The Theory of A_p^q Spaces

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Contents

Abstract

The aim of the thesis is to extend the notion of A_p^q space from its historical context in the work of Herz and to recognise such spaces as preduals of spaces of intertwining operators of induced representations as suggested by the work of Rieffel. This generalisation of A_p^q spaces involves considering tensor products of a given norm of L_p spaces of Banach space-valued functions (the spaces of induced representations) and constructing a convolution of functions of such spaces. First, the analysis is carried out when the tensor product space is endowed with the greatest cross-norm, and sufficient conditions for the existence of the integral of the convolution are established. Most of this analysis depends upon an identity we derive of Radon-Nikodym derivatives of measures on homogeneous spaces involved.

The elements of the generalised A_p^q space are shown to be cross-sections of a Banach semi-bundle over the double coset space corresponding to the groups from which the representations are induced, and their properties are duly discussed. In particular, the generalised form of the classical result $L_p * L_q \subseteq L_r$, where $1/r = 1/p + 1/q - 1$, is shown to be true in this situation. The result that the A_p^q space is the predual of the space of intertwining operators is then established, under the condition that the intertwining operators can be approximated, in the ultraweak operator topology, by integral operators.

Sufficient conditions under which the above analysis can be carried out, when the tensor product space is endowed with either p-nuclear norm or the Hilbert-Schmidt norm are then given.

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Chapter 1

Introduction

The study of intertwining operators of induced representations on Hilbert spaces was begun by Mackey [31, 32, 33]. He generalized the notion of a representation of a finite group by linear transformations to the case in which the group is a separable locally compact topological group and the linear transformations are unitary transformations on Hilbert spaces. His main results include a generalized version of the Frobenius Reciprocity Theorem, the Intertwining and the Strong-Intertwining Operator Theorems.

To explain these results, let G be a group, let H and K be two subgroups of G, and let π and γ be representations of H and K, respectively. If G is finite, Mackey's results assert that the **intertwining number** (Sec.2.4) of the two **induced representations** U^π and U^γ of G (Sec.2.5) can be expressed as a sum of intertwining numbers of the representations π^x and γ^y of the subgroups $H^x \cap K^y$, $x, y \in G$. In the case of an infinite group, if the subgroups are open and closed, a similar characterization is possible especially when π and γ are one-dimensional. If the subgroups are closed, Mackey showed that the above criteria for computing the intertwining number holds for the space of those operators which are in the Hilbert-Schmidt class.

Moore [34], in 1962, continued this study. The fact that every continuous linear map of an L_1 space into a separable reflexive space can be better represented as an integral operator led him to extend the concept of induced representations to include the action of a group on a Banach space by isometries. He proved that the Frobenius Reciprocity Theorem remains true under these modifications and the assumption that the corresponding G-coset space possesses an invariant measure.

Among other developments that are important for us, the first is the work of Rieffel[37] on Banach G-modules and their products. He proved, in particular, that

$$(V \otimes_S W)^* \cong Hom_S(V, W^*),$$

where S is a set, V and W are two S-modules, $\otimes_S$ denotes the projective tensor product of V and W and $Hom_S(V, W^*)$ is the space of intertwining operators of the Banach G-modules. Applying this to $L_p(G)$ spaces $(1 \leq p \leq \infty)$ of complex-valued functions defined on a group G, Rieffel obtained the result that, under certain conditions, the corresponding intertwining operators (multipliers) form the dual space of the space of functions A_p^q: a subset of an L_r space (where r is related to p and q as described in Prop.2.7.1) consisting of those functions which can be written as a sum of convolution of functions from L_p and L_q. This is the context in which we shall set our study of intertwining operators, that is, regarding the space of such operators as the dual of a tensor product space.

The next development of importance to us in this regard is that of Herz [25]. He studied the predual of the space of intertwining operators of the regular representations of G on L_p and L_q. He was able to show, in particular, that the tensor product space is an algebra of functions on G and, in some sense, a natural analogue of the space of absolutely convergent Fourier Series. Our aim is to extend the Herz- Rieffel results from regular representations which may be seen as induced representation from the trivial subgroup to arbitrary induced representations.

In order to complete the programme we shall need to go beyond spaces of functions on G to sections of Banach (semi-)bundles on G. The concept of a Banach bundle was developed by Fell in 1977 and we shall use it as the appropriate device for the study of the tensor product spaces. Unfortunately, in the most general case, our semi-bundle will fail to be a bundle in the complete sense, but will be more akin to the objects studied by Dauns and Hofmann[5].

The structure of the thesis is as follows. Our first aim is to construct A_p^q spaces using projective tensor products of L_p spaces of induced representations. The first task in the construction of A_p^q spaces using L_p spaces of Banach space valued functions is to define a 'convolution' formula as the image of a linear map Ψ on $L_p(\pi) \otimes L_{q'}(\gamma^*)$, where q' denotes the conjugate exponent defined by $1/q + 1/q' = 1$. (It is known that $L_p(\mu, X)^* = L_{p'}(\mu, X^*)$ if and only if X^* satisfies the Radon-Nikodym property (Definition 2.5.3) with respect to the measure μ; the spaces involved in our work are assumed to have this property.) As Hörmander has shown in the case of complex-valued functions, we see that, under certain conditions, the space $L_p(\pi) \otimes_G L_{q'}(\gamma^*) \cong \underline{0}$ when $1/p + 1/q' < 1, 1 \leq p, q < \infty$ and G/H and G/K are non-compact. In the case where these homogeneous spaces possess invariant measure, the convolution formula turns out to be

$$(\Psi(\Sigma_{i=1}^{\infty} f_i \otimes g_i))(x, y) = \int_{\frac{G}{H^x \cap K^y}} \Sigma_{i=1}^{\infty} f_i(xt) \otimes_{x,y} g_i(yt) d\mu_{x,y}(t),$$

where $\mu_{x,y}$ is a suitably chosen quasi-invariant measure on $G/H^x \cap K^y$ and $\otimes_{x,y}$ is the tensor product on a quotient Banach space $\mathcal{A}_{x,y}$ of $\mathcal{H}(\pi) \otimes \mathcal{H}(\gamma)$ (Definition 4.1.2). In the absence of invariant measures on homogeneous spaces, we need to modify the integrand in the above formula using corresponding λ-functions (Radon-Nikodym derivatives of measures mentioned above), in order to make it integrable and well defined on $G/(H^x \cap K^y)$. A substantial part of our analysis becomes possible because of an identity we derive regarding the λ-functions:

$$\lambda_H(xts^{-1}, s)\lambda_K(yts^{-1}, s)\lambda_{H^x \cap K^y}(t, s^{-1}) = 1$$

for all $s, t \in G$ and almost all $(x, y) \in (G \times G)/(H \times K)$.

The space A_p^q is defined to be the range of Ψ with the quotient norm (Definition 4.3.3.). For $x, y \in G$ we see that the values of the elements in A_p^q spaces are in the spaces $\mathcal{A}_{x,y}$, and these spaces may not be the same for different (x, y) in general. We show that the collection of these spaces $\mathcal{A}_{x,y}$ forms the bundle space of a Banach semi-bundle (Sec.4.2) . In the general case, we see that the elements of A_p^q are cross-sections of this Banach semi-bundle.

When p-nuclear norms are considered, we see that similar results can be obtained at least in the case where the representation spaces of the subgroups are Lebesgue spaces. Considering Hilbert-Schmidt norms, we derive an isometry

$$A_2(\pi, \rho) \otimes_G A_2(\vartheta, \gamma) \mapsto A_2(\pi \otimes_G \rho, \vartheta \otimes_G \gamma)$$

which is similar to Herz's result (Herz[25],Theorem B)in this regard.

Chapter 2

Basic concepts

This chapter is devoted to some background information involving induced representations and A_p^q spaces sufficient for our purposes.

2.1 Notations and terminology

We shall assume throughout that all the topological spaces under consideration are second countable.

Let X be a locally compact topological space; let $C(X)$ denote the algebra of complex-valued, continuous functions on X. The algebra of functions in $C(X)$ with compact support is denoted by $C_0(X)$. The space of real-valued, positive continuous functions on X with compact support is denoted by $C_0^+(X)$. The space of complex-valued continuous functions vanishing at infinity on X forms a Banach space, denoted by $C_\infty(X)$, with norm $\|f\| = \max_{x \in X} |f(x)|$. $C_0(X, Y)$ denotes the space of continuous functions with compact support on X, mapping to the topological space Y. For $1 \leq p < \infty$, $L_p(X)$ is the Lebesgue space on X while $L_p(X, Y, \mu)$ denotes the set of (equivalence classes of) all μ-measurable functions f on X mapping to the Banach space Y such that the map $x \mapsto \|f(x)\|^p$ is μ-integrable where $\| \ \|$ denotes the norm in the Banach space Y and μ is a positive measure on X. $L_\infty(X, Y, \mu)$ denotes the space of all μ-measurable, essentally bounded Y-valued functions on X. The norm of such a function is its μ-essential supremum. The composition of two mappings f and g is denoted by $f \circ g$, whenever it is defined. If f is a function defined on a group G, then ${}_xf$ denotes a function on G defined by ${}_xf(y) = f(xy)$. The support of a function f, denoted by $\mathrm{supp}(f)$, is the closure of the set of all points x where $f(x) \neq 0$.

The dual pairing between a space V and it dual V^* (see Definition 2.4.2) is denoted by $\langle , \rangle$. For any set of indices J, the closed linear span of a set of vectors $\{u_\alpha : \alpha \in J\}$ in any given vector space is denoted by $\langle \{u_\alpha : \alpha \in J\} \rangle$.

Let G be a locally compact topological group. We denote the right-invariant Haar measure on G by ν_G. e denotes the identity element of the group. For a subgroup H of G, the canonical mapping from G to the set of right-cosets G/H is denoted by p_H.

Let $\mathcal{R}$, $\mathcal{C}$, $\mathcal{Q}$, $\mathcal{Z}$ and $\mathcal{N}$ denote the set of real numbers, the set of complex numbers, the set of rational numbers, the set of integers and the set of positive integers respectively, with their usual algebraic and topological structures. Let I denote a directed set.

For any number p, $1 \leq p \leq \infty$, we let p' denote the conjugate exponent of p defined by $1/p + 1/p' = 1$.

The symbols $\cong$ and $\simeq$ indicate an isometric isomorphism and a topological equivalence, respectively.

2.2 Borel Spaces, G-action and quasi-invariant measures.

Definition 2.2.1 *(cf.Gaal[19],p.234) Let X be a topological space and let G be a locally compact topological group. We say that G* **acts on X on the right** *if X is endowed with an external law of composition $(s, x) \mapsto x.s$ for which G is the set of operators, satisfying the following conditions:*

(a) *the mapping $(s, x) \mapsto x.s$ of $G \times X$ into X is continuous;*

(b) *$(x.t).s = x.(ts)$ for all $s, t \in G$ and $x \in X$;*

(c) *the mapping $x \mapsto x.s$ is a homeomorphism for every $g \in G$.*

If $x.s = x$ for every $s \in G$ and $x \in X$ we say that G **acts trivially** on X. G is said to **act transitively** on X (and X is called a **transitive G-space**) if for any ordered pair (x_1, x_2) there exist an s in G such that $x_1.s = x_2$. For each $x \in X$ the set $x.G = \{x.s : s \in G\}$ is called the **orbit** of x, and the set of all $s \in G$ such that $x.s = x$ is called the **stabilizer** of x which is a subgroup of G. The relation R on X defined by $x \sim y$ iff x and y belong to the same orbit is an equivalence relation on X and the equivalence classes with respect to this relation are the orbits of the points of X. The topological space X/R is called the **orbit space** of X or the **quotient space** of X by the group G and is denoted by X/G. The topology of X/G is the quotient of the topology of X by R.

If H is a closed subgroup in the locally compact topological group G then G acts on the homogeneous space $X = G/H$ of right G-cosets via the mapping $(s, Hx) \mapsto Hx.s$. This action is transitive and has the properties (a) and (b) in

Definition 2.2.1. In particular, if we consider the action of a closed subgroup K of G on the homogeneous space G/H we see that the orbits are in one-to-one correspondence with the double cosets $H : K$. The stabilizer of $Hx \in G/H$ under the action of K is $H^x \cap K$.

Definition 2.2.2 *(cf.Hewitt and Ross[26], p.118) Let X be a topological Hausdorff space. The σ-algebra $A(X)$ generated by the open subsets of X is called the* **Borel σ-algebra** *on X; the Borel subsets of X are those that belong to $A(X)$.*

The following well known results which deal with properties of the Borel structures of G/H and G are of fundamental importance to the development of our work.

Throughout this section, let X denote the transitive G-space G/H.

Lemma 2.2.3 *There exists a Borel set B such that*

(a) *B intersects each right G coset in exactly one point, and*

(b) *for each compact subset K of G, $(p_H^{-1}(p_H(K))) \cap B$ has a compact closure.*

Proof: See Mackey[31], p.102, Lemma 1.1.

□

A set B with the properties described in Lemma 2.2.3 is called a **regular Borel section** of G with respect to H.

Lemma 2.2.4 *A subset E in X is Borel if and only if $p_H^{-1}(E)$ is Borel in G. A function f on X is a Borel function if and only if $f \circ p_H : x \mapsto f(p_H(x))$ is a Borel function on G.*

Proof: See Mackey[31], p.103, Lemma 1.2.

□

A Borel measure μ on X is a countably additive, non-negative, extended real-valued set function defined on $A(X)$ which is finite on compact sets. It is called **quasi-invariant** if the null sets of X are G invariant i.e. $\mu(E) = 0$ if and only if $\mu(E.s) = 0$. In other words, every translated measure μ_s, defined by $\mu_s(E) = \mu(E.s)$, must be equivalent to μ. A detailed study of these quasi-invariant measures on homogeneous spaces is given in [31], including the analytic properties of the Radon-Nikodym derivatives of their translates.

For a given group G and $\sigma \in G$, let us write $\Delta_G(\sigma)$ for the constant Radon-Nikodym derivative of the measure $E \mapsto \nu_G(\sigma E)$ with respect to the measure

$E \mapsto \nu_G(E)$. Δ_G is a continuous homomorphism of G into the group of positive real numbers and is called the **modular function** of G. For a given subgroup H of G, the modular function Δ_H is defined similarly. A group G is called **unimodular** if $\Delta_G(\sigma) = 1$ for all $\sigma \in G$.

Lemma 2.2.5 *Let G be a locally compact group and H a closed subgroup of G. There exists a strictly positive, real-valued continuous function ρ_H on G such that*

$$\rho_H(hx) = (\Delta_H(h)/\Delta_G(h))\rho_H(x) \qquad (2.1)$$

for all $x \in G$ and $h \in H$.

Proof: See Mackey[31], p.104, Lemma 1.4 or Gaal[19], p.260, Proposition 4.

□

A Borel function with properties stated in the above Lemma will be called a ρ-function. The existence of a strictly positive ρ-function satisfying the functional equation (2.1) has been established in a number of places in the literature. In particular, it is known that for every closed subgroup H in G there exists a function β on G with $\int_H \beta(hx)d\nu_H(h) = 1$ for all $x \in G$ which gives rise to a ρ-function of the required nature. The details of such a β function are given in the following Lemma.

