[L \to \tau]_0 = \{f \in [L \to \tau];\ f(o) = \phi\}$$

where o is the minimal element of L. This is of course a complete continuous lattice. This follows immediately from the completeness and continuity of $[L \to \tau]$ and the fact that supremum of a subset of $[L \to \tau]_0$ is once again in $[L \to \tau]_0$, that is for $f_i \in [L \to \tau]_0$

$$(\vee_i f_i)(\alpha) = \cup_i f(\alpha).$$

The supremum is defined point wise with respect to τ and hence

$$(\vee_i f_i)(o) = \cup_i f(o) = \cup_i \phi = \phi.$$

If we let τ be a topology on a topological space U we have of course

$$[L \to \tau]_0 \equiv [\Sigma L, \Sigma\tau]$$

and

$$[L \to \tau]_0 \equiv [\Sigma L, \Sigma\tau]_0 \quad \text{(with the obvious interpretation).}$$

We will write $I(A)(\cdot) \equiv \tilde{A}$.

<u>Definition 5.3</u> : A fuzzy topological space is pair $(U,\mathcal{L})$, $\mathcal{L} \subseteq \mathcal{L}_L(U)$ s.t.

(i) $0,1 \in \mathcal{L}$ where 0 is the minimal element of $\mathcal{L}_L(U)$ and 1 the maximal element.

(ii) $\tilde{A}, \tilde{B} \in \mathcal{L}$ implies $\tilde{A} \cap \tilde{B} \in \mathcal{L}$.

(iii) $(\tilde{A}_i)_{i\in I} \subseteq \mathcal{L}$ implies $\cup_{i\in I} \tilde{A}_i \in \mathcal{L}$.

Before we restrict ourselves to the class of fuzzy sets we will consider a slightly more general class, namely, the fuzzy classes which are closed under arbitrary supremums, i.e. $\mathcal{L}'$ is closed under supremums if $\{\tilde{A}_i ; i \in I\} \subseteq \mathcal{L}'$ implies $\cup_{i\in I} \tilde{A}_i \in \mathcal{L}'$. As usual one can define the infimum as follows,

$$\underline{\wedge}_i\, g_i = \vee\{g : g \in \mathcal{L}'; g \leqslant g_i ; \forall_{i\in I}\}$$

and hence derive a complete lattice $\mathcal{L}'$. We denote, for any crisp set A,

$$\alpha I_0(A)(y) = \begin{cases} \alpha ; & y \in A \\ 0 & \text{otherwise.} \end{cases}$$

<u>Proposition 5.1</u> : Suppose L and τ are sup-complete and hence (via the above trick) complete lattices. For $\tau \subseteq \mathcal{P}(U)$, where $\phi \in \tau$ is the minimum element and $U \in \tau$ is the maximal element, we define;

$$\mathcal{L}' = \{f : U \to L \text{ s.t. } f^{-1}(\hat{\uparrow}\alpha) \in \tau; \alpha \in L\}.$$

Then if L and τ are continuous lattice so is $\mathcal{L}'$ and if (U,τ) is a topological space then $\mathcal{L}' = [U, \Sigma L]$. If we have only $\tau \subseteq T$

where (U,T) is a topological space, then

$$\mathcal{L}' \subseteq [U, \Sigma L].$$

Proof : The second assertion follows immediately from the observation that Theorem 1.11 is applicable. Since any Scott open set, by Proposition 1.6(iii),

$$S = \cup\{\hat{\uparrow}\alpha : \alpha \in S\} \equiv \text{int } S, \text{ then}$$

for $f \in \mathcal{L}'$ we have that

$$f^{-1}(S) = \cup\{f^{-1}(\hat{\uparrow}\alpha); \alpha \in S\}.$$

This is obviously open when $\tau \subseteq T$ and hence

$$\mathcal{L}' \subseteq [U, \Sigma L].$$

When τ is a topology we have equality since $\hat{\uparrow}\alpha$ is open and hence for $f \in [U, \Sigma L]$ we have $f^{-1}(\hat{\uparrow}\alpha) \in \tau$, the open sets.

To prove the first assertion we need to characterise the way below relation on $\mathcal{L}'$. We suppose $f \ll g$ in $\mathcal{L}'$, we let $t = \vee\{g(u) : u \in U\}$ and take two directed sets

$$D_1 = \{S : S \ll t\} \subseteq L$$

$$D_2 = \{V \in \tau : V \ll U\} \subseteq \tau.$$

We form a new directed set in $\mathcal{L}'$

$$D_3 = \{S\, I_0(V) : S \in D_1 \text{ and } V \in D_2\}.$$

Obviously $\vee D_3 \geqslant g$ and here there exists $S \in D_1$ and $V \in D_2$ s.t. $S\, I_0(V) \geqslant f$. Since $I_0(V)(u) = 0$ for $u \notin V$ we must have $f(u) = 0$ for $u \notin V$. This prompts us to investigate the functions $\alpha I_0(V)(\cdot)$ for $V \in \tau$.

By definition

$$\sigma_\alpha(f) \equiv f^{-1}(\hat{\uparrow}\alpha) \in \tau; \ \forall \ \alpha.$$

Hence $\alpha I_0(\sigma_\alpha(f))(u) << f(u); \ \forall \ u \in U$ which in turn implies

$$\vee\{\alpha I_0(\sigma_\alpha(f)); \ \alpha \in L\}(u) \leq f(u); \ \forall \ u \in U.$$

We have $u \in \sigma_\beta(f)$ for $\beta << f(u)$. That is

$$\vee\{\alpha I_0(\sigma_\alpha(f)); \ \alpha \in L\}(u) \geq \beta$$

for all $\beta << f(u)$. Hence

$$\vee\{\alpha I_0(\sigma_\alpha(f)) : \alpha \in L\}(u) \geq \vee\{\beta : \beta << f(u)\} \equiv f(u),$$

since L is continuous.

For each α, $\sigma_\alpha(f) \in \tau$ and $\forall \ u \in \sigma_\alpha(f)$ we have $\alpha << f(u)$. This prompts us to look at the functions $\alpha I_0(V)(\cdot)$ s.t. $V \in \tau$, $\alpha \in L$ and

$$\alpha << f(u); \ \forall \ u \in V.$$

We note that

$$\begin{aligned} f(u) &\equiv \vee\{\alpha I_0(\sigma_\alpha(f)): \alpha \in L\}(u) \\ &\leq \vee\{\alpha I_0(V); \ V \in \tau \text{ and } \alpha << \underline{\wedge}\{f(u): u \in V\}\}(u) \\ &\leq f(u) \ ; \ u \in U. \end{aligned}$$

This follows from the observation that

(i) if $\beta \in L$, for $\forall \ u \in \sigma_\beta(f) \in \tau$, we have

$$\beta << f(u) \qquad (\text{i.e. } \beta < f(u); \ \forall \ u \in \sigma_\beta(f))$$

(ii) $\underline{\wedge}\{f(u) : u \in \sigma_\beta(f)\}$

$$\equiv \vee\{x : x \leqslant f(u);\ \forall\, u \in \sigma_\beta(f)\} \text{ implying}$$

$$\beta \leqslant \underline{\wedge}\{f(u) : u \in \sigma_\beta(f)\}$$

because

$$\beta \in \{x : x \leqslant f(u);\ \forall\, u \in \sigma_\beta(f)\} \text{ and}$$

(iii) the fact that L is a continuous lattice allows us to take

$$\alpha \ll \underline{\wedge}\,\{f(u) : u \in \sigma_\beta(f)\}.$$

The functions $\alpha I_0(V) \in \mathcal{L}'$ for $V \in \tau$ since

$$\{u : \alpha I_0(V)(u) \gg \beta\} = \begin{cases} \phi\ ;\ \beta \in (\check{\downarrow}\alpha)^c \\ U\ ;\ \beta \text{ is the minimal element of } L \\ V\ ;\ \text{otherwise.} \end{cases}$$

Finally we characterise the way below relation for such functions. We have $\alpha I_0(V) \ll \beta I_0(M)$ if $\alpha \ll \beta$ and $V \ll M$ in these respective lattices.

Suppose we have a directed set $D = \{h : U \to L;\ h \in \mathcal{L}'\}$ s.t. $\vee D \geqslant \beta I_0(M)$. Suppose also that $\alpha \ll \beta$ and $V \ll M$. If 0 is the minimal element of L we define

$$V_h = \{u : h(u) \neq 0\}$$

$$= \{u : h(u) \in L \setminus \downarrow 0\}.$$

Now $\downarrow 0$ is Scott closed and hence $L \setminus \downarrow 0$ is Scott open and

$$L \setminus \downarrow 0 = \cup\{\hat{\uparrow}x : x \notin \downarrow 0\}.$$

This implies

$$V_h = h^{-1}(L \setminus \downarrow 0) = \cup\{h^{-1}(\hat{\uparrow}x) : x \notin \downarrow 0\} \in \tau.$$

We define also $\alpha_h = \underline{\wedge}\{h(u) : u \in V\}$.

From $VD \geqslant \beta I_0(M)$ we can deduce that $\vee\{V_h : h \in D\} \supseteq M$. First we let $K(\cdot) = \vee\{h : h \in D\}(\cdot)$ and note that $\{u : K(u) \neq 0\} \supseteq M$. Suppose $h(u) = 0, \forall h \in D$. Then $\vee\{h(u) : h \in D\} = 0$, that is, $K(u) = 0$. Hence $K(u) \neq 0$ implies $\exists h \in D$ s.t. $h(u) \neq 0$, i.e.,

$$\vee\{V_h : h \in D\} \supseteq \{u : K(u) \neq 0\} \supseteq M.$$

Since $M \gg V$ and $\vee\{V_h : h \in D\} \supseteq M$ we must have $V_h \supseteq V$ for some $h \in D$.

We can also deduce that $\vee\{\alpha_n : h \in D\} \geqslant \beta$. First we note that

$$\alpha_n = \vee\{x : h(u) \geqslant x; \forall u \in V\}$$

$$= \vee\{x : h(u) \gg x; \forall u \in V\}$$

$$= \vee\{x : \{u : h(u) \in \hat{\uparrow} x\} \supseteq V\},$$

since L is continuous.

