EXACT STATIC-MODEL S-MATRIX SOLUTIONS

AND THE BOOTSTRAP

by

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SUMMARY

This work is, for the most part, concerned with the analysis of a class of static S-matrix models, defined by the requirements of static two-body unitarity, analyticity and crossing symmetry. The crossing-matrices of the models are to be parametrised in as general a way as possible, consistent with general requirements for a crossing-matrix, namely that the matrix be real, its rows sum to unity, and for involutory crossing relations, be involutory.

Using the Rothleitner transformation of the energy plane, the general solution is constructed for a two-channel, non-involutory system, for all values of the matrix parameters. It is then sought to impose a bootstrap condition, together with certain other general 'physical' requirements, upon the solutions corresponding to a general, involutory two-channel matrix. The bootstrap criterion is that of Levinson's Theorem. The nature of this condition as a generator of bootstrap mechanisms, and its relation to other proposed criteria is discussed; in particular, we give a general proof, subject to certain asymptotic conditions, of its equivalence to the N/D bootstrap prescription.

It is shown that the conditions to be imposed on the general, involutory, two-channel solutions, are incompatible unless the crossing-matrix is restricted to a subclass of SU2 crossing-matrices.

For solutions corresponding to this subclass, which may then be interpreted as describing the static scattering of a particle of integer isospin t by an isofermion, the imposition of the bootstrap criterion is found to greatly reduce the arbitrariness of the general solution, and possible bound state distributions. In particular, it is proved that there exists no bootstrap amplitude satisfying an unsubtracted dispersion relation, or without a cut-off function. For the case of S- and P-wave scattering, we construct explicit bootstrap solutions, showing the derived necessary bootstrap conditions to be also sufficient. A calculation is made, showing the numerical dependence of the coupling constants and bound state energies upon the remaining arbitrary parameters, and on t.

We attempt to extend these considerations to higher dimensional models. A method is given of constructing general parametrised crossing matrices of any dimension, and of functions verifying crossing relations defined by these matrices. The imposition of unitarity upon these functions for three-channel models, of which there are two classes according to the sign of the trace of the crossing-matrix, is again effected by means of a transformed energy plane. For one class, we are able to construct exact solutions for all values of the parameters, while for the other, we are able to construct explicit, exact solutions for only a subclass of matrices, which contains again as a proper subclass, 'generalised' SU2 crossing-matrices.

The 'Inverse Scattering Problem' of constructing an equivalent potential for the models, is discussed by means of Verde's relativistic generalisation of the Ge'lfand-Levitan equations. We are unable to solve the integral equations exactly, but present a method of approximate solution. The resulting approximate potentials are found to be of no value for a discussion of the bootstrap restrictions.

STATEMENT

This thesis contains no material which has been accepted for award of any other degree or diploma in any University and that, to the best of my knowledge or belief, the thesis contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

A.A. Cunningham

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CHAPTER ONE

INTRODUCTION

This work is concerned with the solutions and properties of an exactly soluble model, namely the generalised static Chew-Low model, which we will define and discuss shortly. The study of such a model, although perhaps not justified as providing an accurate 'picture' of some physical aspect of nature, may still be very rewarding. The model incorporates many of the accepted required attributes of a more realistic model, while its departure from a truly physical system is reasonably well defined and understood; its study may hence give us insight into a more exact theory. The model is likely to be especially fruitful, as in several important situations, it permits an exact, general solution. It may then serve as a testing ground for approximation methods, and for such concepts as the bootstrap hypothesis, while the method of exact solution may suggest improved computational procedures with which to attack a more realistic model. This latter possibility is especially important since theories, for the most part, have outgrown our computational armoury.

The static models are the simplest models of strong interaction processes, and are formulated with the assumption that a heavy particle, interacting with a light particle, is fixed and does not recoil. They have the feature, greatly simplifying their analysis, that scattering occurs in only a few partial waves.[1] The Chew-Low static model is based upon the Low Equation[2], and as originally set up, pertained to the description of low energy π- N scattering. It was assumed that the effects of nucleon recoil, anti-nucleon effects, direct meson-meson interactions and kaon and hyperon effects are all small, and account of them can, in some approximate sense, be included in a cut-off function. The model enjoys considerable experimental success, predicting with good agreement the (3,3) resonance and low energy phase shift, and with straightforward extensions,[3),4)] photomeson production. A discussion and interpretation of this success, in terms of the dominance of the (3,3) resonance, has been given by Low.[5] It is also of interest to note, that the model as described by the Chew-Low dispersion relations, may be rederived from alternative standpoints. For example, Remler[6] has given a general procedure for constructing 'equivalent' problems in a reduced Hilbert space, whenever an interacting system possesses an Abelian invariance group. For the pion - nucleon system, the procedure leads to an approximate expansion of the hamiltonian, the first term of which may be identified with the static-source theory; it is argued that the approximation need not be regarded as a 'non-relativistic' approximation, not being obtained by expanding in terms of the inverse mass of the nucleon. One might hope then, that certain consequences of the Chew-Low theory would hold good in a fully relativistic treatment. Alternatively, Chew, Goldberger, Low and Nambu[7] have derived the π-N static P-wave theory from the assumptions of a relativistic dispersion relation, and the dominance of the (3,3) resonance. The model is clearly rich in structure, relevant to physics, and worthy of extensive study.

However, our primary interest in the Chew-Low dispersion relations, in the one-meson approximation anyway, is their direct implication of unitarity, hermitian analyticity and crossing symmetry (see, for example, Schweber[8]). Conversely, any amplitude satisfying these three conditions in their static elastic form, will be a solution of the appropriate Chew-Low equation. We need, therefore, not concern ourselves with the specification of an interaction hamiltonian, but will proceed as advocated by Chew[9] himself, in the spirit of S-matrix theory. We will adopt these three conditions as postulates, their specific forms being the limitations of the model. It is not then necessary to specify the number or location of bound state poles, or coupling constants. Also, we shall attempt to work with a crossing matrix parametrised in as general a way as possible, consistent with the general requirements for a crossing matrix (see Appendix to Chapter 3); we are then not even immediately concerned with the specification of possible symmetry groups. It is in this sense that we term the model a 'generalised Chew-Low model'.

The conditions of unitarity, analyticity and crossing symmetry, at least in their general forms, are generally accepted requirements for any realistic S-matrix model of strong interactions. Unitarity is the most well understood condition, being an expression of probability conservation. Crossing symmetry is likewise believed to be of general validity; it does not refer to any particular type of interaction. Wigner[10] has stressed its particular importance in understanding strong interaction physics, by providing a link between geometrical and dynamical principles of invariance. However, it is in the form of the crossing relations that the chief limitation of the static model lies: the crossing matrix relates the amplitude in the s-channel to that in the u-channel, where for the static model, s becomes z, the incoming energy, and u becomes -z. The crossing matrix itself, is a consequence of invariance under some internal symmetry group, which need not necessarily be specified. Analyticity, however, is the least understood and yet, probably the most important requirement of all; without it no dispersion relation could be written down. It is generally thought of as being, in some sense, a consequence of causality. The difficulty in verifying this lies, clearly, in that causality implies some concept of time, which is very difficult, and perhaps even undesirable[9], to incorporate into such a theory. A number of authors have recently written on this problem.[11)-13)] Branson discusses a formalism for introducing an 'idealised' microscopic time, which may be modified to give a macroscopic time. Then, using the usual intuitive concept of causality, he is able to show that for a non-relativistic theory, the scattering amplitude has a continuation into the upper-half energy plane, the continuation being effected by Schwarz's Reflection Principle; one then has 'hermitian analyticity'. For a relativistic theory the position is less clear, and it may be that the condition must remain a plausible axiom. Nevertheless, one is justified in regarding the three stated conditions as largely independent of detailed dynamical assumptions of the sort necessary for the original field-theoretic derivation of the model, and that features of the model and solutions may hold in more realistic models. However, one should note that Aks[14)] has proved that for a four dimensional world, the requirements of crossing symmetry and elastic two-body unitarity in all channels, are incompatible unless the scattering amplitude vanishes identically. Hence, the more realistic model must be either set in a two dimensional world, where Lardner[15)] has shown that the equation describing scattering, neglecting multi-particle intermediate states, must be essentially the Low equation, or, at least three-body unitarity be employed.

We will now proceed to outline the work that has been done on the type of model to be considered, and the extensions which we have attempted to make.

The problem of solving the model may be simply stated as follows: given a suitable crossing-matrix, A, to construct functions $S_i(z)$ having a square root branch point at threshold[16)] which may be taken to be at $z = 1$, and a similar branch point at $z = -1$, satisfying the conditions;-

(a) Unitarity (elastic) so that $S_i(z)$ has no further branch points on the real axis, and is such that its continuation to the second sheet of the threshold branch point, $S_i^{(2)}(z) = 1/S_i(z)$.

(b) Hermitian analyticity, so that $S_i(z)$ is meromorphic in the z-plane cut from 1 to ∞ and from $-\infty$ to -1, and is real in the sense $S_i(z^*) = S_i^*(z)$.

(c) Crossing symmetry, so that $S_i(-z) = \sum_j A_{ij} S_j(z)$.

Castillejo, Dalitz and Dyson[17] were able to obtain exact general solutions to the neutral and charged scalar theories ($A=1$; $A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, respectively). Their method of solution turned essentially on the use of the Generalised R-function, the solution following quite straightforwardly due to the particularly simple form of the crossing-matrices for these models. Ning Hu[18] also constructed solutions for these same models, in the form of finite products of algebraic functions. However, Froissart and Omnes[19] showed that this form of solution was not possible with less simple crossing-matrices, and that in general, the Riemann surfaces of the solutions would have an infinite number of sheets.

Such a solution was constructed by Wanders[20] for the neutral pseudoscalar model ($A = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$), mathematically equivalent to the symmetric scalar model. Wanders' mode of solution was, using crossing symmetry, to decompose the S-matrix functions into symmetric and anti-symmetric functions, and to then construct general solutions of the algebraic equations between these functions and their continuations, which result from the imposition of unitarity. As had been suggested by Froissart and Omnes[10], the solutions have an infinite number of Riemann surfaces corresponding to those of the function arcsin(z). Wanders' method was extended by Martin and McGlinn[21], to yield a general solution with a general involutory crossing-matrix $A = \begin{pmatrix} c & 1-c \\ 1+c & -c \end{pmatrix}$, where c is a real parameter. Martin and McGlinn found that solutions could be constructed for all values of c.

The same conclusions were reached by Rothleitner.[22] Rothleitner proposed a method of solution whereby the problem is transformed into the $w = \cosh^{-1}z$ plane, when it reduces to that of solving a system of coupled difference equations. The method provides a very powerful technique of solution, and general solutions in closed form were obtained for the general involutory two-dimensional problem mentioned above, and special solutions obtained for the particular three-dimensional problem corresponding to the pseudoscalar symmetric -N problem of Chew and Low ($A = -\frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}$). We shall employ the Rothleitner technique extensively later in this work.

The two-channel problem may be further generalised by removing the involutory requirement, so that the most general crossing-matrix becomes a two parameter matrix, $A = \begin{pmatrix} c & 1-c \\ 1-b & b \end{pmatrix}$. Applying crossing symmetry twice does not then return the amplitude to the original channel, so that it is required to construct two pairs of S-matrix functions, $U_i(z)$, $S_i(z)$, where now $U_i(-z) = \sum_j A_{ij} S_j(z)$. A solution of this problem has recently been given by Fairlie,[23] who interprets the model as corresponding to elastic scattering of a quark, transforming as a totally symmetric representation of SU_n, in the static limit. Fairlie used a successive correction technique which, in fact, serves only to produce a subclass of solutions in non-closed form, it being necessary to assume one of the arbitrary functions of the analysis to be odd. Fairlie claimed that the Rothleitner technique fails to treat such a model. In Chapter 3 we shall show this not to be the case, and will construct the general solution in closed form, containing Fairlie's solution as a subclass of solutions. This work has been published[24] by the author.

Also, in Chapter 5, we shall construct, using the Rothleitner transformation, special classes of solutions to the problem with completely general three dimensional crossing matrices. This problem has been examined by Tambasco[25], using the Martin and McGlinn extension of the Wanders method. However, he considers only a subclass of one of the two possible classes of crossing-matrices. This is obtained by setting one of the parameters equal to zero, thereby reducing the problem to a two-channel case, with a further condition to determine the third S-matrix function. Even so, we would like to point out that his 'general' solution is incorrect, being valid for only a particular value of the remaining parameter.

The solutions obtained for all the cases mentioned above have the feature that they involve arbitrary functions and undetermined parameters, so that there exists, in general, an uncountable infinity of solutions; neither the position or number of bound states is determined by the solutions. Castillejo, Dalitz and Dyson[17] argued that this arbitrariness is a consequence of the Low equation containing no information concerning the 'internal structure' of the scatterer. The question now arises of how one might choose the correct, or class of correct, 'physical' solutions. Of course, for the original Chew-Low[1] theory, it was insisted that the nucleon be the only bound state, and the solution was then uniquely determined by the field-theoretic requirement that the solution be the analytic continuation of the perturbation power series. For a strictly S-matrix approach, however, some other functional criterion is clearly necessary. This possibility was considered by Dyson[26], who doubted that such a condition could select a unique physical solution.

Albright and McGlinn[27] suggested the dynamical criterion that the number of zeros of the partial-wave S-matrix elements be equal to the minimum number required to yield finite phase shift behaviour at infinity. Albright and McGlinn[28] subsequently applied this criterion to the Wanders'[20] solutions for the neutral pseudoscalar and symmetric scalar models. The arbitrariness of the solution was reduced to that of two undetermined parameters, one being a cut-off parameter. However, they restricted the possible pole structure of the solution by, in fact, insisting that the only bound state pole occur at the target baryon energy. Also, possible subtractions and alternative forms of cut-off function were not considered, which, we would hope to be also restricted by a suitable criterion.

A most appealing criterion would be one formulated in accordance with the bootstrap philosophy, expressing the notion that all particles are composite states of one another, there being no 'elementary' particles. Huang and Low[29] proposed as such a criterion, Levinson's Theorem of potential scattering[30], which they indicated was equivalent to the usual N/D bootstrap prescription. In Chapter 2, we shall discuss the various other bootstrap criteria that have been proposed, their relation to each other, and possible equivalence; we shall also give a more detailed proof, subject to certain asymptotic conditions, of the general equivalence of Levinson's Theorem and the N/D prescription.

Huang and Low[29] imposed this condition on the solutions to those models with particular crossing matrices corresponding to the neutral charged and symmetric scalar theories, and the neutral pseudoscalar theory. The possibility of an arbitrary number of subtractions was considered (i.e. the asymptotic behaviour of the amplitude was not restricted), while the cut-off function, to a large extent, remained general, although contained in a special class of possible cut-off functions. They found the criterion was very restrictive with respect to possible bound state distributions, and to the cut-off function, there being no possible bootstrap solution without both a cut-off and at least one subtraction. It was suggested that the necessity of a cut-off and subtractions indicated that the existence and properties of bootstrap solutions to the complete S-matrix problem will depend critically upon the details of high energy effects. The bootstrap solutions obtained by Huang and Low all remained dependent upon two arbitrary parameters. A number of mis-statements were made in the paper by Huang and Low, which we discuss later; in particular, an assumption was made concerning the occurence of possible bound state which, in fact, is a consequence of the bootstrap mechanism, and so worthwhile noting.

It is clear from the foregoing that the proposed criterion is extremely restrictive of possible bootstrap mechanisms. The question then,also arises of whether it is capable of predicting, as a dynamical consequence, an internal symmetry. This question, for an S-matrix model, would, of necessity, have to be answered in terms of the crossing-matrix elements. Martin and McGlinn [21] had hoped that, for the generalised two dimensional crossing-matrix, the sole conditions of unitarity, analyticity and crossing symmetry would be sufficient to restrict the allowed class of crossing-matrices. As previously mentioned, this was found to be not the case. However, they observed, as did Rothleitner [22], that for a certain class of crossing matrices, corresponding to a subclass of SU_2 crossing-matrices, the solutions reduced to a particularly simple, rational form. They hypothesised that a dynamical criterion which was sufficient to restrict the exact solutions to this form, would imply a prediction of SU_2 internal symmetry, and could not be a consequence of any overt approximation. In a published paper [31], we imposed the bootstrap criterion of Levinson's Theorem upon the exact general solutions, and found that unless the crossing-matrix was so restricted, an essential singularity at threshold would result, and the phase shifts would be ill-defined. We have since realised that this result could be more easily inferred from the additional dynamical requirements we adopt from the work of Huang and Low. We present this argument in Chapter 4, where we will also construct explicit bootstrap solutions for the general crossing matrix of this subclass.

Huang and Mueller [32], have also considered the problem with general two dimensional crossing-matrix, and by means of a very neat technique avoiding the involved analysis of the explicit solutions, obtained the same qualitative conditions for a bootstrap solution as did Huang and

Low.[29] However, they assumed the phase shifts to be well-defined and analytic functions for all values of the crossing-matrix parameter, and so obtained no restriction on the allowed class of crossing matrix. Also, we point out that Huang and Mueller made the tacit assumption that no bound state pole can occur in more than one channel; actually this assumption is not necessary and may be shown to be a consequence of the bootstrap mechanism. The conditions proved for bootstrap solutions are necessary conditions, however, and without recourse to the explicit solutions, they were unable to show them to be sufficient, (for example, by verifying the coupling constants squared to be positive), or to examine the detailed analytic structure of the bootstrapped solution. We, therefore, construct these explicit solutions in Chapter 4.

Cushing[33] has, likewise, considered a model with crossing symmetry defined by the general one parameter two dimensional crossing-matrix, but for which all two particle intermediate states, both elastic and inelastic may be treated exactly. The consequences of imposing Levinson's Theorem on the solutions with first order corrections to the static model are considered, and he concludes that the class of crossing-matrices are unrestricted. However, the solutions are again not explicitly constructed in closed form, while specific assumptions are made concerning the possible bound state distributions, and convergence of certain infinite series; the analysis would, hence, seem to be incomplete.

We conclude this chapter by summarising the work to be presented in subsequent chapters.

In Chapter 2 we will survey and discuss the various bootstrap criteria that have been suggested, and the evidence for a belief in their equivalence. In particular, a general argument is presented for the equivalence of Levinson's Theorem and the N/D prescription as generators of bootstrap mechanisms.

Chapter 3 will be concerned with the construction, using Rothleitner's transformation technique, of the most general pairs of S-matrix functions satisfying unitarity, analyticity and crossing symmetry defined by the most general, two parameter, two dimensional crossing-matrix. Solutions will be shown to exist for all real values of the matrix parameters. Phase shifts for the solutions corresponding to a subclass of SU_2 crossing-matrices will be calculated for use in the following chapter.

In Chapter 4 we will consider the imposition of a number of general dynamical requirements upon the general solutions to the problem with an involutory two-dimensional crossing-matrix. These conditions will be shown to imply an essential singularity, in the solutions, at threshold unless the crossing-matrix parameter is restricted to values corresponding to a subclass of SU_2 crossing-matrices. For this subclass, which may be interpreted as resulting from the interaction of an isofermion with a particle of integer isospin, and for arbitrary incoming orbital angular momentum, we will impose the bootstrap criterion of Levinson's Theorem. The arbitrariness of the solutions will be found to be greatly restricted, with conclusions similar to those of Huang and Mueller,[32] although we will be able to demonstrate more explicitly the analytic structure of the solutions. We will show the inferred necessary bootstrap conditions to be also sufficient for S and P wave scattering, and for arbitrary incoming integer 'isospin'. A rather lengthy appendix is included concerning the analysis of the zero structure of a certain function, vital to the construction of the solutions and the analysis of the chapter, and concerning which a number of mis-statements were made by Huang and Low.[29]

In Chapter 5 we will demonstrate a method for the construction of general, arbitrary dimensional crossing-matrices, and also, the most general functions satisfying crossing symmetry defined by these matrices. Explicit forms for the three channel crossing-matrices are obtained, of which there are found to be two classes defined by the sign of the trace. With crossing symmetry defined by each of these classes, we will attempt to construct exact S-matrix solutions. Ror one class, which corresponds to that incompletely considered by Tambasco,[25] we will be able to demonstrate the existence of solutions for all values of the parameters, while for the other, we will construct exact explicit solutions for a subclass of two parameter matrices, containing as a subclass, 'generalised' SU_2 crossing-matrices. These 'generalised' SU_2 crossing-matrices are derived in an appendix.

Having obtained a solution for an S-matrix function, it is of interest to determine how much information may be derived concerning the nature of interparticle forces. In Chapter 6, we discuss an approach to this problem usually referred to as the 'Inverse Scattering Problem',i.e. the problem of determining a potential, from the S-matrix which, for a suitable wave equation, would duplicate the scattering. This problem apart from its obvious intrinsic physical interest, might have special relevance in understanding better the bootstrap mechanism. The machinery for determining such a potential for the Klein-Gordan equation, does exist as a generalisation of the well known Ge'lfand-Levitan[34] equations. Unfortunately, these equations are not soluble by any straightforward method. We demonstrate a method for their approximate solution, but the resulting potentials prove to be of little interest with regard to the bootstrap problem.

In Chapter 7 we will discuss more fully our conclusions, and possible further extensions.

CHAPTER TWO

BOOTSTRAP CRITERIA

In this chapter, we present a brief survey and discussion of the various proposed bootstrap criteria, and the evidence for a belief in their equivalence. A general argument for the equivalence of the usual N/D bootstrap prescription and the Levinson's Theorem criterion, in particular, subject to certain asymptotic conditions, is also given.

Section 2.1 The Bootstrap Mechanism and Criteria of Compositeness

The fundamental tenet of the bootstrap philosophy is that the strongly interacting particles are 'composite', each being a bound state of other such particles, with the binding forces provided by 'exchange forces'. When the particles become grouped into multiplets due to the invariance of the system under an internal symmetry group, the bootstrap hypothesis asserts that each multiplet is a bound state of others, the binding being due to the exchange of multiplets. With the ever growing multitude of observed sub-atomic particles and resonances, such an idea is of obvious appeal.

Such a philosophy, it is hoped would lead, could one write down a complete set of amplitudes satisfying such general postulates as unitarity, analyticity, crossing symmetry and Lorentz invariance, to a self consistent system of equations, determining the masses and coupling constants of all particles. The solutions to these hypothetical equations need not be unique, but are perhaps limited to a finite set of possible solutions, or dependent on a few observable parameters. However, such a programme is far from being achieved, and as yet, one must, at best, be content with attempting to 'bootstrap' a given subset of particles. Even then, one or more of the general conditions to be imposed on the amplitudes must usually be violated to some degree.

The first, and by no means trivial, difficulty in carrying through the bootstrap programme, assuming calculational procedures to determine the amplitudes, is the formulation of a mathematical criterion of compositeness. That is, a criterion by means of which one may differentiate between the possibilities of a particle being, in some sense, 'elementary', or alternatively, a composite bound state of one or more particles. The natural distinction between 'elementary' and 'composite' particles of a Lagrangian formulation of Quantum Field Theory, whereby each type of field operator appearing in the Lagrangian is associated with an elementary particle, the composite particles being associated with the one-particle states for which there is no corresponding operator, has no clear parallel for an S-matrix formulation. In S-matrix theory, both elementary and composite particles are manifested as poles of the matrix elements, functions of appropriate kinematical variables. If the particle pole is regarded as elementary, then its residue gives the coupling constant, while if regarded as composite, the asymptotic normalisation of the bound state wave function. A criterion is needed, then, which will enable one to distinguish between elementary and composite particle poles, from a knowledge of the S-matrix functions alone. A perturbation theory formulation is clearly of no use, as it is incapable of describing bound states or resonances. Several such suitable criteria have been proposed, principally by analogy with potential theory or simple field-theoretic models, which we now discuss.

In potential theory, dynamical poles may be defined by means of the Regge Pole concept.[35)] Here, dynamical states are associated with Regge trajectories, represented by poles in the scattering amplitude corresponding to the passing of a Regge trajectory through an integer value of spin. The non-composite, or elementary particle, however, may be added to the scattering amplitude as an explicitly introduced pole, at arbitrary energy, with arbitrary angular momentum, and with arbitrary residue. Such a distinction, it has been argued[36)] , may be extended to the relativistic case so that, here also, dynamical particles are required to lie on Regge trajectories. The bootstrap hypothesis would then become that all poles in ℓ, of the partial wave amplitude, are Regge poles. Such a hypothesis is not easily implemented.

The vanishing of the renormalisation constants, usually denoted by Z_3 has been suggested many times (see, for example, Salam[37)]) as a suitable criterion for compositeness. The renormalisation constant for a scalar particle may be interpreted as a measure of the probability of finding the bare particle in the centre of the cloud for the physical or 'dressed' particle. Hence, the vanishing of Z_3 implies that the bare particle does not exist, and so is not elementary. A more rigorous argument in favour of this criterion has been given by Fried and Jin.[38] The condition has achieved a considerable vogue in recent times, for which a full discussion and bibliography may be found in the paper by Zimmerman.[39]

By far the most widely and frequently used criterion has been that formulated in terms of the N/D method (see, for example, Chew[40]). The scattering amplitude is required to be a meromorphic function of s, the centre of mass energy squared, with a right-hand cut coming from unitarity and "direct" processes, and a left-hand cut coming from "exchange" processes. These analyticity and unitarity properties are automatically satisfied if the amplitude, f(s), is represented as f(s) = N(s)/D(s), where D(s) has the same right-hand cut as does f(s) but no left-hand cut, while N(s) has no right-hand cut but the same left-hand cut as does f(s), and if N(s) and D(s) are related by certain coupled integral equations. In the case of many-channel scattering, N(s) and D(s) may be interpreted as suitable matrix functions.[41] Also, any amplitude which has a bounded phase shift with a finite number of oscillations has such an N/D decomposition.[42] A zero of D is interpreted as a bound state pole of f. A pole of D, however, corresponds to a zero of the amplitude, and is termed a C.D.D. zero.[17] As may be shown (see, for example, Zachariasen[43]) the introduction of such a C.D.D. zero must be accompanied by a zero of D near to the C.D.D. zero, giving rise to a pole of f. This pole is interpreted as corresponding to an 'elementary' particle, there arising two new undetermined parameters which one may associate with the mass and coupling constant of the particle. The number of C.D.D. zeros of the amplitude, or equivalently, the number of poles of D, gives, then, the number of elementary particles present. Hence, the bootstrap condition becomes the prescription that the D function should have no poles. The more general formulation of this criterion for the case of many-channel dispersion relations is given by Franklin.[44)]

The criterion to be used in the subsequent work of this thesis, is that of Levinson's Theorem, proposed by Huang and Low.[29)] In non-relativistic potential scattering, the theorem asserts that $\frac{\Delta\delta}{\pi} = -n_b$, where $\Delta\delta$ is the change in phase shift from threshold to infinity, and n_b is the number of bound state poles of the amplitude. It implies that all bound states are results of the interaction, for the phase shift becomes identically zero when the interaction is turned off. If this interpretation is carried over into the relativistic problem, and we define an elementary particle to be a bound state that persists when the interaction is turned off, then Levinson's Theorem implies that there are no elementary particles. The imposition of Levinson's Theorem would, thus, appear to be a suitable bootstrap criterion. However, it is not a criterion whereby a pole of the scattering amplitude is determined as elementary or not, but rather a condition which, if satisfied, implies that all poles of the scattering amplitude are bound state poles. If the condition is not satisfied, all that can be said is that we do not have a bootstrap solution in the sense specified by the criterion.

In practice, most bootstrap calculation have ignored all such criteria, and leant on the intuitive idea that if the parameters of a particle are determined by those of other particles, then that particle is dynamical (see, for example, Zachariasen and Zemach[45]). This approach has the defect that one can never guarantee that the parameters were not only calculable by virtue of the inevitable approximations. If the approximation were improved, the equations determining the parameters might become mere identities. A typical bootstrap calculation of this kind usually proceeds by the assumption that it is sufficient to consider only one-particle exchange graphs in calculating the left-hand discontinuity of the scattering amplitude. Such an assumption, of course, does great violence to crossing symmetry in particular, and may lead, as demonstrated by Diu and Rubinstein[46], to equations that are in no way related to the exact ones, even if they exist.

The various criteria outlined above are obviously somewhat diverse, and the question may be posed, whether or not they are compatible, or perhaps even equivalent. Some work has been performed, in the context of model calculations, concerning this question. Zachariasen[43] has considered a simple model for which the condition of no C.D.D. pole implies that Z_3, the wave function renormalisation constant for the particle considered, goes to zero, and discusses for the model the relationship between the $Z_3 = 0$, Regge pole and Levinson's Theorem criteria, showing that they are all essentially equivalent. Jin and

Kang[47] have given an exhaustive analysis of a model obtained by a slight extension of the Lee model[8], in order to study the bootstrap mechanism. This model has the virtue of being exactly soluble in full generality. It is found that a bootstrap mechanism is possible, in the sense that meaningful relationships are obtained between the coupling constants and masses, if and only if, in addition to the general requirements of unitarity, analyticity and crossing symmetry, two additional restrictions are imposed. These conditions are that:- (i) the amplitude, $f(w)$, w being the c.m. energy, satisfy the asymptotic condition $\lim_{w\to\infty} w^2 f^{-1}(w) = 0$; and (ii) $f(w)$ should have no C.D.D. zeros. It is shown that condition (i) is equivalent, in the N/D decomposition, to $\lim_{w\to\infty} \frac{N(w)}{w} = 0$, or to $Z_3 = 0$, while conditions (i) and (ii) are equivalent to the Levinson's Theorem criterion. Either conditions (i) or (ii) separately, gives only an inequality between masses and coupling constants. The conclusion here, then, is that the $Z_3 = 0$ and N/D prescriptions are not self-sufficient criteria, but that a further condition is necessary, when they become equivalent. The Levinson Theorem criterion seemingly has the advantage of requiring no additional conditions.

The hope is encouraged by these analyses that it might be possible to prove the essential equivalence of the criteria, in general. Such a general proof would be most enlightening concerning a final mathematical description of the property of being 'elementary' versus 'composite'. Which criterion were adopted would be immaterial, and would be chosen according to the mode of calculation to be considered.

In the next section we present a general argument for the equivalence of Levinson's Theorem with the N/D prescription.

The application of the bootstrap criteria discussed so far, involve detailed dynamical calculations which will invariably resort to some form of approximation. We mention that there is also the possibility of using criteria of a more qualitative nature, abandoning any attempt to determine dynamical parameters such as mass values, but perhaps facilitating an understanding of the quantum numbers of the particle spectrum without solutions to the complete problem. Hopkinson[48)] gives a description of such a criterion in terms of the size and sign of the crossing matrix elements. Hwa and Patil[49)] have suggested the additional restriction that the bootstrap scheme involve the minimum number of particles. Ratios of coupling strengths and mixing parameters may be calculated within such a scheme which agree well with experimental numbers. This approach may, therefore, also prove valuable.

Section 2.2 The N/D Prescription and Levinson's Theorem

Denote by $f(z)$ the partial wave scattering amplitude, z being the centre of mass energy.

$f(z)$ is required to be meromorphic in the z-plane, with a right-hand cut from, say, z_0 , to infinity, and a left-hand cut from -infinity to say, z_1, both cuts being along the real axis. Also, $f(z)$ must be real hermitian, so that

$$f^*(z) = f(z^*) ,$$

and satisfy a possibly subtracted dispersion relation.

As is usual, we define $f(x) = \lim_{\epsilon \to 0+} f(x + i\epsilon)$, for x real, and write $f(x) = \rho(x)\sin\delta(x)e^{i\delta(x)}$ where $\delta(x)$, the phase shift is real for $x > z_0$, and normalised so that $\delta(z_0) = 0$.

Then, as mentioned previously, provided $\delta(x)$ is finite and smooth, we may write

$$f(z) = \frac{N(z)}{D(z)} \tag{2.2.1}$$

where N and D are real hermitian functions, D having only the right-hand cut, and N the left-hand cut, and satisfying suitable dispersion relations.

As shown by Chew and Mandelstam,[50] an N and D satisfying these requirements is given by

$$\mathbb{D}(z) = \exp\left[\frac{-z}{\pi}\int_{z_0}^{\infty}\frac{\delta(x)\,dx}{x(x-z)}\right] \tag{2.2.2}$$

and $$\mathcal{N}(z) = f(z)\,\mathbb{D}(z) \tag{2.2.3}$$

$\mathbb{D}(z)$, termed the Omne's function, has no zeros or poles anywhere, except possibly at infinity.

Sugawara and Kanazawa[51] have shown that, provided $\delta(x)$ has a limit $\delta(\infty)$ as $x \to \infty$, then

$$\frac{\operatorname{Log}\mathbb{D}(z)}{\operatorname{Log} z} \underset{z\to\infty}{\longrightarrow} \frac{\delta(\infty)}{\pi} \tag{2.2.4}$$

Let D(z) be a general D function for the given amplitude, satisfying the required conditions.

Then $\frac{D(z)}{\mathbb{D}(z)}$ is a real hermitian function, and since D and $\mathbb{D}$ have the same phase on the right-hand cut, must be meromorphic everywhere. Hence,

$$\frac{D(z)}{\mathcal{D}(z)} = \frac{P(z)}{Q(z)} \tag{2.2.5}$$

where P(z) and Q(z) are polynomials of order, say, n and m respectively.

The poles of f(z) are then given by the zeros of P(z), and m is the number of C.D.D. zeros of f(z). Therefore, from (2.2.5)

$$\lim_{z\to\infty} \frac{\operatorname{Log}\mathcal{D}(z)}{\operatorname{Log} z} = \lim_{z\to\infty} \frac{\operatorname{Log} D(z)}{\operatorname{Log} z} + \lim_{z\to\infty} \frac{\operatorname{Log} P(z)}{\operatorname{Log} z} - \lim_{z\to\infty} \frac{\operatorname{Log} Q(z)}{\operatorname{Log} z}$$

$$= \lim_{z\to\infty} \frac{\operatorname{Log} D(z)}{\operatorname{Log} z} + (n - m)$$

Hence, by (2.2.4)

$$\lim_{z\to\infty} \frac{\operatorname{Log} D(z)}{\operatorname{Log} z} = \frac{\delta(\infty)}{\pi} - (n - m) \tag{2.2.6}$$

That $\frac{\delta(\infty)}{\pi} = (n - m)$ has been termed the 'Generalised Levinson's Theorem'. It has been proposed by several authors (for example, Frye and Warnock [52]), and has been derived subject to various asymptotic conditions on the amplitude for inelastic scattering.[53] It has also been derived, subject to certain uniqueness conditions, by field-theoretic methods.[54]

However, we see from the above that a necessary and sufficient condition for the Generalised Levinson's Theorem to hold is that

$$\lim_{z\to\infty} \frac{\operatorname{Log} D(z)}{\operatorname{Log} z} = 0 \tag{2.2.7}$$

Levinson's Theorem in the usual form, will then hold if and only if, m = 0, i.e., if and only if f has no C.D.D. zeros.

The condition will certainly hold if, as is usually tacitly assumed in N/D bootstrap calculations, $D \sim$ a constant, as $z \to \infty$.

We conclude, therefore, that provided f(z) is such that D(z) may be chosen subject to (2.2.7), then Levinson's Theorem is equivalent to the usual N/D bootstrap prescription.

We, hence, have confidence in the imposition of Levinson's Theorem on the solution to the static model to be discussed in Chapter 4, as a valid bootstrap criterion, and in the results thereof, as being, within the limitations of the model, a consequence of the bootstrap mechanism.

CHAPTER THREE

EXACT SOLUTIONS OF THE GENERALISED TWO CHANNEL STATIC MODEL

This chapter is concerned with the application of an adapted technique of Rothleitner,[22] to derive the exact solution to the completely generalised two channel static S-matrix model. It is sought to construct two pairs of S-matrix functions of energy, each pair describing the scattering in one of the two channels, to be called the S, and U channels, satisfying only the three basic conditions. The first two of these conditions are those of analyticity and elastic unitarity, with the implied consequence of a square root branch point at threshold (see Zimmerman[16]). These conditions, as discussed in Chapter 1, are generally held to be requirements for more physical strong interaction models. The third condition is that of crossing symmetry, relating the two channel pairs of S-matrix functions by means of a constant, two dimensional matrix. It is this form of crossing relation which mainly typifies static models. The symmetry group to which the crossing-matrix corresponds need not be specified, the only requirement to be imposed upon it being that its rows sum to unity. It will, therefore, depend on two arbitrary parameters.