Lemma 2.2.6 *For every closed subgroup H of a locally compact group G, there exists a function β on G with the following properties:*

(a) *if K is any compact set in G, then β coincides on the strip HK with a function in $C_0^+(G)$;*

(b) $\int_H \beta(hx)d\nu_H(h) = 1$ *for all $x \in G$.*

Proof: See Reiter[35], Chapter 8, section 1.9.

□

A function β on G satisfying the properties stated in Lemma 2.2.6 is called a **Bruhat function** for H.

Given a Bruhat function β for a closed subgroup H, a ρ-function can be obtained by letting

$$\rho_H(x) = \int_H \beta(hx)\Delta_G(h)\Delta_H(h^{-1})d\nu_H(h).$$

Then ρ_H is continuous (cf. (a) and [35], Chapter 3, section 3.2, Remark), $\rho_H(x) > 0$ for all $x \in G$ and ρ_H satisfies (2.1).

For a given ρ-function $\rho(sy)/\rho(s)$ is a Borel function of s and y which is constant on the right $H\times G$ cosets in $G\times G$. Since there is a natural homeomorphism from this coset space to $X\times G$, these ρ-functions give rise to a unique Borel function λ_ρ on $X \times G$ such that

$$\lambda_\rho(p_H(s), y) = \frac{\rho(sy)}{\rho(s)}$$

for all s and y in G.

Lemma 2.2.7 *The function λ_ρ has the following properties:*

(a) *for all $x \in X$ and $s, t \in G$,* $\lambda_\rho(x, st) = \lambda_\rho(x.s, t)\lambda_\rho(x, s)$;

(b) *for all $h \in H, \lambda_\rho(p_H(e), h) = \Delta_H(h)/\Delta_G(h)$;*

(c) *$\lambda_\rho(p_H(e), t)$ is bounded on compact sets as a function of t.*

Proof: See, for example, Gaal[19], p.263, Lemma 10.

Lemma 2.2.8 *Let ρ be an arbitrary ρ-function on G. Then there exists a quasi-invariant measure μ in X such that for all $y \in G$, the corresponding λ-function λ_ρ has the property that $\lambda_\rho(\cdot, y)$ is a Radon-Nikodym derivative of the measure μ_y with respect to the measure μ.*

Proof: See Mackey[31], p.105, Lemma 1.5.

□

Let us write $\mu \succ \lambda$ to mean that for all $y \in G$, $\lambda(\cdot, y)$ is a Radon-Nikodym derivative of the measure μ_y with respect to μ.

Theorem 2.2.9 *There are quasi-invariant measures on X. Any two have the same null sets and hence are mutually absolutely continuous. A Borel set E in X is a null set if and only if $p_H^{-1}(E)$ has Haar measure zero. The relations $\mu \succ \lambda$ and $\lambda = \lambda_\rho$ between quasi-invariant measures, λ-functions and ρ-functions have the following properties:*

(a) *Every λ-function is of the form λ_ρ; $\lambda_{\rho_1} = \lambda_{\rho_2}$ if and only if ρ_1/ρ_2 is a constant.*

(b) *For every λ-function there is a quasi-invariant measure μ such that $\mu \succ \lambda$; if $\mu_1 \succ \lambda$ and $\mu_2 \succ \lambda$ then μ_1 is a constant multiple of μ_2.*

(c) *For every quasi-invariant measure μ there is a λ-function such that $\mu \succ \lambda$. If $\mu \succ \lambda_1$ and $\mu \succ \lambda_2$ then for all t, $\lambda_1(\cdot, t) = \lambda_2(\cdot, t)$ almost everywhere in X.*

(d) *If $\mu \succ \lambda_{\rho_1}$ and $\mu \succ \lambda_{\rho_2}$ then ρ_1/ρ_2 is constant almost everywhere.*

Proof: See Mackey[31], p.106, Theorem 1.1.

□

The quasi-invariant measure on the homogeneous space G/H of a subgroup H of a group G will be denoted by μ_H and the Radon-Nikodym derivative of the measure $E \mapsto \mu_H([E]y)$ with respect to the measure μ_H is denoted by $\lambda_H(.,y)$.

Given a ρ-function ρ_H for a closed subgroup H of G, it is possible to construct a ρ-function for a conjugate subgroup H^x $(x \in G)$ in an obvious manner. Following Lemma has the details.

Lemma 2.2.10 *Let $\rho_H : G \mapsto (0,\infty)$ be a continuous ρ-function for the subgroup H. Then, for $x \in G$,*

$$\rho_{H^x}(y) = \rho_H(xy)\Delta_G(x), y \in G, \tag{2.2}$$

defines a positive continuous ρ-function for H^x.

Proof: See Gaal[19], Chapter VI, Sec.10, Lemma 3.

□

For simplicity of notation, $\lambda_H(p_H(x), y)$ will be written as $\lambda_H(x,y)$, or by $\lambda(x,y)$ if the subgroup H is clearly understood.

Corollary 2.2.11 *For $x \in G$ let $\dot{x}= p_H(x)$. If μ denotes the quasi-invariant measure corresponding to the function ρ then*

$$\int_G f(x)\rho(x)d\nu_G(x) = \int_{\frac{G}{H}} \int_H f(hx)d\nu_H(h)d\mu(\dot{x}), \quad f \in C_0(G).$$

Proof:See Gaal[19], p.263, Corollary to Theorem 9.

□

Finally we shall discuss the notion of **disintegration of measures** which has been dealt with in a number of places in the literature (see, for example, Mackey[31], Halmos[23]).

Let μ be a finite measure on X and suppose that there is an equivalence relation R given on X. For $x \in X$ let $r(x) \in X/R$ be the equivalence class to which x belongs. The equivalence relation is said to be **measurable** if there exists a countable family $E_1, E_2, \ldots\ldots\ldots$ of subsets of X/R such that $r^{-1}(E_i)$ is measurable for each i and such that each point in X/R is the intersection of the E_i which contain it.

Let H and K be closed subgroups of G. We say that H and K are **discretely related** if there exists a subset of G whose complement has Haar measure zero and which is itself the union of countably many $H : K$ double cosets. H and K are said to be **regularly related** if there exists a sequence $E_0, E_1, E_2, \dots\dots\dots$ of measurable subsets of G each of which is a union of $H : K$ double cosets such that E_0 has Haar measure zero and each double coset not in E_0 is the intersection of the E_i which contain it. Hence H and K are regularly related if and only if the orbits of G/H under the action of K, outside a certain set of measure zero, form the equivalence classes of a measurable equivalence relation.

The following Lemma states that a measure μ defined on X may be decomposed as an integral over X/R of measures μ_y concentrated in the equivalence classes.

Lemma 2.2.12 *Let $\tilde{\mu}$ be the measure in X/R such that $E \subseteq X/R$ is measurable if and only if $r^{-1}(E)$ is μ measurable and that $\tilde{\mu}(E) = \mu(r^{-1}(E))$. Then for each y in X/R there exists a finite Borel measure μ_y in X such that $\mu_y(X - r^{-1}(\{y\})) = 0$ and*

$$\int f(y) \int g(x) d\mu_y(x) d\tilde{\mu}(y) = \int f(r(x)) g(x) d\mu(x), \tag{2.3}$$

whenever $f \in L_1(X/R, \tilde{\mu})$ and g is bounded and measurable on X.

Proof: See Mackey[31], p.124, Lemma 11.1 or Effros[13], Lemma 4.4.

□

Lemma 2.2.13 *Let the measure μ on X be quasi-invariant. Then, in the disintegration of μ as in Lemma 2.2.12, almost all of the μ_y are also quasi-invariant under the action of G.*

Proof: See Mackey[31], p.126, Lemma 11.5.

□

2.3 Banach Bundles

The following definitions and the results in terms of Banach bundles are due to Fell (see [15], Chapter 2 and [16]).

Definition 2.3.1 *A* **bundle** *$\underline{\mathcal{B}}$ over a Hausdorff space X is a pair $(\mathcal{B}, \theta)$ such that $\mathcal{B}$ is a Hausdorff space called the* **bundle space** *of $\underline{\mathcal{B}}$ and $\theta : \mathcal{B} \mapsto X$ is a continuous open surjection called the* **bundle projection** *of $\underline{\mathcal{B}}$. X is called the* **base space** *of $\underline{\mathcal{B}}$, and for $x \in X$, $\theta^{-1}(x) = \{\xi : \theta(\xi) = x, \xi \in \mathcal{B}\}$ is called the* **fibre** *over X and is denoted by $\mathcal{B}_x$.*

Definition 2.3.2 *A bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ over X is a* **Banach semi-bundle** *over X if we can define a norm making each fibre $\mathcal{B}_x$ into a Banach space satisfying the following conditions:*

(a) *$\xi \mapsto \|\xi\|$ is upper semi-continuous on $\mathcal{B}$ to $\mathcal{R}$.*

(b) *The operation $+$ is continuous on the set $\{(\xi, \eta) \in \mathcal{B} \times \mathcal{B} : \theta(\xi) = \theta(\eta)\}$ to $\mathcal{B}$.*

(c) *For each λ in $\mathcal{C}$, the map $\xi \mapsto \lambda.\xi$ is continuous on $\mathcal{B}$ to $\mathcal{B}$.*

(d) *If $x \in X$ and $\{\xi_i\}$ is a net of elements of $\mathcal{B}$ such that $\|\xi_i\| \to 0$ and $\theta(\xi_i) \to x$, then $\xi_i \to 0_x$, where 0_x denotes the zero element of the Banach space $\mathcal{B}_x$.*

A bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ over X is called a **Banach bundle** *if it satisfies (b), (c) and (d) above together with the condition that*

(ã) *$\xi \mapsto \|\xi\|$ is continuous on $\mathcal{B}$ to $\mathcal{R}$.*

Given a Banach space A and a Hausdorff space X, it is easy construct a Banach bundle by letting $\mathcal{B} = A \times X$ and $\theta(\xi, x) = x$. Then $(\mathcal{B}, \theta)$ is a bundle over X and if we equip each fibre $A \times \{x\}$ with the Banach space structure making $\xi \mapsto (\xi, x)$ an isometric isomorphism, then it becomes a Banach bundle. The Banach bundle $(\mathcal{B}, \theta)$ so constructed is called a **trivial Banach bundle**.

Let X and Y be any two Hausdorff spaces and $\phi : Y \mapsto X$ be a continuous map. Suppose $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ is a Banach (semi-)bundle over X. Let $\mathcal{B}^{\#}$ be the topological subspace $\{(y, \xi) : y \in Y, \xi \in \mathcal{B}, \phi(y) = \theta(\xi)\}$ of $Y \times \mathcal{B}$; and define $\theta^{\#} : \mathcal{B}^{\#} \mapsto Y$ by $\theta^{\#}(y, \xi) = y$. Then $\theta^{\#}$ is a continuous open surjection since θ is open. Hence $(\mathcal{B}^{\#}, \theta^{\#})$ is a bundle over Y. For $y \in Y$, we make $\mathcal{B}^{\#}_y = {\theta^{\#}}^{-1}(y)$ into a Banach space in such a way that the bijection $\xi \mapsto (y, \xi)$ of $\mathcal{B}_{\phi(y)}$ onto $\mathcal{B}^{\#}_y$ becomes a linear isometry. Then $(\mathcal{B}^{\#}, \theta^{\#})$, denoted by $\underline{\mathcal{B}}^{\#}$, becomes a Banach (semi-)bundle which is called the **Banach (semi-)bundle retraction** of $\underline{\mathcal{B}}$ by ϕ.

Let $i^{\#} : \mathcal{B}^{\#} \mapsto \mathcal{B}$ be the surjection given by $i^{\#}(y, \xi) = \xi$. Then, we have the following diagram:

$$\begin{array}{ccc} \mathcal{B}^{\#} & \overset{i^{\#}}{\longmapsto} & \mathcal{B} \\ \theta^{\#} \downarrow & & \downarrow \theta \\ Y & \overset{\phi}{\mapsto} & X \end{array}$$

Since $\theta(i^{\#}(y, \xi)) = \theta(\xi) = \phi(y) = \phi(\theta^{\#}(y, \xi))$, we have $\theta i^{\#} = \phi \theta^{\#}$, and the diagram commutes.

Suppose $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ and $\underline{\mathcal{D}} = (\mathcal{D}, \vartheta)$ are Banach (semi-)bundles over the same base space X. Let $u : \mathcal{B} \mapsto \mathcal{D}$ be a map for which the diagram

$$\begin{array}{ccc} \mathcal{B} & \xmapsto{u} & \mathcal{D} \\ & \theta \searrow \swarrow \vartheta & \\ & X & \end{array}$$

commutes, so that $\theta(\xi) = \vartheta(u(\xi))$ for $\xi \in \mathcal{B}$. Let Y be another Hausdorff space and $\phi : Y \mapsto X$ be a continuous map. Let $\mathcal{B}^{\#}$ and $\mathcal{D}^{\#}$ be the retractions of $\mathcal{B}$ and $\mathcal{D}$ by ϕ respectively . Define the map $j^{\#}(u) : \mathcal{B}^{\#} \mapsto \mathcal{D}^{\#}$ by

$$j^{\#}(u)(y, \xi) = (y, u(\xi)).$$

Then

$$\vartheta^{\#}(j^{\#}(u)((y,\xi))) = \vartheta^{\#}((y, u(\xi)) = y = \theta^{\#}(y,\xi),$$

for $(y, \xi) \in \mathcal{B}^{\#}$, so that the diagram;

$$\begin{array}{ccc} \mathcal{B}^{\#} & \xmapsto{j^{\#}(u)} & \mathcal{D}^{\#} \\ \theta^{\#} \searrow & & \swarrow \vartheta^{\#} \\ & Y & \end{array}$$

commutes.

Suppose $u : \mathcal{B} \mapsto \mathcal{D}$ is a continuous and open map. It is clear that the map $j^{\#}(u)$ is the restriction of the map $(j, u) : Y \times \mathcal{B} \mapsto Y \times \mathcal{D}$, where j is the identity map from Y to itself and $(j,u)(y, \xi) = (y, u(\xi))$. Clearly, (j, u) is a continuous, open map. Let $\tilde{U} \subset \mathcal{B}^{\#}$ be an open set. Then there exists an open set $U \subseteq Y \times \mathcal{B}$ such that $\tilde{U} = U \cap \mathcal{B}^{\#}$. Let $j^{\#}(u)\tilde{U} = \tilde{V}$ and $(j,u)(U) = V$. Now V is an open set in $Y \times \mathcal{D}$ and $\tilde{V} \subseteq V \cap \mathcal{D}^{\#}$. Note that if $(y, \xi) \notin \mathcal{B}^{\#}$, then $\phi(y) \neq \theta(\xi)$, and therefore $\vartheta(u(\xi)) = \theta(\xi) \neq \phi(y)$, which implies that $(y, u(\xi)) \notin \mathcal{D}^{\#}$. Therefore, if $x \in V \cap \mathcal{D}^{\#}$ is the image of $z \in U$, then z cannot be outside of $\mathcal{B}^{\#}$. This implies that $\tilde{V} = V \cap \mathcal{D}^{\#}$, which shows that $\tilde{V}$ is an open set in $\mathcal{D}^{\#}$. Hence $j^{\#}(u)$ is an open map.