Now if $U_D\{u : h(u) \in \hat{\uparrow} x\} \gg V$, then $\exists h \in D$ s.t. $\{u : h(u) \in \hat{\uparrow} x\} \supseteq V$, since $\{u : h(u) \in \hat{\uparrow} x\} \in \tau$ is directed. Hence $\exists h \in D$ s.t.

$$\{x : U_D\{u : h(u) \in \hat{\uparrow} x\} \gg V\}$$

$$\subseteq \{x : \{u : h(u) \in \hat{\uparrow} x\} \supseteq V\},$$

that is,

$$\vee\{x : U_D\{u : h(u) \in \hat{\uparrow} x\} \gg V\}$$

$$\leqslant \vee_D \vee\{x : \{u : h(u) \in \hat{\uparrow} x\} \supseteq V\}$$

$$= \vee_D \alpha_n .$$

Since $V_D\ h(u) \in \hat{\uparrow}\ x$ implies $\exists\ h \in D$ s.t. $h(u) \gg x$ (see Proposition 1.4) we have $\{u : V_D\ h(u) \in \hat{\uparrow}\ x\} \subseteq U_D\{u : h(u) \geqslant x\}$. Hence

$$\{x : \{u : V_D\ h(u) \in \hat{\uparrow}\ x\} \gg V\}$$

$$\subseteq \{x : U_D\{u : h(u) \gg x\} \gg V\}.$$

This implies

$$V\{x : \{u : V_D\ h(u) \in \hat{\uparrow}\ x\} \gg V\}$$

$$\leqslant V\{x : U_D\{u : h(u) \gg x\} \gg V\}$$

$$\equiv V\{x : U_D\{u : h(u) \in \hat{\uparrow}\ x\} \gg V\}\ ,$$

since L is continuous.

Finally since

$$K(u) = V_D\ h(u) \geqslant \beta \text{ for all } u \in M \gg V,$$

and we have

$$\{x : \{u : K(u) \in \hat{\uparrow}\ x\} \gg V\}$$

$$\supseteq \{x : \beta \gg x\}\ .$$

We can deduce that

$$\beta = V\{x : \beta \gg x\} \leqslant V\{x : \{u : K(u) \in \hat{\uparrow}\ x\} \gg V\}$$

$$\leqslant V\{x : U_D\{u : h(u) \in \hat{\uparrow}\ x\} \gg V\}$$

$$\leqslant V_D \alpha_h\ .$$

The relation $\beta \gg \alpha$ implies the existence of a $h' \in D$ s.t. $\alpha_n{}' \geqslant \alpha$. We have already shown that there exists a $h'' \in D$ s.t.

$$V_{h'} = \{u : h''(u) \neq 0\} \supseteq V.$$

Since D is directed we have $h \geqslant h'Vh''$, $h \in D$ s.t.

$$\alpha I_0(V)(u) < \alpha_h I_0(V)(u) \leqslant h(u); \; \forall\, u \in U,$$

that is,

$$\alpha I_0(V) \leqslant h$$

in $\mathcal{L}'$. This implies

$$\alpha I_0(V) << \beta I_0(M) \text{ in } \mathcal{L}'.$$

The continuity of $\mathcal{L}'$ follows from the continuity of L and τ, after noting that

$$f = V\{\alpha I_0(V) : V \in \tau \text{ and } \alpha << \underline{\wedge}\{f(u) : u \in V\}\}$$

$$= V\{\alpha I_0(V) : V,M \in \tau \text{ and } V << M \text{ where } \alpha << \underline{\wedge}\{f(u) : u \in M\}=\beta\}$$

that is,

$$f = V\{\alpha I_0(V) : \alpha I_0(V) \in \mathcal{L}' \text{ and } \alpha I_0(V) << f\}$$

$$= V\{g : g \in \mathcal{L}' \text{ and } g << f\}.$$ □

It is rarely the case that L and L_{ops}, the lattice induced by reversing the order on L, are both continuous lattices. It is true for [0,1] with the complement of $r \in [0,1]$ being $r' = 1 - r$ since $[0,1]' \equiv [0,1]$. It is true for $L = (R^*)^n$ with the order reversing operation of multiplication by -1.

The way below relation on these lattices differs slightly from strictly less than in the following sense. For [0,1] or R* we let 0 be the 'minimal' element and note that $x << y$ iff either $x < y$ or $x = y = 0$. In $(R^*)^n$ we have $(x_1,..,x_n) << (y_1,..,y_n)$ iff $x_i << y_i \; \forall\, i = 1,..,n$. We always have $0 << \beta; \; \forall\, \beta \in L$.

Proposition 5.2 : Suppose τ, L_{ops} and L are supremum complete continuous lattices for which τ satisfies,

(E) If $u \in K \in \tau$ then $\exists\, O \in \tau$ s.t. $u \in O << K$;

and L satisfies,

(F) (i) If $\beta \neq 0$ then $\alpha >> \beta$ in L implies $\beta <<_{ops} \alpha$ in L_{ops}

(ii) $\beta > 0$ in L iff $\beta <<_{ops} 0$ in L_{ops}.

Then there is a dual isomorphism of complete continuous lattices between $[L_{ops} \to \tau]_0$ and $\mathcal{L}'$.

Proof : We will refer to the order on L_{ops} as $\leqslant_{ops}$ which generates V_{ops} $<<_{ops}$ and $\underset{\sim}{\wedge}_{ops}$. We define $\Phi(\tilde{A}) = f$ by

$$f(\alpha) = \sigma_\alpha(A) = \{u \in U : I(\tilde{A})(u) >> \alpha\} \in \tau \quad \text{for } \alpha \neq 0$$

and

$$f(0) = \{u \in U : I(\tilde{A})(u) > 0\}$$

$$= \{u \in U : I(\tilde{A})(u) \neq 0\}$$

$$= \{u \in U : I(\tilde{A})(u) \in L \setminus \downarrow 0\} \in \tau$$

for any given $\tilde{A} \in \mathcal{L}'$. We note that

$$f(1) = \{u \in U : 1 << I(\tilde{A})(u)\} = \phi,$$

since 1 is the maximal element of L.

In terms of L_{ops}, 1 is the minimal element and hence all we need to do to show that

$$f(\cdot) \in [L_{ops}, \tau]_0$$

is to investigate $\{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\}$ when $\underline{\wedge}\alpha_i \neq 0$ and $\{u : \underline{\wedge}\alpha_i < I(\tilde{A})(u)\}$ when $\underline{\wedge}\alpha_i = 0$.

Now

$$\underline{\wedge}\alpha_i = V\{\beta : \beta \leqslant \alpha_i ; \forall i\}$$

$$= V\{\beta : \beta \geqslant_{ops} \alpha_i ; \forall i\}$$

$$\geqslant_{ops} \alpha_i ; \forall i.$$

Hence

$$\underline{\wedge}\alpha_i \geqslant_{ops} V_{ops}\,\alpha_i \quad \text{or} \quad \underline{\wedge}\alpha_i \leqslant V_{ops}\alpha_i .$$

From

$$V_{ops}\,\alpha_i \geqslant_{ops} \alpha_i ; \forall i$$

we have

$$V_{ops}\,\alpha_i \leqslant \alpha_i ; \forall i,$$

which in turn implies $V_{ops}\,\alpha_i \leqslant \underline{\wedge}\alpha_i$. Hence $V_{ops}\,\alpha_i = \underline{\wedge}\alpha_i$.

From this it follows that if $\underline{\wedge}\alpha_i \neq 0$, then $\{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\} = f(V_{ops}\,\alpha_i)$ and $\{u : \underline{\wedge}\alpha_i < I(\tilde{A})(u)\} = f(V_{ops}\,\alpha_i)$ if $\underline{\wedge}\alpha_i = 0$.

In the first case if

$$u \in \{u : \alpha_i << I(\tilde{A})(u)\} \quad \text{for some } i,$$

then

$$\underline{\wedge}\alpha_i \leqslant \alpha_i << I(\tilde{A})(u)$$

and

$$u \in \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\},$$

so

$$\bigcup_i \{u : \alpha_i << I(\tilde{A})(u)\}$$

$$\subseteq \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\}.$$

We have the following three cases.

(i) When $\underline{\wedge}\alpha_i = 0$ and $\alpha_i \neq 0, \forall i$ then

$$f(\alpha_i) = \{u : \alpha_i << I(\tilde{A})(u)\}$$

$$\subseteq \{u : 0 < I(\tilde{A})(u)\} = f(0)$$

That is we have $0 < \alpha_i << I(\tilde{A})(u)$, implying

$$\bigcup_i \sigma_{\alpha_i}(\tilde{A}) \subseteq \{u : 0 < I(\tilde{A})(u)\}.$$

(ii) If $\alpha_i = 0$ for any i then

$$\bigcup_i f(\alpha_i) = f(0) = \{u : 0 < I(\tilde{A})(u)\}$$

$$= f(V_{ops}\, \alpha_i).$$

(iii) Next we show

$$\bigcup\{\sigma_\beta(\tilde{A}) : \beta >> \underline{\wedge}\alpha_i\}$$

$$= \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\}$$

when $\underline{\wedge}\alpha_i \neq 0$, where $\sigma_\beta(\tilde{A}) = \{u : \beta << I(\tilde{A})(u)\}$. Obviously

$$\bigcup\{\sigma_\beta(\tilde{A}) : \beta >> \underline{\wedge}\alpha_i\}$$

$$\subseteq \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\} .$$

If $u \in \{u \in U : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\}$, then by the strong interpolation property (Proposition 1.3) there exists a $\beta >> \underline{\wedge}\alpha_i$ s.t.

$$\underline{\wedge}\alpha_i << \beta << I(\tilde{A})(u);\ \underline{\wedge}\alpha_i \neq \beta ,$$

i.e.,

$$u \in \{u : \beta << I(\tilde{A})(u)\}.$$

Now if $\underline{\wedge}\alpha_i = 0$ we show

$$\cup\{\sigma_\beta(\tilde{A}) : \beta > 0\}$$

$$= \{u : 0 < I(\tilde{A})(u)\}.$$

It is always the case that

$$\sigma_\beta(\tilde{A}) = \{u : \beta << I(\tilde{A})(u)\}$$

$$\subseteq \{u : 0 < I(\tilde{A})(u)\},$$

since $0 < \beta << I(\tilde{A})(u)$ implies $0 < I(\tilde{A})(u)$.

If $0 < I(\tilde{A})(u)$, then $0 << I(\tilde{A})(u)$ and $0 \neq I(\tilde{A})(u)$.

By the strong interpolation property we have the existence of a β s.t. $0 << \beta << I(\tilde{A})(u)$ and $0 \neq \beta$. Now $0 << \beta$ implies $0 \leqslant \beta$ but since $\beta \neq 0$ we have $0 < \beta$. Hence

$$\cup\{\sigma_\beta(\tilde{A}) : \beta > 0\}$$

$$\supseteq \{u : 0 < I(\tilde{A})(u)\}.$$

Suppose $\underline{\wedge}\alpha_i \neq 0$ and $u \in \{u : \wedge\alpha_i << I(\tilde{A})(u)\} \in \tau$. Then by property (E) $\exists$ $0 \in \tau$ s.t.

$$u \in 0 << \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\} \in \tau.$$

Since each $\sigma_\beta(\tilde{A}) \in \tau$, there must exist $\beta >> \underline{\wedge}\alpha_i$ s.t.

$$u \in 0 \leqslant \sigma_\beta(\tilde{A}).$$

By property F(i), $\beta >> \underline{\wedge}\alpha_i$ implies $\beta <<_{ops} \underline{\wedge}\alpha_i \equiv \vee_{ops}\, \alpha_i$ and the directedness of $\{\alpha_i : i \in I\}$ implies the existence of α_i s.t. $\beta \leqslant_{ops} \alpha_i$, i.e. $\beta \in \uparrow \alpha_i$. That is,

$$u \in 0 \leqslant \sigma_\beta(\tilde{A}) \leqslant \sigma_{\alpha_i}(\tilde{A})$$

and

$$\cup_i \{u : \alpha_i << I(\tilde{A})(u)\} \supseteq \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\}$$

implying $f(V_{ops}\, \alpha_i) = \cup_i f(\alpha_i)$.