In Section 3.1 the problem is formulated, and in Section 3.2

the exact solutions obtained. In Section 3.3, the crossing-matrix is restricted to that corresponding to a generalised SU_N symmetry, depending on one continuous parameter, and for this case, the phase shifts calculated. A discussion and interpretation of this crossing-matrix is given by Fairlie.[23)] We conclude by writing down the bootstrap conditions on these phase shifts, in the form of Levinson's Theorem. An appendix is included, concerning the properties of the crossing-matrix.

Section 3.1 Formulation of the Problem

The mathematical problem may be stated as follows. We wish to construct two pairs of functions of a complex variable z, which we denote by $\underline{U}(z) = \begin{pmatrix} U_1(z) \\ U_2(z) \end{pmatrix}$ and $\underline{S}(z) = \begin{pmatrix} S_1(z) \\ S_2(z) \end{pmatrix}$, meromorphic in the z plane cut from +1 to $+\infty$ and from -1 to $-\infty$, having square root branch points at ± 1, and satisfying the conditions:-

(i) Hermitian analyticity, so that $\underline{U}^*(z) = \underline{U}(z^*)$, $\underline{S}^*(z) = \underline{S}(z^*)$

(* denotes complex conjugation.)

(ii) Unitarity, so that for real $z > 1$, the continuations onto the second sheet of the right-hand branch point are given by

$$U_i^{(2)}(z) = 1 / U_i(z) \;; \quad S_i^{(2)}(z) = 1/ S_i(z)$$

(iii) Crossing symmetry, so that for all z in the cut plane

$$\underline{U}(-z) = A\, \underline{S}(z)$$

Here, A is a 2 X 2 generalised crossing-matrix, having by condition (i), real elements, and which is such that the vector $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of A, with eigenvalue +1 (see Appendix 3A)

A may, therefore, be parametrised in the form

$$A = \begin{pmatrix} c & 1-c \\ 1+b & -b \end{pmatrix} \qquad (3.1.1)$$

b and c being, at the moment, arbitrary but real.

The analyticity requirements for $\underline{U}$ and $\underline{S}$ imply that they may be regarded as functions of variables z and $iq = i(z^2-1)^{\frac{1}{2}}$, q being taken to be real and positive above the cut $(1, \infty)$ on the first sheet of the right-hand branch point.[18] The continuations may then be written as

$$U_i^{(2)}(z,iq) = U_i(z,-iq) \; ; \quad S_i^{(2)}(z,iq) = S_i(z,-iq).$$

We now make the transformation $z \rightarrow w = \cosh^{-1} z$, so that the z plane is mapped onto the strip $0 \leqslant \mathrm{Im}\, w \leqslant \pi i$ in the w plane.

Noting that when $w \rightarrow -w$, $(z,q) \rightarrow (z, -q)$, and that when $w \rightarrow w + \pi i$, $(z,q) \rightarrow (-z, -q)$, the unitarity condition becomes

$$U_i(w)\, U_i(-w) = 1 \; ; \quad S_i(w)\, S_i(-w) = 1, \; w \text{ real} \qquad (3.1.2)$$

Then, using (3.1.2), we obtain the following relations from the crossing symmetry condition (iii):

$$U_1(w + \pi i) = \frac{c}{S_1(w)} + \frac{(1-c)}{S_2(w)} \qquad (3.1.3a)$$

$$U_2(w + \pi i) = \frac{1+b}{S_1(w)} - \frac{b}{S_2(w)} \qquad (3.1.3b)$$

$$\frac{1}{U_1(w)} = c\, S_1(w + \pi i) + (1-c)\, S_2(w + \pi i) \qquad (3.1.3c)$$

$$\frac{1}{U_2(w)} = (1+b)\, S_1(w+\pi i) \;-\; b\, S_2(w+\pi i) \qquad (3.1.3d)$$

The problem thus becomes that of constructing the most general pairs of meromorphic functions $\underline{S}(w)$ and $\underline{U}(w)$ satisfying the coupled difference equations (3.1.3a) - (3.1.3d)

Section 3.2 The Exact General Solutions

S_2/S_1 is an S-matrix element, i.e., is a meromorphic function in the cut z plane, satisfying the unitarity and real hermiticity conditions. We may hence write

$$\frac{S_2}{S_1} = \frac{\bar{K}+1}{\bar{K}-1} \qquad (3.2.1)$$

where $\bar{K}$ is a real hermitian meromorphic function in the cut z plane, satisfying

$$\bar{K}(z) + \bar{K}^{(2)}(z) = 0$$

Substituting (3.2.1) into equations (3.1.3), we get

$$U_1(w+\pi i) = \frac{1}{S_1(w)}\,\frac{\bar{K}(w)+(2c-1)}{\bar{K}(w)+1} \qquad (3.2.2a)$$

$$U_2(w+\pi i) = \frac{1}{S_1(w)}\,\frac{\bar{K}(w)+(2b+1)}{\bar{K}(w)+1} \qquad (3.2.2b)$$

$$\frac{1}{U_1(w)} = S_1(w+\pi i)\,\frac{\bar{K}(w+\pi i)-(2c-1)}{\bar{K}(w+\pi i)-1} \qquad (3.2.2c)$$

$$\frac{1}{U_2(w)} = S_1(w+\pi i)\,\frac{\bar{K}(w+\pi i)-(2b+1)}{\bar{K}(w+\pi i)-1} \qquad (3.2.2d)$$

Therefore, from (3.2.2a) and (3.2.2b)

$$\frac{U_1(w+\pi i)}{U_2(w+\pi i)} = \frac{\bar{K}(w) + (2c-1)}{\bar{K}(w) + (2b+1)} \tag{3.2.3}$$

while from (3.2.2c) and (3.2.2d)

$$\frac{U_1(w)}{U_2(w)} = \frac{\bar{K}(w+\pi i) - (2b+1)}{\bar{K}(w+\pi i) - (2c-1)} \tag{3.2.4}$$

Hence, from (3.2.3) and (3.2.4), we must have that, by continuation,

$$\frac{\bar{K}(w+2\pi i) - (2b+1)}{\bar{K}(w+2\pi i) - (2c-1)} = \frac{\bar{K}(w) + (2c-1)}{\bar{K}(w) + (2b+1)}$$

i.e. $2(b-c+1)\left[\bar{K}(w+2\pi i) - \bar{K}(w)\right] = 4(b-c+1)(b+c)$

The crossing-matrix must be non-singular, and so $\det A = (c-b-1) \neq 0$.

Therefore,

$$\bar{K}(w+2\pi i) - \bar{K}(w) = 2(b+c) \tag{3.2.5}$$

We suppose $b+c \neq 0$, otherwise we note from equations (3.1.3), the problem is invariant under the interchange $1 \leftrightarrow 2$, and becomes degenerate.

Put $-(b+c)\,K(w) = \bar{K}(w)$ (3.2.6)

Therefore,

$$K(w+2\pi i) - K(w) = -2$$

giving the solution

$$K(w) = \frac{iw}{\pi} + f(w) \tag{3.2.7}$$

where $f(w)$ is an arbitrary, real meromorphic, $2\pi i$-periodic function

$$f(w+2\pi i) = f(w) \tag{3.2.8}$$

Now, from (3.2.2a) and (3.2.2c),

$$\frac{S_1(w + 2\pi i)}{S_1(w)} = \frac{\bar{K}(w+2\pi i) - 1}{\bar{K}(w+2\pi i) - (2c-1)} \quad \frac{\bar{K}(w) + 1}{\bar{K}(w) + (2c-1)}$$

$$= \frac{\bar{K}(w) + 2(b+c) - 1}{\bar{K}(w) + (2b+1)} \quad \frac{\bar{K}(w) + 1}{\bar{K}(w) + (2c-1)}$$

$$= \frac{K(w) - r}{K(w) - (1-s-r)} \quad \frac{K(w) - (2-r)}{K(w) - (1+r+s)} \qquad (3.2.9)$$

where $r = \dfrac{1}{b+c}$; $s = \dfrac{b-c}{b+c}$ (3.2.10)

Hence, if we can solve the difference equation (3.2.9) generally, for $S_1(w)$, we would then by (3.2.1) have the general expression for $S_2(w)$, and by the crossing relations, determine $U_1(w)$ and $U_2(w)$. We would have solved the problem.

The solution of (3.2.9) is unique to within a factor of a function of period $2\pi i$. We, therefore, attempt to find a solution of the form

$$S(w) = F[K(w)]$$

so that $F[K(w + 2\pi i)] = F[K(w) - 2]$

Therefore, $$\frac{F[K-2]}{F[K]} = \frac{K - r}{K - (1-s-r)} \cdot \frac{K - (2-r)}{K - (1+r+s)} \qquad (3.2.11)$$

Taking Logarithms we thence obtain a linear difference equation:

$$\operatorname{Log}F[K-2] - \operatorname{Log}F[K] = \operatorname{Log}[K-r] + \operatorname{Log}[K-(2-r)] - \operatorname{Log}[K-(1-s-r)] - \operatorname{Log}[K-(1+r+s)] \qquad (3.2.12)$$

We now prove the following Lemma.

<u>Lemma</u> The solution of the difference equation

$$\operatorname{Log} \bar{F}(\xi - 2) - \operatorname{Log} \bar{F}(\xi) = \operatorname{Log}(\xi - \alpha) \quad \text{is}$$

$$\bar{F}(\xi) = \frac{\bar{\sigma}(\xi/2)}{2^{\xi/2}\,\Gamma\left[\xi/2 - c/2 + 1\right]}$$

where $\bar{\sigma}(\xi - 1) = \bar{\sigma}(\xi)$

Proof Put $\xi = 2\eta$

Therefore, $\operatorname{Log} \bar{F}\left[2(\eta - 1)\right] - \operatorname{Log} \bar{F}\left[2\eta\right] = \operatorname{Log}2 + \operatorname{Log}\left[\eta - \alpha/2\right]$

Put $G(\eta) = \operatorname{Log} \bar{F}(2\eta)$

$$\therefore \quad G(\eta - 1) - G(\eta) = \operatorname{Log}2 + \operatorname{Log}\left[\eta - \alpha/2\right]$$

Put $G(\eta) = -\eta \operatorname{Log}2 + H(\eta)$

$$\therefore \quad H(\eta - 1) - H(\eta) = \operatorname{Log}\left[\eta - \alpha/2\right]$$

This has the particular solution

$$H_o(\eta) = -\operatorname{Log}\left[\Gamma(\eta - \alpha/2 + 1)\right]$$

and so the general solution is

$$H(\eta) = -\operatorname{Log}\left[\bar{\omega}(\eta)\,\Gamma(\eta - \alpha/2 + 1)\right]$$

where $\bar{\omega}(\eta - 1) = \bar{\omega}(\eta)$

$$\therefore \quad G(\eta) = -\operatorname{Log}\left[2^{\eta}\bar{\omega}(\eta)\,\Gamma(\eta - \alpha/2 + 1)\right]$$

i.e. $$\bar{F}(\xi) = \frac{\bar{\sigma}(\xi/2)}{2^{\xi/2}\,\Gamma(\xi/2 - \alpha/2 + 1)}$$

where $$\bar{\sigma}(\xi/2) = \frac{1}{\bar{\omega}(\xi/2)}$$

This proves the Lemma.

The solution to (3.2.12) is therefore seen to be

$$F[K] = \bar{\sigma}(K/2) \frac{\Gamma\left[\frac{K}{2} + \frac{r+s+1}{2}\right] \Gamma\left[\frac{K}{2} - \frac{r+s-1}{2}\right]}{\Gamma\left[\frac{K}{2} - \frac{r}{2} + 1\right] \Gamma\left[\frac{K}{2} + \frac{r}{2}\right]} \qquad (3.2.13)$$

$$\equiv \bar{\sigma}(K/2) \; \bar{F}(k)$$

where $\bar{\sigma}\left(\frac{K-2}{2}\right) = \bar{\sigma}\left(\frac{K}{2}\right)$

We put the further restriction of unitarity upon this particular solution, which, since by (3.2.1) $K(w) + K(-w) = 0$, takes the form,

$$F[K] \; F[-K] = 1.$$

Now, $\bar{F}[K] \; \bar{F}[-K] =$

$$\frac{\Gamma\left[\frac{K}{2} + \frac{r+s+1}{2}\right] \Gamma\left[1-\left(\frac{K}{2}+\frac{r+s+1}{2}\right)\right] \Gamma\left[\frac{K}{2}-\frac{r+s-1}{2}\right] \Gamma\left[1-\left(\frac{K}{2}-\frac{r+s-1}{2}\right)\right]}{\Gamma\left[\frac{K}{2} - \frac{r+1}{2}\right] \Gamma\left[-\left(\frac{K}{2}-\frac{r}{2}\right)\right] \Gamma\left[\frac{K}{2}+\frac{r}{2}\right] \Gamma\left[1 - \left(\frac{K}{2} + \frac{r}{2}\right)\right]}$$

$$= -\frac{\sin\pi\left[\frac{K}{2} + \frac{r}{2}\right] \sin\pi\left[\frac{K}{2} - \frac{r}{2}\right]}{\sin\pi\left[\frac{K}{2} - \frac{r+s-1}{2}\right] \sin\pi\left[\frac{K}{2} + \frac{r+s+1}{2}\right]} \qquad (3.2.14)$$

where we have used $\Gamma(\zeta)\,\Gamma(1-\zeta) = \pi \operatorname{cosec}(\pi\zeta)$

The function $\dfrac{\sin\pi\left[\frac{K(w)}{2} + \frac{r}{2}\right]}{\sin\pi\left[\frac{K(w)}{2} + \frac{r+s+1}{2}\right]}$ is clearly of period $2\pi i$.

Therefore, write

$$F[K] = \sigma(k/2) \frac{\sin\pi\left[\frac{K}{2} + \frac{r+s+1}{2}\right]}{\sin\pi\left[\frac{K}{2} + \frac{r}{2}\right]} \bar{F}[K]$$

$$\equiv \sigma(K/2)\, F_0[K] \tag{3.2.15}$$

where

$$F_0[K] = \frac{\Gamma\left[1 - \frac{r}{2} - \frac{K}{2}\right]\Gamma\left[\frac{1-r-s}{2} + \frac{K}{2}\right]}{\Gamma\left[1 - \frac{r}{2} + \frac{K}{2}\right]\Gamma\left[\frac{1-r-s}{2} - \frac{K}{2}\right]} \tag{3.2.16}$$

where we have again used the above identity, used in (3.2.14).
We now have that $\sigma(w) \equiv \sigma[K(w)/2]$ is an arbitrary real meromorphic function of period $2\pi i$ in the w plane, and is also an S-matrix element satisfying

$$\sigma(w)\ \sigma(-w) = 1.$$

The general solution of (3.2.9) is hence

$$S_1(w) = D(w) \cdot F_0[K(w)]$$

where $D(w)\, D(-w) = 1$ (3.2.17)

and $D(w + 2\pi i) = D(w)$ (3.2.18)

We now transform back to the z plane.
(3.2.1) and (3.2.8) imply that

$$f(z) + f^{(2)}(z) = 0, \quad z > 1$$

and $f(-z) + [f(-z)]^{(2)} = 0, \quad z > 1.$

Therefore, we may write $f(z) = \frac{-iz}{q}\beta(z)$ where $\beta(z)$ is a real hermitian meromorphic function of z in the entire z plane. (This particular form for f(z) is chosen for convenience of use in later work).

Using the easily verified identity,

$$\frac{i\cosh^{-1} z}{\pi} = \frac{1}{\pi}\sin^{-1} z - \tfrac{1}{2} \quad , \text{ we have}$$

$$K(z) = B(z) - \tfrac{1}{2} \qquad (3.2.19)$$

where $B(z) = \frac{1}{\pi}\arcsin z - \frac{iz}{q}\beta(z).$

Similarly, the conditions (3.2.17) and (3.2.18) imply that

$$D(z)\,D^{(2)}(z) = 1\ , \quad z > 1 \qquad (3.2.20)$$

and $D(-z)\,D^{(2)}(-z) = 1, \quad z > 1$ (3.2.21)

Write $T_1(z) = D(z)\ ;\quad T_2(z) = D(-z)$

Then T_1 and T_2 are both unitary functions satisfying the conditions of hermitian analyticity in the cut plane, unitarity and the crossing symmetry defined by the crossing-matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The general solution to this problem is that given by Castillejo, Dalitz and Dyson.[17)] The arbitrariness in D(z) then consists of the Generalised R-function, and the form of the 'factor cut-off function'.

Finally then, we have the solutions

$$S_1(z) = F_0\left[K(z)\right]\ D(z)$$

$$S_2(z) = \frac{K(z) - r}{K(z) + r}\, F_0\left[K(z)\right]\ D(z) \qquad (3.2.22)$$

where $K(z)$, F_0 and $D(z)$ are defined by (3.2.19), (3.2.16), (3.2.20) and (3.2.21). $\underline{U}(z)$ is then determined by means of the crossing relations.

The solutions hence depend on the two arbitrary functions $\beta(z)$ and $D(z)$.

Section 3.3 Solution and Phase Shifts for the Involutory Case

As discussed by Fairlie,[23] when the crossing matrix parameters are specialised to the forms

$$b = \frac{1}{N(t+1) - 1} \; ; \qquad c = \frac{N - 1}{N(t+1) - 1} \; ,$$ where $N \geqslant 2$ is a positive integer, the model becomes that corresponding to SU_N scattering if t is also a positive integer.

We shall need the solutions for the case $b = c$ in the following chapter, and so discuss now the case for $N = 2$.

We now have that $\quad r = \dfrac{2t+1}{2} \; ; \quad s = 0$ (3.3.1)

Hence, from (3.2.16),

$$F_0[K] = \frac{\Gamma\left[\frac{3}{4} - \frac{t}{2} - \frac{K}{2}\right]\Gamma\left[\frac{1}{4} - \frac{t}{2} + \frac{K}{2}\right]}{\Gamma\left[\frac{3}{4} - \frac{t}{2} + \frac{K}{2}\right]\Gamma\left[\frac{1}{4} - \frac{t}{2} - \frac{K}{2}\right]} \qquad (3.3.2)$$

The crossing matrix is now involutory and so we must impose the additional condition that $\underline{U} = \underline{S}$.

From (3.2.2a) and (3.2.2c) we then obtain that

$$\bar{K}(w + \pi i) - \bar{K}(w) = (b+c) \qquad (3.3.3)$$

so that f(w) of (3.2.7) must, for this case, be periodic of period πi. $\beta(z)$ of (3.2.19) is then a real meromorphic even function of z in the entire z plane.

Our solution, from (3.2.2a) must now also be such that,

$$S_1(w)\, S_1(w+\pi i) = \frac{K-1+r}{K-r} = \frac{K+t-\frac{1}{2}}{K-t-\frac{1}{2}} \qquad (3.3.4)$$

However, since $K(w+\pi i) = K(w) - 1$, we have that

$$F_0\left[K(w+\pi i)\right] = F_0[K-1] = \frac{\Gamma\left[\left(\frac{1}{4}-t/2-K/2\right)+1\right]}{\Gamma\left[\frac{1}{4}-t/2+K/2\right]}\;\frac{\Gamma\left[\left(\frac{3}{4}-t/2+K/2\right)-1\right]}{\Gamma\left[\left(\frac{3}{4}-t/2-K/2\right]\right.}$$

$$= -\left[\frac{K+t-\frac{1}{2}}{K-t-\frac{1}{2}}\right]\frac{1}{F_0[K]}$$

Therefore, in addition to being periodic of period $2\pi i$, and being an S-matrix function, we require $\sigma(K/2)$ of (3.2.15) to be such that $\sigma(K/2)\;\sigma\left(\frac{K-1}{2}\right) = -1$.

Such a function is $\tan\frac{\pi}{2}(K+\frac{1}{2})$.

We therefore write our solution for the involutory case as

$$S_1 = \tan\frac{\pi}{2}(K+\tfrac{1}{2})\; F_0[K]\; D(z) \qquad (3.3.5a)$$

$$S_2 = \frac{K-t-\frac{1}{2}}{K+t+\frac{1}{2}}\; S_1 \qquad (3.3.5b)$$

where D(z) is an arbitrary symmetric S-matrix element.

Using again the identity employed in (3.2.14) we obtain the more convenient form for our purposes:

$$S_1 = \frac{\tan\pi/2\left[K+\frac{1}{2}\right]}{\tan\pi/2\left[K+\frac{1}{2}+t\right]}\;\frac{\Gamma\left[\frac{1}{4}-t/2+K/2\right]}{\Gamma\left[\frac{1}{4}+t/2+K/2\right]}\;\frac{\Gamma\left[\frac{3}{4}+t/2+K/2\right]}{\Gamma\left[\frac{3}{4}-t/2+K/2\right]}\; D$$

$$= \frac{\tan \pi/2 \left[K + \frac{1}{2}\right]}{\tan \pi/2 \left[K + \frac{1}{2} + t\right]} U_0(K)\, D \qquad (3.3.6)$$

Now suppose t is an integer.

Then the factor $\frac{\tan \pi/2 \left[K+\frac{1}{2}\right]}{\tan \pi/2 \left[K+\frac{1}{2}+t\right]}$ becomes a symmetric S-matrix element, and may be absorbed into D. Also, $U_0(K)$ becomes a rational expression in K, which may be compactly written in the form

$$U_0(K) = \frac{K + \frac{1}{2}}{K + (-1)^t \frac{1}{2}} \prod_{m=1}^{M} \frac{K-t+2m-\frac{1}{2}}{K+t-2m+\frac{1}{2}} \prod_{m=0}^{M-1} \frac{K+t-2m-\frac{1}{2}}{K-t+2m+\frac{1}{2}} \qquad (3.3.7)$$

where $M = t/2$ if t is even

and $M = (t-1)/2$ if t is odd.

We calculate expressions for the phase shifts for S_1 and S_2 for the case where t is integer.

Let us write

$$\left.\begin{aligned} \frac{K(z) - t - \frac{1}{2}}{K(z) + t + \frac{1}{2}} &= e^{-2i\psi(z)} \\ U_0(z) &= e^{2i\varphi(z)} \\ D(z) &= e^{-2i\theta(z)} \end{aligned}\right\} \qquad (3.3.8)$$

Then θ, φ and ψ are, by unitarity, real functions of z, for $z \geqslant 1$, all of which we may take to be zero at $z = 1$.

Now,
$$-\cot 2\psi(z) = \frac{\mathrm{Re}\left[\frac{K(z)-t-\frac{1}{2}}{K(z)+t+\frac{1}{2}}\right]}{\mathrm{Im}\left[\frac{K(z)-t-\frac{1}{2}}{K(z)+t+\frac{1}{2}}\right]}, \quad z \geqslant 1$$

$$= i \frac{\left[\frac{K(z)-t-\frac{1}{2}}{K(z)+t+\frac{1}{2}} + \frac{K^{(2)}(z)-t-\frac{1}{2}}{K^{(2)}(z)+t+\frac{1}{2}}\right]}{\left[\frac{K(z)-t-\frac{1}{2}}{K(z)+t+\frac{1}{2}} - \frac{K^{(2)}(z)-t-\frac{1}{2}}{K^{(2)}(z)+t+\frac{1}{2}}\right]}$$

$$= i \left\{ \frac{\left[K(z)-t-\frac{1}{2}\right]^2 + \left[K(z)+t+\frac{1}{2}\right]^2}{\left[K(z)-t-\frac{1}{2}\right]^2 - \left[K(z)+t+\frac{1}{2}\right]^2} \right\}$$

$$= \frac{1 - \left(\frac{iK(z)}{t+\frac{1}{2}}\right)^2}{\frac{2iK(z)}{t+\frac{1}{2}}}$$

Hence, $\cot \psi(z) = \frac{i}{(t+\frac{1}{2})} K(z)$ (3.3.9)

We shall suppose; as will be shown in the next chapter, must be the case, that $K(z)$ has a pole at threshold. Then, if $\alpha(z)$ is such that $\cot \alpha(z) = \lambda \cot \psi(z)$, we must have that

$$\Delta \alpha \equiv \alpha(\infty) - \alpha(1) = \Delta \psi \quad \text{if } \lambda > 0$$

(3.3.10)

$$= -\Delta \psi \quad \text{if } \lambda < 0$$

Hence, if $e^{-2i\alpha(z)} = \frac{K(z)-\lambda}{K(z)+\lambda}$ then as above,

$$\cot \alpha(z) = \frac{1}{\lambda} \cot \psi(z)$$

and $\Delta \alpha = \frac{\lambda}{|\lambda|} \Delta \psi$.

From (3.3.7) we see, therefore, that since

$t-2m+\frac{1}{2} > 0$ $m=1, \ldots\ldots\ldots, M$ and

$t-2m-\frac{1}{2} < 0$ $m=0, \ldots\ldots\ldots\ldots, M-1$

then $\Delta \varphi = \Delta \psi$ if t is odd and $\Delta\varphi = 0$ if t is even (3.3.11)

Hence, writing $S_\alpha(z) = e^{2i\,\delta_\alpha(z)}$, so that

$$\begin{aligned} \Delta\delta_1 &= \Delta\varphi - \Delta\theta \\ \Delta\delta_2 &= -\Delta\psi + \Delta\varphi - \Delta\theta \end{aligned} \qquad (3.3.12)$$

we have $(\Delta\delta_1, \Delta\delta_2) = (\Delta\psi - \Delta\theta, -\Delta\theta)$ if t is odd (3.3.13)

$$(\Delta\delta_1, \Delta\delta_2) = (-\Delta\theta, -\Delta\psi - \Delta\theta) \text{ if t is even.}$$

If we require S_α to verify Levinson's Theorem, as a bootstrap criterion, i.e.

$\Delta\delta_\alpha/\pi = -b_\alpha$, where b_α is a positive integer or zero, then

$$\left(\frac{\Delta\theta}{\pi}, \frac{\Delta\psi}{\pi}\right) = (b_2, b_2 - b_1) \text{ if t is odd}$$
$$= (b_1, b_2 - b_1) \text{ if t is even.} \qquad (3.3.14)$$

We see that the form of the bootstrap criterion becomes that of conditions on the arbitrary functions of the exact solutions which depend, in turn, on the range and value of the crossing-matrix parameter.

These results will be used extensively in the next chapter.

Appendix 3A. Properties of the Crossing-Matrix

In this appendix we intend, for completeness, to demonstrate the relevant properties used in this and subsequent chapters. The method used follows that presented by Martin and McGlinn.[21] The properties derived are not restricted to the static model, but are limited by the fact that the two physical processes related by the crossing symmetry must be represented by amplitudes of the same general structure. To be more precise, suppose we are relating the processes described by the reactions

$$a + b \rightarrow c + d$$

$$a + \bar{d} \rightarrow c + \bar{b}$$

Then the assunption of elastic scattering means that particles a and c belong to the same irreducible representation of the relevant symmetry group, while b and d likewise transform according to another irreducible representation. The representation, according to the above is then restricted to be self-conjugate. For a fuller discussion of this point, see Taylor.[55]

Let the crossing relation between the two physical processes be written as:

$$M_{cb}(s_i') = M_{bc}(s_i) \qquad (3A.1)$$

where b and c represent the quantum numbers of the appropriate initial and final states, while the s_i are the appropriate kinematical variables.

Consider the expansion of these amplitudes in terms of a complete set of eigenamplitudes corresponding to the different values of some conserved quantity:

$$M_{bc}(s_i) = \sum_{\gamma} h_{\gamma}(s_i)\, P^{\gamma}_{bc} \qquad (3A.2)$$
$$M_{cb}(s'_i) = \sum_{\gamma'} h_{\gamma'}(s'_i)\, P^{\gamma'}_{cb}$$

where the P^{γ}_{bc} are projection operators for the conserved quantity.

The projection operators possess the properties

$$\sum_{\gamma} P^{\gamma}_{bc} = \delta_{bc} \qquad (3A.3)$$

$$\sum_{d} P^{\gamma}_{bd}\, P^{\gamma'}_{dc} = \delta_{\gamma\gamma'} P^{\gamma}_{bc} \qquad (3A.4)$$

and are related, according to crossing symmetry thus:

$$P^{\gamma}_{bc} = \sum_{\gamma'} A_{\gamma'\gamma}\, P^{\gamma'}_{cb} \qquad (3A.5)$$

From (3A.5)

$$P^{\gamma}_{bc} = \Big(\sum_{\gamma'} A_{\gamma'\gamma} \sum_{\gamma''} A_{\gamma''\gamma'}\, P^{\gamma''}_{bc} \Big)$$

$$= \sum_{\gamma''} \Big(\sum_{\gamma'} A_{\gamma''\gamma'} A_{\gamma'\gamma} \Big) P^{\gamma''}_{bc}$$; (assuming, as we must, the sums to be finite dimensional).

Hence, by independence of the operators

$$\sum_{\gamma'} A_{\gamma''\gamma'}\, A_{\gamma'\gamma} = \delta_{\gamma\gamma''} \qquad (3A.6)$$

i.e., the crossing-matrix is its own inverse, or involutory.

From (3A.3)

$$\delta_{bc} = \sum_{\gamma} P^{\gamma}_{bc} = \sum_{\gamma} \sum_{\gamma'} A_{\gamma'\gamma}\, P^{\gamma'}_{cb} = \sum_{\gamma'} P^{\gamma'}_{cb}$$

Therefore, $$\sum_{\gamma} A_{\gamma'\gamma} = 1 \qquad (3A.7)$$

i.e., the sum over each row of the crossing-matrix equals unity.

The crossing relation for the eigenamplitudes is obtained from

(3A.1), (3A.2) and (3A.5),

$$\text{i.e.,} \quad \sum_{\gamma} h_{\gamma}(s_i) P_{bc}^{\gamma} = \sum_{\gamma'} h_{\gamma'}(s_i') P_{cb}^{\gamma'}$$

$$= \sum_{\gamma'\gamma''} h_{\gamma'}(s_i') A_{\gamma''\gamma'} P_{bc}^{\gamma''}$$

$$\therefore \quad h_{\gamma}(s_i) = \sum_{\gamma'} A_{\gamma\gamma'} h_{\gamma'}(s_i') \qquad (3A.8)$$

If the process is such that crossing twice does not bring the system back to the original channel, then we can only conclude that the sum of the rows of the crossing-matrix is unity. This is the case for the completely generalised two channel crossing-matrix considered in this chapter. The crossing-matrix must, however, be non-singular, since the crossed channel must be continuable back to the original channel by a crossing-matrix, which for consistency, is the inverse of the original matrix.

i.e. if $g_{\gamma}(s_i') = \sum_{\gamma'} A_{\gamma\gamma'} h_{\gamma'}(s_i)$

and $h_{\gamma'}(s_i) = \sum_{\gamma''} B_{\gamma'\gamma''} g_{\gamma''}(s_i')$

then

$$\sum_{\gamma'} A_{\gamma\gamma'} B_{\gamma'\gamma''} = \delta_{\gamma\gamma''}$$

i.e. $B = A^{-1}$.

The hermitian analyticity requirement for the S-matrix functions, implies that the crossing matrix be real.

CHAPTER FOUR

BOOTSTRAP RESTRICTIONS OF LOW EQUATION SOLUTIONS

In this chapter, we shall endeavour to impose the bootstrap requirement of Levinson's Theorem, upon the exact solutions to the static model with a two dimensional involutory crossing-matrix. A number of general 'physical' conditions will be imposed upon the general solution which, we will argue, are sufficient to restrict the otherwise arbitrary crossing-matrix parameter to integer values corresponding to an SU_2 internal symmetry group. The non-integer parameter solutions are ruled out, essentially, on the grounds that otherwise, the phase shifts are not well defined. The dependence of our bootstrap solutions upon the explicit form of these 'physical' requirements, for example the form of cut-off function assumed, threshold behaviour etc., will also be investigated in a later chapter.

Appendix 4A consists of the proof of a Lemma employed in the analysis. This Lemma is a generalisation of that incorrectly proved (acknowledged by Huang in a private communication) in an appendix to the paper by Huang and Low.[29)]

In appendix 4B we list the computer program used to calculate various coupling constants and bound state energies as functions of the remaining arbitrary parameters of the model.

As far as possible, we have attempted to preserve the same notation and development as that of Huang and Low, in order to facilitate comparison.

Section 4.1 Formulation of the Model

We denote the two S-matrix elements for the two channel static model by $S_\alpha(z)$, $(\alpha=1,2)$, where z is the energy of the incident particle.

With the incident-particle mass normalised to unity, the momentum of the incident-particle is therefore given by $q = (z^2 - 1)^{\frac{1}{2}}$. The branch cuts of q in the complex z plane are taken along the real axis from +1 to $+\infty$ and from -1 to $-\infty$, and q is chosen to be real and positive above the cut from 1 to ∞, so that iq is a real analytic function in the cut plane.

$S_\alpha(z)$ must satisfy then the following conditions of the model:
(i) analyticity, so that S_α is real meromorphic in the cut z plane.
(ii) elastic unitarity, i.e., S_α has only 1 branch point on the positive real axis, which is of the square root type at z = 1, the threshold point. Also, the analytic continuation of S_α onto the second Riemann surface is given by $S_\alpha^{(2)}(z) = 1/S_\alpha(z)$.
(iii) crossing symmetry, i.e.,

$S_\alpha(-z) = \sum_\beta A_{\alpha\beta} S_\beta(z)$, where $(A_{\alpha\beta})$ is a suitable crossing-matrix, and for the model to be considered has the general form

$$(A_{\alpha\beta}) = \frac{1}{(2t+1)} \begin{pmatrix} -1 & 2t+1 \\ 2t & 1 \end{pmatrix} \qquad (4.1.1)$$

where t is a real parameter, which, without loss of generality, we may take to be positive.

We now write

$$S_\alpha(z) = 1 + 2iq^{2\ell+1} v(z)\, h_\alpha(z) \qquad (4.1.2)$$

where v(z) is a cut-off function, which we take to be of the form

$$v(z) = \frac{\kappa^{2c}}{(q^2+\kappa^2)^c} \quad ; \quad \kappa > 1; \quad c = 0,1,2,\ldots\ldots \qquad (4.1.3)$$

and ℓ is the orbital angular momentum of the incoming particle. $h_\alpha(z)$ is the scattering amplitude for the process. The form (4.1.2) is dictated by the generally supposed threshold behaviour

$$S_\alpha \approx 1 + O(q^{2\ell+1}) \qquad (4.1.4)$$

Proofs of this threshold behaviour have been given by Jin and Martin, based on required analytic properties of the absorbtive part of the scattering amplitude,[56] and also for $\ell \geqslant 2$, assuming a disperion relation and complete crossing symmetry among the mandelstam variables s,t, and u.[57]

The above requirements for S_α are expressed by the following dispersion relation for h_α :

$$h_\alpha(z) = P_\alpha(z) + \frac{1}{\pi}\int_1^\infty dz'\, q'\, v(z')\left[\frac{|h_\alpha(z')|^2}{z'-z} + \sum_\beta A_{\alpha\beta}\frac{|h_\beta(z')|^2}{z'+z}\right] \qquad (4.1.5)$$

(see, for example, Schweber[8]) , where P_α is the sum of the poles located on the real axis between the branch points:

$$P_\alpha(z) = \sum_i\left[\frac{\lambda_{i\alpha}}{z_i - z} + \frac{\sum A_{\alpha\beta}\lambda_{i\beta}}{z_i + z}\right] \quad ; \quad |z_i| < 1 \qquad (4.1.6)$$

The above dispersion relation which is merely a generalised Low equation,[2] may require one or more subtractions to account for high energy effects. The crossing relation ensures that each such subtraction will introduce only one further parameter. We remark that such a dispersion relation may be derived as the static limit of a more general relativistic dispersion relation (see, for example, Bogoliubov and Shirkov[58]).

The following 'physical' conditions are to be imposed on any solution:

(a) $\lambda_{i\alpha} > 0$. $\lambda_{i\alpha}$, being the squared coupling constant of a bound state in channel α, would, if negative, indicate a ghost state.

(b) In order that S_α have the correct threshold behaviour given by (4.1.4), $h_\alpha(1)$ must be finite.