Now we turn to the construction of a particular type of Banach (semi-)bundle. Let the Banach (semi-)bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ over X with $\mathcal{B} = \mathcal{H} \times X$ be such that $\mathcal{H}$ is a Banach space, X is a Hausdorff space and $\theta(\xi, x) = x$. Suppose that there is an equivalence relation R given on X. Let r be the canonical mapping from X to X/R. For $x \in X$, let $r(x) \in X/R$ be the equivalence class to which x belongs. Define $\underline{\mathcal{B}}^R = (\mathcal{B}^R, \theta^R)$ over X/R by letting $\mathcal{B}^R = \mathcal{H} \times X/R$ and $\theta^R(\xi, r(x)) = r(x)$. Clearly, both bundles $\underline{\mathcal{B}}$ and $\underline{\mathcal{B}}^R$ are trivial bundles with constant fibre $\mathcal{H}$ (see Section 2.3, p.10).

Proposition 2.3.3 *The Banach bundle retraction*

$$\underline{\mathcal{B}}^{R\#} = (\mathcal{B}^{R\#}, \theta^{R\#})$$

of $\underline{\mathcal{B}}^R$ *by* r *is topologically equivalent to* $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$.

Proof: The two Banach bundles $\underline{\mathcal{B}}^{R\#}$ and $\underline{\mathcal{B}}$ have the same base space X.

$$\begin{aligned}\mathcal{B}^{R\#} &= \{(x',(\xi,r(x))) : \theta(\xi,r(x)) = r(x'), x', x \in X, \xi \in \mathcal{H}\} \\ &= \{(x',(\xi,r(x))) : x' \in r(x), x', x \in X, \xi \in \mathcal{H}\},\end{aligned}$$

and for $x \in X$, $\mathcal{B}_x = \{(\xi,x) : \xi \in \mathcal{H}\}$, while $\mathcal{B}_x^{R\#} = \{(x,(\xi,r(x))) : \xi \in \mathcal{H}\}$. Clearly, the mapping $(\xi,x) \mapsto (x,(\xi,r(x)))$ is a homeomorphism.

□

Definition 2.3.4 *A* cross-section *of $\underline{\mathcal{B}}$ is a function $f : X \mapsto \mathcal{B}$ such that $f(x) \in \mathcal{B}_x$ for each $x \in X$. The linear space of all continuous cross-sections of $\underline{\mathcal{B}}$ is denoted by $C(\underline{\mathcal{B}})$ and the subspace of $C(\underline{\mathcal{B}})$ consisting of those cross-sections which vanish outside some compact set is denoted by $C_0(\underline{\mathcal{B}})$. The set of all bounded cross-sections is denoted by $B(\underline{\mathcal{B}})$.*

We say that $\underline{\mathcal{B}}$ has enough continuous cross-sections *if for every $\xi \in \mathcal{B}$ there exists a continuous cross-section $f : X \mapsto \mathcal{B}$ for which $f(\theta(\xi)) = \xi$.*

The following unpublished result about the existence of enough continuous cross-sections has been proved by A.Douady and L.dal Soglio-Hérault .

Theorem 2.3.5 *If X is either paracompact or locally compact, every Banach bundle over X has enough continuous cross-sections.*

Proof: See Fell[15], p.324.

□

Definition 2.3.6 *(Gierz[20], p.80) Let $\underline{\mathcal{B}} = (\mathcal{B},\theta)$ be a Banach (semi-)bundle over X. Let $\mathcal{F}$ be a subset of $\mathcal{B}$. Then $(\mathcal{F},\theta)$ is called a* Banach (semi-)subbundle *if*

(a) *$\theta^{-1}(x) \cap \mathcal{F}$ is a subspace of $\mathcal{B}_x$ for every $x \in X$, and*

(b) *given $\xi \in \mathcal{F}$ and $\epsilon > 0$, there is a neighborhood N of $\theta(\xi)$ and a continuous cross-section $f : N \mapsto \mathcal{B}$ such that $f(x) \in \mathcal{F}$ for all $x \in N$ and such that $\|f(\theta(\xi)) - \xi\| < \epsilon$.*

Furthermore, a subbundle $(\mathcal{F},\theta)$ is called fibrewise closed *if $\theta^{-1}(x) \cap \mathcal{F}$ is closed in $\mathcal{B}_x$ for every $x \in X$.*

Definition 2.3.7 *Let $1 \leq p < \infty$. A cross-section of $\underline{\mathcal{B}}$ is said to be* p^{th}-power summable *if it is locally μ-measurable and*

$$\|f\|_p = (\int_X \|f(x)\|^p d\mu(x))^{1/p} < \infty.$$

The space of all p^{th}-power summable cross-sections is denoted by $L_p(\underline{\mathcal{B}};\mu)$.

$L_p(\underline{\mathcal{B}};\mu)$ is a Banach space under the norm $\| \quad \|_p$ defined in Definition 2.3.8.

Definition 2.3.8 *The space $L_\infty(\underline{\mathcal{B}};\mu)$ is defined to be the space of all μ-essentially bounded cross-sections of $\underline{\mathcal{B}}$.*

$L_\infty(\underline{\mathcal{B}};\mu)$ is a Banach space under the norm $\|f\|_\infty = \mu\text{-ess }\sup_{x\in X}\|f(x)\|$.

2.4 Banach Modules and Representations of locally compact groups

Banach modules

Let G be a locally compact group. We let $M(G)$ denote the Banach algebra of all finite complex-valued regular Borel measures on G. The group algebra is denoted by $L(G)$, and is considered to be the two-sided ideal in $M(G)$ consisting of the elements of $M(G)$ which are absolutely continuous with respect to the Haar measures on G (cf.Hewitt and Ross[26], p.269).

Definition 2.4.1 *(cf.Rieffel[37], p.446.)*

(a) *Let V be a Banach space and A be a set. Then a* left Banach A-module *is defined to be the Banach space V together with a map*

$$A \times V \mapsto V$$

$$(a, v) \mapsto av$$

such that for any fixed $a \in A$, the map $v \mapsto av$ is a bounded linear operator. We let $\|a\|_V$ be the bound of this operator.

(b) *If the set A is a locally compact group with identity e, then in addition to (a) we require that*

(1) $ev = v$ for all $v \in V$;

(2) $a(bv) = (ab)v$ for all $a, b \in A$ and $v \in V$; and

(3) the map $A \times V \mapsto V$ be continuous.

If $\|a\|_V = 1$ for every $a \in A$, we say that V is an isometric A-module.

(c) *If the set A is an algebra then in addition to (a) we need V to be a left module over A in the algebraic sense.*

(d) *If A is a Banach algebra then in addition to (c) we require that the bilinear map* $A \times V \mapsto V$ *be continuous, so that there is a constant k such that* $\|av\| \leq k\|a\|\|v\|$ *for all* $a \in A$, $v \in V$; *where* $\|.\|$ *denotes the norm in the corresponding Banach spaces.*

Right Banach A-modules are defined similarly.

If G is a locally compact group and V is a uniformly bounded G-module, then it is well known (see, for example, Hewitt and Ross[26],p.269) that an action of $M(G)$ on V can be defined by

$$mv = \int_G xv\,dm(x)$$

for $m \in M(G), v \in V$. With this action V becomes an $M(G)$-module and so also an $L(G)$-module.

Definition 2.4.2 *(Rieffel[37],p.447) Let A be a set and V and W be A-modules. An* **intertwining operator** *is a continuous A-module homomorphism; that is, a bounded linear operator T from V to W which satisfies* $T(av) = a(T(v))$ *for all* $a \in A, v \in V$. *The Banach space of all intertwining operators (with the operator norm) is denoted by* $Hom_A(V, W)$.

The A-module $Hom(V, C)$ *where C is the complex field, is called the* **dual** *of V, and is denoted by* V^*.

If V and W are A-modules such that $W = V^*$, *then V is called the* **predual** *of W.*

Definition 2.4.3 *(Rieffel[37],p.453) Let A be a Banach algebra and let V be an A-module. Then V is said to be an* **essential** *A***-module** *if* $AV = \{av : a \in A, v \in V\}$ *is dense in V.*

Definition 2.4.4 *(Rieffel[37],p.453) Let A be a Banach algebra. An* **approximate identity** *for A is a net* $\{e_j\}_{j \in J}$, *where J is a directed set, of elements of A having the property that* $\lim_j e_j a = a$ *and* $\lim_j a e_j = a$ *for all* $a \in A$.

Proposition 2.4.5 *If A is a Banach algebra with bounded approximate identity* $\{e_j\}$ *and if V is an A-module, then the following are equivalent:*

(a) *V is an essential A-module;*

(b) $\lim_j e_j v = v$ *for every* $v \in V$.

(c) *given* $v \in V$, *there exist* $v' \in V$ *and* $a \in A$ *such that* $v = av'$.

Proof: See Rieffel[37], Proposition 3.4.

□

Definition 2.4.6 *(Rieffel[37],p.454.) Let A be a Banach algebra with bounded approximate identity and let V be an A-module. Then the closed linear subspace of V spanned by AV is called the* **essential part** *of V and is denoted by V_e.*

REMARK I. (Rieffel[37], p.456.) If G is a locally compact group and Z is a G-module, then the linear subspace of Z^* on which the action of G is strongly continuous is exactly $(Z^*)_e$.

Representations of locally compact groups

Definition 2.4.7 *(cf.Gaal[19], Chapter IV.) Let G be a locally compact group and H a closed subgroup of G. By a* **representation** *π of H on a Banach Space $\mathcal{H}(\pi)$ we mean a homomorphism $h \mapsto \pi(h)$ of the group H into the group $U(\mathcal{H}(\pi))$ of all isometries of $\mathcal{H}(\pi)$ onto itself such that for any $u \in \mathcal{H}(\pi)$ the function $h \mapsto (\pi(h))(u)$ is continuous in the norm topology on $\mathcal{H}(\pi)$. Thus in particular we require $\pi(h)\pi(t) = \pi(ht)$ and $\pi(h^{-1}) = (\pi(h))^{-1}$ for all $h, t \in H$.*

Definition 2.4.8 *(cf.Mackey[33]) Given a representation π of a subgroup H of G on a Banach space $\mathcal{H}(\pi)$, the representation π^x of the subgroup $H^x = x^{-1}Hx$ on the space $\mathcal{H}(\pi)$ is defined by $\pi^x(b) = \pi(xbx^{-1})$ for $b \in H^x$.*

Let π and γ be representations of the locally compact group G. A bounded linear operator T from $\mathcal{H}(\pi)$ to $\mathcal{H}(\gamma)$ is called an **intertwining operator** for π and γ if $\pi(x)T = T\gamma(x)$ for all $x \in G$. The vector space of all intertwining operators is denoted by $Int_G(\pi, \gamma)$ and the dimension (possibly infinite) of this space, called the **intertwining number**, is denoted by $\partial(\pi, \gamma)$.

An operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is called a **Hilbert-Schmidt operator** if for some maximal orthonormal system $\{x_i\}$ ($i \in \mathcal{I}$, a general index set) in $\mathcal{H}$, one has $\sum_i \|Tx_i\|^2 < \infty$. An intertwining operator $T : \mathcal{H}(\pi) \mapsto \mathcal{H}(\gamma)$, where $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$ are Hilbert spaces, is said to be a **strong intertwining operator** if it is an Hilbert-Schmidt operator. The space of all strong intertwining operators for π and γ is denoted by $S.Int_G(\pi, \gamma)$.

Definition 2.4.9 *(cf.Gaal[19],p.152.) Two representations π and γ of the group G are called* **equivalent** *if there exists an isometry Γ of $\mathcal{H}(\pi)$ onto $\mathcal{H}(\gamma)$ such that $\gamma\Gamma = \Gamma\pi$; i.e. $\gamma(x)\Gamma = \Gamma\pi(x)$ for every x in G.*

If π and γ are equivalent we write $\pi \approx \gamma$. Therefore, $\pi \approx \gamma$ if and only if $Int_G(\pi, \gamma)$ contains an isometry of $H(\pi)$ onto $H(\gamma)$.

For a given representation π of a group G on a Banach space $\mathcal{H}(\pi)$ let us define the map $\pi^* : H \mapsto U((\mathcal{H}(\pi))^*)$ such that $\pi^*(h) = (\pi(h^{-1}))^*$. We see that π^* is a representation of G on the Banach space $\mathcal{H}(\pi^*) = (\mathcal{H}(\pi))^*$, if $\mathcal{H}(\pi)$ is reflexive (Proposition 3.3.2). We call π^* the **adjoint representation** of π.

2.5 The induced representations of locally compact groups

Let G be a locally compact group and H a closed subgroup of G. Suppose that π is a representation of H on a Hilbert space $\mathcal{H}(\pi)$. Let μ be any quasi-invariant measure in the homogeneous space $X = G/H$ of right cosets which belongs to a continuous ρ-function. Let us denote by $\mathcal{M}_H$ the set of all continuous functions f from G to $\mathcal{H}(\pi)$ which satisfy the **covariance condition**

$$f(hx) = \pi_h f(x)$$

for all $h \in H$ and $x \in G$ and such that $\|f(\cdot)\|$ has compact support in X. The inner product

$$\langle f_1, f_2 \rangle := \int_X \langle f_1(x), f_2(x) \rangle d\mu(x)$$

can be introduced since the integrand is constant on each right coset Hx and hence defines a function on X. We identify two elements f and g of $\mathcal{M}_H$ if $\|f - g\| = 0$ where $\|\cdot\|$ denotes the norm derived from $\langle \cdot, \cdot \rangle$. The same symbol $\mathcal{M}_H$ will be used to denote the inner product space of equivalence classes.

The complex Hilbert space obtained by completing the inner product space $\mathcal{M}_H$ will be denoted by $L_2(\pi, \mu)$.

Definition 2.5.1 *(cf.Gaal[19], Chapter VI, Sec. 4, Definition 1) The* **induced representation** ${}^{\mu}U_y^{\pi} : G \mapsto U(L_2(\pi, \mu))$ *is defined by*

$$({}^{\mu}U_y^{\pi} f)(x) := \lambda(x, y)^{\frac{1}{2}} f(xy)$$

for each $x, y \in G$ and $f \in L_2(\pi, \mu)$, where $\lambda(\cdot, y)$ is the Radon-Nikodym derivative of the measure μ_y with respect to the measure μ.

The fact that ${}^{\mu}U_y^{\pi}$ is a well defined representation is dealt with in detail in Mackey[31] (p.107) and Gaal[19] (p.348).

Theorem 2.5.2 *Let μ and μ' be quasi-invariant measures on X. Then there exists an isometry W from $L_2(\pi, \mu)$ onto $L_2(\pi, \mu')$ such that $W({}^{\mu}U_y^{\pi}) = ({}^{\mu'}U_y^{\pi})W$ for all $y \in G$. In other words, the two representations ${}^{\mu}U_y^{\pi}$ and ${}^{\mu'}U_y^{\pi}$ are equivalent.*

Proof: See Mackey[31], p.107.

□

In Chapter 3, Section 3, we will be introducing p-induced representations U_p^π which is a generalisation of the above to include L_p spaces as the representation spaces. In order to discuss the properties of the adjoints $U_{p'}^{\pi^*}$, $1/p+1/p' = 1$, of such representations, we need to consider the space $L_{p'}(\pi^*)$. It is known (cf.Gretsky and Uhl[21]) that the space $L_p(X,\mu)^*$, where X is a Banach space, is not necessarily the same as $L_{p'}(X^*,\mu)$; but the equality holds if the space X satisfies the Radon-Nikodym property. First, we will state few definitions of terms which are necessary to define this property.