Now suppose $\underline{\wedge}\alpha_i = 0$ and $u \in \{u : 0 < I(\tilde{A})(u)\} \in \tau$. Then by property (E) $\exists\, 0 \in \tau$ s.t.

$$u \in 0 << \{u : 0 < I(\tilde{A})(u)\} \in \tau.$$

Since each $\sigma_\beta(\tilde{A}) \in \tau$, there must exist $\beta > 0$ s.t. $u \in 0 \subseteq \sigma_\beta(\tilde{A})$. By property F(ii); $\beta > 0$ implies $\beta <<_{ops} 0 = \underline{\wedge}\alpha_i = V_{ops}\, \alpha_i$ and the directedness of $\{\alpha_i : i \in I\}$ implies the existence of α_i s.t. $\beta \leqslant_{ops} \alpha_i$ i.e., $\beta \in \uparrow \alpha_i$. That is to say

$$u \in 0 \leqslant \sigma_\beta(\tilde{A}) \leqslant \sigma_{\alpha_i}(\tilde{A}), \text{ if } \alpha_i \neq 0,$$

and

$$\cup_i \{u : \alpha_i << I(\tilde{A})(u)\}$$

$$\supseteq \{u : \underline{\wedge}\alpha_i << I(\tilde{A})(u)\}.$$ This implies

$$f(V_{ops}\, \alpha_i) = \cup_i f(\alpha_i).$$

If $\alpha_i = 0$ then

$$f(V_{ops}\, \alpha_i) = f(0) = f(\alpha_i) \leqslant \cup_i f(\alpha_i)$$

implying again $f(V_{ops}\, \alpha_i) = \cup_i f(\alpha_i)$.

Hence $f(\cdot) \in [L_{ops}, \tau]_0$. We note that $\alpha \geqslant_{ops} \beta$ implies $f(\alpha) \geqslant f(\beta)$.

Let us show that Φ is onto. If $f \in [L_{ops}, \tau]_0$ we must construct a $\tilde{A} \in \mathcal{L}'$ s.t.

$$\sigma_\alpha(\tilde{A}) = f(\alpha);\ \alpha \neq 0$$

and

$$\{u : 0 < I(\tilde{A})(u)\} = f(0).$$

For $u \in U$ we let

$$I(u) = \{\beta \in L : u \in f(\beta)\}.$$

We define

$$I(\tilde{A})(u) = \vee I(u).$$

Now let us prove that

$$\Phi(\tilde{A}) = f(\alpha).$$

Suppose $\alpha \neq 0$, we wish to show $\sigma_\alpha(\tilde{A}) = f(\alpha)$.

Let $u \in \sigma_\alpha(\tilde{A})$. Then $I(\tilde{A})(u) = \vee I(u) \gg \alpha$. Hence $\exists\ \beta \in I(u)$ s.t. $\beta \geqslant \alpha$, since $I(u) = \downarrow I(u)$ is a directed set. As $\beta \in I(u)$, we have $u \in f(\beta)$ and as $\alpha \leqslant \beta$ we have $\alpha \geqslant_{ops} \beta$ and $u \in f(\beta) \leqslant f(\alpha)$, implying $u \in f(\alpha)$.

If $u \in f(\alpha)$ then $\alpha \in I(u)$ and all that is needed is to show that $I(u) = \downarrow I(\tilde{A})(u)$.

As $f \in [L_{ops} \rightarrow \tau]_0$, we know from Definition 1.17 that $0 \ll f(\alpha)$ iff for some $\beta \ll_{ops} \alpha$ one has $0 \ll f(\beta)$.

Let us suppose $u \in f(\alpha_0)$ where $\alpha_0 = I(\tilde{A})(u)$. Then from property (E) there exists $0 \in \tau$ s.t. $0 \ll f(\alpha_0)$ and $u \in 0$. For some $\beta \ll_{ops} \alpha_0$, one has $u \in 0 \ll f(\beta)$. This contradicts the definition of α_0, namely

$$\alpha_0 = \vee I(u) = \vee\{\beta : u \in f(\beta)\}$$

$$= \wedge_{ops} \{\beta : u \in f(\beta)\},$$

Since the postulate that $u \in f(\alpha_0)$ implies

$$\alpha_0 = \wedge_{ops} \{\beta : u \in f(\beta)\} \leqslant_{ops} \beta \ll \alpha_0 .$$

Suppose $\beta << \alpha_0 = VI(u)$. Then since L is continuous $\exists\ \alpha \in I(u)$ s.t. $\beta \leqslant \alpha$ or $\beta \geqslant_{ops} \alpha$ and

$$u \in f(\alpha) \leqslant f(\beta) \text{ so } \beta \in I(u).$$

Suppose $\alpha\ = 0$ and

$$u \in \{u : 0 < I(A)(u)\} \in \tau .$$

Then we have

$$\cup\{\sigma_\beta(\tilde{A}) : \beta > 0\} = \{u : 0 < I(\tilde{A})(u)\}$$

and (by property (E)) $\exists\ 0 \in \tau$ s.t.

$$u \in 0 << \{u : 0 < I(\tilde{A})(u)\}$$

Thus we must have a $\beta > 0$ s.t. $u \in 0 \subseteq \sigma_\beta(\tilde{A})$.

That is, for $\beta <_{ops} 0$ we have

$$u \in f(\beta) \subseteq f(0).$$

Now suppose $u \in f(0) \in \tau$. By property (E) there exists a set $0 \in \tau$ s.t.

$$u \in 0 << f(0)$$

Hence for some $\beta <<_{ops} 0$, one has $u \in 0 << f(\beta)$. That is, $\beta \in I(u)$ for $\beta <<_{ops} 0$, which according to property F(ii) implies $\beta > 0$. Hence

$$I(\tilde{A})(u) = VI(u) \geqslant \beta > 0$$

and

$$u \in \{u : I(\tilde{A})(u) > 0\}.$$

Finally we show Φ is 1-1. Suppose $A \neq B$ and $\Phi(\tilde{A}) = \Phi(\tilde{B})$. Then $\exists\ \bar{u} \in U$ s.t.

$$I(\tilde{A})(\bar{u}) \neq I(\tilde{B})(\bar{u}).$$

But

(i) $\sigma_\alpha(\tilde{A}) = \sigma_\alpha(\tilde{B})$ for $\alpha \neq 0$ and

(ii) $\{u : I(\tilde{A})(u) > 0\} = \{u : I(\tilde{B})(u) > 0\}$.

We let $\alpha_0 = I(\tilde{A})(\bar{u})$ and suppose first that $\alpha_0 = 0$. Since

$$I(\tilde{B})(\bar{u}) \neq \alpha_0,$$

we must have

$$\bar{u} \in \{u : I(\tilde{B})(u) > 0\} = \{u : I(\tilde{A})(u) > 0\}$$

which implies $I(\tilde{A})(\bar{u}) > 0$, a contradiction.

On the other hand suppose $\alpha_0 = I(\tilde{A})(\bar{u}) \neq 0$.

We note that if $\alpha \ll \alpha_0$ then $\alpha \ll I(\tilde{A})(\bar{u})$. Thus

$$u \in \sigma_\alpha(\tilde{A}) = \sigma_\alpha(\tilde{B}), \text{ i.e.,}$$

$$\alpha \ll I(\tilde{B})(\bar{u});\ \forall\, \alpha \ll \alpha_0,$$

or

$$\Downarrow I(\tilde{B})(\bar{u}) \supseteq \Downarrow \alpha_0.$$

Since L is continuous,

$$I(\tilde{B})(\bar{u}) = \bigvee \Downarrow I(\tilde{B})(\bar{u}) \geqslant \bigvee \Downarrow \alpha_0 = \alpha_0$$

which implies

$$I(\tilde{B})(\bar{u}) \geqslant I(\tilde{A})(\bar{u}) = \alpha_0.$$

In a similar way we can show $I(\tilde{B})(\bar{u}) \leqslant I(\tilde{A})(\bar{u})$ and arrive at a contradiction

$$I(\tilde{B})(\bar{u}) = I(\tilde{A})(\bar{u}).$$ □

The type of lattice we are dealing with here is like $(R^*)^n$ in that it has an order reversing involution, namely multiplication by -1, which preserves the lattice continuity. It also satisfies property (F) since

(i) $\alpha \gg \beta$ and $\beta \neq 0$ implies $\alpha > \beta$ and $\alpha <_{ops} \beta$

(ii) $\beta > 0$ implies $\beta <_{ops} 0$ namely $\beta \ll_{ops} 0$.

The type of lattic we use for τ could be a locally compact topology or, as the next proposition shows, the class of open concave sets in a compact space. If U is a compact convex subset of a locally convex topological vector space, we denote by Con (U) the lattice of all closed convex subsets of U (including the empty set). Recall that Con $(U)_{ops}$ is the lattice with reverse ordering.

<u>Proposition 5.3</u> : The lattice Con $(U)_{ops}$ is a continuous lattice, in which we have $A \ll B$ iff $B \leqslant \text{int } A$, the interior being taken in the relative topology of U.

<u>Proof</u> : Reference [10] Proposition 1.22.1. □

Of course set complementation is an isomorphism of continuous lattices and the proposition implies that the sup complete continuous lattice

$$\tau = \{K \cap U : K^c \subseteq U \text{ is convex, closed}\}$$

has a way below relation which will satisfy property (F). This follows directly from the Hahn-Banach theorem in the case when U is a compact subset of a normed vector space. Of course U, $\phi \in \tau$ since U, $\phi \in$ Con $(U)_{ops}$.

<u>Corollary 5.2</u> : Suppose $U \subseteq (R^*)^n$ is compact convex and let $\tau =$ Con $(U)_{ops}$. Then there is a dual isomorphism of continuous lattices between

$$[(R^*)^n_{ops} \to \tau]_0 = \{f : (R^*)^n_{ops} \to \tau : f(+\infty) = \phi$$

$$f(V_{ops} D) = \cup\{f(d) : d \in D\} \text{ D a directed set in } (R^*)^n_{ops}\}$$

and

$$\mathcal{L}' = \{f : U \to (R^*)^n : f^{-1}(\hat{\uparrow}\alpha) \in \tau; \alpha \in (R^*)^n\}.$$

Proof : This is a direct consequence of Proposition 5.2. □

For $f \in \mathcal{L}'$ we have $\sigma^c_\alpha(f) = \{u \in U : f(u) \leq \alpha\}$ closed and convex $\forall \alpha \in R^m$ and for $\alpha \equiv \infty$ we have $\sigma^c_\alpha(f) \equiv U$ which is closed and convex. Hence f is ℓ.s.c. and a quasi-convex function from U to R^n.