(c) $S_\alpha(z)$ must have the correct high energy behaviour, which will depend on the number of subtractions made in the dispersion relation for $h_\alpha(z)$. We require that $h_\alpha(z)$ does not go to infinity faster than any polynomial. Since the unitarity condition for the unsubtracted case may be written as

$$\operatorname{Im} h_\alpha = q\, v\, |h_\alpha|^2 \text{ , so that } |h_\alpha(z)| \leqslant |q\, v(z)|^{-1}, \ (z \geqslant 1),$$

subtractions are needed only if the cut-off factor is such that

$$\lim_{z\to\infty} q\, v(z) = L \tag{4.1.7}$$

where L is finite.

(d) The target-particle should occur as a bound state in one of the two channels, in order that the incident particle-target particle coupling constant should be non-vanishing.

(e) No bound state should have a smaller mass than that of the target baryon, else there would be an inelastic threshold below the assumed elastic threshold. Hence, no z_i should be negative.

(f) The S-matrix elements should not have an essential singularity on the real axis, in order that the asymptotic phase shifts be determinable.

(g) There should be at most a finite number of bound states and resonances in the scattering.

The solution dictated by analyticity, unitarity and crossing symmetry was obtained in Section 3.3, and we write, changing the previous notation slightly and relabelling the channels,

$$S_1 = U_o \frac{B - t - 1}{B + t} D \tag{4.1.8}$$

$$S_2 = U_o D \tag{4.1.9}$$

where

$$B(z) = \tfrac{1}{2} + i\left[\pi^{-1} \log(z + q) - \frac{z}{q}\beta(z)\right] \tag{4.1.10}$$

and $\beta(z)$ is a real meromorphic and even function in the entire z plane.

$$U_o(B) = \frac{\tan\pi\frac{B}{2}\,\Gamma\left[\frac{B+t+1}{2}\right]\Gamma\left[\frac{B-t}{2}\right]}{\tan\pi\frac{(B+t)}{2}\,\Gamma\left[\frac{B-t+1}{2}\right]\Gamma\left[\frac{B+t}{2}\right]} = \frac{\tan\pi\frac{B}{2}}{\tan\pi\frac{(B+t)}{2}}\,U(B) \tag{4.1.11}$$

D is an arbitrary real symmetric S-matrix element, i.e.

$$D(z) = D(-z) \tag{4.1.12a}$$

$$D^*(z) = D(z^*) \qquad (4.1.12b)$$

$$D^{(2)}(z) = 1/D(z) \qquad (4.1.12c)$$

The solution is then essentially in the same form as that found by Rothleitner,[22] and is found to be most convenient, since for t integer,

$\dfrac{\tan \pi \frac{B}{2}}{\tan \pi \frac{(B+t)}{2}}$ becomes a symmetric S-matrix element and may be absorbed

into D, while U reduces to a finite product of terms.

A necessary condition for S_1 and S_2 to have the correct threshold behaviour (4.1.4), is that the factor

$$\frac{B - t - 1}{B + t} \sim 1 + O(q^{\lambda}) \text{ as } z \to 1 \text{ , where } \lambda \geqslant 2\ell+1.$$

This implies that $\beta(z)$ have a pole at z=1 of order at least 2ℓ .

The required bound state pole at z=0 must occur as either a pole of U_0 or a zero of U_0, for if a pole of D, it would be a double pole. Clearly, from (4.1.11), for t non-integer, a necessary condition for this is that B, which is an odd function, not have a pole at the origin, i.e. that $\beta(0)$ be finite. B(z) has then, poles at $z = \pm 1$ and a zero at z = 0. It then follows that $B(z) - \gamma$ has a real zero between $z = \pm 1$ for all real γ .

For t non-integer, U_0 must then have an infinite number of distinct poles and zeros between $z = \pm 1$, and so by condition (g), D must have an infinite number of corresponding zeros and poles in the interval (-1,+1). The zeros and poles will have limit points in the interval [-1,+1] , which by the analyticity requirements for the S-matrix functions, must be at $z = \pm 1$. Since $|D| = 1$, and $|U_0| = 1$ for $z \geqslant 1$, we conclude that U_o and D have essential singularities at threshold for t non-integer. The singularities can only be wholly cancelled if we may write U_o as a product of a symmetric S-matrix function of B and a rational function of B. This, however, is possible only if t is integer (see also, Martin and McGlinn[21] on this point). We shall, therefore, consider only the cases for t integer.

We may now use expression (3.3.7) to write

$$U = \frac{B}{B-t} \prod_{m=1}^{M} \frac{[B+t-(2m-1)]}{[B+t-2m]} \frac{[B-t+(2m-1)]}{[B-t+2m]} \qquad (4.1.13)$$

where $\quad M = t/2 \quad$ for t even

$\quad = (t-1)/2 \quad$ for t odd.

The postulates (f) and (g) above imply now that D may be represented in the rational form

$$D(z) = \prod_{m} \frac{1-ir_m q}{1+ir_m q} \prod_{n} \frac{(1-a_n q)(1+a_n^* q)}{(1+a_n q)(1-a_n^* q)} \qquad (4.1.14)$$

where $\operatorname{Im} r_m = 0$; $\operatorname{Re} a_n > 0$.

D has been normalised so that $D \rightarrow 1$ as $q \rightarrow 0$, and the threshold condition will then be satisfied provided $\beta(z)$ has a pole of order at least 2ℓ at z=1.

The high energy behaviour is obtained by noting that

$$D(z) \xrightarrow[z \to \infty]{} 1 + d_1/z + d_2/z^2 + \ldots\ldots \qquad (4.1.15)$$

$$B(z) \xrightarrow[z \to \infty]{} i\left[\frac{\operatorname{Log} z}{\pi} - \beta(z)\right] \qquad (4.1.16)$$

$$U(z) \xrightarrow[z \to \infty]{} \left[1 + \frac{t}{B(z)}\right] \qquad (4.1.17)$$

(from 4.1.13) and 4.1.16)

Hence,

$$h_\alpha = \frac{S_\alpha - 1}{2iq^{2\ell+1} v} \longrightarrow z^{2c-2\ell-1}\left[\frac{\mu_\alpha}{B(z)} + \frac{d_1}{z} + \frac{d_2}{z^2} + \ldots\right] \qquad (4.1.18)$$

where μ_α is a constant.

Since $\beta(z)$ is an even meromorphic function

$$\beta(z) \underset{z\to\infty}{\longrightarrow} k\, z^{2n} \qquad (n=0,\pm1,\pm2,\ldots..) \qquad (4.1.19)$$

where k is a constant.

Hence $\quad B(z) \approx \text{Log } z \quad$ if $n \leqslant 0$, and

$\qquad B(z) \approx z^{2n} \quad$ if $n > 0$.

The required number of subtractions, K, is determined by the asymptotic behaviours above, so that

for $K = 0$, $\lim\limits_{z\to\infty} h_\alpha(z) = 0$

for $K \geqslant 1$, $\lim\limits_{z\to\infty} \dfrac{h_\alpha(z)}{(z^K)} = 0$; $\quad \lim\limits_{z\to\infty} \dfrac{h_\alpha(z)}{(z^{K-1})} \neq 0$

Hence, for K=0, we have that if $c \leqslant \ell$, there are no restrictions on β and D, while if $c \geqslant \ell + 1$, $n \geqslant c - \ell$. $\qquad$ (4.1.20)

For $K \geqslant 1$, if $c \leqslant \frac{1}{2}[2\ell + 1 + K]$; no conditions on β and D

$\qquad$ if $c > \frac{1}{2}[2\ell + 1 + K]$; $n > c - \frac{1}{2}[2\ell + 1 + K]$, plus conditions on D. $\qquad$ (4.1.21)

Section 4.2 Phase Shift Expressions

We maintain the notation of Section 3.3, and write

$$S_\alpha(z) = \exp\left[2i\, \delta_\alpha(z)\right] \qquad (4.2.1)$$

$$D(z) = \exp\left[-2i\,\theta(z)\right] \tag{4.2.2}$$

$$\frac{B(z)-t-1}{B(z)+t} = \exp\left[-2i\,\psi(z)\right] \tag{4.2.3}$$

$$U(z) = \exp\left[2i\,\varphi(z)\right] \tag{4.2.4}$$

where $\delta_\alpha(z)$, $\theta(z)$, $\psi(z)$, $\varphi(z)$ are real for $z \geqslant 1$.

The phase shifts for each channel may then be written as

$$\Delta\delta_1 = \Delta\varphi - \Delta\psi - \Delta\theta \tag{4.2.5a}$$

$$\Delta\delta_2 = \Delta\varphi - \Delta\theta$$

Using the representation (4.1.14)

$$\theta(z) = \sum_m \tan^{-1}(r_m q) + \sum_n \tan^{-1} \frac{2q\,\mathrm{Im}\,a_n}{1 - q^2|a_n|^2} \tag{4.2.6}$$

so that $$\frac{\Delta\theta}{\pi} = M_+ - M_- + \tfrac{1}{2}(m_+ - m_-) \tag{4.2.7}$$

where $M_\pm$ = Number of a_n's in (4.1.14) with $\mathrm{Im}\ a_n \gtrless 0$;

$m_\pm$ = Number of r_m's in (4.1.14) with $r_m \gtrless 0$.

Hence we see that each pair of symmetricially positioned real or pure imaginary poles (zeros) of D contributes $\frac{1}{2}$ $(-\frac{1}{2})$ to $\Delta\theta/\pi$, while each quadruple of symmetrically positioned complex poles (zeros) contributes +1 (-1).

Also, by (3.3.9) of the previous chapter,

$$\cot\psi(z) = -\frac{2}{(2t+1)}\left[\pi^{-1}\,\mathrm{Log}(z+q) - \frac{z}{q}\beta(z)\right] \tag{4.2.8}$$

so that $$\frac{\Delta\psi}{\pi} = N_+ - N_- + \sigma \tag{4.2.9}$$

where N_+ and N_- are, respectively, the number of poles of $\beta(z)$ on the real axis, $z > 1$, with positive and negative residues, and σ is as given in the following table:

$n \geq 1$		$n \leq 0$	
$k\beta(1) > 0$	$k\beta(1) < 0$	$\beta(1) > 0$	$\beta(1) < 0$
0	1 $(k > 0)$	1	0
	-1 $(k < 0)$		

Table 4.2

It may then be verified, by the results of Appendix 4A to this chapter, that

$$\frac{\Delta\psi}{\pi} = \frac{K_0 - K_1}{2} \tag{4.2.10}$$

where K_γ denotes the number of roots of $B(z) = \gamma$. $\Delta\varphi$ was calculated in Section 3.3 of the previous chapter, and was shown to be given by

$$\Delta\varphi = 0 \quad \text{if } t \text{ is even} \tag{4.2.11}$$
$$\Delta\varphi = \Delta\psi \quad \text{if } t \text{ is odd.}$$

Section 4.3 Location of Bound State Poles

$S_\alpha(z)$ has the same poles as does $h_\alpha(z)$, plus the poles of the cut-off function, $v(z)$, which are of order c, located at $z = \pm i(\kappa^2 - 1)^{\frac{1}{2}}$, on the imaginary axis.

The poles of $S_\alpha(z)$ can occur only at the poles of D, of U(B), and the roots of $B(z) = -t$.

By (4.1.13), we see that the zeros of $U(B)$ in the physical region, lie at the roots of

$$B(z) = \gamma \quad \text{where} \quad \gamma = \pm\, t-(2m-1) \quad , m=1,2,\ldots,M \tag{4.3.1a}$$

$$\text{and} \quad \gamma = 0 \text{ if } t \text{ is odd} ,$$

while the poles of $U(B)$ in the physical region lie at the roots of

$$B(z) = \gamma \quad \text{where } \gamma = t, \text{ and } \gamma = \pm\,[t-2m] \,, \; m=1,\ldots,M \tag{4.3.1b}$$

Concerning the roots of $B = \gamma$, we shall need the following Lemma proved in Appendix 4A:

<u>Lemma</u>

(i) There is no root of $B = \gamma$ on the imaginary axis except for $\gamma = 0$.

(ii) There is no root on the real axis $|z| \geq 1$ for any $\gamma \neq \frac{1}{2}$

(iii) $K_\gamma = K_{-\gamma}$

(iv) K_γ is independent of γ for $|\gamma| > \frac{1}{2}$ and for $|\gamma| < \frac{1}{2}$.

(v) $K_1 = 1 - \nu + \max(Z_\beta, P_\beta)$

where ν = 1 if $\beta(z)$ has a pole at $z = 0$

0 otherwise

and Z_β, P_β are respectively the total number of zeros and poles of $\beta(z)$ on the physical sheet and so are given by

$P_\beta = 2(N + N_+ + N_-)$

$Z_\beta = 2(n + N + N_+ + N_-)$ (n is as defined in (4.1.19))

Here, 2N is the number of poles of $\beta(z)$ not on the real axis $|z| > 1$.

By statement (i) of the Lemma, U can have no poles on the imaginary axis, unless t is an even integer greater than 2, when U will have poles at the zeros of $B(z) = 0$. We must hence, have that for t odd, the cut-off poles occur in D.

The residue of a bound state pole in $h_\alpha(z)$, may have the opposite sign to the residue of the pole in S_α, since $iq < 0$ for z real and between the branch points. In fact, denoting the effective coupling constant squared by Λ_α, and the residue of a bound state pole z_α in S_α by Λ'_α then Λ_α is given in terms of the actual coupling constant squared by

$$\Lambda'_\alpha = (-1)^\ell \, 2v(z_\alpha)\,(1 - z_\alpha^2)^{\ell+\frac{1}{2}} \lambda_\alpha = (-1)^\ell \Lambda_\alpha \qquad (4.3.2)$$

Hence, by condition (a) of Section 4.1, we require that

$$\begin{aligned} \Lambda'_\alpha &\geqslant 0 \text{ for } \ell \text{ even} \\ \Lambda'_\alpha &\leqslant 0 \text{ for } \ell \text{ odd.} \end{aligned} \qquad (4.3.3)$$

Huang and Low[29] claimed, for the t=1 case, that in order that the pole structure of h_α conform to (4.1.6) with positive λ_α, it was necessary that poles of D occur only at roots of $U = 0$, real roots of $B = t + 1$, and at the cut-off poles, i.e., that bound state poles should not occur in D, uncancelled by zeros of the remaining S-matrix factors of the solution. We shall show that this does not follow.

Suppose that D has real, simple poles at $z = \pm z_0$, where $|z_0| < 1$, and that $\operatorname{Res} D\big|_{z=z_0} = R$, so that $\operatorname{Res} D\big|_{z=-z_0} = -R$. Suppose that these poles are, in fact, uncancelled by zeros of other factors. Then

S_1 and S_2 have simple poles at $z = \pm z_o$ and

$$\Lambda'_{10} = \text{Res } S_1 \Big|_{z=z_o} = U(z_o)\,\frac{B_o - t - 1}{B_o + t}\,R$$

and $$\Lambda'_{20} = \text{Res } S_2 \Big|_{z=z_o} = U(z_o)\,R$$

where $B_o = B(z_o)$. Then Λ'_{10} and Λ'_{20} will have the same sign, if and only if $(B_o-t-1)(B_o+t) > 0$, ie. if either $B_o > t+1$, or $B_o < -t$, when it may easily be verified that $U(z_o) > 0$. Hence, provided the r_m of (4.1.14) have a suitable distribution so that R has the correct sign, $\lambda_{o\alpha} > 0$, $(\alpha = 1,2)$, and no inconsistency will arise. The pole structure (4.1.6) will then be guaranteed by crossing symmetry, and the coupling constants will be positive.

However, we do not disallow the existence of such 'extra poles' of D, each of which will contribute a bound state pole in each channel. It will follow from the ensuing analysis and imposition of a bootstrap criterion that, in fact, there should not be any extra poles. The condition that a bound state pole should not occur in both channels, we note, was assumed, without explicit statement, by Huang and Mueller,[32)] for their qualitative analysis.

To sum up, in addition to 'extra', uncancelled real simple poles of D, poles of D can occur only at:

(i) $\pm i(\kappa^2 - 1)^{\frac{1}{2}}$; the cut-off poles

(ii) Roots of $U = 0$.

(iii) Real roots of $B = t + 1$, lying between $z=\pm 1$.

In addition, there must be zeros of D at all complex poles of U. There may, of course, be 'extra' zeros.

The connection between poles and bound states, and the corresponding effective coupling constants squared, are shown in the following table:

Bound State Energy	Channel	Conditions	Coupling Constants
0	1, or 2, or both	Either $U=\infty, D=0$ or $U=0$, $D=\infty$ and UD has a simple pole.	$\Lambda'_1 - \Lambda'_2 = -\frac{(2t+1)}{2t}$ Res UD
$0<\lvert z_0\rvert<1$	z_0 in 1, $-z_0$ in 2.	Either (a) z_0 is a pole of U, D=0 and UD has a simple pole	$\Lambda'_2 = (2t+1)$ Res UD
		or (b) z_0 is a pole of D, U=0 and UD has a simple pole.	$\Lambda'_1 =$ ResUD $\frac{B_0-t-1}{B_0+t}$ $-2(t+1)$
$0<\lvert z_0\rvert<1$	z_0 in both	z_0 is a simple pole of D.	$\Lambda'_1 = \frac{B_0-t-1}{B_0+t}$ U ResD $\Lambda'_2 =$ U ResD
$0<\lvert z_0\rvert<1$	1	B+t=0 (simple zero) $UD \neq 0$,	$\Lambda'_1 = -(2t+1)$ Res $\frac{UD}{B+t}$
$0<\lvert z_0\rvert<1$	2	B-t-1=0 (simple zero) $D=\infty$ (simple pole)	$\Lambda'_2 = U(B=t+1)$ Res D

Table 4.3

Section 4.4 Necessary Bootstrap Conditions

We now restrict the general solutions of the model by imposing the Levinson's Theorem form of the bootstrap criterion, thereby obtaining necessary bootstrap conditions. We have two cases to consider, according to when t is odd and $\Delta\varphi = \Delta\psi$, and when t is even and $\Delta\varphi = 0$.

The case $\Delta\varphi = \Delta\psi$

By (4.2.5) and (4.2.10), we may write

$$\Delta\delta_1 = -\Delta\theta$$

$$\Delta\delta_2 = \Delta\psi - \Delta\theta$$

so that the bootstrap criterion becomes

$$\frac{\Delta\theta}{\pi} = b_1 \qquad (4.4.1)$$

$$\frac{\Delta\psi}{\pi} = b_1 - b_2 \qquad (4.4.2)$$

where b_1 and b_2 are respectively the number of bound states in Channels 1 and 2.

We further write

$$b_1 = u_1 + b_{1t} + p_1 + Y \qquad (4.4.3)$$

$$b_2 = u_2 + b_{21+t} + p_2 + Y \qquad (4.4.4)$$

where u_1 is the number of bound states in Channel 1 at the roots of U=0

b_{1t} " " " " " " " " 1 " " " " B+t=0

p_1 " " " " " " " " 1 " " poles "(B-t)U

u_2 " " " " " " " " 2 " " roots " U=0

b_{21+t} is the number of bound states in Channel 2 at the roots of B-t-1=0

p_2 " " " " " " " " " " " poles of (B-t)U

and Y is the number of extra poles of D, each of which contributes a bound state to both channels.

Now, for all values of t, U has poles at the roots of B=t, and for all values of t other than a positive even integer has zeros at the roots of B=0. Since we are at present considering the case where t is odd, we write

$$U = \frac{B}{B-t} V(B) \tag{4.4.5}$$

t being odd, the cut-off poles must occur in D, so we may write, by (4.2.7)

$$\frac{\Delta\theta}{\pi} = \tfrac{1}{2}c + (\Delta\theta_0 + \Delta\theta_V + \Delta\theta_{-t} + \Delta\theta_{t+1})/\pi + \tfrac{1}{2}(Y-X) \tag{4.4.6}$$

where

$\frac{\Delta_0}{\pi}$ is the contribution to $\frac{\Delta\theta}{\pi}$ from zeros of B = 0

$\frac{\Delta\theta_V}{\pi}$ " " " " " " " " B+t=0

$\frac{\Delta\theta_{-t}}{\pi}$ " " " " " " " " B-(t+1)=0

$\frac{\Delta\theta_{t+1}}{\pi}$ " " " " " " zeros and poles of V

and $\frac{1}{2}X$ is the contribution to $\frac{\Delta\theta}{\pi}$ from 'extra zeros' of D. X and Y are, of course, either zero or positive integers.

Consider the roots of B=-t.

Let there be K_{-tR} real roots and $2K_{-tC}$ complex roots. Therefore,

$$K_{-tR} + 2K_{-tC} = K_{-t}$$

Every complex root must be cancelled by zeros of D.

Suppose K'_{-tR} real roots are cancelled. Then by (4.2.7),

$$\frac{\Delta\theta_{-t}}{\pi} = -K_{-tC} - \tfrac{1}{2}K'_{-tR} \qquad (4.4.7)$$

Also, $$b_{1t} = K_{-tR} - K'_{-tR} \qquad (4.4.8)$$

Therefore, $$\frac{\Delta\theta_{-t}}{\pi} = \tfrac{1}{2}\left[b_{1t} - K_{-t}\right] \qquad (4.4.9)$$

Consider now the roots of B=t+1 (N.B. $t \neq -1$).

Let there be K_{1+tR} real roots, and $2K_{1+tC}$ complex roots. Therefore,

$$K_{1+tR} + 2K_{1+tC} = K_{1+t} .$$

Poles of D may be placed at the real roots but not at the complex roots. Suppose K'_{1+tR} of the real roots are cancelled by poles of D. Then again, by (4.2.7)

$$\frac{\Delta\theta_{1+t}}{\pi} = \tfrac{1}{2}K'_{1+tR} \qquad (4.4.10)$$

$$b_{21+t} = K'_{1+tR} \qquad (4.4.11)$$

and so $$\frac{\Delta\theta_{1+t}}{\pi} = \tfrac{1}{2}b_{21+t} \qquad (4.4.12)$$

Consider the roots of B=0. There is always a root at z=0, unless $\beta(z)$ has a pole at the origin. We therefore designate the number of roots as follows:-

at z=0 : $(1 + 2m_o)\,\delta_{\nu o}$ where ν is as defined in statement (v) of the Lemma of Section 4.3.

at $z\neq 0$: $2K_{OR}$ real roots

$2K_{OI}$ pure imaginary roots.

$4K_{OC}$ complex roots.

Therefore, $(1+2m_o)\,\delta\nu_o + 2(K_{OR} + K_{OI} + 2K_{OC}) = K_O$ (4.4.13)

Let there be a factor $\left[\frac{1 - iq}{1 + iq}\right]^s$ of D to cancel some of the roots at z = 0, and let there be poles of D such as to cancel $2K'_{OR}$, $2K'_{OI}$; $4K'_{OC}$ of the other roots. Hence, the contribution to $\frac{\Delta\theta}{\pi}$ from the B factor of U is

$$\frac{\Delta\theta_o}{\pi} = \tfrac{1}{2}s + \tfrac{1}{2}(K'_{OR} + K'_{OI}) + K'_{OC} \qquad (4.4.14)$$

Suppose V(B) has $2K_{VR}$ real roots, $4K_{VC}$ complex roots, $2Z_{VR}$ real poles, and $4Z_{VC}$ complex poles. (V(B) has no pure imaginary zeros or poles). D must have zeros at all the complex poles. Suppose that D has zeros at $2Z'_{VR}$ of the real poles, poles at $4K'_{VC}$ of the complex roots, and poles at $4K'_{VR}$ of the real roots.

Then, $$\frac{\Delta\theta_V}{\pi} = \tfrac{1}{2}K'_{VR} + K'_{VC} - \tfrac{1}{2}Z'_{VR} - Z_{VC} \qquad (4.4.15)$$

B V(B) D can have at most a simple pole at real roots, and cannot have a pole at the non-real roots. Therefore,

$$K'_{OC} \leqslant K_{OC} \qquad (4.4.16)$$

$$K'_{OI} \leqslant K_{OI} \qquad (4.4.17)$$

$$K'_{VC} \leqslant K_{VC} \qquad (4.4.18)$$

Let z be the number of pairs of simple poles of B V(B) D located at some or all of the non-zero roots of B=0, V=0, and the poles of V.

Then, $K'_{OR} = K_{OR} + K'_{VR} - K_{VR} + Z_{VR} - Z'_{VR} \leqslant z$ (4.4.19)

and from Table 4.3, $u_1 + u_2 + p_1 + p_2 = 2z + 1$ (4.4.20)

(N.B. we are insisting that the target particle occur as a bound state in one of the channels.)

In the neighbourhood of the origin $BD \sim z^x$ where $z = (1+2m_o)\delta_{\nu o} - 2s$. For the target particle to be a bound state of the model, we must have that $x = -1$. This is satisfied only when

$$s = 1 + m_o \quad (4.4.21)$$

$$\nu = 0 \quad (4.4.22)$$

Using inequalities (4.4.16) - (4.4.19), and the expressions (4.4.14) and (4.4.19), we obtain the inequality,

$$2\left[\frac{\Delta\theta_o}{\pi} + \frac{\Delta\theta_V}{\pi}\right] \leqslant \left[\tfrac{1}{2} + \tfrac{1}{2}K_0 + z\right] + \left[(K_{VR} + 2K_{VC}) - (Z_{VR} + 2Z_{VC})\right] \quad (4.4.23)$$

Now, $\frac{\Delta\psi}{\pi} = b_1 - b_2$

$$= (u_1 + b_{1t} + p_1 + Y) - (u_2 + b_{21+t} + p_2 + Y)$$

$$= (u_1 + 2\frac{\Delta\theta_{-t}}{\pi} + K_{-t} + p_1 + Y) - (u_2 + 2\frac{\Delta\theta_{1+t}}{\pi} + p_2 + Y) \quad (4.4.24)$$

where we have used (4.4.9) and (4.4.12).

But, by (4.4.1), $b_1 = (u_1 + 2\frac{\Delta\theta_{-t}}{\pi} + K_{-t} + p_1 + Y) = \frac{\Delta\theta}{\pi}$

$$= \tfrac{1}{2}c + (\Delta\theta_o + \Delta\theta_V + \Delta\theta_{-t} + \Delta\theta_{t+1})/\pi + \tfrac{1}{2}(Y-X)$$

$$\therefore\ 2\left[\frac{\Delta\theta_{-t}}{\pi} - \frac{\Delta\theta_{t+1}}{\pi}\right] = c + 2\left[\frac{\Delta\theta_o}{\pi} + \frac{\Delta\theta_V}{\pi}\right] - (X+Y) - 2(u_1+p_1) - 2K_{-t}$$

Substituting this into (4.4.24),

$$\frac{\Delta\psi}{\pi} = c + 2\left[\frac{\Delta\theta_o}{\pi} + \frac{\Delta\theta_V}{\pi}\right] - (u_1+u_2+p_1+p_2) - K_{-t} - (X+Y)$$

Therefore, using (4.4.23) and (4.4.20)

$$\frac{\Delta\psi}{\pi} \leqslant c + \tfrac{1}{2}K_0 - z - \tfrac{1}{2} - K_{-t} + \left[(K_{VR}+2K_{VC}) - (Z_{VR}+2Z_{VC})\right] - (X+Y)$$

i.e., using (4.2.10)

$$0 \leqslant c + \tfrac{1}{2}K_1 - K_{-t} - z - \tfrac{1}{2} + \left[(K_{VR}+2K_{VC}) - (Z_{VR}+2Z_{VC})\right] - (X+Y)$$

By statement (iv) of the Lemma of Section 4.3, we must have that $K_{-t} = K_1$, while, from the explicit form for U, (4.1.13), we see that

$$\left[(K_{VR}+2K_{VC}) - (Z_{VR}+2Z_{VC})\right] = 0.$$

Hence, we have the inequality

$$0 \leqslant c - \tfrac{1}{2}K_1 - \tfrac{1}{2} - z - (X+Y) ,$$

or, using statement (v) of the Lemma of Section 4.3,

$$0 \leqslant c - (N + N_+ + N_-) - \max(0,n) - 1 - z - (X+Y) \qquad (4.4.25)$$

This is our basic inequality. Before discussing the consequences of (4.4.25), we show that the same inequality holds for the case of t even.

<u>The case $\Delta\psi = 0$</u>

By (4.2.5) we may now write

$$\Delta\delta_1 = -\Delta\psi - \Delta\theta$$

$$\Delta\delta_2 = -\Delta\theta$$

so that the bootstrap criterion becomes

$$\frac{\Delta\theta}{\pi} = b_2 \qquad (4.4.26)$$

$$\frac{\Delta\psi}{\pi} = b_1 - b_2 \qquad (4.4.27)$$

For the case t = 0, $U(B) \equiv 1$, and

$$S_1 = \frac{B - 1}{B} D \qquad (4.4.28)$$

$$S_2 = D \qquad (4.4.29)$$

But then it is seen that if $z = 0$ is a pole of S_2, it as at least a double pole of both S_1 and S_2, and so cannot be a bound state pole for either channel; we will therefore suppose $t \neq 0$.

Much of the previous analysis will still hold with a suitable change of notation.

U now has poles at the roots of B=0, and so we write

$$U(B) = \frac{1}{B(B-t)} V(B) \qquad (4.4.30)$$

We also, now, have the additional complication that the cut-off poles may occur as zeros of B. Suppose that $2c'$ of these multiple order poles occur as poles of D. Then with our redefinition of V, (4.4.6) is replaced by

$$\frac{\Delta\theta}{\pi} = \tfrac{1}{2}c' + (\Delta\theta_o + \Delta\theta_V + \Delta\theta_{-t} + \Delta\theta_{t+1})/ \quad + \tfrac{1}{2}(Y-X) \qquad (4.4.31)$$

Equations (4.4.7) to (4.4.13) remain valid.

D must now have zeros at all complex roots of B=0, and at all imaginary roots other than possibly occuring cut-off poles.

Hence, $K'_{OC} = K_{OC}$, and we write

$$K_{OI} - K'_{OI} = c'' \qquad (4.4.32)$$

where $c' + c'' = c \qquad (4.4.33)$

Let there be a factor $\left[\frac{1 + iq}{1 - iq}\right]^s$ of D to cancel some of the roots of B=0 at z=0.

Then $\frac{\Delta\theta_o}{\pi} = \frac{1}{2}(K'_{VR} - Z'_{VR}) + (K'_{VC} - Z_{VC})$ (4.4.34)

and, as before,

$$\frac{\Delta\theta_V}{\pi} = \tfrac{1}{2}(K'_{VR} - Z'_{VR}) + (K'_{VC} - Z_{VC}) \qquad (4.4.35)$$

If z is the number of pairs of simple poles of $\frac{V(B)\,D}{B}$ located at some or all of the non-zero real roots of B=0, V=0, and the poles of V, then

$$K_{OR} - K'_{OR} + K'_{VR} - K_{VR} + Z_{VR} - Z'_{VR} \leqslant z \qquad (4.4.36)$$

(4.4.20) remains valid.

Suppose, $\frac{D}{B} \sim z^x$ as $z \sim 0$.

Then $2s - (1 + 2m_o)\delta_{\nu o} = x$.

Again we require that $x = -1$, so that we must have

$$s = m_o \text{ and } \nu = 0 \qquad (4.4.37)$$

From (4.4.34) and (4.4.35),

$$2(\Delta\theta_o + \Delta\theta_V)/\pi = -s-(K'_{OR}+K'_{OI}+2K_{OC})+ \left[(K'_{VR}+2K'_{VC})-(Z'_{VR}+2Z_{VC})\right]$$
$$= -s+c''-(K_{OR}+K_{OI}+2K_{OC})+ \left[(K_{VR}+2K_{VC})-(Z_{VR}+2Z_{VC})\right]$$
$$+ \left[(K_{OR}-K'_{OR})+(K'_{VR}-K_{VR})+(Z_{VR}-Z'_{VR})\right] +2(K'_{VC}-K_{VC})$$

where we have used (4.4.32).

Now, $\left[(K_{VR}+2K_{VC}) - (Z_{VR}+2Z_{VC})\right] = K_1$, by the Lemma of Section 4.3, and so since $K'_{vc} \leqslant K_{VC}$, we have using (4.4.36), (4.4.37), and (4.4.13),

$$2(\Delta\theta_o + \Delta\theta_V)/\pi \leqslant c'' + \tfrac{1}{2} + z - \tfrac{1}{2}K_0 + K_1 \qquad (4.4.38)$$

(4.4.24) remains valid; but now

$$b_2 = (u_2 + 2\frac{\Delta\theta}{\pi}1+t + p_2 + Y) = \frac{\Delta\theta}{\pi}$$

$$= \tfrac{1}{2}c' + (\Delta\theta_o + \Delta\theta_V + \Delta\theta_{-t} + \Delta\theta_{t+1})/\pi + \tfrac{1}{2}(Y - X)$$

$$\therefore \quad 2\left[\frac{\Delta\theta_{-t}}{\pi} - \frac{\Delta\theta_{t+1}}{\pi}\right] = -c' - 2(\Delta\theta_o + \Delta\theta_V)/\pi + 2(u_2+p_2) + (Y+X)$$

Hence, substituting into (4.4.24), we obtain

$$\frac{\Delta\psi}{\pi} = -c' + K_{-t} - 2(\Delta\theta_o + \Delta\theta_V)/\pi + (u_1+u_2+p_1+p_2) + (X+Y)$$

Hence, by (4.2.10), (4.4.38), and (4.4.20),

$$0 \leqslant (c' + c'') - \tfrac{1}{2} - z - \tfrac{1}{2}K_1 - (X + Y)$$

(having used $K_1 = K_{-t}$).

But $(c' + c'') = c$, so that using statement (v) of the Lemma of Section 3.3, we have again the inequality derived for the case of t odd:

$$0 \leqslant c - (N + N_+ + N_-) - \max(0,n) - 1 - z - (X + Y) \qquad (4.4.39)$$

Using this inequality, we now prove generalisations of the theorems proved by Huang and Low, which are valid for all integer values of t and all values of ℓ.

<u>Theorem I</u> There exists no bootstrap solution satisfying an unsubtracted dispersion relation, or without a cut-off function.

<u>Proof</u> This theorem is, of course, merely a statement of the asymptotic behaviour of the bootstrapped amplitude.

We saw in Section 4.1, that in order that S_1 and S_2 have the correct threshold behaviour, $\beta(z)$ must have a pole of order at least 2ℓ at z=1. That is, we must have $N \geqslant \ell$.

Hence, in all cases, we see from (4.4.39) that for a bootstrap solution, $c \geqslant \ell + 1$, so that there must always be a cut-off function.

From (4.1.20) we then have, for no subtraction, the necessary condition $n \geqslant c - \ell > 0$

Then, $c - (N + N_+ + N_-) - \max(0,n) - 1 - z - X - Y$

$$\leqslant (\ell - N) - (N_+ + N_-) - 1 - z - X - Y < 0, \text{ since } N \geqslant \ell .$$

Hence, the necessary bootstrap condition (4.4.39) cannot be satisifed if $h_\alpha(z)$ satisfies an unsubtracted dispersion relation, and the theorem is proved.

Theorem II If we require the amplitudes to satisfy a once subtracted dispersion relation, then the bootstrap solution must be such that $c = \ell + 1$, and the most general form for $\beta(z)$ is

$$\beta(z) = q^{-2\ell}\left[\beta_0 + \beta_2 z^2 + \beta_4 z^4 + \ldots\ldots\ldots + \beta_{2\ell} z^2\right]$$

where the coefficients β_{2k} are real and such that $\sum_{k=0}^{\ell} \beta_{2k} \neq 0$

In addition, the S-matrix function $D(z)$ may have no 'extra' zeros or poles.

Proof We know that, as above, $c \geqslant \ell + 1$.

Suppose that $c > \ell + 1$. Hence, by the high energy condition (4.1.21) with $K = 1$, $n > c - (\ell + 1) > 0$.

So, since $\beta(z)$ must have a pole at $z = 1$ of order at least 2ℓ, $N \geqslant \ell$, and the inequality (4.4.39) is again violated.

Hence, $c = \ell + 1$ (4.4.40)

Rewriting (4.4.39), we then have

$$\ell + 1 \geqslant (N + N_+ + N_-) + \max(0,n) + 1 + z + X + Y > 0$$

From this we see that we must have

$$N = \ell \; ; \; N_+ = N_- = 0 \; ; \; n \leqslant 0 \qquad (4.4.41)$$

But by statement (v) of the Lemma of Section 3.3,

$$Z_\beta = 2(n + N) \geqslant 0$$

and so $0 \geqslant n \geqslant -\ell$ (4.4.42)

The most general meromorphic, even function satisfying (4.4.41), (4.4.42), with poles of order 2ℓ at $z = 1$, is

$$\beta(z) = q^{-2\ell}\left[\beta_0 + \beta_2 z^2 + \beta_4 z^4 + \ldots\ldots\ldots + \beta_{2\ell} z^{2\ell}\right] \qquad (4.4.43)$$

where $\sum_{k=0}^{\ell} \beta_{2k} \neq 0$

We must also have that $z = 0$; $X = Y = 0$ (4.4.44)

The theorem is thus proved.