Definition 2.5.3 *(Dunford and Schwartz[10], p.97) Let μ be a set function defined on the field Σ of subsets of a set S. Then for every E in Σ* the total variation of μ on E, *denoted by $\nu(\mu, E)$, is defined as*

$$\nu(\mu, E) = \sup \sum_{i=1}^{n} |\mu(E_i)|,$$

where the supremum is taken over all finite sequences $\{E_i\}$ of disjoint sets in Σ with $E_i \subseteq E$.

Definition 2.5.4 *(Dunford and Schwartz[10], p.131) Let γ, μ be finitely additive set functions defined on a field Σ. Then γ is said to be* continuous with respect to μ *or simply* μ-continuous, *if*

$$lim_{\nu(\mu,E)\to 0}\gamma(E) = 0.$$

Definition 2.5.5 *(Gretsky and Uhl[21], Chapter III, Sec. 1, Definition 3.) Let (Ω, Σ, μ) be a measure space. A Banach space X is said to have the* Radon-Nikodym property *with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $F : \Sigma \to X$ of bounded variation there exists $g \in L_1(X, \mu)$ such that $F(E) = \int_E g d\mu$ for all $E \in \Sigma$. A Banach space X has the Radon-Nikodym property if X has the Radon-Nikodym property with respect to every finite measure space.*

Theorem 2.5.6 *Let (Ω, Σ, μ) be a σ-finite measure space, $1 \leq p < \infty$, and let X be a Banach space. Then $L_p(\Omega, X, \mu)^* = L_{p'}(\Omega, X^*, \mu)$, where $1/p + 1/p' = 1$, if and only if X^* has the Radon-Nikodym property with respect to μ.*

Proof: See Gretsky and Uhl[21], Chapter IV, Sec. 1, Theorem 1.

□

Theorem 2.5.7 *Let (Ω, Σ, μ) be a nonatomic finite measure space and X be a Banach space. Then $L_p(\Omega, X, \mu)$ has the Radon-Nikodym property if and only if $1 < p < \infty$ and X has the Radon-Nikodym property.*

Proof: See Gretsky and Uhl[21], Chapter V, Sec.4, Theorem 1.

□

2.6 Tensor product spaces

Let X and Y be vector spaces. Then there is a vector space W and a bilinear map $\omega : X \times Y \mapsto W$ such that for any bilinear map $\psi : X \times Y \mapsto V$, where V is a vector space, we can find a linear map ϕ such that the diagram

$$\begin{array}{ccc} X \times Y & \overset{\omega}{\mapsto} & W \\ \psi \downarrow & \swarrow \phi & \\ V & & \end{array}$$

commutes. If X and Y are normed vector spaces then W can be given a norm in such a way that ω is continuous. If ψ is continuous so is ϕ. Evidently in this case

$$\|\psi\| \le \|\phi\| \|\omega\|.$$

Let $Bilin(X \times Y, V)$ denote the space of all (bounded) bilinear operators from $X \times Y$ to V and let $\mathcal{L}(W, V)$ be the set of all continuous linear operators from W to V. Define $\Omega_V : \mathcal{L}(W, V) \mapsto Bilin(X \times Y, V)$ by $\Omega_V(\phi) = \phi\omega$. ($\|\Omega_V\| = \|\omega\|$ unless V is the zero vector space.) We have the following definition:

Definition 2.6.1 *(cf.Gaal[19], Chapter VI, Sec.3.) A* tensor product *of (normed) vector spaces X and Y is a pair (W, ω) where W is a (normed) vector space and $\omega : X \times Y \mapsto W$ is a (bounded) bilinear operator such that for every (normed) vector space V the map Ω_V is (an isometry) a bijection. In other words, (W, ω) is a tensor product of X and Y if and only if for every V and every (bounded) bilinear operator $\psi : X \times Y \mapsto V$ there exists a unique (bounded) linear operator $\phi : W \mapsto V$ such that the above diagram is commutative (and $\|\phi\| = \|\psi\|$).*

Lemma 2.6.2 *Suppose that (W, ω) is a tensor product of X and Y. Then W is the smallest linear subset generated by the elements of the form $\omega(x, y), x \in X, y \in Y$.*

Proof: See, for example, Gaal[19], Chapter VI, Sec.3.

□

Definition 2.6.3 *(Light and Cheney[29], p.2) When X and Y are vector spaces, the above defined tensor product is called the* **algebraic tensor product** *of X and Y and will be denoted by $X \otimes Y$. The element $\omega(x, y)$ is denoted by $x \otimes y$.*

Since ω is bilinear, we have the following computation rules:

$$(x + x') \otimes y = x \otimes y + x' \otimes y, \quad x \otimes (y + y') = x \otimes y + x \otimes y',$$

and

$$(\lambda x) \otimes y = x \otimes (\lambda y) = \lambda(x \otimes y) = \lambda x \otimes y.$$

From Lemma 2.6.2 it is clear that every element of $X \otimes Y$ can be represented in the form $z = \Sigma_i^n x_i \otimes y_i$ where $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_n \in Y$.

Lemma 2.6.4 *Every expression $\sum_{i=1}^m x_i \otimes y_i$ is equivalent to either $0 \otimes 0$ or to an expression $\Sigma_{i=1}^n a_i \otimes b_i$ with $n \leq m$, where $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are linearly independent sets.*

Proof: See Light and Cheney[29], p.2.

□

If X and Y are normed spaces, it is possible to construct norms in $X \otimes Y$ using those in X and Y. A norm α in $X \otimes Y$ is called a **cross-norm** (cf. Light and Cheney[29]) if it satisfies $\alpha(x \otimes y) = \|x\|_X \|y\|_Y$. Among many ways of constructing a norm on $X \otimes Y$, the following four (which are known to be cross-norms,) are of great interest to our work.

Definition 2.6.5 *(cf. Light and Cheney[29]) For $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$,*

(1) *the* **greatest cross-norm** *σ of z is defined by*

$$\sigma(z) := \inf\{\sum_{i=1}^n \|x_i\| \|y_i\| : x_i \in X, y_i \in Y\}$$

where the infimum is taken with respect to all representations of z;

(2) *the* **least cross-norm** *γ of z defined by*

$$\gamma(z) := \sup_{x^*, y^*}\{\sum_{i=1}^n |x^*(x_i) y^*(y_i)| : x^* \in X^*, y^* \in Y^*, \|x^*\| = \|y^*\| = 1\};$$

(3) *for $1 \leq p \leq \infty$, the* **p-nuclear norm** *α_p of z is defined by*

$$\alpha_p(z) := \inf_{x_i, y_i}\{(\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}} \mu_{p'}(y_1, \ldots y_n)\}$$

where $\mu_{p'}(y_1, \dots y_n) := \sup\{(\sum_{i=1}^n |\psi(y_i)|^{p'})^{\frac{1}{p'}} : \psi \in Y^*, \|\psi\| = 1\}$ *for* $1 \leq p' < \infty$, *and* $\mu_\infty(y_1, \dots y_n) := \sup\{ \max_{1 \leq i \leq n} |\psi(y_i)| : \psi \in Y^*, \|\psi\| = 1\}$. *The infimum is taken with respect to all representatons of* z, *and* $(\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}}$ *is understood to mean* $\max_i \|x_i\|$ *when* $p = \infty$;

(4) *when* X *and* Y *are Hilbert spaces, the* **Hilbert-Schmidt norm** $\beta(z)$ *of* z *is defined by*

$$\beta(z) := \{\sum_{i=1}^n \sum_{j=1}^n \langle x_i, x_j\rangle_X \langle y_i, y_j\rangle_Y\}^{\frac{1}{2}},$$

where $\langle\cdot,\cdot\rangle_X$ *and* $\langle\cdot,\cdot\rangle_Y$ *are inner products in* X *and* Y *respectively.*

It is known that the above norms are well defined in their tensor product space (see, for example, Light and Cheney[29]). The completion of $X \otimes Y$ with respect to a norm α is a Banach space (resp. Hilbert space) denoted by $X \otimes^\alpha Y$, whenever X and Y are Banach spaces (resp. Hilbert spaces).

Let V and W be Banach spaces, and let α be a norm on $V \otimes W$. Let $\mathcal{L}(V, W^*)$ be the set of all continuous linear operators from V to W^*. For $A \in \mathcal{L}(V, W^*)$, the supremum,

$$\sup_{x_i, y_i}\{|\sum_{i=1}^n (Ax_i)(y_i)| : \alpha(\sum_{i=1}^n x_i \otimes y_i) = 1, x_i \in V, y_i \in W\},$$

is denoted by $\|A\|_\alpha$. The set of all operators having $\|A\|_\alpha < \infty$ is denoted by $\mathcal{L}_\alpha(V, W^*)$.

It is well known (see, for example, Light and Cheney[29], p.15) that if α is a cross-norm on $V \otimes W$, then

$$(V \otimes^\alpha W)^* = \mathcal{L}_\alpha(V, W^*). \tag{2.4}$$

In particular,

(a) if we choose α to be the greatest cross-norm σ, we have

$$(V \otimes^\sigma W)^* = \mathcal{L}(V, W^*); \tag{2.5}$$

(b) in the case where V and W are Hilbert spaces, if we choose the cross-norm β, we have

$$(V \otimes^\beta W)^* = HS(V, W^*), \tag{2.6}$$

where $HS(V, W^*)$ denotes the space of Hilbert-Schmidt operators from V to W^*.

Let G be a locally compact group and V and W be left Banach G-modules. Let L be the closed linear subspace of $V \otimes^\alpha W$ spanned by elements of the form

$$av \otimes w - v \otimes wa, \quad a \in G, v \in V, w \in W.$$

The quotient Banach space $(V \otimes^{\alpha} W)/L$ is called the G-**module tensor product**, and is denoted by $V \otimes_G^{\alpha} W$. Then we have a natural isometric isomorphism

$$Hom_G(V, W^*) \cong (V \otimes_G^{\alpha} W)^*. \tag{2.7}$$

where $Hom_G(V, W^*)$ is the space of intertwining operators from V to W^*. If the greatest cross-norm is considered (2.7) will be written in the form

$$Int_G(V, W^*) \cong (V \otimes_G^{\gamma} W)^*, \tag{2.8}$$

where $Int_G(V, W^*)$ denotes the Banach space of all continuous intertwining operators from V to W^* with the operator norm. In the case of Hilbert spaces, we have the analogous result

$$(V \otimes_G^{\beta} W)^* \cong HS_G(V, W^*), \tag{2.9}$$

where $HS_G(V, W^*)$ denotes the space of all intertwining operators from V to W^* with finite Hilbert-Schmidt norm (See Rieffel[37], Corollary 2.13 and Light and Cheney[29]).

Let $t \in V \otimes_G W$ with an expansion of the form $t = \sum_{i=1}^{\infty} v_i \otimes w_i$ for $v_i \in V$ and $w_i \in W$ (see Grothendieck[22]). Then the linear functional F on $Int_G(V, W^*)$ which corresponds to t has value $\sum_{i=1}^{\infty} \langle w_i, T v_i \rangle$ at $T \in Int_G(V, W^*)$. Therefore the topology on $Int_G(V, W^*)$ defined by the these linear functionals corresponds to the weak*-topology on $V \otimes_G W$.

Definition 2.6.6 *(Rieffel[36], p.73) The topology on $Int_G(V, W^*)$ which corresponds to the weak*-topology on $V \otimes_G W$ is called the* **ultraweak*-operator topology** .

Finally, we state a few important results regarding p-nuclear norms α_p which will be used in Chapter 5.

Theorem 2.6.7 *Let S be a finite measure space and $1 \leq p < \infty$.*

(a) *If Y is a Banach space such that, under the natural map,*

$$L_p(S) \otimes^{\alpha_p} Y \cong Y \otimes^{\alpha_p} L_p(S),$$

then

$$L_p(S) \otimes^{\alpha_p} Y \cong L_p(S, Y).$$

(b) *If T is a finite measure space then, under the natural maps,*

(1) $L_p(S) \otimes^{\alpha_p} L_p(T) \cong L_p(T) \otimes^{\alpha_p} L_p(S)$; *and*

(2) $L_p(S) \otimes^{\alpha_p} L_p(T) \cong L_p(S \times T)$.

Proof: See Light and Cheney[29], Theorems 1.50, 1.51 and Corollary 1.52.

□

2.7 Theory of A_p^q spaces

In the classical theory, A_p^q spaces were constructed using the usual Lebesgue spaces $L_p(G)$, where G is a locally compact topological group. First we state the following Proposition which motivates the Definition 2.7.2.

Proposition 2.7.1 *Let $f \in L_p(G)$ and $g \in L_q(G)$, where $1 \le p, q \le \infty$.*

(a) *Suppose G is compact. Then the convolution*

$$(f * g)(x) := \int_G f(xy) g(y^{-1}) d\nu_G(y)$$

is defined almost everywhere, $f * g \in L_r(G)$, and

$$\|f * g\|_r \le \|f\|_p \|g\|_q,$$

where r is defined as follows:

(1) *if $1/p + 1/q > 1$, then $1/r = 1/p + 1/q - 1$;*

(2) *if $1/p + 1/q \le 1$, then $r = \infty$. In this case, $f * g \in C(G)$.*

(b) *Suppose that G is not compact and $1/p + 1/q \ge 1, 1 \le p, q < \infty$. Then $f * g$ is defined almost everywhere and*

$$\|(\Delta_G(\cdot))^{\frac{1}{p'}} f * g\|_r \le \|f\|_p \|g\|_q,$$

where $1/r = 1/p + 1/q - 1$. If $1/p + 1/q = 1$, then $(\Delta_G(\cdot))^{\frac{1}{q}}(f * g) \in C_\infty(G)$.

Proof: See Rieffel[36], Prop. 3.1 and 5.3.

□

Let $f \in L_p(G)$ and $g \in L_q(G)$, where $1 \le p, q \le \infty$. Define a bilinear map b from $L_p(G) \times L_q(G)$ into $L_r(G)$ or $C(G)$ by

$$b(f, g) = (\Delta_G(\cdot))^{\frac{1}{p'}} f * g, \quad f \in L_p(G), g \in L_q(G).$$

(Note that this is the same formula as in (a)(2) of the above Proposition, since compact groups are unimodular.) Then $\|b\| \le 1$, and we can lift it to a linear map B from $L_p(G) \otimes^\sigma L_q(G)$ into either $L_r(G)$ or $C(G)$, depending upon the value of r, with $\|B\| \le 1$.

Definition 2.7.2 *(cf.Rieffel[36], Definition 3.2.) The* space *A_p^q is defined to be the range of B, with the quotient norm.*

Considering the fact that an element of $L_p(G) \otimes^\sigma L_q(G)$ has an expansion of the form $\sum_{i=1}^{\infty} f_i \otimes g_i$, we see that the elements of A_p^q consist of those functions h on G which have at least one expansion of the form

$$h = (\Delta_G(\cdot))^{\frac{1}{p'}} \Sigma_{i=1}^{\infty} f_i * g_i,$$

with $f_i \in L_p(G)$, $g_i \in L_q(G)$ and $\Sigma_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q \leq \infty$ with the expansion converging in the norm of $L_r(G)$ or $C(G)$.