The corollary tells us that there is a very close association between these functions and $[(R^*)^n_{ops} \to \tau]_0$. Let us specify a function

$f : (R^*)^n_{ops} \to \tau$ s.t.

(i) $f(+\infty) = \phi$

(ii) $f(\underline{\wedge} D) = \cup\{f(d) : d \in D\}$

for directed sets $D \subseteq (R^*)^n_{ops}$.

Then there corresponds a lower semi-continuous quasi-convex function. In fact there is exactly one!

We could instead specify, of course, $f : (R^*)^n \to \text{Con}(U)$.

$\text{Con}(U) = \{K \subseteq U : K \text{ is closed and convex}\}$ s.t.

(i) $f(+\infty) = U$

(ii) $f(\underline{\wedge} D) = \cap\{f(d) : d \in D\}$

for all filtered sets $D \subseteq (R^*)^n$.

<u>Proposition 5.4</u> : Suppose $\tilde{A} \in \mathcal{L}_L(U), L = [0,1]$ and $\sigma_\alpha(\tilde{A}) = \{u \in U : I(\tilde{A})(u) > \alpha\}$. Define

$$I_0(\sigma_\alpha(\tilde{A})) = \begin{cases} 1 \; ; \; u \in \sigma_\alpha(\tilde{A}) \\ 0 \; ; \; \text{otherwise.} \end{cases}$$

Then $\tilde{A} = \cup\{\alpha \cdot \sigma_\alpha(\tilde{A}) : \alpha \in [0,1]\}$ where $\alpha \cdot \sigma_\alpha(\tilde{A})$ is the fuzzy set given by

$$I(\alpha \cdot \sigma_\alpha(\tilde{A}))(u) = \alpha \cdot I_0(\sigma_\alpha(\tilde{A}))(u).$$

<u>Proof</u> : Reference [34] theorem 3. □

Since [0,1] and R* are homomorphic the same holds in R*. This gives a stronger indication of how the correspondence works. We can exploit this correspondence in a number of ways.

<u>Proposition 5.5</u> : Let f be ℓ.s.c. quasi-convex and $f : U \to R^n$, where $U \subseteq R^n$ is compact. Then $\exists \; f_\delta : U \to R^n$; ℓ.s.c. strictly quasi-convex s.t. $f_\delta \uparrow f$ point-wise as $\delta \to 0$.

<u>Proof</u> : First we note from Theorem 3.6 that if f_δ is quasi-convex and $\Gamma(b) = \{u \in U \; ; \; f_\delta(u) \leq b\}$ is ℓ.s.c. multi-valued, then f_δ is strictly quasi-convex.

From our preamble we know that there is a 1-1 correspondence between f and its b-cuts, namely $\Gamma(b)$ which satisfy

(i) $\Gamma(+\infty) = U$

(ii) $\Gamma(\underline{\wedge} D) = \cap\{\Gamma(d) : d \in D\}$
for any filtered set $D \subseteq R^n$.

Now if $\Gamma(b) < \Gamma_\delta(b) : \forall b$, then

$$f_\delta(\cdot) \leq f(\cdot)$$

Since f is ℓ.s.continuous quasi-convex and $\Gamma(b)$ is compact,then by Theorem 3.5 $\Gamma(\cdot)$ is u.s.c. at $b \in R^n$.

As U is bounded,we can assume the domain $U_1 \subseteq R^n$ of $\Gamma(\cdot)$ is compact and hence the range $U_2 \equiv \cup\{\Gamma(b) : b \in U_1\}$ is also compact.

Using Corollary 2.92 we can conclude that $\exists$ a Hausdorff continuous multi-valued mapping $\wedge_\delta(\cdot) : U_1 \to KV(U_2)$ approximating $\Gamma(b)$ from above, i.e.,

$$\cap_{\delta>0}\wedge_\delta(b) = \Gamma(b)$$

and also approximating $\Gamma(b)$ in graph.

Now

$$\wedge_\delta(b) \supseteq \Gamma(b), \ \forall\ b \in U_1$$

$$\Gamma(+\infty) = U \subseteq U_b \ \wedge_\delta(b) \subseteq \wedge_\delta(+\infty) \subseteq U\ ,$$

This implies $\wedge_\delta(+\infty) = U$. However we don't know whether (ii) holds.

Since $\wedge_\delta(\cdot)$ is continuous it is uniformly ℓ.s.continuous (see Theorem 1.13). Hence $\forall\ \varepsilon > 0$; $\exists\ \bar{\delta} > 0$ independent of $\hat{b}$ s.t.

$$\wedge_\delta(\hat{b}) \subseteq \bar{N}(\wedge_\delta(b),\varepsilon)$$

$$\forall\ b \in N(\hat{b},\bar{\delta}).$$

Let $\bar{b} \in R^n$ be arbitrary. By noting that this holds $\forall\ \hat{b} \geqslant \bar{b}$, we have

$$\cap\{\wedge_\delta(\hat{b}) : \hat{b} \geqslant \bar{b}\}$$

$$\subseteq \cap\{\bar{N}(\wedge_\delta(b),\varepsilon) : b \geqslant b'\}$$

for all $b' \in N(\bar{b},\bar{\delta})$.

So if we call $\Gamma_\delta(\bar{b}) = \cap\{\wedge_\delta(\hat{b});\ \hat{b} \geqslant \bar{b}\}$ we have

$$\Gamma_\delta(\bar{b}) \subseteq \cap\{\overline{N}(\wedge_\delta(\hat{b}),\varepsilon);\ \hat{b} \geqslant b'\}$$

$$= \overline{N}(\cap\{\wedge_\delta(\hat{b});\ \hat{b} \geqslant b'\},\varepsilon)$$

$$= \overline{N}(\Gamma_\delta(b'),\varepsilon)\ \forall\ b' \in N(\bar{b},\bar{\delta}).$$

Hence $b \to \Gamma_\delta(b)$ is lower semi continuous. Since $\wedge_\delta(b) \supseteq \Gamma(b);\ \forall\ \delta, \Gamma_\delta(\bar{b}) = \cap\{\wedge_\delta(b);\ b \geqslant \bar{b}\} \supseteq \cap\{\Gamma(b);\ b \geqslant b\} \equiv \Gamma(\bar{b})$.
Obviously for $b \geqslant \hat{b}, \Gamma_\delta(b) \supseteq \Gamma_\delta(\hat{b})$ and so (i) must be satisfied.
Trivially (ii) holds.
We have $\Gamma_\delta(\cdot)$ corresponding to a unique quasi convex function f_δ, say, which must be strictly quasi-convex due to Theorem 3.6.

As $\forall\ b \in R^n$

$$\Gamma(b) = \cap_{\delta>0}\wedge_\delta(b)$$

$$\supseteq \cap_{\delta>0}\Gamma_\delta(b) \supseteq \Gamma(b),$$

we know that

$$f_\delta \uparrow f \text{ as } \delta \to 0 \text{ pointwise.}$$ □

In our previous proof $\wedge_\delta(\cdot) : U_1 \to KV(U_2)$ approximates $\Gamma(b)$ from above and in graph, i.e.,

$$d^*(G_\delta,G) < \varepsilon$$

for δ sufficiently small, where G_δ is the graph of $\wedge_\delta(\cdot)$ and G is the graph of $\Gamma(\cdot)$.

We define

$$\Gamma_\delta(\bar{b}) = \cap\{\wedge_\delta(\hat{b}) : \hat{b} > \bar{b}\}$$

and so,

$$\Gamma_\delta(\bar{b}) \subseteq \Lambda_\delta(\bar{b});\ \forall\, b.$$

Since $G_\delta' \subseteq G_\delta$, where G_δ' is the graph of $\Gamma_\delta(\cdot)$ we have

$$d^*(G_\delta',G) \leqslant d^*(G_\delta,G) < \varepsilon;$$

for δ small.

In fact since

$$\Gamma_\delta(\bar{b}) \supseteq \Gamma(\bar{b});\ \forall\, \bar{b}, \text{ we have}$$

$$G_\delta' \supseteq G \text{ and } d^*(G,G_\delta') = 0.$$

That is

$$d(G_\delta',G) \equiv d^*(G_\delta',G) < \varepsilon;\ \text{ for } \delta \text{ small},$$

$$G_\delta' = \{(u_2,b);\ f_\delta(u_2) \leqslant b\},$$

$$G = \{(u_2,b);\ f(u_2) \leqslant b\}.$$

This sort of approximation is important in the theory of convex functions and recently has been used to rewrite the Stone Approximation theorem for the lattice of upper-semi-continuous function on a compact metric space (see reference [35]).

For an upper-semi-continuous function $g(\cdot)$, the hypo-graph of g is defined to be

$$\text{hypo } g = \{(u_2,\alpha):\ \alpha \leqslant g(u_2)\}.$$

For a ℓ.s.continuous function f we have

$$G = \{(u_2,b) : f(u_2) \leqslant b\}$$

$$= \{(u_2,b) : -b \leqslant -f(u_2)\}.$$

Hence

$d(G_\delta, G) < \varepsilon$

would imply $d(\text{hypo}(-f_\delta), \text{hypo}(-f)) < \varepsilon$

$d_3(-f_\delta, -f) < \varepsilon$, in the notation of reference [35].

The condition of the Stone theorem that a sublattice Ω of u.s. continuous functions "isolates points" actually characterises the sub-lattice which is "upper dense", i.e., for which each u.s.c. g is in the closure of $\{g' : g' \geqslant g \text{ and } g' \in \Omega\}$.

Theorem 5.1 : Let Ω be a lattice of u.s.c. functions on a compact metric space U_2 that isolates points [i.e. if (u_2, b), (u_2', b') are such that either $u_2 \neq u_2'$ or $u_2 = u_2$ and $b < b'$, there exists $\phi \in \Omega$ such that

$(u_2, b) \in \text{int hypo } \psi$

$(u_2', b') \notin \text{hypo } \psi$].

If g is u.s.c. then there exists $\{h_p\}$ in Ω convergent to f from above in the metric d_3.

Proof : See reference [35], theorem 1, page 8. □

Our Proposition 5.4 can be thought of as a kind of Stone approximation theorem. The general question of what characterises a lattice as being upper or lower dense in another lattice is the general subject at hand. Conversely, in what lattice would the class $\mathcal{L} = \{f : f : U_1 \to R$ continuous and $\text{cl } I(b) = \Gamma(b)$; $\forall\, b \in \text{int } B\}$ be a lower dense sub-lattice?

Due to Proposition 3.2 we actually only require point-wise convergence of $f_\delta \uparrow f$ to derive Corollary 3.9, namely that if

$$f(u_1,u_2) = \sup_{i \in I} f_i(u_1,u_2),$$

$f(u_1,\cdot)$ quasi-convex,

$f_i(u_1,\cdot)$ strictly quasi-convex and $f_i(\cdot,\cdot)$

continuous on the compact set $U_1 \times U_2$ then

$$d(G_m,G) < \varepsilon \text{ for m sufficiently large,}$$

where G_m is the graph of

$$T_m(u_1) = \{u_2 : \sup_{i=1,..,m} f_i(u_1,u_2) \leqslant b\} \text{ and}$$

G the graph of $\Gamma(u_1) = \{u_2 : f(u_1,u_2) \leqslant b\}$.