We note that (4.4.40) has the consequence that as $\ell \to \infty$, the cut-off function goes to zero above threshold, so that, in the limit, there is no scattering.

The conditions we have derived for a bootstrap solution are all necessary conditions. By use of them, we can construct explicit solutions and check the sign of the relevant coupling constants and bound state energies, thereby checking the verification of our remaining conditions. However, to do this for arbitrary values of ℓ, requires a near impossible analysis of the position of the $2\ell + 1$ roots of $B = \gamma$, $|\gamma| > \frac{1}{2}$, and of the roots of $B = 0$. We shall, therefore, content ourselves with the construction of explicit solutions for the cases of S-and P-wave scattering.

Section 4.5 The case $\ell = 0$

In this section we shall construct general and explicit solutions of the bootstrapped model for all integer values of t, and for $\ell = 0$, assuming one subtraction.

By (4.4.43), $\beta(z) = \beta_o$, a non-vanishing constant. We then have from the Lemma of Section 4.3, that

$K_1 = 1$, and from the results of Appendix 4A,

$K_o = 1$ if $\beta_o < 0$

$K_o = 3$ if $\beta_o > 0$

Hence, for all values of t, by (4.2.10)

$$\frac{\Delta\psi}{\pi} = b_1 - b_2 = 0 \text{ if } \beta_o < 0$$
$$= 1 \text{ if } \beta_o > 0 \qquad (4.5.1)$$

and by (4.4.20) and (4.4.44)

$$u_1 + u_2 + p_1 + p_2 = 1 \qquad (4.5.2)$$

For $\beta_o > 0$, $B(z)=0$ has a root at $z = 0$, and two further roots, symmetrically positioned about the origin, either on the real axis, $|z| < 1$, or on the imaginary axis. In fact, since

$$B'(z) = (-iq)\left[\left(\frac{1}{\pi} - \beta_o\right) - \frac{z^2}{\pi}\right], \qquad (4.5.3)$$

and $iq < 0$ for $|z| < 1$, we see that the roots all lie on the real axis for $\beta_o < 1/\pi$, are all at the origin if $\beta_o = 1/\pi$, and are all on the imaginary axis if $\beta_o > 1/\pi$.

We summarise the above observations in the following table:

	K_1	K_o	K_{OR}	K_{OI}	m_o	$b_1 - b_2$
$\beta_o < 0$	1	1	0	0	0	0
$0 < \beta_o < 1/\pi$	1	3	1	0	0	1
$\beta_o = 1/\pi$	1	3	0	0	1	1
$\beta_o > 1/\pi$	1	3	0	1	0	1

Table 4.5

We therefore have several distinct cases to consider, according to whether t is even or odd, and according to the sign of β_o.

<u>The case of t odd</u>

The required bound state pole at z=0 occurs as a zero of U=0, and so from (4.5.2) we must have $p_1 = p_2 = 0$, and $(u_1,u_2)=(1,0)$ or $(0,1)$. Suppose firstly that $\beta_o < 0$, so that $b_1 = b_2$, and the roots of $B(z) = \gamma$ are real and simple for all γ .

Hence, since there are the same number of bound states in each channel, we must have

$$(u_1,u_2) = \begin{matrix}(1,0)\\(0,1)\end{matrix} \Leftrightarrow (b_{1t},b_{21+t}) = \begin{matrix}(0,1)\\(1,0)\end{matrix} \qquad (4.5.4)$$

In either case, $b_1 = b_2 = 1$, i.e., there is only one bound state in each channel.

We now have $b_1 = \frac{\Delta\theta}{\pi} = \frac{1}{2} + \left[\frac{(\Delta\theta_o + \Delta\theta_V)}{\pi} + \frac{1}{2}(b_{1t} + b_{21+t})\right] - \frac{1}{2}$

$\therefore \; (\Delta\theta_o + \Delta\theta_V)/\pi = \frac{1}{2}$

But, by (4.4.21) , $s = 1$ (4.5.5)

so that (4.4.14) gives $\frac{\Delta\theta_o}{\pi} = \frac{1}{2}$, and then

$$\frac{\Delta\theta_V}{\pi} = 0, \text{ or, by (4.4.15)} \quad K'_{VR} = Z'_{VR} .$$

But, we must now have, in order that there be only one bound state pole in each channel, $K'_{VR} \leqslant K_{VR}$ while $Z'_{VR} \geqslant Z_{VR}$, and so, since $K_{VR} = Z_{VR}$, we have

$$K'_{VR} = K_{VR} = Z_{VR} = Z'_{VR} ,$$

i.e., D must have simple poles at all the zeros of V and simple zeros at all the poles of V.

Denote the roots of $B = N$, by z_N , $N = 0, \pm 1, \pm 2, \ldots\ldots$

and write $S_N = (1 - z_N^2)^{\frac{1}{2}}$.

Assuming that $(b_{1t}, b_{2l+t}) = (1,0)$, so that there is a bound state at $z=0$ in Channel 2, and a bound state at z_{2n+1} in Channel 1, where $t = 2n+1$, then

$$D(z) = \frac{(1 - iq)}{(1 + iq)} \frac{(1 - iq/\kappa)}{(1 + iq/\kappa)} \prod_{k=1}^{n} \left[\frac{(1 - iq/S_{2k})\ (1 + iq/S_{2k-1})}{(1 + iq/S_{2k})\ (1 - iq/S_{2k-1})} \right]$$

Calculating the effective incident particle-target particle coupling constant squared given by, from Table 4.3,

$\Lambda_2 = \frac{2t+1}{2t} \operatorname{Res} UD \Big|_{z=0}$, we find that $\Lambda_2 < 0$, and so this solution is unacceptable.

Assuming that $(b_{1t}, b_{2l+t}) = (0,1)$ so that the target particle appears as a bound state in Channel 1, while a bound state at the root of $B = 2n+2$ appears in Channel 2. We now have

$$D(z) = \frac{(1 - iq)\ (1 - iq/\kappa)\ (1 - iq/S_{2n+2})\ (1 + iq/S_{2n})}{(1 + iq)\ (1 + iq/\kappa)\ (1 + iq/S_{2n+2})\ (1 - iq/S_{2n})} \times \prod_{k=1}^{n} \left[\frac{(1 - iq/S_{2k})\ (1 + iq/S_{2k-1})}{(1 + iq/S_{2k})\ (1 - iq/S_{2k-1})} \right] \qquad (4.5.6)$$

From Table 4.3, we then calculate the two effective coupling constants squared to be

$$\Lambda_1 = \frac{2(4n+3)}{(2n+1)^2} \prod_{k=1}^{n} \left(\frac{2k}{2k-1}\right)^2 (1/\pi - \beta_0) \frac{(\kappa+1)(S_{2n+2}+1)(S_{2n+1}-1)}{(\kappa-1)(S_{2n+2}-1)(S_{2n+1}+1)} \times$$

$$\prod_{k=1}^{n} \left[\frac{(1+S_{2k})(1-S_{2k-1})}{(1-S_{2k})(1+S_{2k-1})}\right] \qquad (4.5.7a)$$

$$\Lambda_2 = 4(2n+1) \frac{S_{2n+2}^2}{z_{2n+2}} \frac{(1+S_{2n+2})}{(1-S_{2n+2})} \frac{(\kappa+S_{2n+2})}{(\kappa-S_{2n+2})} \frac{(S_{2n+1}-S_{2n+2})}{(S_{2n+1}+S_{2n+2})}$$

$$\times \prod_{k=1}^{2n} \left(\frac{2k}{2k+1}\right) \prod_{k=1}^{n} \left[\frac{(S_{2k}+S_{2n+2})(S_{2k-1}-S_{2n+2})}{(S_{2k}-S_{2n+2})(S_{2k-1}+S_{2n+2})}\right] \qquad (4.5.7b)$$

Clearly $\Lambda_1 > 0$, while, as seen from (4.5.3), B(z) is monotonically increasing on the interval (0,1) for $\beta_0 < 0$, and so $\Lambda_2 > 0$ with a positive bound state pole in Channel 2. This solution is then an acceptable bootstrap solution.

We now investigate the case when $\beta_0 > 0$, so that there is one more bound state in Channel 1, than in Channel 2.

We must have $(u_1,u_2) = (1,0)$ and either $(b_{1t},b_{21+t}) = (0,0)$ or (1,1), i.e. either $b_1 = 1$, $b_2 = 0$, or $b_1 = 2$, $b_2 = 1$.

In both cases $b_1 - \frac{1}{2}(b_{1t} + b_{21+t}) = 1$, and so from the expression for $\frac{\Delta\theta}{\pi}$, we have

$$\frac{\Delta\theta}{\pi}_0 + \frac{\Delta\theta}{\pi}_V = 1.$$

But, by (4.4.14), $\frac{\Delta\theta}{\pi}_0 = \frac{1}{2}$ or 1, while by (4.4.15) and

$Z'_{VR} \geqslant Z_{VR} = K_{VR} \geqslant K'_{VR}$, $\frac{\Delta\theta}{\pi}V \leqslant 0$.

Therefore, the only possibility is that $\frac{\Delta\theta}{\pi}0 = 1$,

i.e., $K'_{OR} + K'_{OI} = 1$ and $\frac{\Delta\theta}{\pi}V = 0$

$\therefore \quad K'_{VR} = Z'_{VR} = Z_{VR} = K_{VR}$

Again, D has simple poles at all the zeros of V, and has simple zeros at all the poles of V.

For $(b_{1t}, b_{2l+t}) = (1,1)$,

$$D(z) = \frac{(1 - iq)\,(1 - iq/\kappa)\,(1 - iq/S_o)\,(1 - iq/S_{2n+2})}{(1 + iq)\,(1 + iq/\kappa)\,(1 + iq/S_o)\,(1 + iq/S_{2n+2})} \times \prod_{k=1}^{n}\left[\frac{(1 + S_{2k})\cdot(1 - S_{2k-1})}{(1 - S_{2k})\,(1 + S_{2k-1})}\right]$$

For the bound state at $z = z_{2n+1}$, Table 4.3 requires that

$$\Lambda_1 = \frac{-(4n + 3)}{2(2n + 1)} \operatorname{Res} \left.\frac{U\,D}{B + 2n+1}\right|_{z=z_{2n+1}}$$

, which is easily verified to be negative. This solution is therefore rejected.

Consider the case $(b_{1t}, b_{2l+t}) = (0,0)$, so that

$$D(z) = \frac{(1 - iq)\,(1 - iq/\kappa)\,(1 + iq/S_{2n+1})\,(1 - iq/S_o)}{(1 + iq)\,(1 + iq/\kappa)\,(1 - iq/S_{2n+1})\,(1 + iq/S_o)} \times \prod_{k=1}^{n}\left[\frac{(1 - iq/S_{2k})\,(1 + iq/S_{2k-1})}{(1 + iq/S_{2k})\,(1 - iq/S_{2k-1})}\right] \qquad (4.5.8)$$

The effective incident particle-target particle coupling constant squared is then given by

$$\Lambda_1 = \frac{4n+3}{2(2n+1)^2} V(0) \operatorname{Res} BD\Big|_{z=0}$$

$$= \frac{2(4n+3)\left(\frac{1}{\pi}-\beta_0\right)(\kappa+1)}{(2n+1)^2(\kappa-1)} \prod_{k=1}^{n}\left(\frac{2k}{2k-1}\right)^2 \frac{(S_{2n+1}-1)(S_0+1)}{(S_{2n+1}+1)(S_0-1)}$$

$$\times \prod_{k=1}^{n}\left[\frac{(1-iq/S_{2k})(1+iq/S_{2k-1})}{(1+iq/S_{2k})(1-iq/S_{2k-1})}\right] \quad (4.5.9)$$

This is positive since $\frac{\pi^{-1}-\beta_0}{1-S_0} \geqslant 0$ for $\beta_0 > 0$, as shown earlier ($S_0 > 1$ if $\beta_0 > 1/\pi$; $S_0 < 1$ if $\beta_0 < 1/\pi$), and we therefore have an acceptable bootstrap solution.

We have, hence, shown that the target - particle always occurs as a bound state in Channel 1, and that there is at most one bound state in Channel 2.

In the case where there is a bound state in each channel, we see that S_1 has no zeros while S_2 will have one zero at the root of $B = -t$, while if there is no bound state in Channel 2, S_1 has a zero at the root of $B = t + 1$, and S_2 has a zero at the root $B = -t$. These zeros become poles on the second Riemann surface of the unitarity branch cut, and so represent virtual states, being real.

The numerical dependence of the coupling constants and bound state energy of the Channel 2 bound state, upon the value of t and β_0 are shown in Figures (4.5.1) and (4.5.2)

The Case of t Even

Now the required bound state pole at z=0 occurs as a pole of U, and so from (4.5.2) we must have $u_1 = u_2 = 0$, and

$(p_1, p_2) = (1,0)$ or $(0,1)$

Consider the situation for $\beta_o < 0$, so that $b_1 = b_2$.

We must then have

$$(p_1, p_2) = \begin{matrix}(1,0)\\(0,1)\end{matrix} \Longleftrightarrow (b_{1t}, b_{21+t}) = \begin{matrix}(0,1)\\(1,0)\end{matrix} \qquad (4.5.10)$$

Suppose $(b_{1t}, b_{21+t}) = (0,1)$.

$K'_{1+tR} = K_{1+t} = 1$ by (4.4.11)

$K'_{-tR} = K_{-t} = 1$ by (4.4.8)

Also, $K'_{OR} = K'_{OI} = K'_{OC} = 0$; $s = m_o = 0$ (by (4.4.37))

and so $\frac{\Delta\theta_0}{\pi} = 0.$

Hence $\frac{\Delta\theta}{\pi} = 1 = 1/2 + \frac{\Delta\theta_V}{\pi}$

i.e., $\frac{\Delta\theta_V}{\pi} = \frac{1}{2}$ and so $K'_{VR} = Z'_{VR} + 1 = Z_{VR} + 1 = K_{VR}$.

D, therefore, has zeros and poles at all the poles and zeros of V.

Putting $t = 2n$, $(n = 1,2,\ldots..)$, we obtain

$$D = \frac{(1 - iq/\kappa)\,(1 - iq/S_{2n+1})\,(1 + iq/S_{2n})\,(1 - iq/S_1)}{(1 + iq/\kappa)\,(1 + iq/S_{2n+1})\,(1 - iq/S_{2n})\,(1 + iq/S_1)}$$

$$\times \prod_{k=1}^{n-1} \left[\frac{(1 - iq/S_{2k+1})\,(1 + iq/S_{2k})}{(1 + iq/S_{2k+1})\,(1 - iq/S_{2k})}\right] \qquad (4.5.11)$$

By Table 4.3, the effective coupling constant squared for the incident particle and target baryon is given by

$$\Lambda_1 = -\left(\frac{4n+1}{4n}\right) \text{Res } UD\Big|_{z=0}$$

$$= \frac{(4n+1)}{8n^2} \frac{(\kappa+1)}{(\kappa-1)} \frac{1}{(1/\pi - \beta_o)} \frac{(1+S_{2n+1})(1-S_{2n})(1+S_1)}{(1-S_{2n+1})(1+S_{2n})(1-S_1)}$$

$$\times \prod_{k=1}^{n-1} \left(\frac{2k+1}{2k}\right)^2 \frac{(1+S_{2k+1})(1-S_{2k})}{(1-S_{2k+1})(1+S_{2k})} \qquad (4.5.12)$$

which is clearly positive.

Similarly for the bound state in Channel 2 at the root of $B = 2n + 1$,

$$\Lambda_2 = \frac{(4n+1)(\kappa+S_{2n+1})}{(\kappa-S_{2n+1})} \prod_{k=1}^{2n} \left(\frac{2k}{2k+1}\right) \prod_{k=0}^{n-1} \frac{S_{2k+1}+S_{2n+1}}{S_{2k+1}-S_{2n+1}} \prod_{k=1}^{n} \frac{S_{2k}-S_{2n+1}}{S_{2k}+S_{2n+1}}$$

$$(4.5.13)$$

This is again positive, $B(z)$ being a monotonic increasing function on the interval (0,1), and the bound state in Channel 2 will have positive energy. We, hence, have an acceptable bootstrap solution.

We try now $(p_1,p_2) = (0,1)$, $(b_{1t},b_{21+t}) = (1,0)$.

$K'_{-tR} = K'_{1+tR} = 0$

and as before, $K'_{VR} = K_{VR}$; $Z'_{VR} = Z_{VR}$.

$$D = \frac{(1 - iq/\kappa)}{(1 + iq/\kappa)} \prod_{k=0}^{n-1} \frac{(1 + iq/S_{2k-1})}{(1 - iq/S_{2k-1})} \prod_{k=1}^{n-1} \frac{(1 + iq/S_{2k})}{(1 - iq/S_{2k})}$$

The effective coupling constant squared for the bound state in Channel 2 is $\Lambda_2 = \frac{2t+1}{2t} \operatorname{Res} UD\big|_{z=0}$ which is easily shown to be negative. This solution is therefore unacceptable.

Suppose now that $\beta_o > 0$, so that $K_o = 3$, and $b_1 = b_2 + 1$. If $(p_1, p_2) = (1,0)$, $(b_{1t}, b_{21+t}) = (1,1)$, then

$$K'_{-tR} = 0 \; ; \; K'_{1+tR} = 1 = K_{1+t} \, .$$

and $$\frac{\Delta\theta}{\pi} = \frac{c'}{2} + \frac{1}{2} + \left(\frac{\Delta\theta_o}{\pi} + \frac{\Delta\theta_V}{\pi}\right) = b_2 = 1$$

$$\therefore \left(\frac{\Delta\theta_o}{\pi} + \frac{\Delta\theta_V}{\pi}\right) = \frac{1 - c'}{2}$$

But, $$\frac{\Delta\theta_o}{\pi} = -\tfrac{1}{2}s - \tfrac{1}{2}(K'_{OR} + K'_{OI}) - K'_{OC}$$

while $K'_{OC} = 0$ and

for $\beta_o < 1/\pi$, $K_{OR} = 1$ and we must have that $K'_{OR} = K_{OR}$; $c'' = 0$

for $\beta_o = 1/\pi$, $s = m_o = 1$; $c'' = 0$

for $\beta_o > 1/\pi$, $K_{OI} = 1$ so that $K'_{OI} = 1 - c''$.

Hence, in all cases, since $c' + c'' = 1$, we have

$\frac{\Delta\theta_V}{\pi} = \frac{1}{2}$; i.e., $Z'_{VR} = Z_{VR}$; $K'_{VR} = K_{VR}$. Hence,

$$D = \frac{(1 - iq/\kappa)\,(1 - iq/S_{2n+1})\,(1 - iq/S_{2n-1})\,(1 + iq/S_o)}{(1 + iq/\kappa)\,(1 + iq/S_{2n+1})\,(1 + iq/S_{2n-1})\,(1 - iq/S_o)} \quad X$$

$$\times \prod_{k=1}^{n-1} \frac{(1 - iq/S_{2k-1})\ (1 + iq/S_{2k})}{(1 + iq/S_{2k-1})\ (1 - iq/S_{2k})}$$

Calculating from this, the effective coupling constant squared for the bound state at z_{-2n} in Channel 1,

$\Lambda_1 = -(4n + 1) \operatorname{Res} \left.\frac{UD}{B+t}\right|_{z = z_{-2n}}$, we find $\Lambda_1 < 0$, and so the assumed distribtuion of bound state poles gives an unacceptable solution.

Try now $b_1 = 1$, $b_2 = 0$, so that $(p_1, p_2) = (1,0)$ and $(b_{1t}, b_{21+t}) = (0,0)$. Hence,

$K'_{1+tR} = 0$; $K'_{-tR} = 1$, and

$$\frac{\Delta\theta}{\pi} = \frac{c'}{2} - \frac{1}{2} + \frac{\Delta\theta_o + \Delta\theta_V}{\pi} = 0.$$

Therefore, as before, $K'_{VR} = K_{VR}$; $Z'_{VR} = Z_{VR}$. Hence,

$$D(z) = \frac{(1 - iq/\kappa)\ (1 + iq/S_o)}{(1 + iq/\kappa)\ (1 - iq/S_o)} \prod_{k=1}^{n} \frac{(1 - iq/S_{2k-1})\ (1 + iq/S_{2k})}{(1 + iq/S_{2k-1})\ (1 - iq/S_{2k})} \quad (4.5.14)$$

The effective coupling constant squared this time is

$$\Lambda_1 = \frac{-4n + 1}{4n} \operatorname{Res} UD\Big|_{z=0}$$

$$= \frac{(4n + 1)}{8n^2} \frac{(\kappa + 1)\ (1 - S_o)}{(\kappa - 1)\ (1 + S_o)} \frac{1}{(1/\pi - \beta_o)} \prod_{k=1}^{n} \frac{(1 - S_{2k-1})(1 + S_{2k})}{(1 + S_{2k-1})(1 - S_{2k})}$$

(4.5.15)

Since, $\dfrac{1 - S_o}{1/\pi - \beta_o} > 0$, $\Lambda_1 > 0$.

Hence, for this distribution of bound state poles, we have a bootstrap solution.

Qualitatively, therefore, the bound state poles for t even are distributed in the same way as for t odd, as are the S-matrix zeros. No resonances can occur, in either case.

In figures (4.5.1a) and (4.5.1b) we show the numerical dependence of the coupling constants squared, calculated via (4.5.7), (4.5.9), (4.5.11), (4.5.12), (4.5.15) and (4.3.2), for various values of t, as functions of β_o.

Similarly in figure (4.5.2) we show the bound state energy of the bound state in Channel 2 as a function β_o, and for various values of t.

The computer program by means of which these calculations were effected, is shown in Appendix 4B.

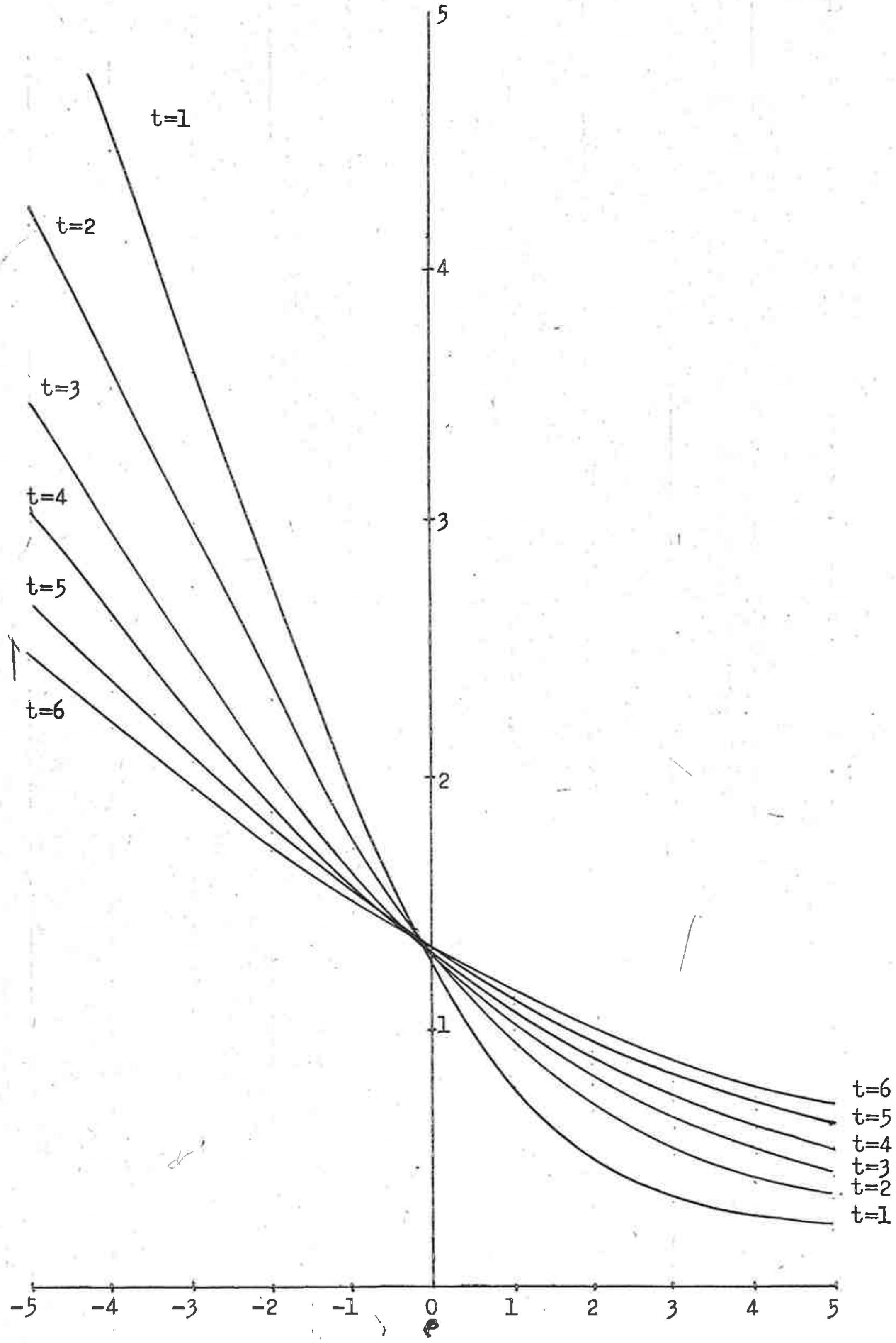


Figure (4.5.1a)

The coupling constant squared for the target baryon as a function of the parameter β , with incident isospin 1,2,3,4,5,6.

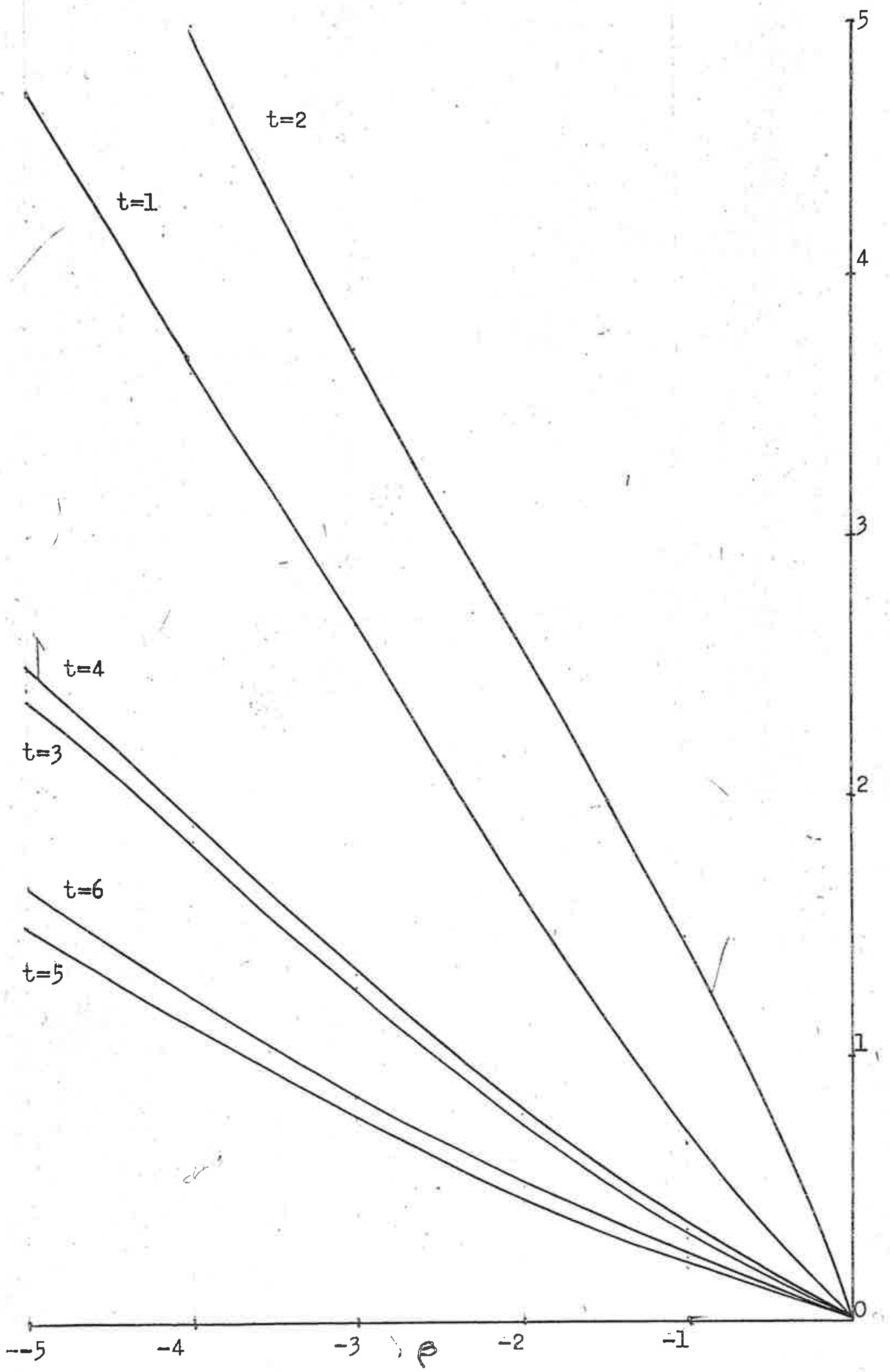
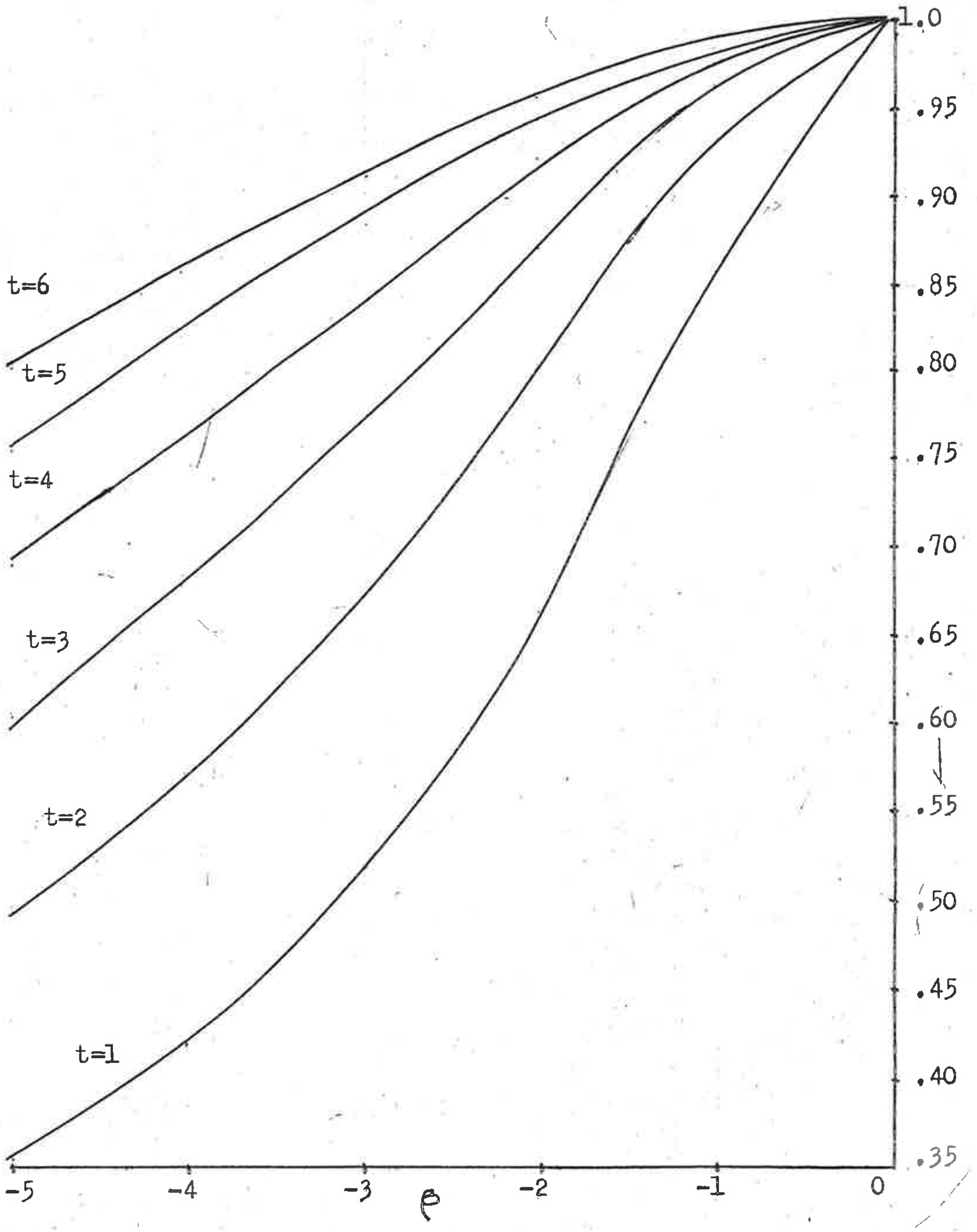


Figure (4.5.1b)

The coupling constant squared for the bound state in Channel 2 as a function of the parameter θ , for incident isospin 1,2,3,4,5,6.



<u>Figure</u> (<u>4.5.2</u>)

The bound state energy of the bound state in Channel 2 as a function of β ,and for incoming isospin 1,2,3,4,5,6.

Section 4.6 The case $\ell = 1$

The procedure for determining the possible bootstrap solutions in the case of P-wave scattering is much the same as that of the previous section.

However, now $K_1 = 3$; and

$$K_o = 3, \quad \beta(1) < 0$$
$$K_o = 5, \quad \beta(1) > 0$$

where $\beta(z) = \dfrac{\beta_o + \beta_1 z^2}{q^2}$, and depending on the relative sign and magnitude of β_o and β_1, there are many more possible pole distributions to investigate. We shall not enter into a full analysis here of all these distinct possibilities, which is is tedious and extremely lengthy, but will merely demonstrate the existence of bootstrap solutions for all integer values of t.

We consider the simplest case when $\beta_o > 0$ and $\beta_1 \geqslant 0$, when $K_o = 5$ so that $b_1 = b_2 + 1$. In addition we suppose that β_o and β_1 are chosen such that $K_{OR} = K_{OI} = 0$; $K_{OC} = 1$ (4.6.1)

Such a distribution is possible (e.g. $\beta_1 = 0$; see Wander's[20] treatment of the t=1 case), but is not, as is easily checked, necessary as asserted by Huang and Low.[29]

We then have that $B(z) = \gamma$, $|\gamma| > \frac{1}{2}$, has one real root in the interval (0,1), and two complex roots.

Consider firstly the case t = 2n + 1, n = 0,1,2,......

As before we have that $p_1 = p_2 = 0$, and

$$(u_1, u_2, b_{1t}, b_{2t+1}) = (1,0,0,0) \quad \text{or} \quad (1,0,1,1) \tag{4.6.2}$$

We show that the former possibility gives an acceptable bootstrap solution.

Now, $m_o = 0$ so that $s = 1$, and then by (4.4.14)

$$\frac{\Delta\theta_o}{\pi} = \frac{1}{2} \quad \text{or} \quad \frac{3}{2}$$

But, $\dfrac{\Delta\theta}{\pi} = b_1 = 1 = \dfrac{c}{2} + \dfrac{\Delta\theta_o + \Delta\theta_V}{\pi} - \dfrac{K_{-t}}{2}$

$$\therefore \quad \frac{\Delta\theta_V}{\pi} = \frac{3}{2} - \frac{\Delta\theta_o}{\pi} \qquad \text{since } c = 2,$$

while $\dfrac{\Delta\theta_V}{\pi} \leqslant \tfrac{1}{2}K_{VR} + K_{VC} - \tfrac{1}{2}Z_{VR} - Z_{VC} = 0.$

Therefore, we must have $\dfrac{\Delta\theta_o}{\pi} = \dfrac{3}{2}$, i.e. $K'_{OC} = K_{OC}$, and

$\dfrac{\Delta\theta_V}{\pi} = 0$, i.e., $K'_{VR} = Z'_{VR} = Z_{VR} = K_{VR}$, and

$$K'_{VC} = K_{VC} = Z_{VC} = Z'_{VC}.$$

Again, D has poles at all the zeros of V, and zeros at all the poles of V.