The following is a summary of some well known, important results in this regard (cf.Rieffel[36]).

Theorem 2.7.3 *Let L be the closed subspace of $L_p(G) \otimes^\sigma L_q(G)$ spanned by elements of the form $(\phi * f) \otimes g - f \otimes (\tilde{\phi} * g)$, where $\phi \in L_1(G), f \in L_p(G), g \in L_q(G)$ and $\tilde{\phi}(x) := \Delta(x^{-1})\phi(x^{-1})$. Let $L_p(G) \otimes_G^\sigma L_q(G)$ denote the quotient Banach space $L_p(G) \otimes^\sigma L_q(G)/L$. Then,*

(1) *if G is compact, $1 < p, q \leq \infty$, and either $p < \infty$ or $q < \infty$, we have the isometric isomorphism*

$$L_p(G) \otimes_G^\sigma L_q(G) \cong A_p^q;$$

(2) *if G is non-compact and $1/p + 1/q < 1, p < \infty, q < \infty$, then*

$$L_p(G) \otimes_G^\sigma L_q(G) = \{\underline{0}\};$$

(3) *if G is non-compact and $1/p + 1/q \geq 1, 1 < p, q < \infty$, then $L_p(G) \otimes_G^\sigma L_q(G) \cong A_p^q$ if and only if every element of $Hom_G(L_p(G), L_q(G))$ can be approximated in the ultraweak operator topology by operators of the form $T_\phi : f \mapsto f * \phi, f \in L_p(G), \phi \in C_0(G)$, the space of continuous functions with compact support.*

Proof: See Rieffel[36], Theorems 3.3, 4.1 and 5.5.

□

It is known that if G is an Abelian or compact, every element of $Hom_G(L_p(G), L_q(G))$ can be approximated in the ultraweak operator topology by operators of the form T_ϕ (see Figa-Talamanca and Gaudry[18], Theorem 1 and Rieffel[36], p.79).

Theorem 2.7.4 *Let G be an Abelian group. Let M_p^q denote the space of bounded operators T on $L_p(G)$ to $L_q(G)$, $(1 \leq p, q < \infty)$, which commutes with translations; that is, $\tau_x T = T\tau_x$, for all $x \in G$, where $\tau_x f(y) := f(x + y)$. Then the space M_p^q is isometrically isomorphic to $(A_p^q)^*$, the (topological) dual of A_p^q.*

Proof: See Figá-Talamanca[17, 18].

□

Herz[25] defined A_p^q spaces using Banach spaces of representations of a locally compact group as follows. Let π be a representation of a group G and $\mathcal{H}(\pi)$ be the corresponding Banach space. Let the map $\Pi(\pi) : \mathcal{H}(\pi) \otimes (\mathcal{H}(\pi))^* \rightarrow C_u(G)$, where $C_u(G)$ is the space of bounded uniformly continuous functions on G in the supremum norm, be defined by $\Pi(\pi) : (f \otimes g)(x) = \langle \pi(x)f, g \rangle$. The quotient of $\mathcal{H}(\pi) \otimes (\mathcal{H}(\pi))^*$ by the kernel of $\Pi(\pi)$ is called the space of π-representative functions, and is denoted by $A(\pi)$.

In particular, if the right (resp.left) regular representation $\lambda_p(\mathcal{C})$ (in Herz's notation in [25],) on an L_p space is considered, the corresponding $A(\lambda_p(\mathcal{C}))$ space is denoted by A_p.

The following are the main results in [25], regarding A_p spaces:

Theorem 2.7.5

(a) *If π is a representation in a p-space $\mathcal{H}(\pi)$, then multiplication of funtions gives a morphism*

$$A_p \otimes A(\pi) \mapsto A_p.$$

(b) *A_p is a Banach algebra under pointwise addition and multiplication of functions.*

(c) *If $p \leq q \leq 2$ or $p \geq q \geq 2$ then multiplication of functions gives a morphism*

$$A_p \otimes A_q \mapsto A_p.$$

Proof: See Herz[25].

□

Chapter 3

Preliminaries

This Chapter has two main aims:

(a) to present some new definitions, notations and results which are essential for later use; and

(b) to state and prove some interesting results on p-induced representations which came to light as a consequence of our theory.

The contents of the three sections (which are independent of each other) are as follows: Section 3.1 is devoted to proving some important results regarding λ-functions (see Sec.2.2) which are used as essential tools in the calculations throughout our work. Lemma 3.1.3 describes an identity among λ-functions of a particular set of subgroups of a given group, which plays an important role in the development of the theory. Section 3.2 deals with some new notations and results on Banach (semi-)bundles. Section 3.3 consists of results on adjoints of representations and p-induced representations. In particular, Moore's version of Frobenius Reciprocity Theorem (see Moore[34]) is dealt with in a general setting, where the corresponding coset spaces do not have to possess an invariant measure.

3.1 Some important results on λ-functions

Lemma 3.1.1 *Let G be a locally compact group. Let H and K be closed subgroups of G with $K \subseteq H$. Then there exist positive quasi-invariant measures μ_K on G/K, μ_H on G/H and $\tilde{\mu}$ on H/K such that, for $F \in C_0(G/K)$,*

$$\int_{\frac{G}{K}} F(z)d\mu_K(z) = \int_{\frac{G}{H}} \left(\int_{\frac{H}{K}} \frac{\lambda_K(y,t)}{\lambda_H(y,t)} F(yt)d\tilde{\mu}(y) \right) d\mu_H(t), \tag{3.1}$$

whenever the integrals exist.

Proof: By Reiter[35], p.158,(see also Mackey[31]), there exists a continuous, strictly positive function ρ_K on G and a positive measure μ_K on G/K such that

$$\int_G f(u)d\nu_G(u) = \int_{\frac{G}{K}}\Big(\int_K \frac{1}{\rho_K(sz)}f(sz)d\nu_K(s)\Big)d\mu_K(z), \tag{3.2}$$

for $f \in C_0(G)$.

Also, by the same reasoning, there exists a continuous, strictly positive function ρ_H on G and a positive measure μ_H on G/H such that

$$\int_G f(u)d\nu_G(u) = \int_{\frac{G}{H}}\Big(\int_H \frac{1}{\rho_H(ht)}f(ht)d\nu_H(h)\Big)d\mu_H(t).$$

Let $\tilde{\rho} = \rho_K/\rho_H$. We see that

$$\tilde{\rho}(sx) = \rho_K(sx)/\rho_H(sx) = (\Delta_K(s)/\Delta_H(s))\tilde{\rho}(x),$$

for $s \in K$ and $x \in G$. Thus $\tilde{\rho}$, restricted to H, is a ρ-function for the homogeneous space H/K. If we let $\tilde{\mu}$ be a quasi-invariant measure associated with this ρ-function, we have

$$\int_G f(u)d\nu_G(u) = \int_{\frac{G}{H}}\int_{\frac{H}{K}}\Big(\int_K \frac{\rho_H(sy)}{\rho_K(sy)\rho_H(syt)}f(syt)d\nu_K(s)\Big)d\tilde{\mu}(y)\mu_H(t). \tag{3.3}$$

By Reiter[35], p.165, for a given $F \in C(G/K)$, there exists a function $f \in C(G)$ such that

$$F(\dot{z}) = \int_K \frac{1}{\rho_K(sz)}f(sz)d\nu_K(s), \tag{3.4}$$

where $\dot{z} = p_K(z)$. Comparing equations (3.2) and (3.3), and using (3.4), we see that

$$\begin{aligned}\int_{\frac{G}{K}} F(z)d\mu_K(z) &= \int_{\frac{G}{H}}\int_{\frac{H}{K}} \frac{\rho_K(yt)\rho_H(y)}{\rho_K(y)\rho_H(yt)}F(yt)d\tilde{\mu}(y)\mu_H(t)\\ &= \int_{\frac{G}{H}}\left(\int_{\frac{H}{K}} \frac{\lambda_K(y,t)}{\lambda_H(y,t)}F(yt)d\tilde{\mu}(y)\right)d\mu_H(t),\end{aligned}$$

for any $F \in C_0(G/K)$, and (3.1) is proved.

□

Let μ_H be a given quasi-invariant measure on G/H with the corresponding λ−function λ_H. Consider the homeomorphism $\phi_x : G/H^x \mapsto G/H$ given by $\phi_x(u) = xu$. Define a measure μ_{H^x} on G/H^x by $\mu_{H^x}(E) = \mu_H(\phi_x(E))$ whenever E is such that $x.E$ is measurable. Clearly, μ_{H^x} is quasi-invariant if and only if μ_H is. The corresponding λ−function of μ_{H^x} is denoted by λ_{H^x}. The following result states the relationship between λ_H and λ_{H^x}.

Lemma 3.1.2 *For $x, t \in G$ and for almost all $v \in G/H$*

$$\lambda_{H^x}(x^{-1}v, t) = \lambda_H(v, t). \tag{3.5}$$

Proof: For $f \in C_0(G/H)$, we have

$$\begin{aligned} \int_{\frac{G}{H}} f(v) d\mu_H(v) &= \int_{\frac{G}{H^x}} f(xu) d\mu_H(\phi_x(u)) \\ &= \int_{\frac{G}{H^x}} f(xu) d\mu_{H^x}(u). \end{aligned} \tag{3.6}$$

Changing variables $u \mapsto ut$, the above gives

$$\begin{aligned} \int_{\frac{G}{H}} f(v) d\mu_H(v) &= \int_{\frac{G}{H^x}} \lambda_{H^x}(u, t) f(xut) d\mu_{H^x}(u) \\ &= \int_{\frac{G}{H}} \lambda_{H^x}(x^{-1}v, t) f(vt) d\mu_H(v). \end{aligned} \tag{3.7}$$

On the other hand, changing variables $v \mapsto vt$ in the integral $\int_{\frac{G}{H}} f(v) d\mu_H(v)$, we obtain

$$\int_{\frac{G}{H}} f(v) d\mu_H(v) = \int_{\frac{G}{H}} \lambda_H(v, t) f(vt) d\mu_H(v). \tag{3.8}$$

Hence, comparing equations (3.7) and (3.8), we have

$$\lambda_{H^x}(x^{-1}v, t) = \lambda_H(v, t)$$

for $x, t \in G$ and for almost all $v \in G/H$, as required.

□

Let us turn to the most important result in this section; namely, an identity among λ-functions of certain subgroups of a given group G.

Let $\Delta = \{(x, x) : x \in G\}$ be the diagonal subgroup of $G \times G$. Consider the right action of Δ on the coset space $(G \times G)/(H \times K)$. The stabilizer of the coset (Hx, Ky) is $(H \times K)^{(x,y)} \cap \Delta$ and the orbit is the double coset $(H \times K)(x, y)\Delta$. Let Υ be the set of all double cosets $(H \times K) : \Delta$ of $G \times G$; that is, the set of all orbits. For each $(x, y) \in G \times G$, let $k(x, y)$ denote the $(H \times K) : \Delta$ double coset to which (x, y) belongs. If ν_0 is any finite measure in $G \times G$ with the same null sets as Haar measure we define a measure $\mu_{(H,K)}$ on Υ by $\mu_{(H,K)}(F) = \nu_0(k^{-1}(F))$ whenever F is such that $k^{-1}(F)$ is measurable. Using Mackey's terminology, we call such a measure an **admissible measure** in Υ. We obtain the following result as a consequence of Lemma 2.2.12.

Lemma 3.1.3 *Suppose that H and K are regularly related. Let Δ be the diagonal subgroup of $G \times G$ and Υ denote the set of all $(H \times K) : \Delta$ double cosets in $G \times G$.*

Then for each double coset $D(x,y) = H \times K(x,y)\Delta$ there exists a quasi-invariant measure $\mu_{x,y}$ on $G/(H^x \cap K^y)$, $x, y \in G$, and $\lambda_{H^x \cap K^y}$ with $\mu_{x,y} \succ \lambda_{H^x \cap K^y}$ such that

$$\lambda_H(xts^{-1}, s)\lambda_K(yts^{-1}, s)\lambda_{H^x\cap K^y}(\underline{t}, \underline{s}^{-1}) \;=\; 1, \qquad (3.9)$$

for all $s, t \in G$, and for almost all $(x,y) \in (G \times G)/(H \times K)$. Moreover, $\lambda_{H^x\cap K^y}(t,s)$ is defined everywhere and continuous on $(G/(H^x \cap K^y)) \times G$.

Proof: Choose two quasi-invariant measures μ_H and μ_K on G/H and G/K respectively, which correspond to two continuous ρ-functions. Define a measure $\mu_{H\times K}$ in $(G \times G)/(H \times K)$ by $\mu_{H\times K} = \mu_H \times \mu_K$ (see, for example, Halmos[24], p.144). Obviously, $\mu_{H\times K}$ is quasi-invariant to the action of Δ. Let ν_0 be the measure in $(G \times G)$ defined by $\nu_0(p^{-1}_{H\times K}(F)) = \mu_{H\times K}(F)$. Let $\mu_{H,K}$ be an admissible measure in Υ corresponding to ν_0.

Let f be a function defined on $(G/H) \times (G/K)$. Suppose $\int_{\frac{G}{H}} \int_{\frac{G}{K}} f(x,y)d\mu_H(x)d\mu_K(y)$ is integrable. Changing the variables $x \mapsto xs$ and $y \mapsto ys$, we get

$$\begin{aligned}
&\int_{\frac{G}{H}} \int_{\frac{G}{K}} f(x,y)d\mu_H(x)d\mu_K(y) \\
&= \int_{\frac{G}{H}} \int_{\frac{G}{K}} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_H(x)d\mu_K(y) \\
&= \int_{\frac{G\times G}{H\times K}} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y).
\end{aligned}$$

For each (x,y) in $(G \times G)/(H \times K)$ let $r(x,y) = k(p^{-1}_{H\times K}(x,y))$. If H and K are regularly related then r defines a measurable equivalence relation(see Sec.2.2). Then, by Lemma 2.2.12, $\mu_{H\times K}$ is an integral of measures $\mu_{x,y}$, where $D(x,y) \in \Upsilon$, with respect to the measure $\mu_{H,K}$ in Υ. By Lemma 2.2.13, each $\mu_{x,y}$ is a quasi-invariant measure on the orbit $r^{-1}(D(x,y))$. Using this decomposition, we have

$$\begin{aligned}
&\int_{\frac{G\times G}{H\times K}} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y) \\
&= \int_{D\in\Upsilon} \int_{\underline{t}\in\frac{\Delta}{(H\times K)^{(x,y)}\cap\Delta}} \lambda_H(xt,s)\lambda_K(yt,s)f(xts,yts)d\mu_{x,y}(\underline{t})d\mu_{H,K}(D),
\end{aligned}$$

where (x,y) is the coset representative of the coset $D(x,y)$. Identifying the space $\Delta/((H \times K)^{(x,y)} \cap \Delta)$ with $G/(H^x \cap K^y)$ we can regard $\mu_{x,y}$ as a measure on $G/(H^x \cap K^y)$. Then we have

$$\begin{aligned}
&\int_{\frac{G\times G}{H\times K}} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y) \\
&= \int_{D\in\Upsilon} \int_{t\in\frac{G}{H^x\cap K^y}} \lambda_H(xt,s)\lambda_K(yt,s)f(xts,yts)d\mu_{x,y}(t)d\mu_{H,K}(D),
\end{aligned}$$