Convexity seems important in passing the graph approximation properties of $\Gamma(\bar{u}_1,\cdot)$, considered as a function of b, across to $\Gamma(\cdot,\bar{b})$ considered as a function of u_1.

Proposition 3.2 dealt with approximation of

$$\Gamma(\cdot) \in [U_1, \Sigma C \Phi_{ops}(U_2)]$$

where $L = C\Phi_{ops}(U_2)$ is the continuous lattice of complements of Φ-convex sets on a compact Hausdorff space. It is interesting to consider this problem in the case when the Φ-convex sets are closed under finite union. In this case $[U_1, \Sigma C \Phi_{op}(U_2)]$ can be considered to be a fuzzy topological space. It is always closed under arbitrary supremums and will be closed with respect to finite infimums in this case. This follows from Theorem 1.12 (i) and the fact that $[U_1, \Sigma C \Phi_{ops}(U_2)]$ will consist of i.s. continuous functions, the complements of u.s.continuous functions.

This condition will be fulfilled if Φ defines a fuzzy topology itself, in which case Φ will be closed under finite infimums. That is, given

$$f(\cdot) = \vee\{f_i(\cdot) \in \Phi;\ i \in I\},$$

$$g(\cdot) = \vee\{g_i(\cdot) \in \Phi;\ j \in J\},$$

we have that

$$f(\cdot)\wedge g(\cdot) = \vee\{f_i(\cdot)\wedge g_i(\cdot) \in \Phi;\ i \in I;\ j \in J\}$$

is Φ- convex.

Essentially Proposition 3.2 states that given an open fuzzy set $U(\cdot) \in [U_1, \Sigma\, C\, \Phi_{o\,ps}(U_2)]$ containing a closed fuzzy set $K(\cdot)$, there exists an open-closed fuzzy set $T(\cdot)$ s.t.

$$K(\cdot) \subseteq T(\cdot) \subseteq U(\cdot).$$

The set $U(\cdot)$ is i.s.continuous and as a consequence $K(\cdot)$ is upper-semi-continuous. The set $T(\cdot)$ is open-closed and hence $T(\cdot)$ considered as a multi-valued mapping is continuous. In the proposition $\forall\, u_1$

$$K^c(u_1),\ T(u_1),\ U(u_1) \in C\, \Phi_{o\,ps}(U_2).$$

Since a "closed" set is the complement of an "open" set for

$$\Gamma(\cdot) \in [U_1, \Sigma\, C\, \Phi_{ops}(U_2)],$$

we have

$$\Gamma^c(u_1) \in C\Phi(U_2).$$

If a closed fuzzy set $\wedge(\cdot)$ can be approximated from above by a countable intersection of open sets $\Gamma_i(\cdot)$; $i \in I$, then any finite intersection will be open since $\cap_{i=1}^{r} \Gamma_i(\cdot)$ will be open. This follows from Proposition 1.9. The topology defined by this lattice must be perfectly normal since any "closed" $C\,\Phi(U_2)$ set is the countable intersection of "open" $C\,\Phi_{ops}(U_2)$ sets.

The "fuzzy" topology defined by

$$[U_1, \Sigma C \Phi_{ops}(U_2)]$$

can be considered perfectly normal as well. Instead of treating the question of lower denseness of continuous multi-valued mappings,we conclude this chapter with a brief discussion of fuzzy normality.

This topic differs from the question of lower approximation in that going from a sup-complete lattice Φ to a fuzzy topology one doubts whether in general we can infer the existence of a generating class Φ s.t.

$$T(u_1) = \{u_2 : \psi(u_1,u_2) > a\}$$

is Hausdorff continuous. We know that $T(\cdot)$ is i.s.c. and hence a finite intersection is i.s.c., i.e.,

$$T_1(u_1) \cap T_2(u_1) = \{u_2 : \psi_1(u_1,u_2) \wedge \psi_2(u_1,u_2) > a\}$$

is the complement of a u.s.c. mapping

$$\{u_2 : \psi_1(u_1,u_2) \wedge \psi_2(u_1,u_2) \leqslant a\}.$$

However, we can't be sure that this mapping is ℓ.s.c..

Proposition 5.6 : Suppose Φ consists of functions $\psi : U_1 \times U_2 \to R$ continuously, U_2 is a compact subset of R^n and U_1 is metric. For $\psi \in \Phi$ let

$$I(\bar{b}) = \{u_2 : \psi(\bar{u}_1,u_2) < \bar{b}\} \neq \phi.$$

Define $\Gamma(\bar{b}) = \{u_2 : \psi(\bar{u}_1,u_2) \lesssim \bar{b}\}$.

Then cl $I(\bar{b}) = \Gamma(\bar{b})$

implies $T(u_1) = \{u_2 : \psi(u_1,u_2) \leqslant \bar{b}\}$ is ℓ.s.c. at $\bar{u}_1$.

Proof : First we show that $\{\psi(u_1,u_2) : u_2 \in U_2\}$ is an equi-continuous class of continuous mappings $u_1 \to \psi(u_1,u_2)$.

We define for a given $\varepsilon > 0$

$$\delta_\varepsilon(u_2) = \sup\{\delta > 0 : |\psi(u_1,u_2) - \psi(\bar{u}_1,u_2)| < \varepsilon \text{ whenever } d(u_1,\bar{u}_1) < \delta\}$$

and show $\delta_\varepsilon(u_2)$ is bounded away from zero on U_2. If we suppose not, then $\exists\ u_2^n \in U_2$ s.t. $\delta_\varepsilon(u_2^n) < \frac{1}{n}$ and since U_2 is compact there exists a convergent subsequence. After renumbering we can say $u_2^n \to \bar{u}_2 \in U_2$. For any $\varepsilon > 0$ and $u_2 \in U_2$ we have $\delta_\varepsilon(u_2) > 0$. We arrive at a contraction by showing

$$\delta_\varepsilon(u_2^n) > \delta > 0 \qquad \text{for } n \text{ large}$$

where

$$0 < \delta < \delta_{\varepsilon/4}(\bar{u}_1).$$

Now

$$\begin{aligned}
&|\psi(u_1,u_2^n) - \psi(\bar{u}_1,u_2^n)| \\
&\quad \leqslant |\psi(u_1,u_2^n) - \psi(u_1,\bar{u}_2)| \\
&\quad + |\psi(\bar{u}_1,\bar{u}_2) - \psi(\bar{u}_1,u_2^n)| \\
&\quad + |\psi(u_1,\bar{u}_2) - \psi(\bar{u}_1,\bar{u}_2)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

for n sufficiently large. Also $d(u_1,\bar{u}_1) < \delta$ where $0 < \delta < \delta_{\varepsilon/4}(\bar{u}_1)$.

Theorem 3.3(b) implies $\Gamma(b)$ is ℓ.s.c. at $\bar{b}$ and Theorem 3.4(b) implies $T(u_1)$ is ℓ.s.c. at $\bar{u}_1$ in the metric space

$$G(\bar{b},\bar{u}_1) = \{u_1 : \{u_2 : \psi(u_1,u_2) < \bar{b}\} \neq \phi$$

$$\sup\{|\psi(u_1,u_2) - \psi(\bar{u}_1,u_2)| : u_2 \in U_2\} < \infty\}$$

with the metric

$$d(u_1,\hat{u}_1) = \sup\{|\psi(u_1,u_2) - \psi(\hat{u}_1,u_2)| : u_2 \in U_2\}.$$

The ℓ.s.c. of $T(u_1)$ in the metric of U_1 follows from the fact that $\forall\ \delta > 0$,

$$d(u_1,\bar{u}_1) < \delta'$$

implies

$$\sup\{|\psi(u_1,u_2) - \psi(\bar{u}_1,u_2)|: u_2 \in U_2\} < \delta \quad \text{for } \delta' > 0 \text{ sufficiently small.}$$ □

As we have seen the concept of convexity is essential when attempting to ensure cl $I(\bar{b}) = \Gamma(\bar{b})$. One cannot be certain that

$$\text{cl}\{u_2 : \psi_1(u_1,u_2) \wedge \psi_2(u_1,u_2) < a\}$$

$$= \{u_2 : \psi_1(u_1,u_2) \wedge \psi_2(u_1,u_2) \leqslant a\}$$

even though

$$\text{cl } I_1(a) = \Gamma_1(a) \text{ and}$$
$$\text{cl } I_2(a) = \Gamma_2(a).$$

This is certainly not the case for strictly quasi-convex functions. It remains an unanswered question as to whether criteria for cl $I(b) = \Gamma(b)$ can be found which does not involve convexity (of the usual type) of the b-cuts.

§5.2 Fuzzy Normality

We note that if S and L are continuous lattices then $[S \to L]$ is a continuous lattice. Moreover, the functions which are elements of $[S \to L]$ are monotone. Bruce Hutton in reference [36] found it necessary to define a "fuzzy unit interval" in order to prove an equivalent statement of the Urysohn lemma. He defined it as follows.

Definition 5.4 : The fuzzy unit interval $[0,1]$ (L) is the set of all monotonically decreasing maps $\lambda : R \to L$ satisfying:

(1) $\lambda(t) = 1$ for $t < 0$; $t \in R$

(2) $\lambda(t) = 0$ for $t > 1$; $t \in R$

after the identification of $\lambda : R \to L$ and $u : R \to L$ if for every $t \in R$

$$\lambda(t-) = \inf\{\lambda(s);\ s < t\} = u(t-)$$

and

$$\lambda(t+) = \sup\{\lambda(s);\ s > t\} = u(t+).$$

We can define a slight variation of this.

Definition 5.5 : The right open fuzzy intervals $[0,1]_R(L)$ are the set of Scott continuous mappings in $[[0,1]_{ops} \to L]$ extended to R via (1) and (2)' $\lambda(t) = 0$; $t \geq 1$.

The continuous lattice $[0,1]_{ops}$ is the unit interval $[0,1]$ endowed with the reversed ordering. We identify $u(\cdot), \lambda(\cdot) \in [0,1]L$ if for all $t \in R$;

$$u(t) = \lambda(t).$$

Since $u(\cdot)$ and $\lambda(\cdot)$ are Scott continuous we have

$$\lambda(t) = \sup\{\lambda(S) : S \ll_{ops} t\}$$

and

$$u(t) = \sup\{u(S) : S \ll_{ops} t\}.$$

The interval $[0,1]$ being a chain means that $S \ll_{ops} t$ iff $S > t$ or $S = t = 1$. This in turn implies

$$\lambda(t) = \sup\{\lambda(S) : S > t\} = \lambda(t+) \quad \text{for } t \neq 1$$

and

$$\lambda(1) = \sup\{\lambda(S) : S \geqslant 1\} = 0.$$

This class is a subset of monotone mappings on $[0,1]$ consisting of those continuous from the right for all points in the interval $[0,1]$ and also satisfying (1) and (2)'. We identify $\lambda(\cdot)$ and $u(\cdot)$ via the criteria $\lambda(t+) = u(t+)(\equiv u(t))$, which is only one sided.