As before, $K'_{-tR} = K_{-tR} = 1$ and $K'_{1+tR} = 0$.

We are now able to construct the function D(z)

Denote the complex roots of B=0 by $\pm\chi_o$; $\pm\chi_o^*$.

Denote the real root of B = N by z_N ; and the complex roots by χ_N; χ_N^*.

Write
$$t_o = (\chi_o^2 - 1)^{\frac{1}{2}}, \quad \text{Im } t_o > 0$$
$$t_N = (\chi_o^2 - 1)^{\frac{1}{2}}, \quad \text{Im } t_N > 0$$
$$S_N = (1 - z_N^2)^{\frac{1}{2}}$$

Then,

$$D(z) = \frac{(1-iq/\kappa)^2(1+iq/S_{2n+1})(1-q/t_{2n+1})(1+q/t^*_{2n+1})(1+q/t_o)(1-q/t^*_o)}{(1+iq/\kappa)^2(1-iq/S_{2n+1})(1+q/t_{2n+1})(1-q/t^*_{2n+1})(1-q/t_o)(1+q/t^*_o)}$$

$$\times \frac{(1-iq)}{(1+iq)}\prod_{k=1}^{n}\frac{(1-iq/S_{2k})(1+iq/S_{2k-1})(1+q/t_{2k})(1-q/t^*_{2k})(1-q/t_{2k-1})(1+q/t^*_{2k-1})}{(1+iq/S_{2k})(1-iq/S_{2k-1})(1-q/t_{2k})(1+q/t^*_{2k})(1+q/t_{2k-1})(1-q/t^*_{2k-1})} \tag{4.6.3}$$

Using (4.6.3), we calculate, from Table 4.3, and (4.3.2), the effective coupling constant squared to be

$$\Lambda_1 = \frac{2(4n+3)}{(2n+1)^2}\left(\frac{1}{\pi}+\beta_o\right)\frac{(\kappa+1)^2(1-S_{2n+1})}{(\kappa-1)^2(1+S_{2n+1})}\left|\frac{t_{2n+1}+i}{t_{2n+1}-i}\right|^2\left|\frac{t_o+i}{t_o-i}\right|^2$$

$$\times \prod_{k=1}^{n}\left(\frac{2k}{2k-1}\right)^2\frac{(1+S_{2k})(1-S_{2k-1})}{(1-S_{2k})(1+S_{2k-1})}\left|\frac{t_{2k}+i}{t_{2k}-i}\right|^2\left|\frac{t_{2k-1}-i}{t_{2k-1}+i}\right|^2 \tag{4.6.4}$$

$\therefore \Lambda_1 \quad 0.$

We have thus shown the existence of P-wave bootstrap solutions for odd integer values of the crossing-matrix parameter.

Consider now the case $t=2n$. We demonstrate that, once again, the possible bound state distribution $b_1 = 1$, $b_2 = 0$, gives a valid solution.

We have now, $u_1 = u_2 = 0$, and $(p_1, p_2, b_{1t}, b_{21+t}) = (1,0,0,0)$. Again, $m_o = 0$ so that $s = 0$, and then by (4.4.34),

$$\frac{\Delta\theta_o}{\pi} = -1 .$$

$$\therefore \quad \frac{\Delta\theta}{\pi} = b_2 = 1 - 1 + \frac{\Delta\theta_V}{\pi} + \tfrac{1}{2}(b_{1t} + b_{21+t}) - \tfrac{1}{2}K_{-t}$$

i.e. $$\frac{\Delta\theta_V}{\pi} = \tfrac{1}{2}(K'_{VR} - Z_{VR}) + (K'_{VC} - Z_{VC}) = 3/2$$

But, $K'_{VC} \leqslant K_{VC} = Z_{VC} + 1$; $K'_{VR} \leqslant K_{VR} = Z_{VR} + 1$

We must, therefore, have that $K'_{VR} = K_{VR}$; $K'_{VC} = K_{VC}$, i.e., all zeros and poles of $V(B)$ are cancelled by poles and zeros of D.

We also have immediately that: $K'_{-tR} = K_{-tR}$; $K'_{1+tR} = 0$.
Hence, with the same notation as previously,

$$D(z) = \frac{(1-iq/\kappa)(1+iq/S_{2n})(1-q/t_o)(1+q/t_o^*)}{(1+iq/\kappa)(1-iq/S_{2n})(1+q/t_o)(1-q/t_o^*)} \times$$

$$\prod_{k=1}^{n} \frac{(1+q/t_{2k-1})(1-q/t^*_{2k-1})(1-q/t_{2k})(1+q/t^*_{2k})}{(1-q/t_{2k-1})(1+q/t^*_{2k-1})(1+q/t_{2k})(1-q/t^*_{2k})} \qquad (4.6.5)$$

giving,

$$\Lambda_1 = \frac{4n+1}{8n^2} \prod_{k=1}^{n-1} \left(\frac{2k+1}{2k}\right)^2 \frac{1}{1/\pi + \beta_o} \frac{(\kappa+1)^2}{(\kappa-1)^2} \left|\frac{t_o - i}{t_o + i}\right|^2$$

$$\times \prod_{k=1}^{n} \frac{(1+S_{2k-1})(1-S_{2k})}{(1-S_{2k-1})(1+S_{2k})} \left|\frac{t_{2k-1}-i}{t_{2k-1}+i}\right|^2 \left|\frac{t_{2k}-i}{t_{2k}+i}\right|^2 \qquad (4.6.6)$$

Once again, $\Lambda_1 > 0$, and we have an acceptable solution.

Hence, for all integer values of t we have shown the existence of bootstrap solutions satisfying all required conditions.

Precisely similar analyses may be made of all other possible bound state distributions corresponding to the simple case presented above, and for other root distributions of the fundamental function B(z). It is found that the target particle, and only the target particle occurs as a bound state in Channel 1(a consequence of taking t to be positive), and that there is at most one bound state pole, at a root of B(z) = t + 1, in Channel 2. The possibility of a bound state in Channel 2 was erroneously denied by Huang and Low[29] for the case t = 1 (see footnote 8 of Huang and Mueller.[32]).

Appendix 4A. Proof of the Lemma

The Lemma stated in Section 4.3 has been extensively used throughout the chapter, and so now we present a detailed proof. Huang and Low[29] sketched a means of proving this Lemma in Appendix B of their paper, but their Table VI is, in fact, incorrect, although the errors cancel out, leaving the final conclusions intact.

For convenience we firstly restate the lemma:

(i) There is no root of $B = \gamma$ on the imaginary axis except for $\gamma = 0$.

(ii) There is no root on the real axis $|z| \geqslant 1$ for any $\gamma \neq \frac{1}{2}$

(iii) $K_\gamma = K_{-\gamma}$

(iv) K_γ is independent of γ for $|\gamma| > \frac{1}{2}$ and for $|\gamma| < \frac{1}{2}$.

(v) $K_1 = 1 - \nu + \mathrm{Max}(Z_\beta, P_\beta)$

where $\nu = 1$ if $\beta(z)$ has a pole at $z = 0$

$\quad$ 0 otherwise.

and Z_β, P_β are respectively the total number of zeros and poles of $\beta(z)$ on the physical sheet, and so are given by

$$P_\beta = 2(N + N_+ + N_-)$$

$$Z_\beta = 2(n + N + N_+ + N_-)$$ where 2N is the number of poles of $\beta(z)$ not on the real axis $|z| > 1$.

If $z = iu$ is a root of $B(z) = \gamma$, where γ is real, then by the real hermitian property of $B(z)$, we must have

$$B^*(iu) = B(-iu) = \gamma .$$

But $B(z)$ is an odd function, and so $\gamma = -\gamma$, i.e., $\gamma = 0$. This proves statement (i).

Also, since B(z) is odd, if z_o is a root of B(z) = γ, $-z_o$ is a root of B(z) = $-\gamma$, proving statement (ii).

Since, $B(z) + B^{(2)}(z) = 1$, we have,

$\mathrm{Re}\, B(z) = \frac{1}{2}$ for $z \geqslant 1$.

Hence, only for $\gamma = \frac{1}{2}$ can roots occur on the real axis $z \geqslant 1$; note that we are not able to assert, as do Huang and Low, that for an arbitrary $\beta(z)$, real roots of $B(z) = \gamma$ cannot occur on the real axis above threshold for any γ. Statement (iii) will then follow.

We now explicitly compute the number of zeros of $B(z) = \gamma$ in terms of the parameters of the function $\beta(z)$.

Write $F(z) \equiv B(z) - \gamma = e^{i\eta(z)}|B(z) - \gamma|$ (4A.1)

Since $B(z) - \gamma = (\frac{1}{2} - \gamma) + i\left[\pi^{-1}\mathrm{Log}(z+q) - \frac{z}{q}\beta(z)\right]$, (4A.2)

F(z) will have $2N - \nu$ poles not on the real axis $|z| > 1$, where 2N is the number of poles of $\beta(z)$ not on the real axis $|z|>1$, and $\nu = 1$ if $\beta(z)$ has a pole at z=0, and is otherwise zero.

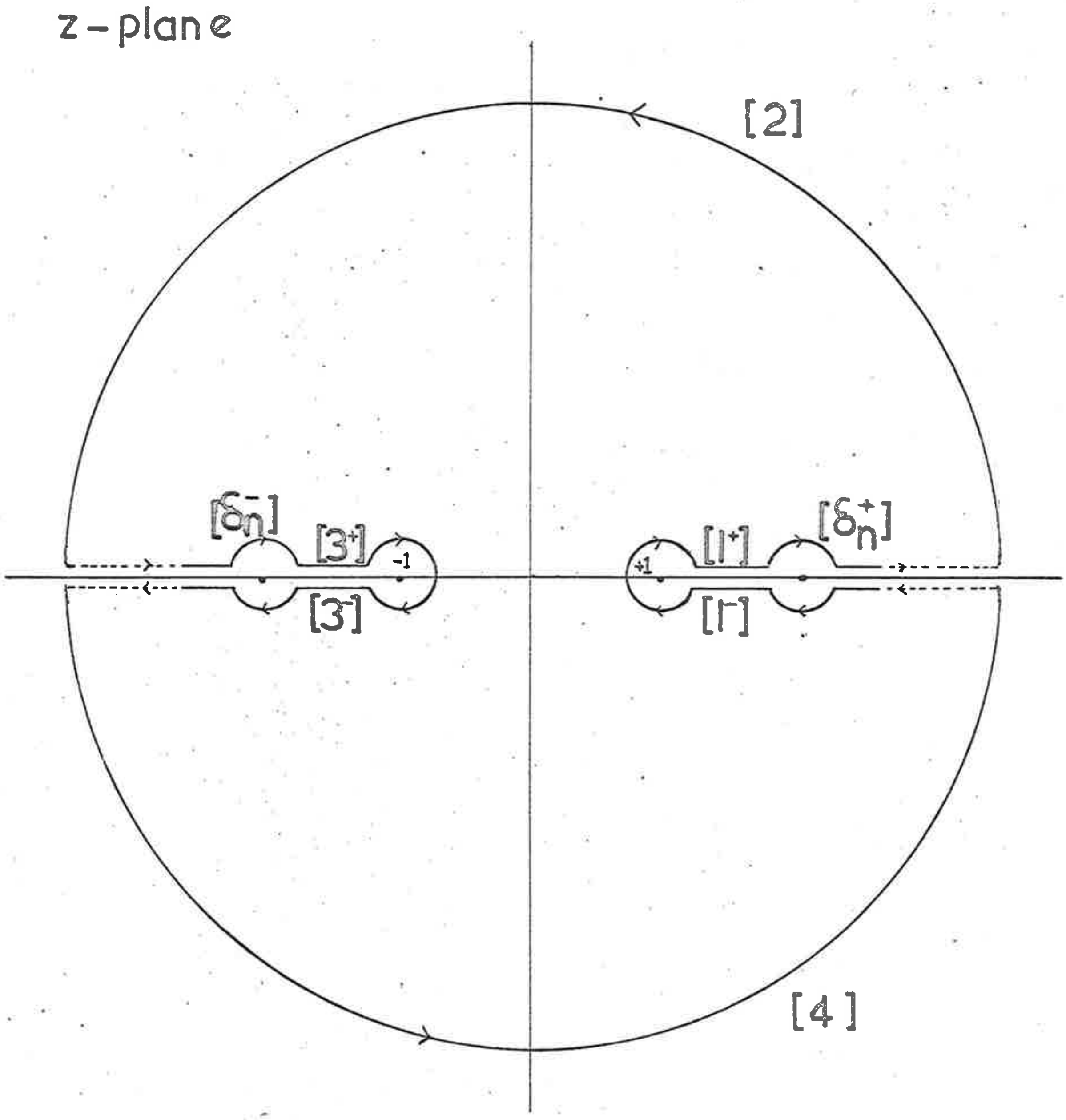
We may then apply the 'principle of the argument'[59], to give

$$\Delta_C \arg F(z) = \Delta_C \eta(z) = 2\pi\left[K_\gamma - (2N - \nu)\right] \qquad (4A.3)$$

where $\Delta_C \eta(z)$ is the change in $\eta(z)$ when z describes the contour C, shown overleaf, in the positively oriented direction.

We then have that $$K_\gamma = \frac{\Delta_C\, \eta(z)}{2} + 2N - \nu \qquad (4A.4)$$

The contour C, as shown, consists of contours $[\delta_n^\pm]$ around each



The Contour C

pole on the real axis $z > 1$ and $z < -1$ respectively, $[1^{\pm}]$ above and below the right-hand cut, $[3^{\pm}]$ above and below the left-hand cut, $[\lambda_1]$ and $[\lambda_2]$ around, respectively, the right and left-hand branch points, and C is closed with arbitrarily large semicircles $[2]$ and $[4]$ in the upper and lower half planes.

We shall assume for simplicity but without loss of generality, as suggested by Huang and Low [29], that $\beta(z)$ has no pole at $z = \pm 1$; if $\beta(z)$ does have poles at the branch points, we may displace them infinitesimally towards the origin and include their count in N.

We easily obtain from (4A.2) that

$$\tan \eta(z) = \frac{\pi^{-1} \operatorname{Log}|z+q| - \operatorname{Re}[z\beta(z)/q]}{\frac{1}{2} - \gamma - \pi^{-1}\arg(z+q) + \operatorname{Im}[z\beta(z)/q]} \qquad (4A.5)$$

which is single valued and continuous at all junctions of the segments of C. The contributions to $\Delta_C \eta$, hence, come only from poles of $\tan\eta(z)$ on each segment, each pole with positive or negative residue additively contributing $-\pi$ or $+\pi$ to $\Delta_C \eta$.

Consider firstly the segment $[2]$, when we may write

$$z = |z| e^{i\theta}, \; 0 < \theta < \pi \;; \; q \sim z \;; \; \beta(z) \sim k z^{2n} .$$

$$\therefore \tan\eta(z) \quad \frac{\pi^{-1} \log 2|z| - k|z|^{2n} \cos 2n\theta}{\frac{1}{2} - \gamma - \theta/\pi + k|z|^{2n} \sin 2n\theta}$$

Hence $\tan\eta(z)$ has poles where

$$f(\theta) \equiv \tfrac{1}{2} - \gamma - \theta/\pi + k|z|^{2n} \sin 2n\theta = 0.$$

Take $n \gg 1$, so that $f(\theta_m) = 0$, where

$$\sin 2n\theta_m = \frac{-1}{k|z|^{2n}}\left[\tfrac{1}{2} - \gamma - \frac{\theta_m}{\pi}\right]$$

and $\theta_m = \dfrac{m\pi}{2n} + \varphi_m$, $0 \leqslant m \leqslant 2n$; $|\varphi_m| \ll 1.$

Substituting into the above expression for $\sin 2n\theta_m$, we have

$$\varphi_m \doteq \frac{\tfrac{1}{2} - \gamma - m/2n}{1/\pi - (-1)^m\, 2n\, k\, |z|^{2n}}$$

The residue of a pole at θ_m is

$$\frac{\pi^{-1} \log 2|z| - k|z|^{2n}\cos 2n\theta_m}{f'(\theta_m)}$$

$$= \frac{\pi^{-1} \log 2|z| - k|z|^{2n}(-1)^m}{\tfrac{1}{2} - \gamma - 1/\pi + (-1)^m k|z|^{2n}}$$

which is clearly negative.

There will be a pole at θ_0 if $\varphi_0 > 0$,

i.e. if $\dfrac{\tfrac{1}{2} - \gamma}{1/\pi - 2nk|z|^{2n}}$, or $(\gamma - \tfrac{1}{2})\, k > 0$

There will be a pole at θ_{2n} if $\varphi_{2n} < 0$

i.e. if $\dfrac{-\tfrac{1}{2} - \gamma}{1/\pi - 2nk|z|^{2n}}$ or $(\gamma + \tfrac{1}{2})k < 0.$

Hence, for $n > 0$, the contribution to $\Delta\eta/\pi$ from [2] is

$$\begin{array}{lll} 2n & \text{if} & |\gamma| \quad \tfrac{1}{2},\ k > 0 \\ 2n-1 & \text{if} & |\gamma| \quad \tfrac{1}{2}.\ k > 0 \\ 2n & \text{if} & |\gamma| \quad \tfrac{1}{2}.\ k < 0 \\ 2n+1 & \text{if} & |\gamma| \quad \tfrac{1}{2}\ \ k < 0 \end{array} \qquad (4A.6)$$

Suppose now $n < 0$.

$\therefore \quad f(\theta) = \frac{1}{2} - \gamma - \theta/\pi$.

and there will be one pole of negative residue, if

$0 < \frac{1}{2} - \gamma < 1$, i.e. $\gamma < \frac{1}{2}$, and no contribution

otherwise.

On contour [4] , $z = |z|\,e^{i\theta'}$; $\pi < \theta' < 2\pi$

Hence, $q \sim -|z|\,e^{i\theta'}$ (Im $q \geqslant 0$ on the physical sheet).

In fact, $q = -|z|e^{i\theta'}\left[1 - \frac{1}{2}\frac{e^{-2i\theta'}}{|z|^2}\right] + O(|z|^{-3})$

$$z + q = \frac{e^{-i\theta'}}{2|z|} + O(z^{-3})$$

Hence, $|z + q| = \frac{1}{2|z|} + O\left(\frac{1}{|z|^3}\right)$

and $\arg(z + q) = 2\pi - \theta'$

Put $\theta' = \pi + \theta$; $0 < \theta < \pi$, so that $\arg(z + q) = \pi - \theta$

We then have $\tan \eta(z) \sim \dfrac{\pi^{-1}\log 2|z| - k|z|^{2n}\cos 2n\theta}{\frac{1}{2} + \gamma - \theta/\pi + k|z|^{2n}\sin 2n\theta}$

which is identical to the expression for $\tan\eta(z)$ for z on [2], except for the change of sign of γ . Therefore, for $n > 0$, the contributions from [4] are given by (4A.6) , and for $n < 0$, as from [2] with change of sign of γ .

We then have that the contribution to $\Delta\eta/\pi$ from [2] and [4] is

$$\begin{aligned}
\text{for } n > 0:\quad & 2n && \text{if } |\gamma| > \tfrac{1}{2} \\
& 2n-1 && \text{if } |\gamma| < \tfrac{1}{2} \text{ and } k > 0 \\
& 2n+1 && \text{if } |\gamma| < \tfrac{1}{2} \text{ and } k < 0 \qquad \text{(4A.7)} \\
\text{for } n \leqslant 0:\quad & 0 && \text{if } |\gamma| > \tfrac{1}{2} \\
& 1 && \text{if } |\gamma| < \tfrac{1}{2}
\end{aligned}$$

We now evaluate the contribution from $[\delta_n^+]$.

We write $$\beta(z) = \frac{R_n}{z - z_n} + f(z) ,$$

where f(z) is real analytic and finite in a neighbourhood of the pole at z_n .

Put $z = z_n + \epsilon e^{i\theta}$ where $0 < \theta < \pi$; $0 < \epsilon \ll 1$

$$\therefore \quad \beta(z) = \frac{R_n e^{-i\theta}}{\epsilon} + f(z_n) + O(\epsilon) \tag{4A.8}$$

$$q = (z_n^2 - 1)^{\frac{1}{2}}\left[1 + \frac{z_n \epsilon e^{i\theta}}{z_n^2 - 1}\right] + O(\epsilon^2) \quad \text{above the cut.}$$

Hence, $$z + q = \left[z_n + (z_n^2 - 1)^{\frac{1}{2}}\right]\left[1 + \frac{\epsilon e^{i\theta}}{(z_n^2 - 1)^{\frac{1}{2}}}\right] + O(\epsilon^2)$$

$$\therefore \quad |z + q| = \left[z_n + (z_n^2 - 1)^{\frac{1}{2}}\right]\left[1 + \frac{\epsilon \cos\theta}{(z_n^2 - 1)^{\frac{1}{2}}}\right] + O(\epsilon^2)$$

and $$\log|z + q| = \log\left[z_n + (z_n^2 - 1)^{\frac{1}{2}}\right] + \frac{\epsilon \cos\theta}{(z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon^2)$$

Put $\chi_n^+ = \arg (z + q)$

$$\therefore \quad \tan \chi_n^+ = \frac{\epsilon \sin\theta}{(z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon^2)$$

$$\therefore \quad \chi_n^+ = \frac{\epsilon \sin\theta}{(z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon^2)$$

Also, $$\frac{z}{q} = \frac{1}{(z_n^2 - 1)^{\frac{1}{2}}}\left[z_n - \frac{\epsilon e^{i\theta}}{z_n^2 - 1}\right]$$

We therefore calculate, using (4A.8)

$$\frac{z}{q}\beta(z) = \frac{1}{(z_n^2-1)^{\frac{1}{2}}}\left[\frac{z_nR_n}{\epsilon}e^{-i\theta} + z_n f(z_n) - \frac{R_n}{(z_n^2-1)}\right] + O(\epsilon)$$

$$\therefore \operatorname{Re}\frac{z}{q}\beta(z) = \frac{1}{(z_n^2-1)^{\frac{1}{2}}}\left[\frac{z_nR_n}{\epsilon}\cos\theta + z_nf(z_n) - \frac{R_n}{(z_n^2-1)}\right] + O(\epsilon)$$

$$\operatorname{Im}\frac{z}{q}\beta(z) = \frac{-\sin\theta}{(z_n^2-1)^{\frac{1}{2}}}\,\frac{z_nR_n}{\epsilon} + O(\epsilon)$$

Hence, above the cut,

$$\tan\eta_n^+ = \frac{\dfrac{-z_nR_n}{(z_n^2-1)^{\frac{1}{2}}}\cos\theta + O(\epsilon)}{\left[(\tfrac{1}{2}-\gamma)\epsilon - \dfrac{z_nR_n}{(z_n^2-1)^{\frac{1}{2}}}\sin\theta\right]}$$

Therefore, if $R_n > 0$, there will be 2 poles if $\gamma < \frac{1}{2}$

0 poles if $\gamma > \frac{1}{2}$

if $R_n < 0$, there will be 2 poles if $\gamma > \frac{1}{2}$

0 poles if $\gamma < \frac{1}{2}$

The contributions at the poles are in all cases positive, since θ is decreasing from π to 0.

Below the cut, put $z = z_n + \epsilon e^{-i\theta}$; $0 < \theta < \pi$

and so $q = -(z_n^2-1)^{\frac{1}{2}}\left[1 + \dfrac{z_n\epsilon e^{-i\theta}}{z_n^2-1}\right]$

$$\therefore \log|z+q| = \log\left[z_n - (z_n^2-1)^{\frac{1}{2}}\right] - \frac{\epsilon\cos\theta}{(z_n^2-1)^{\frac{1}{2}}}$$

If $\chi_n^- = \arg(z + q)$, below the cut, we obtain, as before

$$\tan \chi_n^- = \frac{\epsilon \sin \theta}{(z_n^2 - 1)^{\frac{1}{2}}}$$

$$\therefore \frac{\chi_n^-}{\pi} = \frac{\epsilon \sin \theta}{\pi (z_n^2 - 1)^{\frac{1}{2}}}$$

Also, $\frac{z}{q} = \frac{-1}{(z_n^2 - 1)^{\frac{1}{2}}} \left[z_n - \frac{\epsilon e^{-i\theta}}{z_n^2 - 1} \right]$

$$\beta(z) = \frac{R_n}{\epsilon} e^{i\theta} + f(z_n) + O(\epsilon)$$

$$\text{Re } \frac{z}{q} \beta(z) = \frac{-1}{(z_n^2 - 1)^{\frac{1}{2}}} \left[\frac{z_n R_n}{\epsilon} \cos\theta + z_n f(z_n) - \frac{R_n}{z_n^2 - 1} \right] + O(\epsilon)$$

$$\text{Im } \frac{z}{q} \cdot \beta(z) = \frac{-\sin \theta}{(z_n^2 - 1)^{\frac{1}{2}}} \frac{z_n R_n}{\epsilon} + O(\epsilon)$$

$$\therefore \tan \eta_n^- = \frac{\dfrac{z_n R_n \cos \theta}{(z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon)}{\epsilon (\frac{1}{2} - \gamma) - \dfrac{z_n R_n \sin \theta}{(z_n^2 - 1)^{\frac{1}{2}}}}$$

As before, if $R_n > 0$, there will be 2 poles if $\gamma < \frac{1}{2}$

0 poles if $\gamma > \frac{1}{2}$

if $R_n < 0$, there will be 2 poles if $\gamma > \frac{1}{2}$

0 poles if $\gamma < \frac{1}{2}$

θ is, this time, increasing and so the contributions will be positive.

We now calculate the contribution from $[\delta_n]$, above the cut.

Write, as before, $z = -z_n + \epsilon e^{i\theta}$; $0 < \theta < \pi$

$$q = -(z_n^2 - 1)^{\frac{1}{2}} \left[1 - \frac{z_n \epsilon e^{i\theta}}{z_n^2 - 1} \right] + O(\epsilon^2)$$

$$\therefore \quad z + q = -\left[z_n + (z_n^2 - 1)^{\frac{1}{2}} \right] \left[1 - \frac{\epsilon e^{i\theta}}{(z_n^2 - 1)^{\frac{1}{2}}} \right] + O(\epsilon^2)$$

$$\log |z + q| = \log \left[z_n + (z_n^2 - 1)^{\frac{1}{2}} \right] - \frac{\epsilon \cos \theta}{(z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon^2)$$

If $\chi_{-n}^{+} = \arg(z + q)$,

$$\tan \chi_{-n}^{+} = -\frac{\epsilon \sin \theta}{(z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon^2)$$

$$\frac{\chi_{-n}^{+}}{\pi} = 1 - \frac{\epsilon \sin \theta}{\pi (z_n^2 - 1)^{\frac{1}{2}}} + O(\epsilon^2)$$

Also, $\dfrac{z}{q} = \dfrac{1}{(z_n^2 - 1)^{\frac{1}{2}}} \left[z_n + \dfrac{\epsilon e^{i\theta}}{(z_n^2 - 1)^{\frac{1}{2}}} \right] + O(\epsilon^2)$

and $\beta(z) = \dfrac{-R_n}{z + z_n} + g(z)$

$$= \frac{-R_n e^{-i\theta}}{\epsilon} + g(-z_n) + O(\epsilon)$$

($\beta(z)$ is even, and so the residue at $-z_n$ is the negative of the residue at $+z_n$).

$$\therefore \quad \text{Re}\ \frac{z}{q}\ \beta(z) = \frac{1}{(z_n^2-1)^{\frac{1}{2}}}\left[\frac{-z_n R_n \cos\theta}{\epsilon} + z_n\, g(-z_n) \frac{-R_n}{z_n^2-1}\right] + 0(\epsilon)$$

$$\text{Im}\ \frac{z}{q}\ \beta(z) = \frac{z_n R_n \sin\theta}{\epsilon\,(z_n^2-1)^{\frac{1}{2}}} + 0(\epsilon)$$

$$\therefore \quad \tan \eta^{+}_{-n} = \frac{z_n R_n \cos\theta + 0(\epsilon)}{\left[(z_n^2-1)^{\frac{1}{2}}\ (-\tfrac{1}{2}-\gamma) + \dfrac{z_n R_n \sin\theta}{(z_n^2-1)^{\frac{1}{2}}}\right]}$$

$\therefore$ If $R_n > 0$, there will be two poles if $\gamma > -\frac{1}{2}$.

and 0 poles if $\gamma < -\frac{1}{2}$

If $R_n < 0$, there will be two poles if $\gamma < -\frac{1}{2}$

and 0 poles if $\gamma > -\frac{1}{2}$

θ is decreasing and so the contributions are positive.

As before, an exactly similar contribution will arise from the contour below the real axis.

We may conclude that:

Each pole of positive residue contributes 4 to $\Delta\eta/2\pi$ if $|\gamma| < \frac{1}{2}$

" " " " " " 2 " " " $|\gamma| > \frac{1}{2}$

" " " negative " " 2 " " " $|\gamma| > \frac{1}{2}$

" " " " " " 0 " " " $|\gamma| < \frac{1}{2}$

It remains to calculate the contributions from the contours around the branch points.

For $[\lambda_1]$, write $z = 1 + \epsilon e^{i\theta}$; $0 < \theta < 2\pi$

$$\therefore\ q = \left[2\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}\right]^{\frac{1}{2}}$$
$$= 2^{\frac{1}{2}}\epsilon^{\frac{1}{2}} e^{i\theta/2} + O(\epsilon^2)$$

$$z + q = 1 + \epsilon e^{i\theta} + 2^{\frac{1}{2}}\epsilon^{\frac{1}{2}} e^{i\theta/2} + O(\epsilon^2)$$

$$\therefore\ |z+q| = 1 + \epsilon(\cos\theta + 1) + 2^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\cos\theta/2 + O(\epsilon^{3/2})$$

$$\therefore\ \log|z+q| = \epsilon(\cos\theta + 1) + 2^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\cos\theta/2 + O(\epsilon^{3/2})$$

Putting $\chi = \arg(z+q)$,

$$\tan\chi = 2^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\sin\theta/2 + O(\epsilon^{3/2})$$

$$\therefore\ \chi = 2^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\sin\theta/2 + O(\epsilon)$$

We are assuming $\beta(1)$ to be finite, and so write

$$\beta(z) = \beta(1) + \epsilon e^{i\theta}\beta'(1) + O(\epsilon^2)$$

$$\therefore\ \frac{z}{q}\beta(z) = \frac{\beta(1) + \epsilon e^{i\theta}\beta'(1)}{2^{\frac{1}{2}}\epsilon^{\frac{1}{2}} e^{i\theta/2}} + O(\epsilon^{3/2})$$

$$\therefore\ \mathrm{Re}\,\frac{z}{q}\beta(z) = \frac{1}{2^{\frac{1}{2}}\epsilon^{\frac{1}{2}}}\left[\beta(1)\cos\theta/2 + \epsilon\beta'(1)\cos\theta/2\right] + O(\epsilon^{3/2})$$

$$\mathrm{Im}\,\frac{z}{q}\beta(z) = \frac{1}{2^{\frac{1}{2}}\epsilon^{\frac{1}{2}}}\left[-\beta(1)\sin\theta/2 + \beta'(1)\sin\theta/2\right] + O(\epsilon^{3/2})$$

$$\therefore\ \tan\eta(z) = \frac{-\beta(1)\cos\theta/2}{(\frac{1}{2}-\gamma)\epsilon^{\frac{1}{2}} - \beta(1)\sin\theta/2} + O(\epsilon)$$

$\therefore$ if $\beta(1) > 0$, there are two poles if $\gamma < \frac{1}{2}$

" $\beta(1) > 0$, " " 0 " " $\gamma > \frac{1}{2}$

" $\beta(1) < 0$, " " 0 " " $\gamma < \frac{1}{2}$

" $\beta(1) > 0$, " " two " " $\gamma > \frac{1}{2}$

The residue at all poles is negative since θ is decreasing around $[\lambda_1]$.

Around $[\lambda_2]$ put $z = -1 + \epsilon e^{i\theta}$; $-\pi < \theta < \pi$

$$\therefore \quad q = i\, 2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} e^{i\theta/2} + O(\epsilon^{3/2})$$

and
$$\frac{z}{q} = \frac{i(1 - \epsilon e^{i\theta})}{2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} e^{i\theta/2}}$$

$$z + q = -1 + \epsilon e^{i\theta} + i\, 2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} e^{i\theta/2}$$

$$\therefore \quad |z + q| = 1 + \epsilon(1 - \cos\theta) + 2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \sin\theta/2 + O(\epsilon^{3/2})$$

so
$$\log|z + q| = \epsilon(1 - \cos\theta) + 2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \sin\theta/2 + O(\epsilon^{3/2})$$

Putting $\chi = \arg(z + q)$,

$$\tan\chi = -2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \cos\theta/2 + O(\epsilon^{3/2})$$

$$\therefore \quad \frac{\chi}{\pi} = 1 - \frac{2^{\frac{1}{2}} \epsilon^{\frac{1}{2}}}{\pi} \cos\theta/2 + O(\epsilon^{3/2})$$

$$\beta(z) = \beta(-1) + O(\epsilon) = \beta(1) + O(\epsilon)$$

$$\therefore \quad \frac{z}{q}\beta(z) = \frac{i(1 - \epsilon e^{i\theta})}{2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} e^{i\theta/2}} \beta(1) + O(\epsilon^{\frac{1}{2}})$$

$$\mathrm{Re}\, \frac{z}{q}\beta(z) = \frac{\beta(1)\sin\theta/2}{2^{\frac{1}{2}} \epsilon^{\frac{1}{2}}} + O(\epsilon^{\frac{1}{2}})$$

$$\mathrm{Im}\, \frac{z}{q}\beta(z) = \frac{\beta(1)\cos\theta/2}{2^{\frac{1}{2}} \epsilon^{\frac{1}{2}}} + O(\epsilon^{\frac{1}{2}})$$

$$\tan\eta(z) = \frac{-\beta(1)\sin\theta/2}{(-\frac{1}{2} - \gamma)2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} + \beta(1)\cos\theta/2} + O(\epsilon^{\frac{1}{2}})$$

if $\beta(1) > 0$,	there are	two	poles if	$\gamma > -\frac{1}{2}$
" " "	" "	0	" "	$\gamma < -\frac{1}{2}$
if $\beta(1) < 0$,	" "	two	" "	$\gamma < -\frac{1}{2}$
" " "	" "	0	" "	$\gamma > -\frac{1}{2}$

The poles are again of negative residue.

Hence, the contribution to $\Delta_C \eta /2$ from $[\lambda_1]$ and $[\lambda_2]$ is:

if $\beta(1) > 0$,	2	if $\lvert\gamma\rvert < \frac{1}{2}$
	1	if $\lvert\gamma\rvert > \frac{1}{2}$
if $\beta(1) < 0$,	0	if $\lvert\gamma\rvert < \frac{1}{2}$
	1	if $\lvert\gamma\rvert > \frac{1}{2}$

There are no other contributions.

Our final result for the value of $\Delta_C \eta /2\pi$ around the entire contour C, are tabulated in the following table, proving the required results.

