# BAER STRUCTURES, UNITALS AND ASSOCIATED FINITE GEOMETRIES

by

Catherine Therese Quinn, B.Sc.(Ma.) (Hons.) (Adelaide)

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Department of Pure Mathematics

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# Contents

# Abstract


In this thesis we study the representation of finite translation planes in projective spaces introduced by André [1]. This theory was also developed by Bruck and Bose [21, 22] in a distinct but equivalent form. Throughout this thesis we refer to this representation as the *Bruck and Bose* representation or simply *Bruck-Bose.* Of particular importance is the representation of Baer subplanes of translation planes $\pi_{q^2}$ of order $q^2$; the importance is due to the crucial role Baer subplanes have in the characterisation of various substructures, including unitals and maximal arcs, of projective planes, as will be evident in the text.

In Chapter 1 we present the necessary preliminary material required for the later chapters. In particular we present in detail the Bruck and Bose representation [21, 22] of the Desarguesian plane $PG(2, q^h)$ and the associated coordinatisation.

In Chapter 2 we begin by reviewing the known results concerning the representation of Baer subplanes of $PG(2, q^2)$ in the Bruck and Bose representation in $PG(4, q)$. We provide a new proof of the result of Vincenti [90] and Bose, Freeman and Glynn [19], that the non-affine Baer subplanes of $PG(2, q^2)$ are represented in Bruck-Bose by certain ruled cubic surfaces in $PG(4, q)$ which we term *Baer ruled cubic surfaces.* We characterise Baer ruled cubic surfaces in $PG(4, q)$ for a general fixed Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$. We determine that non-degenerate conics in Baer subplanes of $PG(2, q^2)$ are represented in Bruck-Bose by normal rational curves; a normal rational curve which arises in this way is of order 2, 3 or 4 and is therefore properly contained in a plane, hyperplane or no hyperplane of $PG(4, q)$ respectively. We apply these results to prove the existence of certain $(q^2 + 1)-$caps in $PG(4, q)$ which are not contained in any hyperplane of $PG(4, q)$ and which contain many normal rational curves of order 4. Further properties of these caps are determined in Chapter 3. We also include a discussion of the ruled cubic surface obtained as the projection from a point $P$ of the Veronese Surface in $PG(5, q)$ onto a hyperplane not containing $P$; in this setting we determine some alternative proofs for our results and prove some extensions.

In Chapter 3 we investigate the Bruck and Bose representation in $PG(n, q)$ with $n > 4$. We prove various results concerning the regular $(h-1)-$spreads of $PG(2h-1, q)$ which determine the Bruck and Bose representation of $PG(2, q^h)$ in $PG(2h, q)$, treating the

case $h = 4$ in greater detail. In particular, we prove the existence of *induced* spreads and show how the induced spreads are closely related to Bruck and Bose representation of the Baer substructures of $PG(2,q^h)$. To obtain further properties of the higher dimensional Bruck-Bose representation of the non-affine Baer substructures of the Desarguesian plane, we make use of the Bose representation [18] of $PG(2,q^2)$. In this chapter, we also prove results concerning the Bruck and Bose representation of non-degenerate conics in $PG(2,q^2)$ and we discuss the relationship between these results and the Bruck-Bose representation of non-affine Baer sublines of $PG(2,q^4)$ in $PG(8,q)$.

In Chapter 4 we investigate Baer subplanes and Buekenhout-Metz unitals in $PG(2,q^2)$. In particular we improve the known results by showing that in $PG(2,q^2)$, with $q > 13$, a Baer subplane and a Buekenhout-Metz unital with elliptic quadric as base have at least 1 point and at most $2q+1$ distinct points in their intersection. Our method of proof makes use of the Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$ and the properties of a certain irreducible sextic curve in $PG(4,q)$. We also prove that the non-classical Buekenhout-Metz unitals, with an elliptic quadric base, in $PG(2,q^2)$ are inherited from the classical unitals in $PG(2,q^2)$ by a certain procedure of swapping regular $1-$spreads of $PG(3,q)$ in the Bruck and Bose representation of $PG(2,q^2)$.

In Chapter 5 we prove that a unital in $PG(2,q^2)$ is a Buekenhout-Metz unital if and only if there exists a point $T$ of the unital such that each secant line of the unital through $T$ intersects the unital in a Baer subline. This is an improvement of the characterisation of Lefèvre-Percsy [56] and an improvement of the characterisation of Casse, O'Keefe and Penttila [26] for the cases $q > 3$.

In the final chapter we investigate the relationships between Thas maximal arcs, the generalized quadrangle $T_3(\mathcal{O})$ and egglike inversive planes. This work was motivated by the approach of Barwick and O'Keefe [13] in investigating the relationship between Buekenhout-Metz unitals and inversive planes (see also [6, Section 5.] and [92]). We attempt to characterise the Thas maximal arcs in those translation planes where they exist using two configurational properties; we do not succeed in this, but prove a characterisation of Thas maximal arcs in $PG(2,q^2)$ for certain values of $q$.