The Scott topology on $[0,1]_{ops}$ consists of the sets

$$\tau = \{[0,\alpha) \; ; \; \alpha \in [0,1) \text{ and } [0,1]\}$$

which is an ordinary topology of half open intervals on $[0,1]$. We may consider $([0,1], \tau)$ a topological space in which the open sets τ form a continuous lattice. From our discussion after Definition 1.18 we note that we can associate

$$[[0,1]_{ops} \to L] \equiv [[0,1], \Sigma L],$$

where $[0,1]$ is considered as a topological space endowed with the topology τ. From Theorem 1.11 we know that this is a continuous lattice itself as long as L is continuous. The lattice $[0,1]_R(L)$ is an associated lattice with the ordering induced by the pointwise order on L and as a consequence is also continuous.

The crisp intervals are embedded in the usual way by letting for $r \in [0,1]$

$$R(t) = 1;\ t \leqslant r$$

and

$$R(t) > 0;\ t \geqslant r.$$

We note that for any continuous lattice, $[0,1]_R(L)$ is obviously closed with respect to unions and is in fact closed with respect to finite intersections if L is also.

Proposition 5.7 : Suppose λ, $u \in [0,1]_R(L)$ then the pointwise (with respect to the ordering on L) infimum

$$\gamma(\cdot) = \lambda(\cdot) \wedge u(\cdot) \in [0,1]_R(L)$$

if L is closed with respect to finite infimums.

Proof : Since we always have

(1) $\lambda(t) = 1 = u(t)$ for $t < 0$

and

(2) $\lambda(t) = 0 = u(t)$ for $t > 1$

we also have

$$\lambda(t) \wedge u(t) = 1 \quad \text{for } t < 0$$

$$\lambda(t) \wedge u(t) = 0 \quad \text{for } t > 1.$$

It only remains to verify the Scott continuity in the interval $[0,1]$.

When $t = 1$,

$$0 = \lambda(1) \wedge u(1) = \gamma(1) = \sup\{\lambda(S) \wedge u(S);\ S \geqslant 1\}$$

so we only need to verify right continuity at points $t \in [0,1)$.

Given S, w > t, because of the monotonicity of λ, u we have

$$\lambda(\ell) \wedge u(\ell) \geqslant \lambda(S) \wedge u(w)$$

where

$$\ell = S \wedge w > t.$$

Hence

$$\sup\{\lambda(\ell) \wedge u(\ell); \ell > t\} \geqslant \lambda(S) \wedge u(w) \quad \text{for all } S, w > t$$

Since $\lim_{S \downarrow t} \lambda(S) = \lambda(t+) = \lambda(t)$, we have by, letting $S \downarrow t$ and then $w \downarrow t$, that

$$\sup\{\lambda(\ell) \wedge u(\ell) : \ell > t\} \geqslant \lambda(t) \wedge u(t).$$

Of course we always have $u(t) \geqslant u(\ell)$ and $\lambda(t) \geqslant u(\ell)$ for $\ell > t$ so that

$$\lambda(t) \wedge u(t) \geqslant u(\ell) \wedge \lambda(\ell); \text{ for } \ell > t,$$

that is,

$$\lambda(t) \wedge u(t) \geqslant \sup\{u(\ell) \wedge \lambda(\ell) : \ell > t\}.$$

Thus for $t \in [0,1)$ we have

$$\gamma(t) = \lambda(t) \wedge u(t) = \sup\{u(\ell) \wedge \lambda(\ell) : \ell > t\},$$

implying $\gamma(\cdot)$ is right continuous. □

We now consider the situation when $L = C\,\Phi_{ops}(U_2)$, where the Φ-convex sets are compact.

Proposition 5.8 : Suppose $L = C\,\Phi_{ops}(U_2)$ forms a continuous lattice of open sets in the Euclidean topology of $U_2 \subseteq \mathbb{R}^n$, a Φ-convex set, where the Φ-convex sets are compact in the Euclidean topology.

Then

$$\lambda : [0,1) \quad C\,\Phi_{\alpha ps}\,(U_2),$$

considered as a multi-valued mapping into U_2 endowed with the metric Euclidean topology is i.s.c. Moreover, there is a function

$$f \in \mathcal{L}' = \{f : U_2 \to [0,1] \text{ s.t. } \forall\, \alpha \in [0,1]$$

$$f^{-1}(\hat{\uparrow}\alpha) \in C\,\Phi_{ops}\,(U_2)\}$$

such that

$$\lambda(\alpha) = \{u_2 : f(u_2) > \alpha\} \in C_{ops}\,(U_2).$$

Proof :

If λ is right continuous, monotonically decreasing, then λ^c is monotonically increasing and right continuous, i.e.,

$$\lambda^c(t^+) = \lim_{s \downarrow t} \lambda^c(s) = \cap\{\lambda^c(s) : s > t\}$$

$$= [\cup\{\lambda(s) : s > t\}]^c$$

$$= \lambda^c(t).$$

If we can show $\lambda^c(\cdot)$ is u.s. continuous then we have the i.s. continuity of $\lambda(\cdot)$. For any given $\varepsilon > 0$ there exists a $h > 0$ s.t. $t \leqslant s < t + h$ implies

$$\lambda^c(s) \subseteq N(\lambda^c(t),\varepsilon).$$

This follows from the right continuity of $\lambda^c(\cdot)$ and the fact that $\lambda^c(\cdot)$ is a closed set in the Euclidean topology.

Now suppose $s \in (t-h,t+h)$, and define

$$s' = \begin{cases} s & : s \geqslant t \\ 2t-s & : s < t \end{cases}.$$

Note that $t \leqslant s' < t + h$ and hence $\lambda^c(s') \subseteq N(\lambda^c(t),\varepsilon)$. However, since $s \leqslant s'$, we have

$$\lambda^c(s) \leqslant \lambda^c(s') \subseteq N(\lambda^c(t),\varepsilon).$$

Finally condition (F) of Proposition 5.2 is satisfied by [0,1] since $[0,1]_{ops}$ and [0,1] are both continuous lattices and for $\beta \neq 0$, $\alpha \gg \beta$ is equivalent to $\alpha > \beta$ in [0,1], namely $\alpha <_{ops} \beta$ in $[0,1]_{ops}$. Condition (E) is satisfied, since the $C\,\Phi_{ops}(U_2)$ sets are open in the Euclidean (and hence Hausdorff) topology and also form a continuous lattice.

The lattice $[[0,1]_{ops} \to C\Phi_{ops}(U_2)]_0$ is just the lattice of functions $\lambda(\cdot) \in [0,1]_R(L)$ restricted to [0,1]. Proposition 5.2 is applicable and we conclude that the above lattice is equivalent to

$$\mathcal{L}' = \{f(\cdot) : U_2 \to [0,1];\ f^{-1}(\hat{\uparrow}\alpha) \in C\,\Phi_{ops}(U_2) \text{ for all } \alpha \in [0,1]\}.$$

For $\alpha \in [0,1]$ we have

$$\Phi(f) = \lambda(\alpha) = \{u_2 : f(u_2) > \alpha\} \in C\,\Phi_{ops}(U_2), \qquad \square$$

since $\ll$ is equivalent to $<$.

Proposition 5.8 does not assume that $L = C\,\Phi_{ops}(U_2)$ is closed with respect finite infimums. However, in the case when $C\,\Phi_{ops}(U_2)$ forms a topology, by Proposition 5.1 we have

$$f \in [U_2,\ \Sigma[0,1]].$$

Definition 5.6 :

Suppose

$$\Gamma(\cdot) : U_1 \to \mathcal{P}(U_2) \text{ is multi-valued mapping.}$$

We define the interior to be

$$\Gamma^0(\cdot) = \cup\{\Lambda(\cdot) : \Lambda(\cdot) \subseteq \Gamma(\cdot) \text{ and } \Lambda(\cdot) \in [U_1,\ \Sigma C\,\Phi_{ops}(U_2)]\},$$

and the closure to be

$$\overline{\Gamma}(\cdot) = \cap\{\Lambda(\cdot) : \Lambda(\cdot) \supseteq \Gamma(\cdot) \text{ and } \Lambda^c(\cdot) \in [U_1, \Sigma C\, \Phi_{ops}(U_2)]\}.$$

Obviously

$$\Gamma^0(\cdot) \subseteq \Gamma(\cdot) \subseteq \overline{\Gamma}(\cdot) \text{ and}$$

$u_1 \to \Gamma^0(u_1)$ is Scott continuous (Proposition 1.10) implying

$$\Gamma^0(\cdot) \in [U_1, \Sigma\, C\, \Phi_{ops}(U_2)],$$

the lattice being sup complete. Similarly,

$$\overline{\Gamma}(\cdot) \equiv [(\Gamma^c(\cdot))^0]^c$$

is upper-semi-continuous whenever the Scott continuous functions are inner-semi-continuous. We note also in passing that Proposition 1.9 implies that

$$[\Gamma_1(\cdot) \cap \Gamma_2(\cdot)]^0$$

$$= \cup\{\Lambda(\cdot) \in [U_1, \Sigma\, C\, \Phi_{ops}(U_2)] : \Lambda(\cdot) \subseteq \Gamma_1(\cdot) \cap \Gamma_2(\cdot)\}$$

$$= \cup\{\Lambda_1(\cdot) \cap \Lambda_2(\cdot) \in [U_1, \Sigma\, C\, \Phi_{ops}(U_2)] :$$

$$\Lambda_1(\cdot) \subseteq \Gamma_1(\cdot)$$

$$\Lambda_2(\cdot) \subseteq \Gamma_2(\cdot)\}$$

$$= \Gamma_1^0(\cdot) \cap \Gamma_2^0(\cdot)$$

Recall that a normal space is one such that for every closed set $K(\cdot)$ contained in an open set $M(\cdot)$ there exists a set $V(\cdot)$ s.t. $K \subseteq V^0 \subseteq \bar{V} \subseteq M$. In reference [36] Bruce Hutton proves the following:

<u>Theorem 5.2</u> :

A fuzzy topological space is perfectly normal if and only if it is normal and every closed set is a countable interection of open sets.

Proof :

Reference [36] Theorem 2. □

In our situation we have a fuzzy topology

$$[U_1, \Sigma C \Phi_{ops}(U_2)] \subseteq \{\Gamma(\cdot) : U_1 \to P(U_2)\}.$$

We are interested in the situation when it is a perfectly normal fuzzy topology and hence any i.s.c. mapping in this topology is the intersection of a countable collection of upper-semi-continuous Φ-convex imaged set valued mappings.

We can define a fuzzy topology on $[0,1]_R(L)$ as follows.