$n \geqslant 1$

	$k > 0$		$k < 0$	
	$\beta(1) > 0$	$\beta(1) < 0$	$\beta(1) > 0$	$\beta(1) < 0$
$\lvert\gamma\rvert < \frac{1}{2}$:	$2n+1+4N_+$	$2n-1+4N_+$	$2n+3+4N_+$	$2n+1+4N_+$
$\lvert\gamma\rvert > \frac{1}{2}$:	$2n+1+2(N_++N_-)$	$2n+1+2(N_++N_-)$	$2n+1+2(N_++N_-)$	$2n+1+2(N_++N_-)$

$n \leqslant 0$

	$\beta(1) > 0$	$\beta(1) < 0$
$\lvert\gamma\rvert < \frac{1}{2}$:	$3 + 4N_+$	$1 + 4N_+$
$\lvert\gamma\rvert > \frac{1}{2}$:	$2(N_+ + N_-) + 1$	$2(N_+ + N_-) + 1$

Appendix 4B Computer Program

The S-wave coupling constants squared, and bound state energies were computed for values of the crossing-matrix parameter $t = 1$ to 6, and for a range of values of the parameter β from -5 to +5. The following program was used to effect this calculation, and was written for the University of Adelaide, C.S.I.R.O., C.D.C. 3200 machine in Fortran IV code.

```
      PROGRAM CPLCNST
      COMMON B,P,X
      P=.3183
      X=7.5
C     X IS THE CUTOFF PARAMETER KAPPA.  P IS INVERSE PI. B IS BETA.
      WRITE(61,17)
   17 FORMAT(1H140HCALC OF CPLNGCNSTS AND BDSTATE ENERGIES)
      DO 22 I=1,9,1
      B=-I
      B=B/10.
      WRITE(61,23) B
   23 FORMAT(1X9HB EQUALS F4.1)
   22 CALL CALCNEG
      DO 24 I=1,5
      B=-I
      WRITE(61,25) B
```

```
25   FORMAT(1X9HB EQUALS F3.0)
24   CALL CALCNEG
     DO 18 I=1,5
     WRITE(61,19) I
19   FORMAT(1X9HB EQUALS I2)
     B=I
18   CALL CALCPOS
     DO 20 I=1,9
     B=I
     B=B/10.
     WRITE(61,21) B
21   FORMAT(1X9HB EQUALS F3.1)
20   CALL CALCPOS
     STOP
     END

     FUNCTION WF(K)
C    TO CALCULATE REAL ROOTS OF B=K BY SEARCH METHOD
     COMMON B,P
     COMMON/DATA/E
     DATA(B=1.E-4)
     AK=K
     IF(B.LT.O.)1,2
1    W1=1.0E-2
     W2=.9999
```

```
      GO TO 3
    2 W1=-1.0E-2
      W2=-.9999
      F1=FN(W1)-AK
      F2=FN(W2)-AK
C     FN IS A PROGRAM FUNCTION FOR B(Z)
      IF(F2.LE.O.) 15,14
   15 X=W2
      GO TO 6
   14 X=(W1+W2)*.5
      F=FN(X)-AK
      IF(F) 5,6,7
    5 IF(F.GE.-E) 6,8
    7 IF(F.LE.E) 6,81
    8 IF(F1) 11,10,9
   81 IF(F1) 9,10,11
    9 W2=X
      F2=F
      GO TO 14
   11 W1=X
      F1=F
      GO TO 14
   10 WF=W1
    6 WF=X
      RETURN
      END
```

```
      FUNCTION FN(W)
C     TO COMPUTE FUNCTION VALUES OF B(Z)
      COMMON B,P
      A=SQRT(1.-W*W)
      FN=P*ASIN(W)-W*B/A
      RETURN
      END

      FUNCTION SF(K)
      A=WF(K)
      SF=SQRT(1.-A*A)
      RETURN
      END

      FUNCTION TF(K,L)
      X=SF(K)
      Y=SF(L)
      TF=(X+Y)/(X-Y)
      RETURN
      END

      FUNCTION FACT(N)
      S1=SF(N)
      S2=SF(N+1)
      FACT=(1.-S1)*(1.+S2)/((1.-S2)*(1.+S1))
      RETURN
      END
```

```
      SUBROUTINE CALCPOS
C     TO CALC CHANNEL 1 CPLNG CONST FOR BETA POSITIVE
      COMMON B,P,X
      REAL M1,M2,M3,M4,M5,M6
C     MI IS CHANNEL 1 CPLNG CONST,T=I
C     TO FIND IMAGINARY ROOTS OF B=0 BY NEWTONS METHOD
      IF(B.LE..3) 8,9
    9 U1=EXP(B/P)*.5
   10 W=1.+U1*U1
      V=SQRT(W)
      U2=U1-(P*ALOG(U1+V)*V-U1*V-U1*B)/(P-B/W)
      IF(ABS(U2-U1).LT.1.0E-5) 11,12
   12 U1=U2
      GO TO 10
    8 S=SF(0)
      GO TO 13
   11 S=SQRT(1.+U2*U2)
   13 A=(1.+S)/(1.-S)
      Y=(1.+1./X)
      Q=P-B
      S=SF(1)
      T=(1.-S)/(1.+S)
      Z=Y*Y
      M1=3.* Q*T*A*Z
      M2=5./(16.*Q*A*FACT(1))*Z
```

```
      M3=28./27./FACT(2)*M1
      M4=81./80./FACT(3)*M2
      M5=176./175./FACT(4)*M3
      M6=325./324./FACT(5)*M4
C     OUTPUT CPLCNSTS
      WRITE(61,1) M1,M2,M3
    1 FORMAT(1X3HM1=E15.8,1X3HM2=E15.8,1X3HM3=E15.8)
      WRITE(61,2) M4,M5,M6
    2 FORMAT(1X3HM4=E15.8,1X3HM5=E15.8,1X3HM6=E15.8)
      RETURN
      END

      SUBROUTINE CALCNEG
C     TO CALCULATE COUPLING CONSTS AND ENERGIES FOR NEGATIVE BETA
      REAL L1,L2,L3,L4,L5,L6,M1,M2,M3,M4,M5,M6
C     LI,I=1,6 IS SECOND CHANNEL CPLNG CONST,T=I
C     MI,I=2,7 IS FIRST CHANNEL CPLNG CONST,T=I-1
C     WI,I=1,6 IS SECOND CHANNEL BD STATE ENERGY,T=I
      S1=SF(1) $ S2=SF(2) $ S3=SF(3) $ S4=SF(4) $ S5=SF(5) $ S6=SF(6)
      S7=SF(7) $ S8=SF(8) $ W2=WF(2) $ W3=WF(3) $ W4=WF(4) $ W5=WF(5)
      W6=WF(6) $ W7=WF(7)
      Y=(1.+S2/X)
      L1=2.*S2/W2*(1.+S2)/(1.-S2)/TF(1,2)*Y*Y
      Y=(1.+S3/X)
      A=(S1+S3)*(S2-S3)
      C=(S2+S3)*(S1-S3)
```

```
L2=8./3.*S3/W3*A/C*Y*Y
Y=(1.+S4/X)
L3=S4/W4*3.2*(1+S4)/(1-S4)*TF(2,4)/(TF(3,4)*TF(1,4))*Y*Y
A=(S1+S5)*(S2-S5)*(S3+S5)*(S4-S5)
C=(S1-S5)*(S2+S5)*(S3-S5)*(S4+S5)
Y=(1.+S5/X)
L4=384./85.*S5/W5*A/C*Y*Y
Y=(1.+S6/X)
T=TF(2,6)*TF(4,6)/(TF(5,6)*TF(1,6)*TF(3,6))
L5=256./63.*S6/W6*(1.+S6)/(1.-S6)*T*Y*Y
Y=(1.+S7/X)
A=(S1+S7)*(S2-S7)*(S3+S7)*(S4-S7)*(S5+S7)*(S6-S7)
C=(S1-S7)*(S2+S7)*(S3-S7)*(S4+S7)*(S5-S7)*(S6+S7)
L6=3072./561.*S7/W7*A/C*Y*Y
Q=P-B
Y=(1.+1./X)
M1=3.*Q*FACT(1)*Y*Y
M2=5./(16.*Q)*FACT(2)*(1.+S1)/(1.-S1)*Y*Y
M3=28./27.*FACT(3)*M1
M4=81./80.*FACT(4)*M2
M5=176./175.*FACT(5)*M3
M6=325./324.*FACT(6)*M4
WRITE(61,1) M1,L1,W1
WRITE(61,2)M2,L2,W2
WRITE(61,3) M3,L3,W3
```

```
      WRITE(61,4) M4,L4,W4
      WRITE(61,5) M5,L5,W5
      WRITE(61,6) M6,L6,W6
    1 FORMAT(1X3HM1=E15.8,1X3HL1=E15.8,1X3HW1=E15.8)
    2 FORMAT(1X3HM2=E15.8,1X3HL2=E15.8,1X3HW2=E15.8)
    3 FORMAT(1X3HM3=E15.8,1X3HL3=E15.8,1X3HW3=E15.8)
    4 FORMAT(1X3HM4=E15.8,1X3HL4=E15.8,1X3HW4=E15.8)
    5 FORMAT(1X3HM5=E15.8,1X3HL5=E15.8,1X3HW5=E15.8)
    6 FORMAT(1X3HM6=E15.8,1X3HL6=E15.8,1X3HW6=E15.8)
      RETURN
      END
```

CHAPTER FIVE

HIGHER DIMENSIONAL CROSSING RELATIONS

We propose now to consider the possibility of obtaining exact general solutions to higher dimensional problems, i.e., problems admitting a greater number of channels than previously considered. The conditions to be satisfied are the same as before, namely those of elastic unitarity, hermitian analyticity and crossing symmetry, but where now, we have higher dimensional crossing-matrices. In Section 5.1 we demonstrate a procedure for the construction of an arbitrary n-dimensional crossing-matrix, and the then implied symmetry properties of the S-matrix functions. In Section 5.2, we shall apply this procedure to the explicit case of three channel scattering. We shall find that there are two distinct crossing-matrices, which we term Type I and Type II. The problem corresponding to the Type I matrix is considered in Section 5.3, and that corresponding to Type II, which contains as a subclass, 'generalised' 3X3 SU_2 crossing-matrices, in Section 6.4. Such problems will be shown, in general, to reduce to coupled non-linear difference equations, for which known mathematical techniques are inadequate to solve in full generality. An appendix is included on the derivation of 'generalised' three dimensional SU_2 crossing-matrices.

Section 5.1 The General n-dimensional Case

$$\text{Let } X = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \qquad (5.1.1)$$

having n components.

Then, we must construct an n X n matrix, A , with the properties:

$$A^2 = I \qquad (5.1.2)$$

$$A X = X \qquad (5.1.3)$$

$$A^* = A \qquad (5.1.4)$$

These properties are proved in the appendix to Chapter 3.

By (5.1.2), A has eigenvalues equal to only either ± 1 , and has a minimal polynomial $(x^2 - 1) = (x + 1)(x - 1)$, (or is otherwise the null or unit matrix) , which is a product of real, distinct linear factors. A is therefore similar to a diagonal matrix.[60)]

Hence, there exists a real, non-singular matrix B, such that, denoting the r-dimensional unit matrix by I_r ,

$$B^{-1} A B = \begin{pmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{pmatrix} \equiv I_{nm} \qquad (5.1.5)$$

having assumed A to have m eigenvectors corresponding to the eigenvalue +1, and so, n-m eigenvectors corresponding to the eigenvalue -1.

$$A X = B I_{nm} B^{-1} X = X, \quad \text{by } (5.1.3)$$

i.e., $I_{nm} (B^{-1} X) = (B^{-1} X)$,

so that $B^{-1} X$ is an eigenvector with eigenvalue +1, of the matrix I_{nm}.

Without loss of gnerality, we may therefore take

$$B^{-1} X = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{\epsilon_1}$$

i.e., $X = B \underline{\epsilon_1}$.

Hence $B = (X \; B_{nn-1})$ (5.1.6)

where B_{nn-1} is an arbitrary $n \times (n-1)$ matrix whose columns, with X, for a real basis for $V_n(R)$.

Similarly, denoting the set of linearly independent eigenvectors of A corresponding to eigenvalue +1 by $\underline{a_1}, \ldots\ldots, \underline{a_m}$, and those corresponding to eigenvalue -1 by $\underline{b_1}, \ldots\ldots, \underline{b_{n-m}}$, we may take

$$\underline{a_i} = B \underline{\epsilon_i} \;, \quad i=1,\ldots\ldots,m \tag{5.1.7}$$

$$\underline{b_j} = B \underline{\epsilon_{j+m}} \;, \quad j=1,\ldots\ldots,n-m \tag{5.1.8}$$

where $\underline{\epsilon_i}$ is the n-component column vector with j^{th} component equal to δ_{ij}, and we have taken $\underline{a_1} = X$.

The crossing matrix, A, then is determined by (5.1.5),

i.e. $A = B \, I_{nm} \, B^{-1}$ (5.1.9)

and its entries will be functions of the $n(n-1)$ entries of B_{nn-1}.

Denote the n channel S-matrix functions by $S_1(z), \ldots, S_n(z)$

and write $\underline{S}(z) = \begin{pmatrix} S_1(z) \\ \vdots \\ S_n(z) \end{pmatrix}$ (5.1.10)

The crossing relations to be satisfied are, hence,

$$\underline{S}(-z) = A \, \underline{S}(z) \tag{5.1.11}$$

Using (5.1.2), we have also

$$\underline{S}(z) = A \, \underline{S}(-z) \tag{5.1.12}$$

$$\therefore \quad [\underline{S}(z) + \underline{S}(-z)] = A\,[\underline{S}(z) + \underline{S}(-z)] \tag{5.1.13}$$

i.e., $[\underline{S}(z) + \underline{S}(-z)]$ is an eigenvector of A with eigenvalue +1, for all z. We may therefore write

$$\underline{E}(z) \equiv \underline{S}(z) + \underline{S}(-z) = 2\,p_1(z)\,\underline{a_1} + 2\,p_2(z)\,\underline{a_2} + \ldots + 2\,p_m(z)\,\underline{a_m} \tag{5.1.14}$$

where the $p_i(z)$ are real, meromorphic and even functions of z in the cut z plane, and the $\underline{a_i}$ are as defined by (5.1.7).

Similarly, from (5.1.11) and (5.1.12), $[\underline{S}(z) - \underline{S}(-z)] \equiv \underline{N}(z)$, is an eigenvector of A with eigenvalue -1, for all z, and so

$$\underline{N}(z) = 2\,q_1(z)\,\underline{b_1} + 2\,q_2(z)\,\underline{b_2} + \ldots\ldots + 2\,q_{n-m}\,\underline{b_{n-m}} \tag{5.1.15}$$

where the $q_i(z)$ are real, meromorphic and odd functions of z in the cut z plane, and the $\underline{b_i}$ are defined by (5.1.8).

$$\begin{aligned}
\text{But } \underline{S}(z) &= \tfrac{1}{2}[\underline{E}(z) + \underline{N}(z)] \\
&= [p_1(z)\underline{a_1} + \ldots\ldots p_m(z)\underline{a_m}] + [q_1(z)\underline{b_1} + \ldots\ldots + q_{n-m}(z)\underline{b_{n-m}}] \\
&= B\,\{[p_1(z)\,\underline{\epsilon\,1} + \ldots + p_m(z)\,\underline{\epsilon\,m}] + [q_1(z)\,\underline{\epsilon\,m{+}1} + \ldots \\
&\qquad + q_{n-m}(z)\,\underline{\epsilon\,n}]\}
\end{aligned} \tag{5.1.16}$$

Crossing has thus been satisfied with full generality.

The $p_i(z)$ and $q_i(z)$ must now be constructed so as to satisfy the remaining condition of unitarity. This is the most difficult aspect of the problem, and admits no simple and general formulation. Special devices must usually be employed, as will be illustrated later for the three dimensional case.

Section 5.2. The Three Channel Problem

We firstly construct the most general crossing matrices for the case n=3, according to the procedure of the previous section.

We now have that m = 0,1,2, or 3. The cases m = 0 and m = 3 are trivial in the sense that then $A = \pm I_3$, so that the channels are decoupled and the S-matrix functions are symmetric or anti-symmetric. We consider the cases then, when m = 1 so that Tr A = -1, and m = 2 so that Tr A = +1. These two cases lead to distinct general forms for A, which we will denote as Type I and Type II matrices respectively.

In either case, with the notation of the preceeding section,

$$B = \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{bmatrix} \qquad (5.2.1)$$

and so $$B^{-1} = \frac{1}{\gamma}\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \qquad (5.2.2)$$

where:

$$\begin{aligned} \gamma_1 &= a_2b_3-a_3b_2 \ ; & \gamma_2 &= a_3b_1-a_1b_3 \ ; & \gamma_3 &= a_1b_2-a_2b_1 \\ \beta_1 &= b_2 - b_3 \ ; & \beta_2 &= b_3 - b_1 \ ; & \beta_3 &= b_1 - b_2 \\ \alpha_1 &= a_3 - a_2 \ ; & \alpha_2 &= a_1 - a_3 \ ; & \alpha_3 &= a_2 - a_1 \end{aligned} \qquad (5.2.3)$$

and $\gamma = \det B = \gamma_1 + \gamma_2 + \gamma_3 \neq 0$

Therefore, for Type I,

$$A = B \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix} B^{-1}$$

$$= \frac{1}{\gamma}\begin{bmatrix} 2\gamma_1-\gamma & 2\gamma_2 & 2\gamma_3 \\ 2\gamma_1 & 2\gamma_2-\gamma & 2\gamma_3 \\ 2\gamma_1 & 2\gamma_2 & 2\gamma_3-\gamma \end{bmatrix}$$

$$= \begin{bmatrix} 2c_1-1 & 2c_2 & 2c_3 \\ 2c_1 & 2c_2-1 & 2c_3 \\ 2c_1 & 2c_2 & 2c_3-1 \end{bmatrix} \qquad (5.2.4a)$$

where $c_i = \dfrac{\gamma_i}{\gamma}$ and so $c_1 + c_2 + c_3 = 1$ (5.2.4b)

Type I crossing-matrices are, therefore, essentially a two parameter family of matrices.

For Type II,

$$A = B\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} B^{-1}$$

$$= \frac{1}{\gamma}\begin{bmatrix} \gamma-2\alpha_1 b_1 & -2\alpha_2 b_1 & -2\alpha_3 b_1 \\ -2\alpha_1 b_2 & \gamma-2\alpha_2 b_2 & -2\alpha_3 b_2 \\ -2\alpha_1 b_3 & -2\alpha_2 b_3 & \gamma-2\alpha_3 b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2b_1d_1 & -2b_1d_2 & -2b_1d_3 \\ -2b_2d_1 & 1-2b_2d_2 & -2b_2d_3 \\ -2b_3d_1 & -2b_3d_2 & 1-2b_3d_3 \end{bmatrix} \qquad (5.2.5)$$

where $d_i = \alpha_i/\gamma$ and so by (5.2.3)

$$\underline{b}.\underline{d} = 1 \; ; \; \underline{d}.\underline{X} = 0 \qquad (5.2.6)$$

We see, therefore, that Type II crossing-matrices are essentially a three parameter family of matrices, which contains as a subclass, the generalised 3-dimensional SU_2 crossing-matrices discussed in Appendix 5A.

Denote the S-matrix functions as $S_1(z)$, $S_2(z)$, $S_3(z)$ and consider firstly the problem with Type I crossing-matrix. A has only a one-dimensional space of eigenvectors with eigenvalue +1 , and so

$$S_i(z) + S_i(-z) \equiv 2\, s(z) \quad , i=1-3 \tag{5.2.7}$$

The two-dimensional space of eigenvectors with eigenvalue -1 is spanned by $B\ \underline{\epsilon}\,2 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $B\ \underline{\epsilon}\,3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, or using (5.2.3) and (5.2.4), by $\begin{pmatrix} c_2 \\ -(c_1 + c_3) \\ c_2 \end{pmatrix}$ and $\begin{pmatrix} -c_3 \\ -c_3 \\ (c_1 + c_2) \end{pmatrix}$

The solution of the crossing symmetry, may therefore be written in symmetrical form as:

$$\begin{aligned} S_1(z) &= [s(z) + c_2\, r_3(z) - c_3\, r_2(z)] \\ S_2(z) &= [s(z) - c_1\, r_3(z) + c_3\, r_1(z)] \\ S_3(z) &= [s(z) + c_1\, r_2(z) - c_2\, r_1(z)] \end{aligned} \tag{5.2.8}$$

where the $r_i(z)$ are odd functions of z, and such that

$$r_1(z) + r_2(z) + r_3(z) = 0 \tag{5.2.9}$$

In fact, $r_1(z) = S_2(z) - S_3(z)$; $r_2(z) = S_3(z) - S_1(z)$;
$r_3(z) = S_3(z) - S_1(z)$.

s(z) and the $r_i(z)$ are, of course, required to be real meromorphic functions in the cut plane.

Writing $\underline{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$; $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, the solution (5.2.8) may be expressed in the compact form:

$$\underline{S}(z) = s(z)\, X - \underline{r}(z) \wedge \underline{c} \tag{5.2.10}$$

Consider now the problem with Type II crossing-matrix, which has the two eigenvectors $X = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $B \underline{\epsilon_2} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ corresponding to eigenvalue +1, and one eigenvector $B \underline{\epsilon_3} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ corresponding to eigenvalue -1.

The solution to crossing symmetry may hence be written as

$$S_i(z) = s(z) + a_i s'(z) + b_i p(z) \qquad (5.2.11)$$

where $s(z)$ and $s'(z)$ are even real meromorphic functions in the cut plane, while $p(z)$ is an odd function.

We attempt to impose the condition of unitarity on these solutions in subsequent sections.

Section 5.3 Unitarity for the Type I Problem

Attempting to impose the condition of unitarity directly upon the solution of crossing symmetry (5.2.8), leads immediately to lengthy algebraic equations involving the functions $s(z)$, $r_i(z)$ and their continuations to the second sheet of the right-hand branch point. It is difficult to see any simplifying structure in these expressions, by means of which they might be disentangled, and so we prefer to employ the technique presented in Chapter 3 for the two channel case. We therefore transform the problem into the $w = \cosh^{-1} z$ plane.

The problem of satisfying both unitarity and crossing symmetry, then becomes that of solving the coupled difference equations:

$$S_1(w+\pi i) = \frac{2c_1-1}{S_1(w)} + \frac{2c_2}{S_2(w)} + \frac{2c_3}{S_3(w)}$$

$$S_2(w+\pi i) = \frac{2c_1}{S_1(w)} + \frac{2c_2-1}{S_2(w)} + \frac{2c_3}{S_3(w)} \qquad (5.3.1)$$

$$S_3(w+\pi i) = \frac{2c_1}{S_1(w)} + \frac{2c_2}{S_2(w)} + \frac{2c_3-1}{S_3(w)}$$

Proceeding as in Chapter 3, we introduce functions K_1 and K_2 defined by

$$S_1 = \frac{K_1 - 1}{K_1 + 1} S_3 \quad ; \quad S_2 = \frac{K_2 - 1}{K_2 + 1} S_3 \qquad (5.3.2)$$

K_1 and K_2 are therefore, real meromorphic functions in the cut z plane, and by unitarity, must be such that

$$K_i(z) + K_i^{(2)}(z) = 0, \; i=1,2 \qquad (5.3.3)$$

On transforming to the w plane, (5.3.3) gives that K_1 and K_2 are odd functions of w.

For ease of notation and writing down the ensuing, somewhat cumbersome, expressions, we denote $S_3(w+\pi i)$, $K_1(w+\pi i)$, $K_2(w+\pi i)$, by $\overline{S}_3$, $\overline{K}_1$, and $\overline{K}_2$, dropping explicit insertion of the argument, w.

Substituting (5.3.2) into (5.3.1), we obtain after some simplification and use of (5.2.4b),

$$S_3\bar{S}_3\,\frac{\bar{K}_1-1}{\bar{K}_1+1} = \frac{K_1K_2 + K_1(4c_2-1) + K_2(4c_1-3) + (4c_3-1)}{(K_1-1)(K_2-1)}$$

$$S_3\bar{S}_3\,\frac{\bar{K}_2-1}{\bar{K}_2+1} = \frac{K_1K_2 + K_1(4c_2-3) + K_2(4c_1-1) + (4c_3-1)}{(K_1-1)(K_2-1)} \qquad (5.3.4)$$

$$S_3\bar{S}_3 = \frac{K_1K_2 + K_1(4c_2-1) + K_2(4c_1-1) + (4c_3-1)}{(K_1-1)(K_2-1)}$$

We may eliminate $S_3\bar{S}_3$ from these equations, giving, after some manipulation, the following pair of coupled difference equations for K_1 and K_2:

$$\left[\bar{K}_1 - K_1 + 2(2c_3-1)\right]\left[K_2 - 1\right] = 4c_2(K_1 - K_2)$$
$$\left[\bar{K}_2 - K_2 + 2(2c_3-1)\right]\left[K_1 - 1\right] = 4c_1(K_2 - K_1) \qquad (5.3.5)$$

Eliminating either K_1 or K_2 from these equations gives a non-linear, inhomogeneous difference equation in two increments. No mathematical techniques are known that are adequate for a general solution of such an equation. However, we see immediately that (5.3.5) admits the special class of solutions given by

$$\bar{K}_1 - K_1 + 2(2c_3-1) = 0 \; ; \quad K_2 = K_1 \qquad (5.3.6)$$

This, in fact, is the only class of solutions we are able to construct, but, unfortunately, it is a trivial solution in the sense that $S_1 = S_2$, and the problem reduces to a two channel problem with crossing matrix $\begin{bmatrix} 2c_1+2c_2-1 & 2c_3 \\ 2c_1+2c_2 & 2c_3-1 \end{bmatrix} = \begin{bmatrix} 1-2c_3 & 2c_3 \\ -2c_3+2 & 2c_3-1 \end{bmatrix}$

The general solution within this class of solutions is then as given in Chapter 3. We may conclude, however, that solutions do exist for all values of the crossing matrix parameters, although the special solutions we are able to construct depend on only one of the two independent parameters.

Section 5.4 Unitarity for the Type II Problem

We proceed now to attempt to impose the two conditions of crossing symmetry and unitarity upon the S-matrix functions corresponding to a Type II matrix. Because of the increased complexity of this problem, we find it again preferable to employ the technique presented in Chapter 3 for the two channel problem and to transform the problem into the $w = \cosh^{-1} z$ plane.

Unitarity and crossing symmetry then give the relations:

$$\begin{aligned} S_1(w+\pi i) &= \frac{(1-2b_1d_1)}{S_1(w)} - \frac{2b_1d_2}{S_2(w)} - \frac{2b_1d_3}{S_3(w)} \\ S_2(w+\pi i) &= \frac{-2b_2d_1}{S_1(w)} + \frac{(1-2b_2d_2)}{S_2(w)} - \frac{2b_2d_3}{S_3(w)} \qquad (5.4.1) \\ S_3(w+\pi i) &= \frac{-2b_3d_1}{S_1(w)} - \frac{2b_3d_2}{S_2(w)} + \frac{(1-2b_3d_3)}{S_3(w)} \end{aligned}$$

where, as shown in section 5.2,

$$d_1 + d_2 + d_3 = 0 \qquad (5.4.2)$$

and $b_1d_1 + b_2d_2 + b_3d_3 = 0$ (5.4.3)

Write, proceeding analogously to the two channel case,

$$S_1(w) = \frac{K_1(w) - 1}{K_1(w) + 1} S_2(w) \tag{5.4.4}$$

$$S_3(w) = \frac{K_3(w) - 1}{K_3(w) + 1} S_2(w) \tag{5.4.5}$$

By unitarity, we require that

$$K_1(w) + K_1(-w) = 0 = K_3(w) + K_3(-w) \tag{5.4.6}$$

As before, we denote $S_2(w+\pi i)$, $K_1(w+\pi i)$, $K_3(w+\pi i)$, by $\bar{S}_2$, $\bar{K}_1$, $\bar{K}_3$, again dropping explicit insertion of the argument.

Substituting (5.4.4) and (5.4.5) into (5.4.1), and using (5.4.2) and (5.4.3), we obtain that

$$S_2\bar{S}_2 \frac{\bar{K}_1-1}{\bar{K}+1} = \frac{K_1K_3 - (1+4b_1d_3)K_1 + (1-4b_1d_1)K_3 - (1+4b_1d_2)}{(K_1-1)(K_3-1)}$$

$$S_2\bar{S}_2 = \frac{K_1K_3 - (1+4b_2d_3)K_1 - (1+4b_2d_1) + (1-4b_2d_2)}{(K_1-1)(K_3-1)} \tag{5.4.7}$$

$$S_2\bar{S}_2 \frac{\bar{K}_3-1}{\bar{K}_3+1} = \frac{K_1K_3 + K_1(1-4b_3d_3) - K_3(1+4b_3d_1) - (1+4b_3d_2)}{(K_1-1)(K_3-1)}$$

$S_2\bar{S}_2$ may be eliminated from (5.4.7), giving two difference equations, which after use of (5.4.2) and (5.4.3) and some straight forward algebra, may be written as:

$$2\left[d_3K_1 + d_1K_3 + d_2\right]\left[(b_1-b_2)\overline{K}_1 + (b_1+b_2)\right] = (K_3-1)\left[\overline{K}_1+K_1\right] \qquad (5.4.8)$$

$$2\left[d_3K_1 + d_1K_3 + d_2\right]\left[(b_3-b_2)\overline{K}_3 + (b_3+b_2)\right] = (K_1-1)\left[\overline{K}_3+K_3\right] \qquad (5.4.9)$$

The problem has thus been essentially reduced to the solution of these two coupled non-linear equations. The elimination of say, K_3 , leads to a non-homogeneous, non-linear difference equation in two increments for $K_1(w)$. As before, we are unable to solve such an equation in full generality, and so we content ourselves with the derivation of a wide, but special class, of exact solutions.

To this end then, we attempt to find solutions of the form

$$\begin{aligned} K_1(w + \pi i) &= K_1(w) + \lambda(w) \\ K_3(w) &= \mu(w)\, K_1(w) \end{aligned} \qquad (5.4.10)$$

where λ and μ are periodic functions of period πi.

By (5.4.6), we must have that $\mu(w) = \mu(-w)$ (5.4.11)

Also, since $K_1(-w+\pi i) = -K_1(w) + \lambda(-w)$

then $-K_1(w-\pi i) = -K_1(w) + \lambda(-w)$

$\therefore\ -K_1(w) + K_1(w+\pi i) = \lambda(-w-\pi i) = \lambda(-w)$

Hence, $\lambda(w) = \lambda(-w)$ (5.4.12)

From (5.4.8) and (5.4.9) ,

$$\frac{(b_1-b_2)\,\overline{K}_1 + (b_1+b_2)}{(b_3-b_2)\,\overline{K}_3 + (b_2+b_3)} = \frac{(K_3-1)(\overline{K}_1 + K_1)}{(K_1-1)(\overline{K}_3 + K_3)}$$

Substituting (5.4.10) and $\overline{K}_3 = K_3 + \lambda\mu$, we obtain , on simplification that,

$$\mu K_1^2\left[(b_1-b_2) - \mu(b_3-b_2)\right] + \lambda\mu K_1\left[(b_1-b_2) - \mu(b_3-b_2)\right] + \left[\lambda\mu(b_3-b_1) - \mu(b_1+b_2) + (b_2+b_3)\right] = 0 \qquad (5.4.13)$$

Making the transformation $w \to -w$ and using (5.4.6), (5.4.11) and (5.4.12), we have

$$\mu K_1^2\left[(b_1-b_2)-\mu(b_3-b_2)\right]-\lambda\mu K_1\left[(b_1-b_2)-\mu(b_3-b_2)\right]+$$
$$\left[\lambda\mu(b_3-b_1)-\mu(b_1+b_2)+(b_2+b_3)\right]=0 \quad (5.4.14)$$

Subtracting (5.4.14) from (5.4.13),

$$\lambda\mu K_1\left[(b_1-b_2)-\mu(b_3-b_2)\right] = 0 \quad (5.4.15)$$

If $K_1 \equiv 0$, then (5.4.8), K_3 is a constant which must also be zero, K_3 being odd. Similarly if $\mu \equiv 0$, then $K_3 \equiv 0$ and $K_1 \equiv 0$. If $\lambda \equiv 0$, then $\bar{K}_1 = K_1$; $\bar{K}_3 = K_3$, so that K_3 may be eliminated from (5.4.8) and (5.4.9) giving an algebraic equation for K_1. Such an equation may have only the solution $K_1 \equiv 0$. We see, then, that unless

$$\mu = (b_1 - b_2)/(b_3 - b_2) \quad (5.4.16)$$

(assuming $b_3 \neq b_2$, else $b_1 = b_2 = b_3$) we are lead to a trivial solution.

Taking $\mu = (b_1 - b_2)/(b_3 - b_2)$, we see from (5.4.14) that

$$\lambda = \frac{\mu(b_1 + b_2) - (b_2 + b_3)}{\mu(b_3 - b_1)} = -\frac{[b_1 + b_3]}{[b_1 - b_2]} \quad (5.4.17)$$

We must now ensure consistency of this solution with either (5.4.8) or (5.4.9). Substituting (5.4.10) into (5.4.8) gives

$$2\left[d_3K_1+d_1\mu K_1+d_2\right]\left[(b_1-b_2)(K_1+\lambda)+(b_1+b_2)\right] = \left[\mu K_1-1\right]\left[2K_1+\lambda\right]$$

Using (5.4.16) and (5.4.17), this reduces to

$$\left[2d_2 - (b_1 + b_3)/(b_1 - b_2)\right]\left[K_1 - 1\right] = 0$$

For consistency then, we must have

$$d_2 = \frac{(b_1 + b_3)}{2(b_1 - b_2)} \tag{5.4.18}$$

This condition is a restriction on the form of the crossing-matrix elements. In fact, using this expression for d_2, we have, by (5.4.2) and (5.4.3)

$$d_1 = \frac{b_1 - 2b_2 - b_3}{2(b_1-b_2)(b_1-b_3)} \quad ; \quad d_3 = \frac{b_1 + 2b_2 - b_3}{2(b_2-b_3)(b_3-b_1)}$$

The Type II crossing-matrix then becomes

$$A = \begin{bmatrix} \frac{(u+v)}{(u-1)(u-v)} & \frac{-u(u+v)}{(u-1)(v-1)} & \frac{-u(u-v+2)}{(v-1)(u-v)} \\ -\frac{(u-v-2)}{(u-1)(u-v)} & \frac{(1+uv)}{(u-1)(v-1)} & -\frac{(u-v+2)}{(v-1)(u-v)} \\ -\frac{v(u-v-2)}{(u-1)(u-v)} & \frac{v(u+v)}{(u-1)(v-1)} & -\frac{(u+v)}{(v-1)(u-v)} \end{bmatrix} \tag{5.4.19}$$

where we have put $u = b_1/b_2$; $v = b_3/b_2$

The crossing-matrix is thus restricted to the less general, two parameter form (5.4.19). This reduces to the generalised SU_2 form when $u + v = 1$, and we put $u = T + 1$, $v = -T$ (see Appendix 5A). This class of matrices also contains the three dimensional crossing-matrix for the pseudoscalar symmetric π -N case,[1)] when $u = 4$, $v = -2$.