Changing variables $t \mapsto ts^{-1}$, in the integral on the right-hand side, we get

$$\int_{\frac{G\times G}{H\times K}} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y)$$
$$= \int_{D\in\Upsilon}\int_{t\in\frac{G}{H^x\cap K^y}} \lambda_H(xts^{-1},s)\lambda_K(yts^{-1},s)f(xt,yt)$$
$$\lambda_{H^x\cap K^y}(t,s^{-1})d\mu_{x,y}(t)d\mu_{H,K}(D). \qquad (3.10)$$

On the other hand, if we start with $\int\int_{\frac{G\times G}{H\times K}} f(x,y)d\mu_{H\times K}(x,y)$ and use Lemma 2.2.12, we have

$$\int\int_{\frac{G\times G}{H\times K}} f(x,y)d\mu_{H\times K}(x,y)$$
$$= \int_{D\in\Upsilon}\int_{\underline{t}\in\frac{\Delta}{(H\times K)^{(x,y)}\cap\Delta}} f(xt,yt)d\mu_{x,y}(\underline{t})d\mu_{(H,K)}(D),$$
$$= \int_{D\in\Upsilon}\int_{t\in\frac{G}{H^x\cap K^y}} f(xt,yt)d\mu_{x,y}(t)d\mu_{(H,K)}(D). \qquad (3.11)$$

Hence from (3.10) and (3.11) we have

$$\lambda_H(xts^{-1},s)\lambda_K(yts^{-1},s)\lambda_{H^x\cap K^y}(t,s^{-1}) \;=\; 1,$$

for all $s\in G$, for almost all $t\in G/(H^x\cap K^y)$ and for almost all $(x,y)\in(G\times G)/(H\times K)$. For each such $(x_0,y_0)\in(G\times G)/(H\times K)$,

$$\lambda_H(x_0ts^{-1},s)\lambda_K(y_0ts^{-1},s)\lambda_{H^{x_0}\cap K^{y_0}}(t,s^{-1}) \;=\; 1. \qquad (3.12)$$

By continuity of λ_H and λ_K, we see that (3.12) is true for all $t\in G/(H^{x_0}\cap K^{y_0})$. Furthermore, (3.12) implies that $\lambda_{H^x\cap K^y}(t,s)$ is defined everywhere and continuous on $(G/(H^x\cap K^y))\times G$, which proves the Lemma.

□

The following result is a consequence of Lemma 3.1.3.

Corollary 3.1.4 *Let $(x,y)\in G\times G$ such that the identity (3.9) holds. Then for $s\in H^x\cap K^y$,*

$$\frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(s)\Delta_{H^x\cap K^y}(s)} \;=\; 1, \qquad (3.13)$$

where $h = xsx^{-1}$ and $k = ysy^{-1}$.

Proof: Let $t = s$ in the identity (3.9). Then we have

$$\lambda_H(x,s)\lambda_K(y,s)\lambda_{H^x\cap K^y}(s,s^{-1}) \;=\; 1. \qquad (3.14)$$

By Lemma 2.2.7 (a) this simplifies to

$$\lambda_H(x,s)\lambda_K(y,s) \;=\; \lambda_{H^x\cap K^y}(e,s). \qquad (3.15)$$

Consider $s \in H^x \cap K^y$. Then $s = x^{-1}hx = y^{-1}ky$ for some $h \in H$ and $k \in K$. For such an s, we have by Lemma 2.2.7 (a),

$$\begin{aligned}
\lambda_H(x,s) &= \lambda_H(x, x^{-1}hx), \\
&= \lambda_H(h,x)\lambda_H(x,x^{-1}h), \\
&= \lambda_H(e,x)\lambda_H(e,h)\lambda_H(x,x^{-1}), \\
&= \lambda_H(e,h), \\
&= \frac{\Delta_H(h)}{\Delta_G(h)},
\end{aligned} \tag{3.16}$$

Similarly,

$$\lambda_K(y,s) = \frac{\Delta_K(k)}{\Delta_G(k)}, \tag{3.17}$$

and

$$\lambda_{H^x \cap K^y}(e,s) = \frac{\Delta_{H^x \cap K^y}(s)}{\Delta_G(s)}. \tag{3.18}$$

Using (3.15),(3.16),(3.17) and (3.18), we obtain

$$\frac{\Delta_H(h)}{\Delta_G(h)} \frac{\Delta_K(k)}{\Delta_G(k)} = \frac{\Delta_{H^x \cap K^y}(s)}{\Delta_G(s)}. \tag{3.19}$$

But $\Delta_G(h) = \Delta_G(x^{-1}hx) = \Delta_G(s) = \Delta_G(y^{-1}ky) = \Delta_G(k)$, hence (3.19) simplifies to

$$\frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(s)\Delta_{H^x \cap K^y}(s)} = 1, \tag{3.20}$$

as required.

3.2 Some special Banach bundles and Banach semi-bundles

The principal objective in this section is to consider a special type of Banach (semi-)bundle which arises in the construction of A_q^p spaces, in Sec.4.2. Let the Banach (semi-)bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ over X with $\mathcal{B} = \mathcal{H} \times X$ be such that $\mathcal{H}$ is a Banach space, X is a Hausdorff space and $\theta(\xi, x) = x$. Let R be an equivalence relation defined on X. For $x \in X$ let $r(x) \in X/R$ be the equivalence class to which x belongs. Suppose the Banach space $\mathcal{H}$ is such that there exists a collection of its closed subspaces that can be indexed by the elements of X; that is, a collection that can be expressed as $\{\mathcal{H}_x : x \in X\}$. Suppose further that $\mathcal{H}_x = \mathcal{H}_{x'}$ if and only if $x' \in r(x)$. For each $x \in X$, let $\mathcal{A}_x = \mathcal{H}/\mathcal{H}_x$. We want to look at a particular type of Banach (semi-)bundle constructed in the following manner. First, let us introduce some new notations. Let

$$\begin{aligned}
\mathcal{B}_0 &= \mathcal{H} \times X, \\
\mathcal{B}_0^R &= \mathcal{H} \times X/R, \\
\mathcal{B}_1 &= \cup_{x\in X}\{\mathcal{H}_x \times \{x\}\}, \\
\mathcal{B}_1^R &= \cup_{x\in X}\{\mathcal{H}_x \times \{r(x)\}\}, \\
\mathcal{B}_2 &= \cup_{x\in X}\{\mathcal{A}_x \times \{x\}\}, \text{ and} \\
\mathcal{B}_2^R &= \cup_{x\in X}\{\mathcal{A}_x \times \{r(x)\}\}.
\end{aligned}$$

It is clear that $\mathcal{B}_1$ is a subspace of $\mathcal{B}_0$, and $\mathcal{B}_1^R$ is a subspace of $\mathcal{B}_0^R$.

With j denoting any one of $\{0,1,2\}$, let $\theta_j : \mathcal{B}_j \mapsto X$ and $\theta_j^R : \mathcal{B}_j^R \mapsto X/R$ be defined by $\theta_j(\zeta, x) = x$ and $\theta_j^R(\zeta, r(x)) = r(x)$, respectively, where ζ belongs to the corresponding Banach space. Let $q : \mathcal{B}_0 \mapsto \mathcal{B}_2$ be the quotient map defined by $q(h,x) = (\{\mathcal{H}_x + h\}, x)$. Similarly the quotient map $q_R : \mathcal{B}_0^R \mapsto \mathcal{B}_2^R$ is defined by $q_R(h, r(x)) = (\{\mathcal{H}_x + h\}, r(x))$. We topologize $\mathcal{B}_2$ and $\mathcal{B}_2^R$ so that the map q and q_R are continuous and open.

Lemma 3.2.1 *Suppose X/R is Hausdorff. Then, $\mathcal{B}_j$ and $\mathcal{B}_j^R$ are Hausdorff for each $j \in \{0,1,2\}$.*

Proof: Clearly, $\mathcal{B}_0$ and $\mathcal{B}_0^R$ are Hausdorff, being products of such spaces. The result is valid in the cases of $\mathcal{B}_1$ and $\mathcal{B}_1^R$, since they are subspaces of $\mathcal{B}_0$ and $\mathcal{B}_0^R$ respectively.

Let us show that $\mathcal{B}_2^R$ is Hausdorff. The proof for the case of $\mathcal{B}_2$ is similar.

Let $q : \mathcal{B}_0^R \mapsto \mathcal{B}_2^R$ be defined by $q(\zeta, x) = (\{\mathcal{H}_x + \zeta\}, x)$. Consider $(\zeta', x) \neq (\vartheta', y)$ in $\mathcal{B}_2^R$. Then there exist $(\zeta, x), (\vartheta, x) \in \mathcal{B}_0^R$ such that $q(\zeta, x) = (\zeta', x)$ and $q(\vartheta, y) = (\vartheta', y)$ with $(\zeta, x) \neq (\vartheta, y)$. We have the following cases:

(i) If $x \neq y$, consider disjoint open sets around x and y (this is possible since the space X/R is Hausdorff) which will give rise to open sets $(\theta_2^R)^{-1}(x)$ and $(\theta_2^R)^{-1}(y)$ which are disjoint.

(ii) If $x = y$, choose open sets U and V of $\mathcal{H}/\mathcal{H}_x$ such that $\mathcal{H}_\alpha + \zeta \subset U$ and $\mathcal{H}_\alpha + \vartheta \subset V$. Since the canonical map $p_x : \mathcal{H} \mapsto \mathcal{H}/\mathcal{H}_x$ is continuous, there exist U' and V' in $\mathcal{H}$ such that $p_x U' \subseteq U$ and $p_x V' \subseteq V$. Then, $\tilde{U} = U' \times X/R \subset \mathcal{B}_0^R$ and $\tilde{V} = V' \times X/R \subset \mathcal{B}_0^R$ with $q(\tilde{U}) \cap q(\tilde{V}) = \emptyset$.

Hence $\mathcal{B}_2^R$ is Hausdorff. A similar argument works for $\mathcal{B}_2$.

□

Define $\underline{\mathcal{B}}_j = (\mathcal{B}_j, \theta_j)$ and $\underline{\mathcal{B}}_j^R = (\mathcal{B}_j^R, \theta_j^R)$, for $j = 0, 2$.

Theorem 3.2.2 *$\underline{\mathcal{B}}_j$ and $\underline{\mathcal{B}}_j^R$, for $j = 0, 2$, are bundles over X and X/R, respectively.*

Proof: Clearly, $\theta_0 : \mathcal{B}_0 \mapsto X$ and $\theta_0^R : \mathcal{B}_0^R \mapsto X/R$ are continuous open surjections. Consider $\theta_2 : \mathcal{B}_2 \mapsto X$. We have the diagram

$$\begin{array}{ccc} \mathcal{B}_0 & \overset{q}{\mapsto} & \mathcal{B}_2 \\ \theta_0 \searrow & & \swarrow \theta_2 \\ & X & \end{array}$$

First we show that θ_2 is open. For any open set U in $\mathcal{B}_2$, $q^{-1}(U)$ is an open set in $\mathcal{B}_0$ by definition of the topology of $\mathcal{B}_2$. Then, θ_0 being an open map, we see that $\theta_2(U) = \theta_0(q^{-1}(U))$ is an open set in X. Hence θ_2 is open.

To show θ_2 is continuous let $V \subset X$ be an open set. Then, θ_0 being continuous, $\theta_0^{-1}(V)$ is open. By definition of the topology of $\mathcal{B}_2$, $q(\theta_0^{-1}(V)) = \theta_2^{-1}(V)$ is open, giving the required result.

Similarly, θ_2^R is a continuous open map. Now the result follows from Lemma 3.2.1.

□

It is clear that $\underline{\mathcal{B}}_0$ and $\underline{\mathcal{B}}_0^R$ are trivial Banach bundles (see Sec.2.3). Let us explore the situation in $\underline{\mathcal{B}}_2$ and $\underline{\mathcal{B}}_2^R$. For each $z = r(x) \in X/R$, the fibre of $\mathcal{B}_2^R$ over z is $\mathcal{B}_{2,z}^R = \{\mathcal{A}_x \times \{z\}\}$. We see that $\mathcal{B}_{2,z}^R$, is a Banach space with the norm $\|(\eta, z)\|_{\mathcal{B}_{2,z}^R}$ defined by $\|(\eta, z)\|_{\mathcal{B}_{2,z}^R} = \|\eta\|$ where $\|\eta\|$ means the norm in $\mathcal{A}_x$. The operations + and . in $\mathcal{B}_{2,z}^R$ are defined, in an obvious manner, using + and . in $\mathcal{A}_x$. We can define and topologize the fibres $\mathcal{B}_{2,z}$ in $\mathcal{B}_2$ and define the operations + and . in a similar manner.

Lemma 3.2.3 *$(\eta, z) \mapsto \|(\eta, z)\|_{\mathcal{B}_{2,z}^R}$ is upper semi-continuous on $\mathcal{B}_2^R$ to $\mathcal{R}$. A similar result holds in the case of $\mathcal{B}_2$.*

Proof: Let $\{(\eta_i, z_i) : i \in I\}$ be a net of elements in $\mathcal{B}_2^R$ with $(\eta_i, z_i) \to (\eta, z)$. Then there exist a sequence $\{(\varphi_i, u_i)\}$ and an element (φ, u) in $\mathcal{B}_0^R$ such that $q_R((\varphi_i, u_i)) = (\eta_i, z_i)$ for all $i \in I$, $q_R((\varphi, u)) = (\eta, z)$ and $(\varphi_i, u_i) \to (\varphi, u)$. Now since $\|(\eta, z)\|_{\mathcal{B}_z} = \|\eta\| = inf_{h \in \mathcal{H}_x} \|\varphi + h\|$, without loss of generality we can choose φ such that, for a given $\epsilon > 0$, we have

$$\|\varphi\| \ < \ \|\eta\| + \epsilon. \tag{3.21}$$

Also,

$$\|\eta_i\| \ \leq \ \|\varphi_i\| \tag{3.22}$$

for all $i \in I$. Since $\|\varphi_i\| \to \|\varphi\|$, then from (3.21) and (3.22) we have

$$\|\eta_i\| \leq \|\eta\| + \epsilon,$$

for i sufficiently large.

The proof is similar in the case of $\mathcal{B}_2$.

□

Lemma 3.2.4 *The operation + is continuous on $\mathcal{B}^R_{2,z} \times \mathcal{B}^R_{2,z}$ to $\mathcal{B}^R_{2,z}$, and for each λ in C, the map $b \mapsto \lambda b$ is continuous on $\mathcal{B}^R_2$ to $\mathcal{B}^R_2$. A similar results hold in $\mathcal{B}_2$.*

Proof: Since the topology induced from $\mathcal{B}^R_2$ on its fibres is just the Banach space topology, the operations + and . are continuous.

Similarly for $\mathcal{B}_2$.