Let $L_t(\lambda) = \lambda_t(t^-)$ and $R_t(\lambda) = \lambda(t)$ and take a sub-base $\{R_t , L_t : t \in R\}$ to generate a topology $\mathcal{L}_L$ on $[0,1]_R(L)$. For $W \in \mathcal{L}_L$ we have $W : [0,1]_R(L) \to P(U_2)$.

Definition 5.7 :

If (X,τ_1) and (Y,τ_2) are fuzzy topological spaces then a mapping $f : X \to Y$ is said to be continuous if for every τ_2 open set W

$$f^{-1}(W)(\cdot) = W(f(\cdot)) \in \tau_1.$$

We note that both the sub-bases are fuzzy topologies on $[0,1]_R(L)$. Take $\{R_t : t \in R\}$. We note that for $\lambda \in [0,1]_R(L)$ and $\delta < 0$ we have $\lambda(t) \wedge \lambda(t+\delta) = \lambda(t)$ because $\lambda(t+\delta) \geqslant \lambda(t)$.

If we take

$$R_p , R_\ell \in \{R_t : t \in R\}$$

and suppose $p < \ell$ i.e., $p - \ell < 0$, then

$$R_\ell(\lambda) \wedge R_p(\lambda) = (R_\ell \wedge R_p)(\lambda)$$

$$= \lambda(\ell) \wedge \lambda(\ell + (p-\ell))$$

$$= \lambda(\ell) \in [0,1]_R(L),$$

that is, $R_\ell \wedge R_p \in \{R_t : t \in R\}$. Finally if $T \subseteq R$ we can define

$$S = \{\ell : \ell > t;\ t \in T\}$$

$$= U_T\{\ell : \ell > t\}$$

$$= \{\ell : \ell > \wedge\ T\}.$$

Since $\lambda(\cdot)$ is right continuous we have

$$V_T\lambda(t) = V_T\{\lambda(\ell) : \ell > t\}$$

$$= V_S\lambda(\ell)$$

$$= V\{\lambda(\ell) : \ell > \wedge T\}$$

$$= \lambda(\wedge T).$$

This in turn implies

$$V_{t\in T}R_t(\lambda) = V_{t\in T}\lambda(t) = \lambda(\wedge T)$$

and

$$V_{t\in T}R_t(\cdot) \in \{R_t : t \in R\}.$$

A similar argument using the left continuity of $L_t(\lambda) = \lambda^c(t^-)$ establishes that $\{L_t(\cdot) : t \in R\}$ is a fuzzy topology on $[0,1]_R(L)$.

Proposition 5.9 :

Suppose $U_2 \subseteq R^n$, $f : U_2 \to [0,1]$ is lower semi-continuous and that $\lambda^c(t) = \{u_2 : f(u_2) \leq t\}$ is compact valued. Then the lower semi-continuity of $\lambda^c(\cdot)$ at t, as a multi-valued mapping, implies, in the case when $\{u_2 : f(u_2) < t\} \neq \phi$, that $cl\{u_2 : f(u_2) < \hat{t}\} = \lambda^c(\hat{t})$.

Proof:

This follows via a direct adaptation of the second part of Theorem 2 of reference [13]. The multi-valued mapping $\lambda^c(\cdot)$ is closed valued due to the lower semi-continuity of $f(\cdot)$ and hence $cl\{u_2 : f(u_2) < t\} \subseteq \lambda^c(t)$. Since $\lambda^c(t)$ is compact valued all definitions of semi-continuity coincide (see comment after Theorem 1.8) and we may treat $\lambda^c(t)$ as being l.-H-semi-continuous. If $\hat{u}_2 \in \lambda^c(\hat{t})$ then either

$$\hat{u}_2 \in I(\tilde{t}) = \{u_2 : f(u_2) < \hat{t}\}, \text{ implying } \hat{u}_2 \in cl\ I(\hat{t}),$$

or

$$f(\hat{u}_2) = \hat{t}.$$

Suppose $\hat{u}_2 \notin I(\hat{t})$ and select $\varepsilon > 0$. Since $I(\hat{t}) \neq \phi$ then for n sufficiently large

$$\Gamma(\hat{t} - \frac{1}{n}) \neq \phi.$$

The l.s. continuity of $\lambda^c(t)$ at $\hat{t}$ implies

$$\lambda^c(t) \subseteq N(\lambda^c(\hat{t} - \frac{1}{n}), \varepsilon) \text{ for n large.}$$

This means that $\exists\ \bar{u}_2 \in \lambda^c(\hat{t} - \frac{1}{n})$ such that $\hat{u}_2 \in N(\bar{u}_2,\varepsilon)$, that is $\bar{u}_2 \in N(\hat{u}_2,\varepsilon)$. Thus in every neighbourhood of $\hat{u}_2$, there is a $\bar{u}_2 \in I(\hat{t})$ which implies $\hat{u}_2 \in cl\ I(\hat{t})$. □

We now argue in a similar fashion to Bruce Hutton in reference [36].

Proposition 5.10:

Suppose U_1 is a topological space and $U_2 \subseteq R^n$ a Φ-convex set. Suppose also that the Φ-convex sets are compact and that $C\Phi_{ops}(U_2)$ forms a topology coarser than the Euclidean topology on R^n. We consider U_1 to be a fuzzy topological space with the topology $\mathcal{L}' = [U_1, \Sigma C\Phi_{ops}(U_2)]$ (and $[0,1]_R(L)$ a fuzzy topological space endowed with $\mathcal{L}_L$).

Then the fuzzy topology $\mathcal{L}'$ is normal iff for every closed set $K(\cdot)$ and open set $M(\cdot)$ such that $K \subseteq M$ there is a fuzzy continuous function $h : U_1 \to [0,1]_R(L)$ such that for every $u_1 \in U_1$,

$$K(u_1) \subseteq h(u_1)(1-) \subseteq h(u_1)(0+) \subseteq M(u_1).$$

Furthermore for any fuzzy continuous function $h(\cdot)$ satisfying the above we have the existence of

$$f(u_1)(\cdot) \in [U_2, \Sigma[0,1]] \text{ s.t.}$$

$$h(u_1)(t) = \{u_2 \in U_2 : f(u_1)(u_2) > t\}$$

where

$$h^c(u_1)(t) = \{u_2 : f(u_1)(u_2) \leqslant t\}$$

is a continuous multi-valued mapping at each u_1 s.t. $h^c(u_1)(t-) \neq \phi$.

Proof:

Suppose we have a continuous $h : U_1 \to [0,1]_R(L)$. By proposition 5.8 for each $u_1 \in U_1$ there must exist a function

$$f(u_1)(\cdot) \in [U_2, \Sigma[0,1]] \text{ s.t.}$$

$$h(u_1)(t) = \{u_2 : f(u_1)(u_2) > t\}.$$

We also note that $t \to h(u_1)(t)$ is i.s.c. for $t \in [0,1)$ and that $h^c(u_1)(t)$ is compact in the Euclidean topology for $t \in [0,1]$. Since the topology generated by $C\,\Phi_{ops}(U_2)$ is coarser than the

Euclidean topology the compactness of $h^c(u_1)(t)$ in Euclidean topology implies the compactness in the topology $C\ \Phi_{ops}(U_2)$. Similarly, since $f \in [U_2, \Sigma\ [0,1]]$ implies l.s.c. from U_2 (endowed with the topology $C\ \Phi_{ops}(U_2)$) to $[0,1]$, we must also have l.s.c. with respect to the Euclidean topology.

Now if

$$K(u_1) \subseteq h(u_1)(1-) \subseteq h(u_1)(0+) \subseteq M(u_1),$$

we have for any $t \in (0,1)$ that

$$K(u_1) \subseteq h(u_1)(t) \subseteq h(u_1)(t-) \subseteq M(u_1).$$

Now

$$h^{-1}(L_t^c)(u_1) = L_t^c(h(u_1)(\cdot))$$

$$= h(u_1)(t-)$$

and

$$h^{-1}(R_t)(u_1) = R_t(h(u_1)(\cdot))$$

$$= h(u_1)(t).$$

Since f is continuous we have $f^{-1}(L_t^c)$ is closed and hence is the complement of an inner semi-continuous mapping, that is, it is upper-semi-continuous. Similarly, $f^{-1}(R_t)$ is open and hence inner-semi-continuous (implying l.s.c.). Now

$$h(u_1)(t-) = \cap_{s<t}\{u_2 : f(u_1)(u_2) > s\}$$

$$= \{u_2 : f(u_1)(u_2) \geqslant t\}$$

is upper-semi-continuous, implying that $h^c(u_1)(t-)$ is i.s.continuous. All the conditions of Proposition 5.9 are satisfied and hence

$$\mathrm{cl}\{u_2 \in U_2 : f(u_1)(u_2) < t\} = h^c(u_1)(t)$$

whenever

$$\{u_2 : f(u_1)(u_2) < t\} = h^c(u_1)(t-) \neq \phi.$$

Due to Theorem 1.10(i) we can deduce the l.s.continuity of $\mathrm{cl}\, h^c(u_1)(t-) = h^c(u_1)(t)$. By construction $h^c(u_1)(t)$ is always u.s.continuous and hence is continuous in this case.

In any case we have

$$K(u_1) \subseteq h^{-1}(R_t)(u_1) \subseteq h^{-1}(L^c_t)(u_1) \subseteq M(u_1),$$

implying $[U_2, \Sigma\, C\, \Phi_{ops}(U_2)]$ is a normal topology.

Let us now suppose $[U_1, \Sigma C\, \Phi_{ops}(U_2)]$ is normal. This allows us to contruct $\{V_r : r \in (0,1)\}$ such that

$$K(\cdot) \subseteq V_r(\cdot) \subseteq M(\cdot),$$

where for $r < s, \overline{V}_s \subseteq V^0_r$ we define

$$f(u_1)(t) = \bigcup_{r>t} V^0_r(u_1).$$

By Proposition 1.10 we know that $u_1 \to f(u_1)(t)$ is Scott continuous and hence

$$f^{-1}(R_t)(\cdot) = \bigcup_{s>t} f(u_1)(s)$$

$$= \bigcup_{s>t} \bigcup_{r>s} V^0_r(u_1)$$

$$= \bigcup_{r>t} V^0_r(u_1) = f(u_1)(t).$$

Now for $s > r$

$$V^0_s(\cdot) \subseteq \overline{V}_s(\cdot) \subseteq V^0_r(\cdot) \subseteq \overline{V}_r(\cdot)\ ,$$

implying

$$\bigcap_{r<t} \bigcup_{s>r} V^0_s(\cdot) \subseteq \bigcap_{r<t} \overline{V}_r(\cdot).$$

For $r < s < t$ there must exist ℓ s.t. $r < s < \ell < t$. Hence

$$V_s^0(\cdot) \supseteq \overline{V}_\ell(\cdot)$$

implies

$$\bigcup_{s>r} V_s^0(\cdot) \supseteq \overline{V}_\ell(\cdot)$$

for $r < \ell < t$.