We must now solve (5.4.7) for S_2 , which becomes

$$S_2(w)S_2(w+\pi i) = \frac{\mu K_1^2 - (1+4b_2d_3)K_1 - (1+4b_2d_1)\mu K_1 + (1-4b_2d_2)}{(K_1 - 1)(\mu K_1 - 1)}$$

$$= \frac{[K_1 - (b_1+b_2)/(b_1-b_2)][K_1 - (b_2+b_3)/(b_1-b_2)]}{[K_1 - 1][K_1 - (b_3-b_2)/(b_1-b_2)]}$$

It is convenient to put $K(w) = \dfrac{[b_1 - b_2]}{[b_1 + b_3]} K_1(w)$ (5.4.20)

so that $K(w + \pi i) = K(w) - 1$ (5.4.21)

Put, also, $\dfrac{[b_1 + b_2]}{[b_1 + b_3]} = r$; $\dfrac{[b_1 - b_2]}{[b_1 + b_3]} = s$ (5.4.22)

then

$$S_2(w)\, S_2(w + \pi i) = \frac{[K(w) - r]\,[K(w) - (1-s)]}{[K(w) - s]\,[K(w) - (1-r)]} \qquad (5.4.23)$$

Iterating this equation once, we obtain

$$\frac{S_2(w + 2\pi i)}{S_2(w)} = \frac{[K-(r+1)]\,[K-(2-s)]\,[K-s]\,[K-(1-r)]}{[K-(s+1)]\,[K-(2-r)]\,[K-r]\,[K-(1-s)]} \qquad (5.4.24)$$

The solution of this difference equation is unique to within a factor of an arbitrary function of period $2\pi i$. We therefore try to find a solution of the form $S_2(w) = F[K(w)]$

$\therefore$ $S_2(w+2\pi i) = F[K - 2]$

$$\therefore \quad \frac{F[K - 2]}{F[K]} = \frac{[K-(r+1)]\,[K-(2-s)]\,[K-s]\,[K-(1-r)]}{[K-(s+1)]\,[K-(2-r)]\,[K-r]\,[K-(1-s)]} \qquad (5.4.25)$$

The solution of this equation is, using the Lemma proved in Section 3.2, :-

$$F[K] = \overline{\sigma}[K/2] \frac{\Gamma[K/2-\frac{1}{2}(s+1)+1]\,\Gamma[K/2-\frac{1}{2}(2-r)+1]\,\Gamma[K/2-\frac{1}{2}r+1]\,\Gamma[K/2-\frac{1}{2}(1-s)+1]}{\Gamma[K/2-\frac{1}{2}(r+1)+1]\,\Gamma[K/2-\frac{1}{2}(2-s)+1]\,\Gamma[K/2-\frac{1}{2}s+1]\,\Gamma[K/2-\frac{1}{2}(1-r)+1]}$$

$$= \overline{\sigma}[K/2] \frac{\Gamma[K/2+\frac{1}{2}(1-s)]\,\Gamma[K/2+\frac{1}{2}r]\,\Gamma[K/2-\frac{1}{2}r+1]\,\Gamma[K/2+\frac{1}{2}(1+s)]}{\Gamma[K/2+\frac{1}{2}(1-r)]\,\Gamma[K/2+\frac{1}{2}s]\,\Gamma[K/2-\frac{1}{2}s+1]\,\Gamma[K/2+\frac{1}{2}(1+r)]} \qquad (5.4.26)$$

where $\overline{\sigma}\left[\frac{(K-2)}{2}\right] = \overline{\sigma}[K]$ (5.4.27)

Now,

$$F[K-1] = \overline{\sigma}\left[\frac{K-1}{2}\right] \frac{\Gamma[K/2-\frac{1}{2}s]\,\Gamma[K/2+\frac{1}{2}(r-1)]\,\Gamma[K/2+\frac{1}{2}(1-r)]\,\Gamma[K/2+\frac{1}{2}s]}{\Gamma[K/2-\frac{1}{2}r]\,\Gamma[K/2+\frac{1}{2}(s-1)]\,\Gamma[K/2+\frac{1}{2}(1-s)]\,\Gamma[K/2+\frac{1}{2}r]}$$

$$= \frac{\overline{\sigma}\left[\frac{K-1}{2}\right]\,\overline{\sigma}\left[\frac{K}{2}\right]}{F[K]} \; \frac{[K/2-\frac{1}{2}r]\;[K/2+\frac{1}{2}(s-1)]}{[K/2-\frac{1}{2}s]\;[K/2+\frac{1}{2}(r-1)]}$$

We see, therefore, that F[K] is a solution of the difference equation (5.4.22), provided

$$\overline{\sigma}\left[\frac{K}{2}\right]\,\overline{\sigma}\left[\frac{K-1}{2}\right] = 1 \qquad (5.4.28)$$

Using $\Gamma[\xi]\,\Gamma[1-\xi] = \pi/\sin\pi\xi$, we may write (5.4.26) as

$$F[K] = \overline{\sigma}[K/2] \frac{\Gamma[\frac{1}{2}(1-s)+k/2]\,\Gamma[\frac{1}{2}r+K/2]\,\Gamma[\frac{1}{2}(1-r)-K/2]\,\Gamma[\frac{1}{2}s-K/2]}{\Gamma[\frac{1}{2}(1-s)-K/2]\,\Gamma[\frac{1}{2}r-K/2]\,\Gamma[\frac{1}{2}(1-r)+K/2]\,\Gamma[\frac{1}{2}s+K/2]}$$

$$\times \; \frac{\sin\pi[\frac{1}{2}s-K/2]\;\sin\pi[\frac{1}{2}(1-r)-K/2]}{\sin\pi[\frac{1}{2}r-K/2]\;\sin\pi[\frac{1}{2}(1-s)-K/2]}$$

We require that this particular solution satisfy unitarity, i.e. that $F[K]\;F[-K] = 1$

We therefore see that we must have

$$\overline{\sigma}[K/2]\ \overline{\sigma}[-K/2] = \frac{\tan\pi\left[\tfrac{1}{2}(1-s)-K/2\right]\ \tan\pi\left[\tfrac{1}{2}(1-s)+K/2\right]}{\tan\pi\left[\tfrac{1}{2}(1-r)-K/2\right]\ \tan\pi\left[\tfrac{1}{2}(1-r)+K/2\right]}$$

Consider the function $\overline{\sigma}_o[K/2] = \dfrac{\tan\pi\left[\tfrac{1}{2}(1-s)-K/2\right]}{\tan\pi\left[\tfrac{1}{2}(1-r)-K/2\right]}$

Clearly $\overline{\sigma}_o\left[\frac{K-2}{2}\right] = \overline{\sigma}_o\left[\frac{K}{2}\right]$

while $\overline{\sigma}_o\left[\frac{K}{2}\right]\ \overline{\sigma}_o\left[\frac{K-1}{2}\right] = 1.$

so that $\overline{\sigma}_0$ verifies (5.4.27) and (5.4.28), and so furnishes a solution to our problem.

The general solution is thus

$$S_2(w) = \frac{\sin\pi\left[\tfrac{1}{2}s-K/2\right]\ \sin\left[\tfrac{1}{2}r+K/2\right]\pi}{\sin\pi\left[\tfrac{1}{2}s+K/2\right]\ \sin\left[\tfrac{1}{2}r-K/2\right]\pi}$$

$$\times \frac{\Gamma\left[\tfrac{1}{2}(1-s)+K/2\right]\Gamma\left[\tfrac{1}{2}r+K/2\right]\Gamma\left[\tfrac{1}{2}(1-r)-K/2\right]\Gamma\left[\tfrac{1}{2}s-K/2\right]}{\Gamma\left[\tfrac{1}{2}(1-s)-K/2\right]\Gamma\left[\tfrac{1}{2}r-K/2\right]\Gamma\left[\tfrac{1}{2}(1-r)+K/2\right]\Gamma\left[\tfrac{1}{2}s+K/2\right]}\ D(w) \qquad (5.4.29)$$

$$\equiv F_o\left[K(w)\right]\ D(w).$$

where

$$D(w)\ D(-w) = 1$$

$$D(w)\ D(w+\pi i) = 1 \qquad (5.4.30)$$

$$D(w) = D(w+2\pi i) = 1$$

and the general solution of (5.4.21) for $K(w)$ is

$$K(w) = \frac{i\,w}{\pi} + \overline{\beta}(w) \qquad \text{where} \qquad \overline{\beta}(w) = \overline{\beta}(w+\pi i)$$

and

$$\overline{\beta}(w) + \overline{\beta}(-w) = 0.$$

Therefore, transforming back to the z plane,

$$K(z) = B(z) - \tfrac{1}{2} \qquad (5.4.31)$$

and $B(z) = \pi^{-1}\arcsin z + \frac{iq}{z}\,\beta(z)$,

$\beta(z)$ being an arbitrary real, meromorphic and even function in the entire z plane.

Also, conditions (5.4.30) give

$$D(z) = D(-z) \quad \text{and } D(z)\,D^{(2)}(z) = 1 \qquad (5.4.32)$$

Our class of solutions now read

$$S_1(z) = \frac{K(z) - s}{K(z) + s}\, F_o\left[K(z)\right] D(z)$$

$$S_2(z) = F_o\left[K(z)\right] D(z) \qquad (5.4.33)$$

$$S_3(z) = \frac{K(z) - (1-r)}{K(z) + (1-r)}\, F_o\left[K(z)\right] D(z)$$

If we use the generalised SU_2 parameters, namely $u = b_1/b_2 = T+1$ and $v = b_3/b_2 = -T$, so that

$r = \frac{u+1}{u+v} = T+2$; $s = \frac{u-1}{u+v} = T$, then

$$F_o\left[K(z)\right] = \frac{\sin\pi\left[\frac{1}{2}(T-K)\right]\ \sin\pi\left[\frac{1}{2}(T+K)\right]}{\sin\pi\left[\frac{1}{2}(T+K)\right]\ \sin\pi\left[\frac{1}{2}(T-K)\right]}$$

$$\times\ \frac{\Gamma\left[\frac{1}{2}(1-T)+K/2\right]\Gamma\left[1+\frac{1}{2}T+K/2\right]\Gamma\left[-\frac{1}{2}(1+T)-K/2\right]\Gamma\left[\frac{1}{2}T-K/2\right]}{\Gamma\left[\frac{1}{2}(1-T)-K/2\right]\Gamma\left[1+\frac{1}{2}T+K/2\right]\Gamma\left[-\frac{1}{2}(1+T)+K/2\right]\Gamma\left[\frac{1}{2}T+K/2\right]}$$

$$= \frac{T+K}{T-K}\ \frac{T+1-K}{T+1+K}$$

Our special class of solutions will, therefore, reduce to a finite product of terms, irrespective of the value of the parameter T. The solutions are, in fact,

$$S_1(z) = \frac{K - T - 1}{K + T + 1} D(z)$$

$$S_2(z) = \frac{K + T}{K - T} \frac{K - T - 1}{K + T + 1} D(z) \qquad (5.4.34)$$

$$S_3(z) = \frac{K + T}{K - T} D(z)$$

It would appear that the application of the bootstrap hypothesis to these solutions, as was done for the two channel case in Chapter 4, would result in little information concerning the range of T.

Appendix 5A Three Channel SU_2 Crossing-Matrices

In this appendix, we discuss and derive, briefly, the three dimensional SU_2 crossing-matrices, using the terminology of isospin scattering.

Consider the scattering of a particle of isospin t_1, by one of isospin t_2. The the possible channels are labelled by the number of integer steps from $t_1 + t_2$ to $t_1 - t_2$, of which there are, of course, $2t_2 + 1$, having supposed, without loss of generality, that $t_1 \geqslant t_2$. Hence, for a three channel process, we must have $t_2 = 1$.

We are considering, therefore, two body elastic reactions of the type

$$T + 1 \rightarrow T + 1 \quad \text{(s-channel)} \tag{5A.1}$$

The anti-particles will, as is usual, be denoted by $\overline{T}$ and $\overline{1}$, respectively, so that the crossed reactions are

$$T + \overline{1} \rightarrow T + \overline{1}. \quad \text{(u-channel)} \tag{5A.2}$$

and

$$T + \overline{T} \rightarrow 1 + \overline{1}. \quad \text{(t-channel)} \tag{5A.3}$$

Let $M_v(v')$ denote the scattering amplitude for scattering in a state of isospin v' in channel v. The crossing-matrices are then defined by

$$M_u(u') = \sum_{s'} X_{u's'} M_s(s') \tag{5A.4}$$

$$M_t(t') = \sum_{s'} X_{t's'} M_s(s') \tag{5A.5}$$

As discussed by Carruthers and Krisch[61)], there are two possible types of fields to consider:-

(i) Self-conjugate fields for which a particle and its anti-particle belong to the same isotopic multiplet.

(ii) Pair conjugate fields, for which the anti-particles of an isotopic multiplet form a distinct isotopic multiplet.

Hence there are three distinct types of crossing-matrix according to whether:-

A Both particles are of type (i)

B Both particles are of type (ii)

C One particle is of type (i) and one of type (ii)

The corresponding crossing-matrices may be shown to be [61)]

$$X^A_{us} = X^B_{us} = X^C_{us} = (2s+1)\, W(T\ 1\ 1\ T;\ s\ u) \qquad (5A.6)$$

$$X^A_{ts} = X^B_{ts} = X^C_{ts} = (2s+1)\, W(T\ s\ t\ 1;\ 1\ T) \qquad (5A.7)$$

so that for the three channel case, there is no distinction between the crossing-matrices for the three types of reaction. W is the usual Racah coefficient as defined by Rose.[62)]

To evaluate the crossing-matrix elements, we use the 6 - j symbols,[63)] whereby,

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = (-1)^{a+b+c+d}\, W(a\ b\ d\ c:\ e\ f) \qquad (5A.8)$$

We must now evaluate

$$W(T\ 1\ 1\ T;\ s\ u) = \begin{Bmatrix} 1 & T & s \\ 1 & T & u \end{Bmatrix} \quad \text{(supposing T to be integer)}$$

for $s\ ,\ u = (T-1),\ T,\ (T+1)$

For s = u=T - 1

$$W = \begin{Bmatrix} 1 & T-1 & T \\ 1 & T-1 & T \end{Bmatrix} = \left[\frac{2!\ (2T-2)!\ (2T+1)!\ 2!\ (2T-2)!\ 2!\ (2T-2)!}{(2T+1)!\ (2T-2)!\ (2T+1)!\ 2!\ (2T+1)!}\right]^{\frac{1}{2}}$$

$$= \frac{1}{T(4T^2 - 1)}$$

(having used equation 2.24, P.19, of reference 64)

For s = T - 1, u = T

$$W = \begin{Bmatrix} 1 & T-1 & T \\ 1 & T & T \end{Bmatrix} = \begin{Bmatrix} 1 & T & T \\ 1 & T-1 & T \end{Bmatrix} = -\left[\frac{2\ (2T+1)\ (2T-1)\ (1)\ (2)}{2T\ (2T=1)\ (2T+2)\ (2T-1)\ 2T\ (2T+1)}\right]^{\frac{1}{2}}$$

$$= -\frac{1}{T(2T+1)}$$ (having used equation 2.23, reference 64)

For s = T - 1 , u = T + 1

$$W = \begin{Bmatrix} 1 & T & T-1 \\ 1 & T & T+1 \end{Bmatrix} = \begin{matrix} 1 & T & T+1 \\ 1 & T & T-1 \end{matrix} = \left[\frac{2(T+1)(2T+3)(2T)(2T-1)}{(2T-1)(2T)(2T+1)(2T+1)2(T+1)(2T+3)}\right]^{\frac{1}{2}}$$

$$= \frac{1}{2T+1}$$ (having used equation 2.23, reference 64)

For s = T, u = T

$$W = \begin{Bmatrix} 1 & T & T \\ 1 & T & T \end{Bmatrix} = \frac{[2\ (2T(T+1)\ -\ 2)]}{[2T(2T+1)2(T+1)2T(2T+1)2(T+1)]^{\frac{1}{2}}}$$

$$= \frac{T(T+1) - 1}{T(T+1)(2T+1)}$$ (having used equation 2.25, reference 64)

For s = T, u = T + 1

$$W = \begin{Bmatrix} 1 & T & T \\ 1 & T & T+1 \end{Bmatrix} = \begin{Bmatrix} 1 & T & T+1 \\ 1 & T & T \end{Bmatrix} \left[\frac{2(T+1)(2T+3)2T(2T-1)}{(2T-1)2T(2T+1)(2T+1)2(T+1)(2T+3)}\right]^{\frac{1}{2}}$$

$$= \frac{1}{(T+1)(2T+1)}$$ (having used equation 2.23, reference 64)

For s = T + 1, u = T + 1

$$W = \begin{Bmatrix} 1 & T & T+1 \\ 1 & T & T+1 \end{Bmatrix} = \left[\frac{2.2}{(2T+1)2(T+1)(2T+3)(2T+1)2(T+1)(2T+3)} \right]^{\frac{1}{2}}$$

$$= \frac{1}{(2T+1)(T+1)(2T+3)}$$ (having used equation 2.2 , reference64)

Using the symmetry of the 6 - j symbol in the form

$\begin{Bmatrix} 1 & T & s \\ 1 & T & u \end{Bmatrix} = \begin{Bmatrix} 1 & T & u \\ 1 & T & s \end{Bmatrix}$, we have, collecting the above results,

by (5A.6), the crossing matrix

$$A = X_{us} = \begin{bmatrix} \frac{1}{T(2T+1)} & \frac{-1}{T} & \frac{2T+3}{2T+1} \\ \frac{-(2T-1)}{T(2T+1)} & \frac{T(T+1)-1}{T(T+1)} & \frac{2T+3}{(2T+1)(T+1)} \\ \frac{2T-1}{2T+1} & \frac{1}{T+1} & \frac{1}{(2T+1)(T+1)} \end{bmatrix} \qquad (5A.9)$$

$T = 1$ is the only kinematically permissible incoming isospin which can couple the s and t channels, in which case $X_{us} = X_{st}$.

We need, therefore, only consider the matrix $A = X_{us}$, derived above.

It is easily seen that, irrespective of the value of T (other than $T = -\frac{1}{2}$), the sum of the rows of A is unity, that $A^2 = I_3$, and that Tr A = +1. A is, therefore, a Type II matrix of the kind considered previously. We see, by inspection that we may write

$A = \left[\delta_{us} - 2b_u d_s \right]$ where, for example,

$b_{T-1} = \frac{T+1}{T}$; $b_T = \frac{1}{T}$; $b_{T+1} = -1$

$d_{T-1} = \frac{(2T-1)}{2(2T+1)}$; $d_T = \frac{1}{2(T+1)}$; $d_{T+1} = \frac{-(2T+3)T}{2(T+1)(2T+1)}$

It is then seen that

$$\frac{1}{2}\frac{\left[b_{T-1} \div b_{T+1}\right]}{\left[b_T - b_{T+1}\right]\left[b_{T-1} - b_T\right]} = \frac{1}{2}\frac{\frac{1}{T}}{\frac{T+1}{T} \cdot 1} = \frac{1}{2(T+1)} = d_T$$

which is the consistency requirement of Section 5.4.

When T is regarded as a continuous parameter, we refer to the matrix A, of (5A.9) as a 'generalised' SU_2 crossing-matrix.

CHAPTER SIX

THE INVERSE SCATTERING PROBLEM

The model we have discussed, has described elastic scattering and not involved production processes. The model is also static, with a fixed scattering centre. We would expect such a situation, with the semi-relativistic kinematics incorporated into the model, to be described by the Klein-Gordan equation.

In this chapter, we therefore discuss the following problem: having solved for the S-matrix elements, to what extent may a spherically symmetric potential be determined, which will produce the same scattering when substituted into the Klein-Gordan equation.

This work was motivated by the hope that the existence or non-existence of the potential, or perhaps an asymptotic property of the determined potential, would give an alternative formulation of the bootstrap conditions. The hope was not realised, however, since computational procedures are inadequate to provide an exact solution to the problem.

In Section 6.1 we describe briefly the inversion procedure applicable to the Klein-Gordan equation. In Section 6.2 we illustrate with a simple example, the computational difficulties inherent in the method. These difficulties are circumvented by an approximation for certain integral equation kernels, which we describe in Section 6.3. We then present a method of our own for the actual solution of the resulting integral equations in Section 6.4. Approximate potentials so derived are shown to be of no use for a discussion of the bootstrap criterion. A possible alternative procedure is briefly discussed in Section 6.5.

Section 6.1 The Relativistic Ge'lfand-Levitan Equations.

The Klein-Gordan equation in the time-independent form, may be written, for a spherically symmetric potential, thus:

$$\left[k^2 + (V^2 - 2EV) - \frac{\ell(\ell+1)}{r^2} + \frac{\partial^2}{\partial r^2}\right] \varphi_\ell(E,r) = 0 \quad (6.1.1)$$

where ℓ is the orbital angular momentum of the scattered particle. This equation is, of course, very similar to the Schrödinger equation, being identical to the latter except for the substitution of the energy dependent term

$$\check{V}(r,k) = \left[2(k^2 + 1)^{\frac{1}{2}} - V(r)\right] V(r) \text{ for the potential term.}$$

It has, therefore, many features in common with the Schrödinger equation, (see, for example, Corinaldesi[65]) and an inversion procedure analogous to that of Ge'lfand and Levitan for the non-relativistic case may be set up. For a review of the work on non-relativistic Ge'lfand-Levitan equations, see Faddeyev[34].

Let $\varphi_\ell(E,r)$ be a solution of (6.1.1) such that $\varphi_\ell(E,r)$ behaves like $r^{\ell+1}$ at $r = 0$. $\varphi_\ell(E,r)$ is then an integral function of E. We also define the two independent solution $f_\ell(k,r)$ and $f_\ell(-k,r)$, having the asymptotic behaviour

$$f_\ell(k,r) \sim e^{-ikr} \quad ; \quad f_\ell(-k,r) \sim e^{ikr} \quad \text{for large } r.$$

Then $\varphi_\ell(E,r) = \frac{1}{2ik}\left[f_\ell(k) f_\ell(-k,r) - (-1)^\ell f_\ell(-k)\, f_\ell(k,r)\right]$ (6.1.2)

where $f_\ell(k) = \lim_{r \to 0+} (2\ell+1) r^\ell f_\ell(k,r)$, is the Jost function[66] for the problem.

The S-matrix and phase shift are then given by

$$S_\ell(k) = e^{-2i\delta_\ell} = (-1)^\ell \frac{f_\ell(k)}{f_\ell(-k)}, \quad k \geqslant 0,$$

bound state energies being determined by the zeros of $f_\ell(k)$ which lie in the half-plane $\text{Im}\, k \leqslant 0$.

$f_\ell(k)$ is clearly a two-valued function of E, having branch points at $E = \pm 1$. Consider the function

$$M(E) = f_\ell(k)\, f_\ell(-k) \tag{6.1.3}$$

$M^{(2)}(E) = M(E)$, so that M(E) is a one-valued function of E.

Let $\varphi_\ell^o(E,r)$ be the solution of (6.1.1), corresponding to a potential $V_o(r)$, and distinguish all other quantities corresponding to V_o by a superscript o.

Then, by a technique employing complex integration in the k-plane, Verde[67] is able to show that φ_ℓ^o and φ_ℓ are related by the integral equation:

$$\Phi_\alpha(E,r) = \Phi_\alpha^o(E,r) \cos\left[\int_o^r (V(r') - V_o(r'))dr'\right]$$

$$+ \int_o^r \bar{K}_{\alpha\gamma}(r,t)\, \Phi_\gamma^o(E,t)dt \tag{6.1.4}$$

where $$\begin{pmatrix} \varphi_{\ell,1}(E,r) \\ \varphi_{\ell,2}(E,r) \end{pmatrix} \equiv \underline{\Phi_\ell(E,r)} \equiv \begin{pmatrix} \varphi_\ell(E,r) \\ (E-V)\varphi_\ell(E,r) \end{pmatrix} \tag{6.1.5}$$

$$\bar{K}_{\alpha\beta}(r,s) = \int_{-\infty}^{\infty} d[\rho^{\circ} - \rho] \; \Phi_{\alpha}(E,r) \; \hat{\bar{\Phi}}^{\circ}_{\beta}(E,s) \qquad (6.1.6)$$

$$\underline{\Phi} = \sigma_1 \underline{\bar{\Phi}} \quad \text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (6.1.7)$$

$$\frac{d\rho}{dE} = \frac{k}{\pi M(E)} \; , \quad |E| \geqslant 1 \qquad (6.1.8)$$

$$= \sum_n C_n \, \delta(E - E_n) \; , \quad |E| < 1$$

where $C_n = \left[2 \int_0^{\infty} (E_n - V) \varphi_n^2(r) \, dr \right]^{-1}$

and E_n , and φ_n denote the energy and eigenfunctions of a bound state.

Define $\quad K^{\circ}_{\alpha\beta}(t,s) = \int_{-\infty}^{\infty} d[\rho^{\circ} - \rho] \; \bar{\Phi}^{\circ}_{\alpha}(E,t) \; \hat{\Phi}^{\circ}_{\beta}(E,s) \qquad (6.1.9)$

Hence, multiplying equation (6.1.4) by $\hat{\Phi}^{\circ}_{\beta}(E,s)$ and integrating with respect to the weight $d[\rho^{\circ} - \rho]$, we obtain

$$K_{\alpha\beta}(r,s) = K^{\circ}_{\alpha\beta}(r,s) + \int_0^r K_{\alpha\gamma}(r,t) \, K^{\circ}_{\gamma\beta}(t,s) dt \qquad (6.1.10)$$

where $\quad K_{\alpha\beta}(r,s) = \dfrac{\bar{K}_{\alpha\beta}(r,s)}{\left\{ \cos\left[\int_0^r (V(r') - V_0(r'))dr' \right] \right\}} \qquad (6.1.11)$

It may then be shown that [67)]

$$\frac{K_{11}(r,r)}{K_{12}(r,r)} = \tfrac{1}{2}\left[V(r) - V_0(r)\right] + (\ell + \tfrac{1}{2})\left[V(0) - V_0(0)\right] \qquad (6.1.12)$$

The procedure for solving the inverse scattering problem by this method, may be summarised as follows:

Given the S-matrix $S_\ell(k)$, we form the function M(E), and find the bound state zeros. We thus have the function $\rho(E)$, except for the constants C_n. The wave equation (6.1.1) is solved for a known potential $V_o(r)$, and we may then compute the kernel $K^o_{\alpha\beta}(t,s)$. We must then solve the coupled integral equations (6.1.10), and having obtained the functions $K_{11}(r,s)$ and $K_{12}(r,s)$, compute V(r) by means of (6.1.12). We note from (6.1.12), that we must insist that V(r) has no singularity at the origin, else it remains undetermined by this procedure.

Section 6.2 Derivation of the Kernels, $K^o_{\alpha\beta}$

Assuming that M(E) has been obtained, expressions must be derived for the kernels, $K^o_{\alpha\beta}$. This is the first, and very severe, difficulty of the actual procedure.

We illustrate the difficulty by considering the case of S-wave scattering, for the simplest non-trivial Jost function we can write down:

$$f(k) = \frac{k + i\alpha}{k + i\beta} \quad ; \quad \alpha, \beta > 0 \tag{6.2.1}$$

There are no bound states.

We may take $V_o(r) \equiv 0$ so that

$$\varphi^o(E,r) = \frac{\sin kr}{k} \quad ; \quad f^o(k) \equiv 1 \tag{6.2.2}$$

$$\underline{\Phi}^o = \begin{pmatrix} \dfrac{\sin kr}{k} \\ \dfrac{E\sin kr}{k} \end{pmatrix} ; \quad \underline{\hat{\Phi}}^o = \begin{pmatrix} \dfrac{E\sin kr}{k} \\ \dfrac{\sin kr}{k} \end{pmatrix} \tag{6.2.3}$$

Now, $\frac{d\rho^{o}(E)}{dE} - \frac{d\rho(E)}{dE} = \frac{k}{\pi}\left(1 - \frac{1}{M(E)}\right) = \frac{k}{\pi}\,\frac{(\beta^2 - \alpha^2)}{k^2 + \alpha^2}$, $|E| \geqslant 1$

$\therefore$ Using (6.1.9)

$$K^{o}_{11}(r,s) = K^{o}_{22}(r,s) = \int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{k^2}{E}\,\frac{\alpha^2 - \beta^{-2}}{k^2+\alpha^2}\,\frac{\sin kr}{k}\,\frac{E\sin ks}{k}\,dk$$

$$= \frac{\alpha^2 - \beta^2}{\pi}\, e^{-\alpha r}\sinh\alpha s \;,\quad r > s$$

$$= \frac{\alpha^2 - \beta^2}{\pi}\, e^{-\alpha s}\sinh\alpha r \;,\quad r < s$$

Similarly, $K^{o}_{12}(r,s) = \frac{\alpha^2 - \beta^2}{\pi}\int_{-\infty}^{\infty} \frac{\sin kr\,\sin ks}{E\,(k^2 + \alpha^2)}\,dk$

$$K^{o}_{21}(r,s) = \frac{\alpha^2 - \beta^2}{\pi}\int_{-\infty}^{\infty} \frac{E\,\sin kr\,\sin ks}{(k^2 + \alpha^2)}\,dk$$

Consider $I_1 = \frac{\pi K^{0}_{12}}{\alpha^2 - \beta^2} = \int_{-\infty}^{\infty} \frac{\sin kr\,\sin ks}{\sqrt{k^2+1}\,(k^2+\alpha^2)}\,dk$

$$= \frac{i}{4\alpha}\int_{-\infty}^{\infty}\frac{1}{\sqrt{k^2+1}}\left[e^{ik(r+s)} - e^{-k(r-s)}\right]\left[\frac{1}{k-i\alpha} - \frac{1}{k+i\alpha}\right]dk$$

$\therefore$ Consider $I_1^{+}(\lambda) = \int_{-\infty}^{\infty} \frac{e^{ik\lambda}}{\sqrt{(k^2+1)}\,(k+i\alpha)}\,dk$

We attempt to evaluate this integral by variation of parameters.

$$\frac{d}{d\lambda} I_1^{+}(\lambda) = i\int_{-\infty}^{\infty}\frac{k e^{ik\lambda}}{\sqrt{k^2+1}}\,\frac{dk}{(k+i\alpha)}$$

$$= i\int_{-\infty}^{\infty}\frac{e^{ik\lambda}}{\sqrt{k^2+1}}\,dk + \alpha I_1^{+}(\lambda)$$

$$= 2i\,K_o(|\lambda|) + \alpha I_1^{+}(\lambda)$$ (see p.433 of reference 68)

$$\frac{d}{d\lambda}\left[e^{-\alpha\lambda} I_1^{+}(\lambda)\right] = 2i\, e^{-\alpha\lambda} K_o(|\lambda|)$$

We may suppose $\lambda \geqslant 0$.

We must, therefore, obtain an expression for

$$\int^{\lambda} e^{-\alpha\mu} K_o(\mu)\, d\mu$$

It seems impossible to obtain an expression for this integral, in closed, manageable form, so as to make amenable the solution of the coupled integral equations. Employing alternative methods for the evaluation of the kernels K^o_{12}, and K^o_{21}, leads eventually to the necessity of obtaining a closed expression for the above integral. We are therefore unable to carry through the procedure for this simple case, which, with non-relativistic kinematics and the ordinary Ge'lfand-Levitan equation leads to a very simple potential.[34)]

The situation becomes many times worse for more complicated Jost functions, and we despair of being able to invert the problem corresponding to the S-wave S-matrix of Chapter 4.

Of course, with the simple Jost function, and for general rational Jost functions, the procedure is perfectly amenable to a machine solution by standard numerical methods.

Section 6.3 Approximate Expressions for the Kernels

In view of the difficulties encountered in calculating the necessary kernels, as illustrated in the preceeding section, it seems worthwhile to develop a method which will give approximate expressions for the kernels, in a form facilitating the solution of the integral equations. Such a method has been given by De Alfaro[69], being an extension of that of Regge[70] for the non-relativistic, or Schrödinger case. We outline this method here, and in the next section demonstrate a method for the practical solution of the integral equations.

We now restrict ourselves to S-wave scattering, and consider the problem of determining a potential, V(r), with finite range a, which reproduces the scattering as closely as possible. The S-wave restriction is necessary for the ensuing mathematical development, while the finite range assumption may be generalised to that of a potential which decreases sufficiently fast, namely faster than any exponential (see Regge[70]). However, it is argued that the extreme tail of a potential makes little contribution to physically measureable quantities, particularly to bound state energies and low lying resonances.

Because of the formal equivalence of the Schrödinger and Klein-Gordan equations, the arguments of Regge[71], for the non-relativistic case, may be immediately generalised to the present situation, to show that M(E) is an entire function of E of type 2a,[72] having an infinite number of zeros. It may also be shown that as the moduli of the zeros increases, the zeros diverge asymptotically from the real axis.

Let $\{E_n\}$ denote the set of zeros of M(E). De Alfaro[69] is then able to show that the set $\{\underline{\Phi}(E_n,r)\}$ is both a complete and independent set in the interval $[0,2a]$, by using the general theory of entire functions. It then follows by (6.1.4) that the same is true for the set $\left\{ \underline{\bar{\Phi}}^{o} (E_n,r) \right\}$.

We may, therefore, expand the kernel $K^{o}_{\alpha\beta}(r,s)$ in terms of the $\underline{\bar{\Phi}}^{o} (E_n,r)$:

$$K^{o}_{\alpha\beta}(r,s) = \sum_n \omega_{n\alpha}(r) \; \underline{\hat{\bar{\Phi}}}^{o}_{\beta} (E_n,s) \tag{6.3.1}$$

Consider the function $M_n(E) = \frac{M(E)}{E - E_n}$ (6.3.2)

By a generalisation of the Paley-Weiner Theorem [73], one may show that $M_n(E)$ admits the integral representation

$$M_n(E) = \int_0^\infty \theta_{n\alpha}(r) \; \hat{\bar{\Phi}}^{o}_{\alpha} (E,r) \, dr \tag{6.3.3}$$

where $\theta_{n\alpha}(r) = 0$ for $r > 2a$,

$$\int_0^{2a} \theta_{n\alpha}(r) \; \hat{\bar{\Phi}}^{o}_{\alpha}(E_m,r) \, dr = \delta_{mn} \, M'(E_m) \tag{6.3.4}$$

Using the orthogonality relation, of the twice-complete solutions of the Klein-Gordan equation:

$$\delta_{\alpha\beta}(r - s) = \int_{-\infty}^{\infty} d\rho_o \,(E) \; \bar{\Phi}^{o}_{\alpha} (E,r) \; \hat{\bar{\Phi}}^{o}_{\beta} (E,s) \tag{6.3.5}$$

we invert (6.3.3), to give

$$\theta_{n\alpha}(r) = \int_{-\infty}^{\infty} d\rho_o \,(E) \, \frac{M(E)}{E - E_n} \; \bar{\Phi}^{o}_{\alpha} (E,r) \tag{6.3.6}$$

Using (6.3.4), we obtain from (6.3.1), that

$$\omega_{n\alpha}(r) = \frac{1}{M'(E_n)} \int_0^{2a} \theta_{n\beta}(s) \; K^{o}_{\alpha\beta}(r,s) \, ds$$

$$= \frac{1}{M'(E_n)} \int_0^{2a} \theta_{n\beta}(s) \int_{-\infty}^{\infty} \left[d[\rho^{\circ} - \rho] \, \underline{\Phi}^{\circ}_{\alpha}(E,r) \, \hat{\underline{\Phi}}^{\circ}_{\beta}(E,s) \right] ds$$

$$= \frac{1}{M'(E_n)} \int_{-\infty}^{\infty} d\,[\rho^{\circ} - \rho] \quad \underline{\Phi}^{\circ}_{\alpha}(E,r) \frac{M(E)}{E - E_n}$$

$$= \frac{1}{M'(E_n)} \, \theta_{n\alpha}(r) - \frac{1}{M'(E_n)} \int_{-\infty}^{\infty} d\,\rho(E) \, \frac{M(E)}{E - E_n} \, \underline{\Phi}^{\circ}_{\alpha}(E,r) \qquad (6.3.7)$$

Using (6.1.2), (6.1.8) and, as may be shown, for example by using the results of the appendix of a paper by Croinaldesi,[73]

$$C_n = \frac{2ik_n}{M'(E_n)} \qquad (6.3.8)$$

then the integral on the right-hand side of (6.3.7) may be evaluated to give for $r > 0$,

$$\omega_{n\alpha}(r) = \frac{1}{M'(E_n)} \, \theta_{n\alpha}(r) + f^{\circ}(-k_n,r) \qquad (6.3.9)$$

where
$$\underline{f}^{\circ}(k_n,r) = \begin{pmatrix} e^{-k_n r i} \\ E_n \, e^{-ik_n r} \end{pmatrix} \qquad (6.3.10)$$

$\therefore$ from (6.3.1),

$$K^{\circ}_{\alpha\beta}(r,s) = \sum_n \frac{1}{M'(E_n)} \left[\theta_{n\alpha}(r) + f^{\circ}(-k_n,r) \right] \hat{\underline{\Phi}}^{\circ}_{\beta}(s)$$

But, as follows from (6.3.4) and the completeness of the set $\hat{\underline{\Phi}}^{\circ}(E_n,r)$

$$\sum_n \frac{1}{M'(E_n)} \, \theta_{n\alpha}(r) \, \hat{\underline{\Phi}}^{\circ}_{\beta}(s) = \delta_{\alpha\beta}(r-s) = 0 \quad \text{if } r > s.$$

Hence, we have the following expressions for the kernels:

$$K^{\circ}_{\alpha\beta}(r,s) = \sum_n \frac{1}{M'(E_n)} \, f^{\circ}_{\alpha}(-k_n,r) \, \hat{\underline{\Phi}}^{\circ}_{\beta}(s) \quad , \; r > s \qquad (6.3.11)$$

These expressions for the class of finite range potentials are, of course, exact when the sums are taken over the then infinite number of zeros of $M(E)$. The sums may be shown to be convergent.

Given the S-matrix $S = f(k)/f(-k)$, the approximation consists of forming the function $M(E) = f(k)\, f(-k)$, and then a finite sum (6.3.11), summed over only the bound state and resonance zeros of $M(E)$. It is, therefore implicitly assumed that $M(E)$ may be approximated as closely as we like, by an entire function with the same zeros, but such that other zeros would make a negligible contribution to (6.3.11). This assumption would be extremely difficult to justify in full mathematical rigour, but seems plausible from a physical point of view, as mentioned previously.