□

Lemma 3.2.5 *If $z \in X/R$ and $\{b_i : i \in I\}$, is any net of elements in $\mathcal{B}^R_2$ such that $\|b_i\| \to 0$ and $\theta^R_2(b_i) \to z$, then $b_i \to 0_z$ where 0_z is the zero element in $\mathcal{B}^R_{2,z}$. A similar result holds in $\mathcal{B}_2$.*

Proof: Any element $b_i \in \mathcal{B}^R_2$ is of the form $b_i = (\omega_i + \mathcal{H}_{x_i}, r(x_i))$ where $x_i \in X$. Since $\|b_i\| = \inf\{\|\omega_i + h\| : h \in \mathcal{H}_{x_i}\}$, with $\omega_i \in \mathcal{H}$, there exists an $h_i \in \mathcal{H}_{x_i}$ such that

$$\|\omega_i + h_i\| < \|b_i\| + 1/2^i$$

for all $i \in I$. This implies that the net of elements $\omega_i + h_i$ in $\mathcal{H}$ has the property that $\omega_i + h_i \to \underline{0}$, $\underline{0}$ being the zero element in $\mathcal{H}$. If $\theta^R_2(b_i) = r(x_i) \to z$, this means that $b_i \to 0_z$.

□

Lemma 3.2.6 *$\underline{\mathcal{B}}^R_2$ and $\underline{\mathcal{B}}_2$ are Banach semi-bundles over X/R and X respectively.*

Proof: The result follows from Theorem 3.2.2, Lemmas 3.2.3, 3.2.4 and 3.2.5.

□

Proposition 3.2.7 *The Banach semi-bundle retraction*

$$\underline{\mathcal{B}}^{R\#}_2 = (\mathcal{B}^{R\#}_2, \theta^{R\#}_2)$$

of $\underline{\mathcal{B}}^R_2$ by r is topologically equivalent to $\underline{\mathcal{B}}_2$.

Proof: Consider the diagram

$$\begin{array}{ccccc}
\mathcal{B}_0^{R\#} & \overset{q_R^\#}{\mapsto} & \mathcal{B}_2^{R\#} & & \\
\imath\uparrow & & \uparrow\jmath & & \\
\mathcal{B}_0 \overset{q}{\mapsto} \mathcal{B}_2 & & & & \mathcal{B}_0^R \overset{q_R}{\mapsto} \mathcal{B}_2^R \\
\theta_0 \searrow\swarrow \theta_2 & & & & \theta_0^R \searrow\swarrow \theta_2^R \\
X & & \xrightarrow{q} & & X/R
\end{array}$$

where $q_R^\# = \jmath^\#(q_R)$ (see Sec.2.3, p.11) and $\jmath$ is defined so that $q_R^\# \circ \imath = \jmath \circ q$, $\imath$ being the homeomorphism stated in Proposition 2.3.3. It is clear that $q_R^\#$ is the quotient map. Hence, (also by the discussion on page 11,) $q_R^\#$ is continuous and open. Obviously, $\jmath$ defines a bijection from $\mathcal{B}_2$ onto $\mathcal{B}_2^{R\#}$. We need to show that $\jmath$ and its inverse are continuous. Now q_R is open by the definition of the topology of $\mathcal{B}_2^R$ and the right hand side of the above diagram commutes. Since the maps $\imath$, $q_R^\#$ and q are continuous and open it is clear that $\jmath$ is continuous and open, as required.

□

Proposition 3.2.8 *Let $f : X \mapsto \mathcal{B}_2$ be a continuous cross-section which is constant on equivalence classes. Then the function g defined by $g(p(x)) = i^\#(f(x))$, where $x \in X$, is a continuous cross-section from X/R to $\mathcal{B}_2^R$.*

Proof:By Proposition 3.2.7, a continuous cross-section f of $\underline{\mathcal{B}}_2$ can be regarded as a cross-section of $\underline{\mathcal{B}}_2^{R\#}$. Define $g' : X \mapsto \mathcal{B}_2^R$ so that

$$g'(x) = i^\#(f(x)) \text{ for all } x \in X.$$

(See the diagram below.)

$$\begin{array}{ccc}
\mathcal{B}_2 = \mathcal{B}_2^{R\#} & \overset{i^\#}{\longmapsto} & \mathcal{B}_2^R \\
f\nwarrow & & \nearrow g' \\
& X &
\end{array}$$

Consider the function $g : X/R \mapsto \mathcal{B}_2^R$ which factors through the diagram

$$\begin{array}{ccc}
& & \mathcal{B}_2^R \\
& g' \nearrow & \uparrow g \\
X & \overset{p}{\longmapsto} & X/R
\end{array}$$

It is clear that g is well defined since f is constant on the equivalence classes. Also, $g(p(x)) = g'(x)$ for any $x \in X$ and we see that $g(z) \in \mathcal{B}_{2,z}^R$ for any $z \in X/R$. Hence g is a cross-section of $\underline{\mathcal{B}}_2^R$. Moreover, it is continuous since p is open.

□

We need the following results, proved by Fell[16] for Banach bundles, in the context of semi-bundles.

Proposition 3.2.9 *Let $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ be a Banach semi-bundle. Suppose that $\{s_i : i \in I\}$, is a net of elements of $\mathcal{B}$, and $\theta(s_i) \to \theta(s)$ in X. Suppose further that for each $\epsilon > 0$ we can find a net $\{u_i\}$ of elements of $\mathcal{B}$ (indexed by the same I) and an element u of $\mathcal{B}$ such that: (a) $u_i \to u$ in $\mathcal{B}$, (b) $\theta(u_i) = \theta(s_i)$ for each i, (c) $\|s - u\| < \epsilon$ and (d) $\|s_i - u_i\| < \epsilon$ for all large enough i. Then $s_i \to s$ in $\mathcal{B}$.*

Proof: This is proved by Fell only for Banach bundles. We give the proof for the case of a Banach semi-bundle which follows the same reasoning as in Fell[16], Proposition 1.4, p.12.

It is enough to prove that some sub-net of $\{s_i\}$ converges to s. Since θ is open, for each i we can find an element t_i such that $\theta(t_i) = \theta(s_i)$ and $t_i \to s$. Let $\epsilon > 0$ be given and choose $\{u_i\}$ and u in $\mathcal{B}$ satisfying the conditions (a) to (d). Since addition is continuous in $\mathcal{B}$, we have $t_i - u_i \to s - u$. Since $\|s - u\| < \epsilon$, by the upper semi-continuity of the norm, we have $\|t_i - u_i\| < 2\epsilon$ for i large. This implies

$$\|t_i - s_i\| < \|t_i - u_i\| + \|u_i - s_i\| < 3\epsilon,$$

for i large. Hence $\|t_i - s_i\| \to 0$, therefore $t_i - s_i \to 0_{\theta(s)}$. Consequently $s_i \to s$.

□

Corollary 3.2.10 *Let g be a cross-section for a Banach semi-bundle $\underline{\mathcal{B}}$ such that, for each $x \in X$ and each $\epsilon > 0$, there is a continuous cross-section f and an x-neighborhood U of x such that $\|g(y) - f(y)\| < \epsilon$ for all y in U. Then g is a continuous cross-section.*

Proof: We give the proof in the case of a Banach semi-bundle which is similar to that of a Banach bundle as given in Fell[16], Corollary 1, p.13.

Let $\{x_i : i \in I\}$ be a net of elements of X such that $x_i \to x$. Let $g(x_i) = s_i$ and $g(x) = s$. Then $\theta(s_i) = x_i \to x = \theta(s)$. For a given $\epsilon > 0$, let f be a continuous cross-section and U be an x-neighborhood of x such that $\|g(y) - f(y)\| < \epsilon$ for all y in U. This means that we can find $u_i = f(x_i)$ with $u_i \to u = f(x)$ so that $\|u_i - s_i\| < \epsilon$, $\theta(u_i) = x_i = \theta(s_i)$ and $\|s - u\| < \epsilon$. Hence by Proposition 3.2.9, we have $s_i \to s$, proving g is continuous.

□

Let Y be another locally compact Hausdorff space with a regular Borel measure ν. Let $\kappa : X \times Y \mapsto X$ be the surjection $(x, y) \mapsto x$. Then the Banach (semi-)bundle retraction $\underline{\mathcal{E}} = (\mathcal{E}, \rho)$ by κ is a bundle over $X \times Y$ whose bundle space $\mathcal{E}$ can be identified with $\mathcal{B} \times Y$. The bundle projection is given by $\rho : (\xi, y) \to (\theta(\xi), y)$. For each $x \in X$, $\underline{\mathcal{E}}_{\{x\}\times Y}$ is the trivial bundle with constant fibre $\mathcal{B}_x$. Therefore, for a given $h \in \mathcal{C}_0(\underline{\mathcal{E}})$ and for each x in X, the Bochner integral $\int_Y h(x, y) d\nu(y)$ exists and will belong to $\mathcal{B}_x$.

Lemma 3.2.11 *For each $h \in \mathcal{C}_0(\underline{\mathcal{E}})$ the map $\ell(x) = \int_Y h(x,y)d\nu(y)$ is a continuous cross-section of the Banach semi-bundle $\underline{\mathcal{B}}$.*

Proof: We use a similar argument to that of Fell[16], Lemma 2.3, p.27, where the result is shown to be true in the case of a Banach bundle.

Let $\epsilon > 0$ be given. We show that ℓ is continuous at a fixed point $x_0 \in X$. Let the compact support of h be $D \times E$ in $X \times Y$. Since h is continuous with compact support, $\{h(x_0, y) : y \in Y\}$ is norm-compact in $\mathcal{B}_{x_0}$. So there are finitely many continuous cross-sections $r_1, \dots, r_n$ of $\mathcal{B}$ such that for each $y \in Y$ we have

$$\|h(x_0, y) - r_i(x_0)\| < \epsilon$$

for some $i \in \{1, \dots, n\}$. Therefore, letting $k_i(x,y) = \|h(x,y) - r_i(x)\|$ for $x \in X$ and $y \in Y$, we see that we can subdivide E into disjoint Borel sets $E_1, \dots, E_n$ such that

$$k_i(x_0, y) \leq \epsilon < 2\epsilon$$

for all $y \in \overline{E_i}$. Now by upper semi-continuity of the norm, for all $y \in \overline{E_i}$ there exist neighborhoods $V_y = W_y \times U_y$ of (x_0, y) such that $k_i(z, y) < \epsilon$ for $(z, y) \in V_y$. Since $\overline{E_i}$'s are compact there exist neighborhoods $U_{y_1}, \dots, U_{y_t}$ covering E_i. Now the set $\cap_i W_{y_i} \times \cup_i U_{y_i}$ is an open set containing $x_0 \times E$, and on this set $k_i(z, y) < \epsilon$.
Let $\cap_i W_{y_i} = W$. For $x \in W$,

$$\begin{aligned} \| \int_Y h(x,y)d\nu(y) - \Sigma_i \nu(E_i) r_i(x) \| &\leq \Sigma_i \| \int_{E_i} h(x,y)d\nu(y) - \int_{E_i} r_i(x)d\nu(y) \| \\ &\leq \Sigma_i \int_{E_i} k_i(x,y)d\nu(y) \\ &< \Sigma_i \nu(E_i)\epsilon. \end{aligned}$$

Let $f(x) = \Sigma_i \nu(E_i) r_i(x)$ for all x. Then f is a continuous cross-section, and

$$\|\ell(x) - f(x)\| \leq \nu(E)\epsilon$$

for all $x \in W$. Hence by Corollary 3.2.10, ℓ is continuous.

□

3.3 Adjoints of representations, p-induced representations and intertwining operators

3.3.1 Adjoints of representations

Let π be a representation of a group G on a Banach space $\mathcal{H}(\pi)$. In Section 2.3, we defined the map $\pi^* : G \mapsto U((\mathcal{H}(\pi))^*)$ by letting $\pi^*(x) = (\pi(x^{-1}))^*$, and claimed that π^* is a representation of G on the Banach space $\mathcal{H}(\pi^*) = (\mathcal{H}(\pi))^*$, when $\mathcal{H}(\pi)$ is reflexive.

In order to prove that π^* satisfies the required properties of a representation, we need to prove that the function $h \mapsto \pi^*(h)u^*$ for $u^* \in (\mathcal{H}(\pi))^*$ is continuous in the norm topology. We use the following result in this regard:

Proposition 3.3.1 *$\mathcal{H}(\pi)$ is an essential $L_1(G)$-module. If $\mathcal{H}(\pi)$ is reflexive, then $\mathcal{H}(\pi)^*$ is also an essential $L_1(G)$-module.*

Proof: Let $\aleph$ be a directed set of symmetric neighborhoods N converging to the identity of G. Let $\{e_N\}_{N\in\aleph}$ be a bounded approximate identity in $L_1(G)$ constructed in such a way that $\int_N e_N(x)d\nu_G(x) = 1$ and $e_N(x) = 0$ for $x \notin N$. For a given $u \in \mathcal{H}(\pi)$, we want to show that $e_N u \to u$ (cf.Proposition 2.4.5 (b)). Now

$$\begin{aligned} \|e_N u - u\| &= \|\int_N \pi(x)ue_N(x)d\nu_G(x) - u\| \\ &= \|\int_N (\pi(x)u - u)e_N(x)d\nu_G(x)\|. \end{aligned}$$

For a given $\epsilon > 0$, there exists a neighborhood N_ϵ such that $\|(\pi(x)u - u)\| < \epsilon$ for all $x \in N_\epsilon$. Since $\int_N e_N(x)d\nu_G(x) = 1$, we have $\|e_N u - u\| < \epsilon$ for all $N \subset N_\epsilon$. Hence $\mathcal{H}(\pi)$ is an essential $L_1(G)$-module.

Now suppose $\mathcal{H}(\pi)$ is reflexive. We want to show that the closure of $L_1(G).\mathcal{H}(\pi)^*$ is the same as $\mathcal{H}(\pi)^*$. But since $\mathcal{H}(\pi)$ is reflexive, this is the same as considering the weak closure of $L_1(G).\mathcal{H}(\pi)^*$. Let $u \in \mathcal{H}(\pi)$ and $u^* \in \mathcal{H}(\pi)^*$. Then

$$\|\langle u, (e_N u^* - u^*)\rangle\| = \|\langle e_N u - u, u^*\rangle\|,$$

since the neighborhoods $N \in \aleph$ are symmetric. The right-hand side of the above equality can be made as small as we wish since $\mathcal{H}(\pi)$ is an essential $L_1(G)$-module.

Hence $\mathcal{H}(\pi)^*$ is also an essential $L_1(G)$-module.

□

Proposition 3.3.2 *Suppose $\mathcal{H}(\pi)$ is reflexive. Then π^* is a representation of H on the Banach space $\mathcal{H}(\pi^*) = (\mathcal{H}(\pi))^*$.*

Proof: The fact that $h \mapsto \pi^*(h)u^*$ for $u^* \in (\mathcal{H}(\pi))^*$ is continuous in the norm topology is evident from the Remark(1) in Sec.2.4 and the Proposition 3.3.1. To complete the proof, note that for h_1, h_2 and h in H,

$$\pi^*(h_1h_2) = \pi((h_1h_2)^{-1})^* = \pi(h_2^{-1}h_1^{-1})^* = (\pi(h_1^{-1}))^*(\pi(h_2^{-1}))^* = \pi^*(h_1)\pi^*(h_2)$$

and

$$\pi^*(h)\pi^*(h^{-1}) = (\pi(h^{-1}))^*(\pi(h))^* = (\pi(h)\pi(h^{-1}))^* = I.$$

□

3.3.2 The p-induced representations of locally compact groups and $L_p(\pi)$ spaces

Let G be a locally compact group and let H be a closed subgroup of G. Suppose that π is a representation of H on a Banach space $\mathcal{H}(\pi)$. Let μ be any quasi-invariant measure, in the homogeneous space $X = G/H$ of right cosets, which belongs to a continuous ρ-function. For $1 \leq p < \infty$, let us denote by $L_p(\pi, \mu)$ the set of all functions f from G to a Banach Space $\mathcal{H}(\pi)$ such that

(1) $\langle f(x), v\rangle$ is a Borel function of x for all $v \in \mathcal{H}(\pi)^*$;

(2) f satisfies the covariance condition $f(hx) = \pi_h f(x)$ for all $h \in H$ and $x \in G$; and

(3) $\|f\|_p = \left(\int_{\frac{G}{H}} \|f(x)\|^p d\mu(z)\right)^{\frac{1}{p}} < \infty.$

Note that the integrand in the above integral is constant on each right coset Hx and hence defines a function on X. When functions equal almost everywhere are identified, $L_p(\pi, \mu)$ becomes a Banach space under the norm defined by (3)(for which we use the same symbol $L_p(\pi, \mu)$).