This in turn shows that

$$\bigcap_{r<t} \bigcup_{s>r} V_s^0(\cdot) \supseteq \bigcap_{\ell<t} \overline{V}_\ell(\cdot)$$

and hence that

$$f^{-1}(L_t^c)(\cdot) = \bigcap_{r<t} f(\cdot)(r)$$

$$= \bigcap_{r<t} \bigcup_{s>r} V_s^0(\cdot) = \bigcap_{r<t} \overline{V}_r(\cdot).$$

Since $u_1 \to \overline{V}_r(u_1)$ is u.s. continuous and has Φ-convex images (i.e. closed and compact) by Theorem 1.12 (iii) we know that $f^{-1}(L_t^c)(\cdot)$ is an u.s.continuous multi-valued mapping.

Clearly $K(u_1) \subseteq f(u_1)(1-) \subseteq f(u_1)(0+) \subseteq M(u_1)$, where $f^{-1}(R_t)(\cdot)$ is open (i.s.c.) and $f^{-1}(L_t^c)(\cdot)$ is closed (u.s.c.), implying f is continuous. □

For more material on this sort of theorem one should consult reference [37].

This shows the intimate connection between the properties of the topology $C\,\Phi_{ops}(U_2)$ and the ability to approximate with continuous mappings. This does not of course imply the existence of a fixed point since, except for when $n = 1$, one is not assured that the approximating continuous function admits a fixed point.

The situation of perfect normality is of interest since this implies that we can approximate from above u.s.c. multi-valued mappings with i.s.c. multi-valued mappings. This in turn under reasonable circumstances, would imply that we can approximate with continuous multi-valued mappings. That is, under the conditions of Proposition 5.10 the normality of $[U_1, \Sigma C\, \Phi_{ops}(U_2)]$ implies the following. If $K(\cdot)$ is closed valued (i.e. Φ-convex) and u.s.c., $M(\cdot) \in [U_1, \Sigma C\, \Phi_{ops}(U_2)]$ and $M^c(\cdot) \subseteq K^c(\cdot)$ there must exist a continuous mapping $h(\cdot)(t)$ for $t \in (0,1)$ such that

$$M^c(u_1) \subseteq h(u_1)(t) \subseteq K^c(u_1);\ \forall\ u_1 \in U_1.$$

If we suppose $K(u_1) \neq \phi$ for all u_1, then

$$h^c(u_1)(t-) \supseteq K(u_1) \neq \phi$$

for any $u_1 \in U_1$ and $t \in (0,1)$. This in turn means $u_1 \to h^c(u_1)(t)$ is a continuous multi-valued mapping for any $t \in (0,1)$ and

$$M(\cdot) \subseteq h^c(\cdot)(t) \subseteq K(\cdot)$$

for any $t \in (0,1)$.

Since we can squeeze a continuous mapping between any u.s.c. mapping contained in an i.s.c. mapping, the ability to approximate by i.s.c. mappings can be duplicated by continuous multi-valued mappings. In the case when Theorem 2.7 is applicable, the ability to approximate $K(\cdot)$ in graph by a l.s.c. multi-valued mapping $K_\varepsilon(\cdot)$ can be mirrored by an i.s.c. multi-valued mapping with open image sets, namely $N(K_\varepsilon(\cdot),\varepsilon)$. The i.s.c. of $N(K_\varepsilon(\cdot),\varepsilon)$ follows from Proposition 1.11.

Arguments along these lines indicate that perfect normality of the fuzzy topological space $[U_1, \Sigma C\Phi_{ops}(U_2)]$ is closely related to our ability to approximate u.s.c. mappings by continuous multi-valued mappings. To deduce the existence of a fixed point we then have to impose some sort of more stringent convexity concept to allow selectivity of the image sets.

CONCLUSION

This thesis represents a preliminary enquiry into the extent to which the concepts of generalized convexity and continuous lattice theory help to unify seemingly unrelated areas of mathematics, under a common theme. To what extent these concepts facilitate such an approach remains unclear, but what this thesis does show is that the properties of "classical" convexity are quite consistent with this approach. Conversely, many questions are generated by the text and demand further investigation. We do show though, that upper semi-continuous, closed and convex imaged, multi-functions behave particularly well.

Under fairly general conditions we can approximate any such multi-function from above and in graph by a continuous, convex imaged multi-function. As was indicated in chapter five, this ability to approximate, in graph, is closely related to the approximation properties of the possible classes of functions, which generate such multi-functions. The quasi-convex functions $f(\cdot)$ are able to generate upper semi-continuous multi-functions, via

$$\Gamma(b) = \{u_2 : f(u_2) \leqslant b\}.$$

In a similar fashion the strictly quasi-convex functions generate continuous multi-functions. The abovementioned ability to approximate multi-functions, in graph, is equivalent to an ability to approximate quasi-concave function, by strictly quasi-concave function, in hypo graph. Conversely, the ability to write any quasi-convex function as the pointwise supremum of a class of strictly quasi-convex functions, implies in very general circumstances, a graph approximation of the corresponding multi-functions generated. In fact if

(i) $f(u_1,u_2) = \sup\{f_i(u_1,u_2) : i \in I\}$

(ii) $f_i(u_1,\cdot)$ strictly quasi-convex, and

(iii) $f_i(\cdot,\cdot)$ continuous on the compact set $U_1 \times U_2$ then

$$d(G_m,G) < \varepsilon$$

where G_m is the graph of

$$T_m(u_1) = \{u_2 : \sup_{i=1,..,m} f_i(u_1,u_2) \leqslant b\}$$

and G is the graph of

$$\Gamma(u_1,b) = \{u_2 : f(u_1,u_2) \leqslant b\}.$$

We obtain in this fashion a graph approximation from a simple point-wise limit. The graph approximation ability of $\Gamma(\bar{u}_1,\cdot)$, considered as a function of b, is carried across to $\Gamma(\cdot,\bar{b})$, considered as a function of u_1. As was demonstrated in chapter four, the continuity properties of $\Gamma(u_1,\cdot)$ are closely related to the continuity properties of $\Gamma(\cdot,\bar{b})$. The classes of functions for which such correspondences exist are of importance. Since, the fixed point problem is, at least in part, related to the ability to approximate multi-functions in graph, the generalized convexity concept which facilitates such a correspondence,as stated above, are of interest.

In this way the Kukutani fixed point theorem and a slightly weaker version can be viewed as a consequence of the selectivity of convex sets. This approach reduces the problem of finding a fixed point of a multi-function, to that of finding a fixed point of a single valued mapping. It also forms a bridge between the area of fixed point theory and the area of non-linear optimization. The degree to which this connection can be used to derive new fixed point theorems is unclear (specifically those involving non-convex image sets) but it seems quite likely that in time, techniques for finding solutions to such problems, in applied contexts, could be wrought using ideas from this area. In particular, the areas of generalized derivatives and generalized Lagrangians could play an important part in this pursuit. The connection between the generalized derivative of the marginal mapping and the solutions to the dual problem of our particular augmented Lagrangian, may prove useful in developing algorithms.

As was indicated in chapter five, we may be able to "pointwise" approximate an upper semi-continuous multi-function with a continuous multi-function, in very general circumstances. The conceptual clarity of formulating this problem in terms of fuzzy set theory indicates the virtue of the approach. Both fuzzy set theory and continuous lattice theory, could provide a framework for a recasting of part of the theory of multi-valued mappings. Both formulations could be more "intuitive" and help gain insights into various anomalies in this area.

A number of questions arise from this work and remain unanswered. I iterate a number of these for the interest of the reader. Does the concept of "way below" as defined by

$$A \gg B \text{ iff } A \supseteq N(B,\varepsilon)$$

for some $\varepsilon > 0$, as compared with the lattice theoretic definition of the usual concept of way below, help compare the concepts of upper Hausdorf semi-continuity and upper semi-continuity? How do the rates of local-uniform upper/lower semi-continuity and δ-upper Hausdorf semi-continuity (at points in the graph of $\Gamma(\cdot)$) compare? Under what conditions do we have the local, uniform, upper semi-continuity of a multi-function, at a uniform rate $q(\cdot)$ which has a continuous inverse? Do we have the lower semi-continuity of the optimal solution set mapping, $b \to \alpha(b,0)$ at a point $\bar{b}$, when the Slater condition holds and $f(\cdot)$, the function being maximized, is convex or strictly convex? Is $b \to \alpha(b,0)$ linearly lower semi-continuous if $g(\cdot)$ satisfies the Slater condition and $-f(\cdot)$ is convex or linear? Do we ever have local linear continuity of $b \to \alpha(b,0)$ in a non-linear context?

Could we use the techniques of non-linear optimization to derive fixed point theorems (even in R^m) which do not rely on the convexity of image sets of multi-valued mappings? In passing we speculate whether there are convexity generating classes Φ, defined on a topological space U, for which one could demonstrate some sort of reflexivity of the space U? In this context this property could determine the topological nature of the space on which a particular generalized convex imaged u.s.c. multi-function might behave well. One wonders whether a stronger connection between Hahn-Banach type theorems and generalized convexity could be wrought.

There are many possible connections between generalized convexity and continuous lattice theory. At the least, continuous lattice theory could provide a very useful tool in the development of the area of generalized convexity. Conversely does the area of fuzzy topology bear any relationship to the area of continuous lattice theory? Could this be useful in determining when $\mathcal{L}' = [U_1, \Sigma\, C\Phi_{ops}(U_2)]$ is normal or perfectly normal? Does the stability of a class of multi-functions imply the continuity of the lattice of functions, generating such multi-functions? In what lattice would

$$\mathcal{L} = \{f : f : U \to R \text{ continuous and } \mathrm{cl}\ I(b) = \Gamma(b) \text{ for } b \in \mathrm{int}\ B\},$$

where

$$I(b) = \{u : f(u) < b\} \text{ and}$$

$$\Gamma(b) = \{u : f(u) \leq b\},$$

be a lower dense set?

Many questions remain unanswered which arise from the work in chapter four. Do there exist non-differentiable constraint qualifications which imply local Lipschitzness of the marginal mapping? Could one show that the marginal mapping

$$M(\bar{u}_1,\bar{b}) = \sup\{f(\bar{u}_1,u_2) : u_2 \in \Gamma(\bar{u}_1,\bar{b})\},$$

where

$$\Gamma(\bar{u}_1,\bar{b}) = \{u_2 : g(\bar{u}_1,u_2) \leq \bar{b}\},$$

has a Clark derivative? If so is it the case that,

$$\partial_1 M(\bar{u}_1,\bar{b}) = \text{co}\{\nabla_1 f(\bar{u}_1,\bar{u}_2) + y'\nabla_1 g(\bar{u}_1,\bar{u}_2) : \bar{u}_2 \in \alpha(\bar{u}_1) \text{ and } y' \in \partial_2 M(\bar{u}_1,\bar{b})\}$$

where

$$\alpha(\bar{u}_1) = \{\bar{u}_2 \in \Gamma(\bar{u}_1,\bar{b}) : f(\bar{u}_1,\bar{u}_2) \geq M(\bar{u}_1,\bar{b})\}?$$

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