The range, a, of the potential remains arbitrary and may be chosen arbitrarily large.

It would be useful and instructive to be able to generalise the above method to arbitrary angular momentum. However, no apparent generalisation of the results on the analytic properties of the corresponding Jost functions exist, since now the effective potential does not decrease sufficiently fast.

Section 6.4 The Solution of the Integral Equations

We now demonstrate a method for the solution of equations (6.1.10), using the approximate and finite expression (6.3.11).

We take $V_o(r) \equiv 0$, so that $\underline{\Phi}^o$ and $\underline{\hat{\Phi}}^o$ are given by (6.2.3). Using the explicit expressions for $\underline{f}^o$ and $\underline{\Phi}^o$, we have

$$K^{o}_{11}(r,s) = \sum_{i=1}^{n} \frac{E_i}{k_i\, M'(E_i)} e^{ik_i r} \sin k_i s, \quad r > s$$

$$K^{o}_{12}(r,s) = \sum_{i=1}^{n} \frac{1}{k_i\, M'(E_i)} e^{ik_i r} \sin k_i s, \quad r > s \qquad (6.4.1)$$

$$K^{o}_{22}(r,s) = K^{o}_{11}(r,s)$$

$$K^{o}_{21}(r,s) = \sum_{i=1}^{n} \frac{E_i^2}{k_i\, M'(E_i)} e^{ik_i r} \sin k_i s, \quad r > s$$

$K^{o}_{\alpha\beta}(r,s)$ is symmetric with respect to interchange of r and s.

Put $F_i = k_i\, M'(E_i)$ (6.4.2)

Writing (6.1.10) as

$$K_{\alpha\beta}(r,s) = K^{o}_{\alpha\beta}(r,s) + \int_0^s K_{\alpha\gamma}(r,t)\, K^{o}_{\gamma\beta}(t,s)\,dt + \int_s^r K_{\alpha\gamma}(r,t) K^{o}_{\gamma\beta}(t,s)\,dt$$

and substituting (6.4.1), we obtain

$$\left.\begin{aligned} K_{11}(r,s) &= \sum_{i=1}^{n} \frac{E_i}{F_i}\, g_i(r,s) \\ K_{12}(r,s) &= \sum_{i=1}^{n} \frac{1}{F_i}\, g_i(r,s) \end{aligned}\right\} \qquad (6.4.3)$$

where for $r > s$,

$$g_i(r,s) = e^{ik_i r} \sin k_i s + e^{ik_i s} \int_0^s \left[K_{11}(r,t) + E_i K_{12}(r,t)\right] \sin k_i t \; dt$$

$$+ \sin k_i s \int_s^r \left[K_{11}(r,t) + E_i K_{12}(r,t)\right] e^{ik_i t} dt \qquad (6.4.4)$$

$g_i(r,s)$ is easily seen to satisfy the partial differential equation:

$$\frac{\partial^2 g_i(r,s)}{\partial s^2} + k_i^2\, g_i(r,s) = -k_i\left[K_{11}(r,s) + E_i\, K_{12}(r,s)\right] \qquad (6.4.5)$$

But, by (6.4.3)

$$K_{11}(r,s) + E_i K_{12}(r,s) = \sum_{j=1}^{n} \frac{(E_i + E_j)}{F_j}\, g_j(r,s) \ , \ i = 1 - n.$$

$$\therefore \quad \frac{\partial^2 g_i(r,s)}{\partial s^2} + k_i^2\, g_i(r,s) = -k_i \sum_{j=1}^{n} \frac{(E_i + E_j)}{F_j}\, g_j(r,s)$$

Write $\underline{g}(r,s) = \begin{pmatrix} g_1(r,s) \\ \vdots \\ g_n(r,s) \end{pmatrix}$ (6.4.6)

Then, this system of differential equations may be written in matrix form:

$$P\,\underline{g} \equiv \left\{ \frac{\partial^2}{\partial s^2} I + \left[\begin{pmatrix} k_1^2 & & \\ & \ddots & \\ & & k_n^2 \end{pmatrix} + E \right] \right\} \underline{g} = 0 \qquad (6.4.7)$$

where $E = (E_{ij}) = \left(\frac{k_i(E_i + E_j)}{F_j} \right)$ (6.4.8)

g_1 , g_2 , ... g_n are then all solutions of the equation

$$(\det P)\, g = 0 \qquad (6.4.9)$$

This is a linear partial differential equation of order 2n, involving only even order differentials.

Write $\left[\begin{pmatrix} k_1^2 & & \\ & \ddots & \\ & & k_n^2 \end{pmatrix} + E \right] = B$ (6.4.10)

$g = e^{i\lambda y}$ is therefore a solution of (6.4.9) if

$$\det\left[B - \lambda^2 I\right] = 0 \qquad (6.4.11)$$

Let λ_1^2 ,........., λ_n^2 be the eigenvalues of B, which, for simplicity, we shall suppose to be distinct. The ensuing method, with suitable straightforward modifications, will treat the degenerate case.

Since by (6.4.4), $g_i(r,0) = 0$, we may write the solution to the coupled differential equations (6.4.7) in the form

$$\underline{g}(r,s) = A(r) \begin{pmatrix} \sin\lambda_1 s \\ \vdots \\ \sin\lambda_n s \end{pmatrix} \qquad (6.4.12)$$

where $A(r)$ is an n X n function matrix.

Substituting (6.4.12) into (6.4.7), we see that the matrix $A(r)$ must be constrained by the algebraic relations

$$\left[A(r) \begin{pmatrix} -\lambda_1^2 & & O \\ & \ddots & \\ O & & -\lambda_n^2 \end{pmatrix} + B\,A(r) \right] \begin{pmatrix} \sin\lambda_1 s \\ \vdots \\ \sin\lambda_n s \end{pmatrix} = \underline{O}$$

i.e.,
$$B\,A(r) = A(r) \begin{pmatrix} \lambda_1^2 & & O \\ & \ddots & \\ O & & \lambda_n^2 \end{pmatrix} \qquad (6.4.13)$$

Let U be the matrix whose i^{th} column is the eigenvector of B with eigenvalue λ_i^2. Then the general solution of (6.4.13) is

$$A(r) = U \begin{pmatrix} a_1(r) & & O \\ & \ddots & \\ O & & a_n(r) \end{pmatrix} \qquad (6.4.14)$$

where $a_i(r)$ is an arbitrary function of r.

We must now ensure consistency of our solution with (6.4.4). From (6.4.5)

$$K_{11}(r,s) + E_i K_{12}(r,s) = -\frac{1}{k_i} \sum_{j=1}^{n} A_{ij}(r)(-\lambda_j^2 + k_i^2)\sin\lambda_j s$$

$$(\text{N.B. } k_i \neq 0)$$

Substituting this expression into (6.4.4),

$$g_i(r,s) = e^{ik_i r}\sin k_i s - \frac{1}{k_i}\left\{\sum_{j=1}^{n} A_{ij}(r)(k_i^2 - \lambda_j^2)\left[e^{ik_i s}\int_0^s \sin\lambda_j t \,\sin k_i t\, dt + \sin k_i s\int_s^r \sin\lambda_j t\, e^{ik_i t}\, dt\right]\right\}$$

Evaluating the integrals and simplifying, this expression reduces to

$$g_i(r,s) = e^{ik_i r}\sin k_i s\left[1 + \frac{1}{k_i}\sum_{j=1}^{n} A_{ij}(r)(ik_i\sin\lambda_j r - \lambda_j\cos\lambda_j r)\right] + g_i(r,s)$$

Hence, $A_{ij}(r)$ must be such that

$$\sum_{j=1}^{n} A_{ij}(r)\,(\lambda_j\cos\lambda_j r - ik_i\sin\lambda_j r) = k_i \qquad (6.4.15)$$

Write $\underline{\Lambda}(r) = \begin{pmatrix}\sin\lambda_1 r\\ \vdots\\ \sin\lambda_n r\end{pmatrix}$; $\underline{\Lambda}'(r) = \begin{pmatrix}\lambda_1\cos\lambda_1 r\\ \vdots\\ \lambda_n\cos\lambda_n r\end{pmatrix}$; $\underline{k} = \begin{pmatrix}k_1\\ \vdots\\ k_n\end{pmatrix}$

(6.4.16)

$$\Lambda(r) = \begin{pmatrix}\sin\lambda_1 r & & 0\\ & \ddots & \\ 0 & & \sin\lambda_n r\end{pmatrix};\ \Lambda'(r) = \begin{pmatrix}\lambda_1\cos\lambda_1 r & & 0\\ & \ddots & \\ 0 & & \lambda_n\cos\lambda_n r\end{pmatrix};\ K = \begin{pmatrix}k_1 & & 0\\ & \ddots & \\ 0 & & k_n\end{pmatrix}$$

(6.4.17)

$\therefore$ (6.4.15) may be written in matrix form as

$$A\,\underline{\Lambda}'(r) - i\,K\,A\,\underline{\Lambda}(r) = \underline{k}$$

i.e., using (6.4.14),

$$U\begin{pmatrix}a_1(r) & & \\ & \ddots & \\ & & a_n(r)\end{pmatrix}\underline{\Lambda}'(r) - iK\,U\begin{pmatrix}a_1(r) & & \\ & \ddots & \\ & & a_n(r)\end{pmatrix}\underline{\Lambda}(r) = \underline{k}$$

$$\therefore \left(U\Lambda'(r) - iK\,U\Lambda(r)\right)\begin{pmatrix}a_1(r)\\ \vdots\\ a_n(r)\end{pmatrix} = \underline{k} \qquad (6.4.18)$$

Putting $W(r) = U\Lambda'(r) - iK\,U\Lambda(r)$ (6.4.19)

then, provided $\det W(r) \neq 0$, we may solve (6.4.18) to give

$$\begin{pmatrix} a_1(r) \\ \vdots \\ a_n(r) \end{pmatrix} = W^{-1}(r)\,\underline{k} \tag{6.4.20}$$

Now, from (6.4.12)

$$\underline{g}(r,s) = U \begin{pmatrix} a_1(r) & & 0 \\ & \ddots & \\ 0 & & a_n(r) \end{pmatrix} \underline{\Lambda}(s)$$

$$= U\Lambda(s)\,W^{-1}(r)\,\underline{k} \tag{6.4.21}$$

Put

$$\underline{p} = \begin{pmatrix} E_1/F_1 \\ \vdots \\ E_n/F_n \end{pmatrix} ; \quad \underline{q} = \begin{pmatrix} 1/F_1 \\ \vdots \\ 1/F_n \end{pmatrix} \tag{6.4.22}$$

Then

$$K_{11}(r,s) = \underline{p} \cdot \left[U\Lambda(s)\,W^{-1}(r)\,\underline{k} \right]$$

$$K_{12}(r,s) = \underline{q} \cdot \left[U\Lambda(s)\,W^{-1}(r)\,\underline{k} \right]$$

$$\therefore \quad \frac{K_{11}(r,r)}{K_{12}(r,r)} = \frac{\underline{p} \cdot \left[U\Lambda(r)\,W^{-1}(r)\,\underline{k} \right]}{\underline{q} \cdot \left[U\Lambda(r)\,W^{-1}(r)\,\underline{k} \right]}$$

The problem of determining the approximate potential according to the method of De Alfaro, has hence been reduced to that of determining the eigenvalues and vectors of a known matrix, and determining the inverse of a then known 'function matrix'. Given the input parameters, the method will fail to produce a non-arbitrary potential, if the inverse to this matrix does not exist, in which case arbitrary functions will appear in the potential. It is seen that, in general, the potential will be a rational function of trigonometrical and exponential functions.

Consider the case of one bound state zero at, say , $E = E_1$. Then from (6.4.23), we have immediately that

$$\tfrac{1}{2}[V(r) + V(0)] = E_1$$

$$\therefore \quad V(r) = \text{constant} = E_1 \quad \text{(assuming } V(0) \text{ to be finite)} \qquad (6.4.24)$$

i.e., we have a square-well potential.

For the case of two bound state zeros, at say $E = E_1$, E_2 , the algebra required is straightforward but tedious.

We obtain an expression of the form

$$\frac{K_{11}(r,r)}{K_{12}(r,r)} = \frac{a_1 \sin\lambda_1 r \sin\lambda_2 r + b_1 \sin\lambda_1 r \cos\lambda_2 r + c_1 \sin\lambda_2 r \cos\lambda_1 r}{a_2 \sin\lambda_1 r \sin\lambda_2 r + b_2 \sin\lambda_1 r \cos\lambda_2 r + c_2 \sin\lambda_2 r \cos\lambda_1 r}$$

where the a, b, c are functions of the input parameters and λ_1^2 , λ_2^2 are determined as described above.

λ_1 , λ_2 may, or may not be real, depending on the input information, although the overall expression for V(r) may be seen to be real.

Barut and Ruei[75] have shown that for arbitrary orbital angular momentum

$$\lim_{k\to\infty} f_\ell(k) = 1 + i \lim_{\lambda\to\infty} \int_0^\infty j_\ell^2(\lambda r) \frac{V(r,\lambda)}{\lambda} dr \qquad (6.4.26)$$

where $V(r,\lambda) = [2(\lambda^2 + 1)^{\frac{1}{2}} - V(r)] V(r)$ and $j_\ell(\rho)$ is the spherical Riccatti-Bessel function, defined in terms of the Bessel function by

$$j_\ell(\rho) = \left(\frac{\pi\rho}{2}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(\rho) \qquad (6.4.27)$$

We may use this result to show the nonvalidity of Levinson's Theorem with the approximate potentials.

Consider the square-well potential

$$V(r) = 0, \quad r > a$$
$$= V_o, \quad r \leqslant a$$

$$\therefore \quad V(r, \lambda) = \left[2(\lambda^2 + 1)^{\frac{1}{2}} - V_o\right] V_o, \quad r \leqslant a$$
$$= 0, \quad r > a$$

Put $$I_\lambda = \int_0^\infty j_\ell^2(\lambda r) \frac{V(r, \lambda)}{\lambda} dr$$

$$= \frac{\left[2(\lambda^2 + 1)^{\frac{1}{2}} - V_o\right] V_o}{\lambda} \int_0^a j_\ell^2(\lambda r)\, dr$$

But $$\int_0^a j_\ell^2(\lambda r)\, dr = \frac{\pi}{2\lambda} \int_0^{\lambda a} u\, J_{\ell+\frac{1}{2}}^2(u)\, du$$

$$= \frac{\pi}{2\lambda} \frac{\lambda^2 a^2}{2} \left[J_{\ell+\frac{1}{2}}^2(\lambda a) - J_{\ell-\frac{1}{2}}(\lambda a)\, J_{\ell+3/2}(\lambda a)\right]$$

(see P. 152, Reference 68)

$$\therefore \lim_{\lambda\to\infty} I_\lambda = 2V_o \frac{\pi}{4} \lambda a^2 \frac{2}{\pi\lambda a} \left\{\cos^2\left[\lambda a - \tfrac{1}{2}(\ell + \tfrac{1}{2})\pi - \frac{\pi}{4}\right] - \cos\left[\lambda a - \tfrac{1}{2}(\ell - \tfrac{1}{2})\pi - \frac{\pi}{4}\right] \cos\left[\lambda a - \tfrac{1}{2}(\ell + \tfrac{3}{2}) - \frac{\pi}{4}\right]\right\}$$

$= V_o a \neq 0$ for a non-trivial potential, and for any value of ℓ.

Hence, for such a potential Levinson's Theorem cannot be satisfied. We must conclude that a potential constructed by this approximate method, will yield no insight into the bootstrap nature of the static model.

Section 7.5 An Alternative Formulation

So far, we have been considering the problem of the determination of an equivalent potential, given a partial wave phase shift for all energies, and for a particular value of the orbital angular momentum. There is another possible approach, namely that of determining a potential, given the phase shifts at a particular, fixed energy and for all angular momentum. A technique for the solution of this problem in the case of the Schrödinger equation, has been proposed by Newton.[76) This approach, in some ways, is to be preferred, since, not relying on high energy behaviour, it is more likely, in a practical application, to lead to an acceptable potential, valid for low energies.

Newton considers the Schrödinger equation, normalised to take the form

$$\left[1 - V(r,k) - \frac{\ell(\ell+1)}{r^2} + \frac{\partial^2}{\partial r^2}\right]\varphi_\ell(E,r) = 0 \qquad (6.5.1)$$

and gives a prescription for the construction of V(r,k), given the phase shifts δ_ℓ , for all ℓ at the same fixed energy E. The method turns essentially, on the existence and construction of the inverse of an infinite matrix, the elements of which are functions of the δ_ℓ . Newton concludes that, provided only the phase shifts tend to zero sufficiently rapidly with increasing ℓ , there always exists an underlying potential, which is not necessarily unique.

Since the whole procedure is performed at constant energy, kinematics play no part in the analysis. The procedure is quite independent of the analytic properties of functions, as functions of energy. The method is therefore applicable to the Klein-Gordan equation, which when written in the form (6.1.1), may be transformed into the form (6.5.1), by putting

$x = rk, \quad V^2(r) - 2EV(r) = -k^2 U(x,k)$, to give

$$\left[1 - U(x,k) - \frac{\ell(\ell+1)}{x^2} + \frac{\partial^2}{\partial x^2}\right] \varphi_\ell(E,x) = 0 \qquad (6.5.2)$$

We may then determine, in principle, by Newton's method,

$$k^2 U(x,k) \equiv E^2 \Omega(r,k), \text{ say} \qquad (6.5.3)$$

$$\therefore \quad V(r) = E \pm \left[E^2 (1 - \Omega(r,k))\right]^{\frac{1}{2}}$$

As a minimal requirement for an acceptable potential, we must have $V(r) \to 0$, as $r \to \infty$ $(E > 1)$, and so only the negative sign need be considered,

$$\text{i.e.,} \quad V(r) = E\left[1 - [1 - \Omega(r,k)]^{1/2}\right] \qquad (6.5.4)$$

since $\Omega(r,k) \to 0$ as $r \to \infty$

The construction of such a potential by this method would be very useful, if only to determine how strongly the expression (6.5.4) varies with energy E, in order to draw some conclusion concerning the existence of an energy independent potential.

It has been shown by Sabatier,[77)] using the formalism of Newton, that if the phase shifts go to zero faster than ℓ^{-3} as ℓ goes to infinity, then there exists one and only one potential which goes to zero faster than $r^{-2+\epsilon}$, and that all equivalent potentials have an oscillating tail which is damped by a factor $r^{-3/2}$. We must have, therefore, subject to the above condition on the phase shifts, that the finite range potential considered previously, be uniquely determined, apart from the range a.

CHAPTER SEVEN

DISCUSSION AND CONCLUSIONS

The Rothleitner technique is clearly a very powerful means of solving the kind of model which we have considered. Exact and general solutions were obtained for the completely general two-dimensional case, while wide classes of special solutions were found for three channel cases. The weakness of the method lies in our inability to solve, in full generality, the resulting coupled difference equations for all but the two-channel model; perhaps more effort should be expended on an analysis of such equations, the mathematical literature being very sparse concerning this kind of problem. It may be, however, that we are, in fact, overlooking some more general transformation which would result in equations, amenable to known techniques of solution, or even that our solutions are the only ones compatible with the required analytic properties of the S-matrix functions. It would be especially valuable to obtain the exact general solution to the problem with the 4 X 4 crossing-matrix of the symmetric-pseudoscalar theory of π-N scattering. Special solutions to this problem are readily written down (see Huang and Low [29]) as products of solutions to the two-channel problem, noticing that this crossing-matrix is merely the tensor product of a pair of two-channel matrices. However, these special solutions are not the 'physical' solutions of most interest, not containing the solution with a bound state at $z = 0$ in in the (1,1) channel, a resonance in the (3,3) channel, and no further bound states or resonances in any channel.

In Chapter 4, we imposed the bootstrap criterion of Huang and Low[29] upon the solutions to the two-channel static model, with the crossing-matrix parameter corresponding to an internal SU_2 symmetry. The model might then be interpreted as describing the scattering of a particle with integer isospin, t, by a fixed isofermion. It was found that for arbitrary ℓ, and for arbitrary integer t, no bootstrap solution exists satisfying an unsubtracted dispersion relation, or without a cut-off. Also, with the assumption of one subtraction, the possible bound state distributions are severely limited, and for the cases $\ell = 0$, and 1, must be limited to one bound state in Channel 1, and to either 1 or 0 bound states in channel 2. The same conclusions were reached by Huang and Mueller[32] for arbitrary ℓ without recourse to the explicit solutions, but they were, of course, unable to check the sign of the coupling constants squared. It is of considerable interest to note that a similar solution to those found in Section 4.5 was constructed by McGlinn and Albright[28] for the case $t = 1, \ell = 1$, by the prescription that S_1 and S_2 have a 'minimum number of zeros'. However, they were assuming the specific form of the cut-off with c = 1, without consideration of possible subtractions, and so their solution did not verify Levinson's Theorem.

The assumption of one subtraction although sufficient, is clearly not necessary for the construction of bootstrap solutions, verifying the other 'physical' requirements. The case of one subtraction is, however, the most restrictive in the arbitrariness of the final solution. Even so, the arbitrariness is not entirely removed, there remaining, in general, two arbitrary parameters. This can be seen as follows: near threshold, the S-matrix function D(z) will have the behaviour,

$$D(z) \sim 1 + \alpha_1 q + \alpha_3 q^3 + \ldots\ldots + \alpha_{2\ell-1} q^{2\ell-1} + O(q^{2\ell+1})$$

For the required threshold behaviour to be preserved, we must have

$$\alpha_1 = \alpha_3 = \ldots\ldots\ldots = \alpha_{2\ell-1} = 0.$$

This is a system of relations involving the $\ell + 1$ parameters of $\beta(z)$ (for K = 1), and the cut-off parameter, κ. Hence, in general, any two of these parameters may be chosen arbitrarily. This arbitrariness, as suggested by Huang and Low[29], may be a manifestation of insufficient account being taken of high energy effects and other channels, only considered by means of the cut-off functions. To take more than one subtraction, would drastically affect the form of the solutions, via the functions $\beta(z)$ and D(z). We require a better understanding of the physical meaning of subtractions, in order to be able to interpret alternative solutions corresponding to different values of K. However, we note that Kinoshita[78] has proved that, under the assumptions of unitarity, analyticity and a quite weak high energy bound, the dispersion relation for a partial wave amplitude requires at most one subtraction for any angular momentum. Also, Oleson[79] has shown that at least one subtraction is a necessary condition for a bootstrap solution with $\ell \geqslant 2$, in the sense that otherwise, there must be at least one C.D.D. pole. (This result is derived subject to the

the amplitude verifying the Diffraction Picture[80]). It seems from these results, as verified by our model calculation, that one subtraction is likely to be a necessary, and preferable, feature of any realistic dispersion theoretic model.

Similarly, for the model considered, the usual definite choice of cut-off function, $v(z) = \dfrac{\kappa^{2c}}{(\kappa^2 + q^2)^c}$, $\kappa > 1$, has been made. We might also enquire to what extent this form is unique, and what effect an alternative form might have on our conclusions concerning the analytic structure of our solutions.

The general properties required to be satisfied by v(z) are:-

(i) v(z) must be a real meromorphic function of z in the entire z plane, so as not to introduce additional cuts and singularities into the solutions.

(ii) v(z) must be an even function of z, in order that S-matrix functions and amplitudes satisfy the same crossing relations. We may, hence, by condition (i), consider v to be a function of q^2 and write $v(z) \equiv v(q^2)$. (See Schweber[8], P.374, who notes that a cut-off function, interpreted as the Fourier transform of a spherically symmetric source function is a function of q^2)

(iii) v(z) must have no poles for $|z| < 1$ and no complex poles. It must also have, at most, a finite number of zeros.

(iv) $v(q^2) \longrightarrow 0$ as $q^2 \rightarrow \infty$

(v) $v(0) = 1$, in order that the threshold behaviour is preserved.

Conditions (i), (ii), and (iii) imply that $v(q^2)$ is a rational function of q^2, with a denominator function

$$D(q^2) = \prod_i (\kappa_i^2 + q^2)^{c_i}$$ where c_i is a positive integer, and κ_i is real. The numerator function must be a polynomial in q^2 of order at most $\sum c_i - 1$, with a constant term $\prod_i \kappa_i^2$ in order to satisfy (iv) and (v). In any event,

$$v(q^2) \sim 1/q^{2c} \quad \text{as } q^2 \sim \infty$$, where c is a positive integer. The analysis of Chapter 4 will then proceed exactly as before, and for the case of one subtraction, we must have $c = \ell + 1$. We would have thus shown that the order of the denominator is higher than that of the numerator by exactly $\ell + 1$. If $v(q^2)$ is permitted to have zeros, then v may assume an infinity of forms, each satisfying our bootstrap requirements; choosing any one particular form, the remaining analysis would then follow through as before, with suitable trivial modifications, and the S-matrix solutions would remain essentially unaltered in analytic form. However, if we require that the cut-off function should have no zeros, so that the numerator function is a constant, then for $\ell = 0$, we must have precisely the form assumed. For arbitrary ℓ, insisting that v should have no zeros, implies that $v(q^2)$ be a product of 'elementary' form factors, $\frac{\kappa_i^2}{\kappa_i^2 + q^2}$. If the κ_i^2 are required to be not distinct, so that the minimum number of cut-off poles are introduced into S_α, then we again obtain precisely the form assumed. We conclude that the assumed form is not necessary for a bootstrap analysis, but that it is determined by the requirements that it introduce the minimum number of least order poles into the S-matrix functions.

There would seem to be little point in attempting to impose the bootstrap criterion upon the solutions to the more general situations which we have considered. For example, the completely general two-channel model solved in Chapter 3 is just too general; the amplitudes satisfy two coupled Low type equations, and although the phase shifts may be computed, there is no implied relation between the pole structure of say, S_1 and that of S_2. In the case of the three-channel solution, equation (5.4.3), corresponding to the generalised SU_2 matrix, we note that unless $T = \frac{1}{2}$, in which case we have a two-channel situation, the solutions cannot have a bound state pole at the target particle energy, $z = 0$. Physical solutions of the form required, either do not exist or, more likely, are just not contained in our class of special solutions.

Having obtained a solution to the bootstrap problem, we would like to be able to interpret the implied analytic structure of the S-matrix functions, and so gain further insight into the bootstrap mechanism. We observe, from the analysis of Chapter 4, that the bootstrap criterion has restricted the solutions, for the models concerned, to the most simple concievable forms, all "extra" zeros and poles being eliminated, and the solutions reducing to a finite product of simple S-matrix functions. This suggests that a suitable interpretation might be made in terms of "cluster decomposition" properties,[81)] expressing the intuitive idea that if a number of scattering events are separated by large spatial distances, then the S-matrix for the complete event should factor into a product of S-matrices, describing the various 'independent' events. Essentially the same idea has been proposed by Goldberger and Watson [82], who argued from consideration of large time intervals. There are obvious difficulties concerning the immediate application of such notions; one must have a clear criteria of 'independence' of events, and with time separations, there is the problem of the order in which the events are supposed to occur. Also, of course, such a factorisation is not unique, although this is not a serious objection if one requires all S-matrix 'units' to cancel their inverses if present. Finally, there is the problem that the S-matrix factors may have multiple poles and zeros, which do not permit any clear physical interpretation. However, Bell and Goebel [83] have shown, by consideration of simple models, that double poles may be associated with non-exponential decays, while Hagen [84] has shown that multiple poles in the S-matrix, are not, in general, incompatible with Lagrangian field theory. If we leave aside these problems for the moment, we may make the qualitative statements that, for the model considered, the bootstrap mechanism is such that the process proceeds by the minimum possible number of events (i.e., the D function has a minimum number of factors), and that, not surprisingly, the process will proceed by a greater number of processes for greater incoming isospin, and for increasing orbital angular momentum, ℓ. Also, the processes involved would appear to increase in 'complexity' for increasing ℓ, as the function $\beta(z)$ gives rise to a greater number of zeros, poles and cancellations among the factors. We may also say that, with this interpretation of S-matrix factors, the restriction of the crossing-matrix parameter, t , to integer values is a prescription that the scattering proceed by a finite number of events. With an increased understanding of cluster decomposition properties, more insight will be obtained from the sort of model calculation we have performed.

For the involutory two-dimensional crossing-matrix of Chapter 4, it was shown that for values of the parameter, t, other than integer, in order that certain of our 'physical' requirements be satisfied, an essential singularity at threshold must be present in the S-matrix factors of the solution, which could not be cancelled. Martin and McGlinn[21)] inferred that such a restriction of the parameter, would amount to a dynamical prediction of symmetry, namely an SU_2 internal symmetry. However, although such a restriction of the parameters is certainly a necessary condition for this symmetry group, we should enquire whether or not it is a sufficient condition. We have attempted to examine this question in the following way:

Consider the outer product space of two spaces, spanned by $\{|r\rangle\}$ and $\{|s\rangle\}$, each of which constitutes a representation space of two, possibly distinct, symmetry groups. Suppose that there is some conserved quantity, taking two, and only two values. Denote the two projection operators on the product space, for each of these quantities by P_1 and P_2 .

Then $$P_i(r,r') \equiv \langle r | P_i | r' \rangle \quad , \; i = 1,2 \qquad (7.1)$$

are operators on the s space only.

We require that the $P_i(r,r')$ be projection operators in the $|s\rangle$

space in the sense that

$$P_1(r,r') + P_2(r,r') = (r,r') \qquad (7.2a)$$

$$\sum_{r'} P_i(r,r')\, P_j(r,r'') = \delta_{ij}\, P_i(r,r'') \qquad (7.2b)$$

We also require a crossing symmetry to hold in the form (see Appendix A of Martin and McGlinn[21]):

$$\begin{aligned} P_1(r,r') &= c\, P_1(r',r) + (1+c)\, P_2(r',r) \\ P_2(r,r') &= (1-c)\, P_1(r',r) - c\, P_2(r',r) \end{aligned} \qquad (7.3)$$

Write $P_i(r,r') = E_i(r,r') + O_i(r,r')$, $i = 1,2$, where $E_i(r,r')$ and $O_i(r,r')$ are respectively even and odd with respect to interchange of r and r', and so easily verified to be hermitian and anti-hermitian operators on the $|s\rangle$ space.

By (7.2a) we have, equating symmetric and anti-symmetric parts, $O_1(r,r') + O_2(r,r') = 0$; $E_1(r,r') + E_2(r,r') = \delta(r,r')$.

Substituting into (7.3) and putting $O_1(r,r') = O(r,r')$, we have that $P_i(r,r')$ must be as given by

$$P_1(r,r') = \left(\frac{1+c}{2}\right) \delta(r,r') + O(r,r') \qquad (7.4a)$$

$$P_2(r,r') = \left(\frac{1-c}{2}\right) \delta(r,r') - O(r,r') \qquad (7.4b)$$

The remaining operator condition (7.2b) then gives that

$$\sum_{r''} O(r,r'')\, O(r'',r') = \frac{1-c^2}{4}\, \delta(r,r') - c\, O(r,r') \qquad (7.5)$$

The problem now becomes that of finding the most general set of operators $O(r,r')$ in the $|s\rangle$ space, satisfying (7.5).

Decomposing (7.5) into symmetric and anti-symmetric parts, we have the following two commutator conditions,

$$\sum_{r''} \left\{ O(r,r'') \; , \; O(r'',r') \right\}_{+} = \frac{1-c^2}{2} \, \delta(r,r') \qquad (7.6)$$

$$\sum_{r''} \left[O(r,r'') \; , \; O(r'',r') \right]_{-} = -2c \; O(r,r') \qquad (7.7)$$

We see that if the dimension of the $|r\rangle$ space is either one or two, then by (7.7), all $O(r,r')$ vanish, which is inconsistent with (7.6). If the $|r\rangle$ space is of dimension three, then writing $S_i = \frac{2i}{c} \varepsilon_{ijk} O(j,k)$, S_i is a hermitian operator and verifies the SU_2 commutator conditions,

$$\left[S_i \; , \; S_j \right] = i \, \varepsilon_{ijk} S_k \qquad (7.8)$$

$$\left\{ S_i \; , \; S_j \right\} = \delta_{ij} \frac{t(t+1)}{2} \qquad (7.9)$$

where we have put $c = -1/(2t+1)$. Then by (7.9) the dimension of the $|s\rangle$ space is two, and $t = 1$. The forgoing formulation is then seen to check with scattering of an isospin 1 particle by a particle of isospin $\frac{1}{2}$, for which $c = -1/3$ and $O(r,r') = \frac{i}{3} \varepsilon_{jrr'} \tau_j$ (see Schweber, P. 385)[8)]

In the general case we may choose a set of n linearly independent operators O_k, and write

$$O(i,j) = \sum_{k=1}^{n} a_{ij}^{(k)} O_k \qquad (7.10)$$

and let A_k be the anti-symmetric matrix $\left\{ a_{ij}^{(k)} \right\}$.

The equations (7.6) and (7.7) then become:

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n}[A_k , A_\ell][O_k , O_\ell] = -2c\sum_{m=1}^{n} A_m O_m$$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} A_k A_\ell \{O_k , O_\ell\} = \left(\frac{1-c^2}{2}\right) I \qquad (7.11)$$

If we assume that the O_i form a three parameter Lie group, (of which there are four; see Hammermesh[85]) then the only possible case consistent with (7.11) is that of SU_2 when the $A_j/2ic$ will generate the same group of weight t, provided $c = 1/(2t+1)$.
This may then be regarded as an alternative derivation (see also Martin and McGlinn[21]) of the two-dimensional SU_2 crossing-matrix.

Efforts to analyse the more general situation have so far proved fruitless. The assumption that the A_i and O_i both generate the same Lie Algebra, even a semi-simple algebra, lead to conditions on structure constants to which the SU_2 case is certainly a solution, but we are unable to prove it to be a unique solution, or to demonstrate a counter example. The imposition of any further conditions is tantamount to specifying an SU_2 symmetry from the outset. It is clear from (7.11) that crossing symmetry does impose some restriction on possible symmetry groups, and that possible representations must be a function of the parameter c. This problem, we feel, should be investigated in full mathematical rigour, by perhaps some other approach, and that without a satisfactory resolution, one is not justified in claiming to have dynamically predicted an internal symmetry by the mere restriction of crossing-matrix elements. The problem is also important in that it may lead to a physical interpretation of the parameters of the general crossing-matrix.

The interpretation of the crossing-matrix parameters might also be provided by settling the question of whether the S-matrix and Hamiltonian approaches are, in essense, equivalent. If a hamiltonian may be constructed such as to reproduce all the observables predicted by a given S-matrix, we would expect the hamiltonian to involve the crossing-matrix parameters, and so perhaps elucidate their interpretation. The question, however, remains an open one. Kirzhnits[86] has compared, for a particular simple model, the approaches of the hamiltonian, axiomatic and S-matrix methods, and argues that axiomatic and S-matrix methods result in successive relaxations of the properties of the scattering amplitude, and increased number of solutions. We would not,then, expect an equivalent hamiltonian, should it exist, to be uniquely determined. Pearson[87] has shown that for a certain class of amplitudes, non-unique equivalent hamiltonians may, in principle, be constructed. Fonda and Ghirardi[88] have shown that given a suitably behaved phase shift, and bound state energies, then an equivalent hamiltonian, corresponding to the Dyson model[26] may be determined. There does seem, therefore, to be some hope of resolving the general equivalence of the two approaches for, at least, some class of S-matrices.

Closely related to this problem is that of the existence of an equivalent potential, which we discussed in Chapter 6. The 'Inverse Scattering Problem', has continued to interest a number of authors.

For example, Cox[89] has extended the Ge'lfand-Levitan formalism to include inelastic scattering, while Cirelli and Prosperi[90] have considered the construction of velocity dependent potentials. However, the greatest need, before such formulations may be usefully implemented, is the development of suitable computational procedures. The method of Newton[76] is most amenable to such development, although we saw from the work of Chapter 4, that even for a simple model, the problem of constructing all partial wave S-matrix functions is itself no easy task.

Finally, there is the question of the extension of our considerations to a realistic, fully relativistic situation where we will of necessity, be concerned with functions of several complex variables. Aks' theorem[14], as discussed in Chapter 1, indicates that the basic requirements of analyticity, unitarity and crossing symmetry will be more restrictive of S-matrix solutions, and the bootstrap hypothesis will be more severely tested. The relativistic model and work of Cushing[33] is a promising starting point for such a program.

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