The following Lemmas 3.3.3, 3.3.4, 3.3.5, 3.3.6 and Proposition 3.3.7 describe a few important results regarding functions in $L_p(\pi)$ spaces with compact support.

Lemma 3.3.3 *Let $f : G \mapsto \mathcal{H}(\pi)$ be a continuous function with compact support. Then the function $\theta_f : G \mapsto \mathcal{H}(\pi)$ defined by*

$$\theta_f(x) := \int_H \pi(h^{-1})f(hx)d\nu_H(h)$$

is continuous and satisfies the covariance condition

$$\theta_f(hx) \quad = \quad \pi(h)\theta_f(x) \tag{3.23}$$

for all $x \in G$, $h \in H$. Moreover, $\|\theta_f(.)\|$ has compact support in $X = G/H$.

Proof:(cf.Gaal[19], Chapter VI, Sec.2, Lemma 2.) Let $p_H : G \to G/H$ be the natural map $x \to Hx$. If S is the support of f then $p_H S$ is the support of $\|\theta_f(.)\|$. Let us prove that θ_f is continuous. For $x_1, x_2 \in G$,

$$\begin{aligned}\| \theta_f(x_1) - \theta_f(x_2) \| &\leq \int_H \| \pi(h^{-1})(f(hx_1) - f(hx_2)) \| d\nu_H(h), \\ &\leq \int_H \| f(hx_1) - f(hx_2) \| d\nu_H(h).\end{aligned}$$

Since f is continuous and has compact support it is uniformly continuous. Therefore, for a given $\epsilon > 0$ there is a symmetric neighborhood N of the identity in G such that $\| f(x) - f(y) \| \leq \epsilon$ whenever $xy^{-1} \in N$. Now we fix x_1 and a compact symmetric neighborhood C of x_1. Then $f(hx) \neq 0$ for some x in C only if $h \in SC$. Hence if $x_2 \in C \cap Nx_1$ then

$$\| \theta_f(x_1) - \theta_f(x_2) \| \leq \epsilon\nu_H(SC),$$

where $\nu_H(SC)$ denotes the Haar measure of $H \cap SC$ in H. Thus θ_f is continuous at x_1.

Now for any $h_1 \in H$ and $x \in G$,

$$\theta_f(h_1x) = \int_H \pi(h^{-1})f(hh_1x)d\nu_H(h).$$

Changing variables $hh_1 \mapsto h$,

$$\begin{aligned}\theta_f(h_1x) &= \int_H \pi(h_1h^{-1})f(hx)d\nu_H(h), \\ &= \pi(h_1)\theta_f(x),\end{aligned}$$

which proves (3.23).

□

Let $\mathcal{M}_H(\pi)$ be the subspace of those continuous functions $\theta : G \mapsto \mathcal{H}(\pi)$ which satisfy the covariance condition and have compact support in $X = G/H$ (cf. Sec.2.5).

Lemma 3.3.4 *Every function in $\mathcal{M}_H(\pi)$ is of the form θ_f for a suitable $f \in C_0(G, \mathcal{H}(\pi))$.*

Proof:(cf.Gaal[19], Chapter VI, Sec.4, Proposition 3.) Let S be the compact support of $\|\theta(.)\|$ in X. Choose a continuous function $\phi : X \mapsto [0,1]$ with compact support such that $\phi(x) = 1$ for all $x \in S$. Then there exists a continuous function $\psi : G \mapsto \mathcal{R}$ with compact support such that

$$\phi(x) = \int_H \psi(hx)dh,$$

(see Gaal[19], theorem V.3.7). Now let $f = \psi\theta$. Then $f \in C_0(G, \mathcal{H}(\pi))$. For $x \in G$,

$$\begin{aligned}\theta_f(x) &= \int_H \pi(h^{-1})f(hx)d\nu_H(h), \\ &= \int_H \psi(hx)\pi(h^{-1})\theta(hx)d\nu_H(h), \\ &= \int_H \psi(hx)\theta(x)d\nu_H(h), \\ &= \theta(x)\phi(x).\end{aligned}$$

But $\phi(x) = 1$ if $\theta(x) \neq 0$, Hence $\theta_f(x) = \theta(x)$ for all $x \in G$.

□

We state the following well known result without proof, since it is appropriate to include it in this section.

We use $C_{01}(G, \mathcal{H}(\pi))$ to denote the set of those f which are of the form $f = \phi\xi$ where $\phi \in C_0(G)$ and $\xi \in \mathcal{H}(\pi)$.

Lemma 3.3.5 *Let $f \in C_0(G, \mathcal{H}(\pi))$ and S be the compact support of f. Then there is a compact set C which contains the support of f and for each $\epsilon > 0$ there is a function $g = \sum_{i=1}^n f_i$, with $f_i \in C_{01}(G, \mathcal{H}(\pi))$, such that C contains the support of g and $\| f - g \|_\infty < \epsilon$.*

Proof: See Gaal[19], Chapter VI, Sec.4, Lemma 4.

□

Lemma 3.3.6 *If $f \in C_0(G, \mathcal{H}(\pi))$ and the symmetric compact set S in G contains the support of f then*

$$\|\theta_f\|_p \ \leq \ \|f\|_\infty \nu_H(H \cap S^2)(\mu(p_H S))^{\frac{1}{p}}. \tag{3.24}$$

Proof: Using the definition of θ_f given in Lemma 3.3.3, we obtain

$$\begin{aligned} \|\theta_f(x)\| &= \| \int_H \pi(h^{-1}) f(hx) d\nu_H(h), \| \\ &\leq \int_H \|f(hx)\| d\nu_H(h), \\ &\leq \|f\|_\infty \nu_H(H \cap Sx^{-1}). \end{aligned}$$

Since $\theta_f(x) = 0$ outside the set SH, we only need to consider x of the form $x = hs$, with $h \in H$, and $s \in S$. For such x, $\theta_f(x) = \theta_f(s)$; and S being symmetric, this implies that

$$\|\theta_f(x)\| \leq \|f\|_\infty \nu_H(H \cap S^2)$$

for every $x \in G$. Now integrating the above inequality with respect to x, we obtain

$$\|\theta_f\|_p \leq \|f\|_\infty \nu_H(H \cap S^2)(\mu(p_H S))^{\frac{1}{p}},$$

as required.

□

Proposition 3.3.7 *If $f \in C_0(G, \mathcal{H}(\pi))$, then the map $x \mapsto {}_x f$ of G into $C_0(G, \mathcal{H}(\pi))$ is uniformly continuous from the right.*

Proof: Let $\epsilon > 0$ be given. By Lemma 3.3.5, there exists a function g of the form $g = \sum_{i=1}^{n} f_i$, with $f_i \in C_{01}(G, \mathcal{H}(\pi))$, and a compact set C such that C contains the support of g and $\| f - g \|_\infty < \epsilon/3$. Next, g being right uniformly continuous on G there is a symmetric neighborhood N of e such that

$$\|g - {}_xg\|_\infty < \epsilon/3,$$

for all $x \in N$. Now

$$\|f - {}_xf\|_\infty \leq \|f - g\|_\infty + \|g - {}_xg\|_\infty + \|{}_xg - {}_xf\|_\infty < \epsilon,$$

for $x \in N$. Thus if $ab^{-1} \in N$ then with $x = ab^{-1}$ we have

$$\|{}_af - {}_bf\|_\infty = \|f - {}_xf\|_\infty < \epsilon.$$

□

For each $x, y \in G$ and $f \in L_p(\pi, \mu)$, let us define a mapping ${}^\mu U_y^\pi$ on $L_p(\pi, \mu)$ by

$$({}^\mu U_y^\pi f)(x) := \lambda(x, y)^{\frac{1}{p}} f(xy), \tag{3.25}$$

where $\lambda(\cdot, y)$ is the Radon-Nikodym derivative of the measure μ_y with respect to the measure μ.

Theorem 3.3.8 *${}^\mu U^\pi$ is a representation of the group G on the Banach space $L_p(\pi, \mu)$.*

Proof: For $y \in G$ and $f \in L_p(\pi, \mu)$, we have

$$\begin{aligned} \| {}^\mu U_y^\pi f \|_p &= \left(\int_{\frac{G}{H}} \| {}^\mu U_y^\pi f(x) \|^p \, d\mu(z) \right)^{\frac{1}{p}}, \\ &= \left(\int_{\frac{G}{H}} \lambda(x, y) \| f(xy) \|^p \, d\mu(z) \right)^{\frac{1}{p}}, \\ &= \left(\int_{\frac{G}{H}} \| f(x) \|^p \, d\mu(z) \right)^{\frac{1}{p}} = \| f \|_p, \end{aligned}$$

so that ${}^\mu U_y^\pi$ is an isometry.

Furthermore, for all $x, y, t \in G$,

$$\begin{aligned} \left({}^\mu U_y^\pi ({}^\mu U_t^\pi (f)) \right)(x) &= \lambda(x, y)^{\frac{1}{p}} ({}^\mu U_t^\pi (f))(xy), \\ &= \lambda(x, y)^{\frac{1}{p}} \lambda(xy, t)^{\frac{1}{p}} f(xyt), \\ &= \lambda(x, yt)^{\frac{1}{p}} f(xyt), \text{ by Lemma 2.2.7(a) }, \\ &= \left({}^\mu U_{yt}^\pi (f) \right)(x). \end{aligned}$$

Also, ${}^{\mu}U_y^{\pi}f$ is a Borel function of y for each $f \in L_p(\pi,\mu)$. Thus, $y \mapsto {}^{\mu}U_y^{\pi}$ is a homomorphism of the group G into the group $U(L_p(\pi,\mu))$ of all isometries of $L_p(\pi,\mu)$ onto itself.

Let us show that $y \mapsto {}^{\mu}U_y^{\pi}(f)$ is continuous from G to $L_p(\pi,\mu)$ for each $f \in L_p(\pi,\mu)$. Given $\epsilon > 0$, we can choose a continuous function $g \in L_p(\pi,\mu)$ which vanishes outside a compact set such that $\|f-g\| \leq \epsilon$.

Now

$$\|{}^{\mu}U_y^{\pi}g - g\|_p = \left(\int_{\frac{G}{H}} \|{}^{\mu}U_y^{\pi}g(x) - g(x)\|^p d\mu(z)\right)^{\frac{1}{p}},$$

$$= \left(\int_{\frac{G}{H}} \|\lambda(x,y)^{\frac{1}{p}}g(xy) - g(x)\|^p d\mu(z)\right)^{\frac{1}{p}},$$

$$= \left(\int_{\frac{G}{H}} \|\lambda(x,y)^{\frac{1}{p}}(g(xy)-g(x)) + (g(x))\left(\lambda(x,y)^{\frac{1}{p}} - \lambda(x,e)^{\frac{1}{p}}\right)\|^p d\mu(z)\right)^{\frac{1}{p}},$$

$$= \left(\int_{\frac{G}{H}} \lambda(x,y)\|g(xy)-g(x)\|^p d\mu(z)\right)^{\frac{1}{p}} + \left(\int_{\frac{G}{H}} \|g(x)\|^p|\lambda(x,y)^{\frac{1}{p}} - \lambda(x,e)^{\frac{1}{p}}|^p d\mu(z)\right)^{\frac{1}{p}},$$

by Minkowski's inequality (Hewitt and Ross[26], p.138).

Since $\lambda(x,y) = \lambda(e,xy)/\lambda(e,x)$ is bounded on compact sets, it is clear that we can find a neighbourhood N_1 of e such that

$$\left(\int_{\frac{G}{H}} \lambda(x,y)\|(g(xy)-g(x))\|^p d\mu(z)\right)^{\frac{1}{p}} \leq \epsilon/2,$$

for $y \in N_1$. Also, since λ is continuous, for a fixed x_i, there exists a product neighborhood $N(x_i,e) \subset (G/H)\times G$ such that if $(x,y) \in N(x_i,e) = N(x_i)\times N_{x_i}(e)$ (say) we have

$$\|\lambda(x,y)^{\frac{1}{p}} - \lambda(x,e)^{\frac{1}{p}}\|^p \leq \frac{\epsilon}{2\|g\|}.$$

Now $\{N(x_i) : x_i \in G/H\}$ covers the support of g; hence there exists a finite subcover $\{N(x_i) : i = 1,\ldots,n\}$. Let $N_2 = \cap_{i=1}^{n} N_{x_i}(e)$. Then, for $y \in N_2$ and $x \in$ supp (g),

$$\|\lambda(x,y)^{\frac{1}{p}} - \lambda(x,e)^{\frac{1}{p}}\|^p \leq \frac{\epsilon}{2\|g\|}.$$

Therefore, for $y \in N_1 \cap N_2$

$$\|{}^{\mu}U_y^{\pi}g - g\|_p \leq \epsilon.$$

Consequently, for $y \in N_1 \cap N_2$,

$$\begin{aligned}\|{}^{\mu}U_y^{\pi}f - f\|_p &= \|{}^{\mu}U_y^{\pi}(f-g) - (f-g) + ({}^{\mu}U_y^{\pi}g - g)\|_p,\\ &\leq \|{}^{\mu}U_y^{\pi}\|\epsilon + \epsilon + \epsilon \leq 3\epsilon.\end{aligned}$$

This proves that $\lim_{y\to e} \|{}^{\mu}U_y^{\pi} f - f\|_p = 0$ which implies that the mapping $y \mapsto {}^{\mu}U_y^{\pi}(f)$ is continuous from G to $L_p(\pi, \mu)$. Therefore the mapping $y \mapsto {}^{\mu} U_y^{\pi}$ is a representation ${}^{\mu}U^{\pi}$ of G.

□

Theorem 3.3.9 *Let μ and μ' be two quasi-invariant measures on X. Then there exists an isometry W from $L_p(\pi, \mu)$ onto $L_p(\pi, \mu')$ such that $W({}^{\mu}U_y^{\pi}) = ({}^{\mu'} U_y^{\pi})W$ for all $y \in G$. In other words, the two representations ${}^{\mu}U_y^{\pi}$ and ${}^{\mu'} U_y^{\pi}$ are equivalent.*

Proof: The proof follows the same argument as in Mackey[31], Theorem 2.1.

Let $p_H : G \mapsto X$ be the canonical mapping of G onto X and let ψ be the Radon-Nikodym derivative of μ with respect to μ'. Let W be the mapping $W : L_p(\pi, \mu) \mapsto L_p(\pi, \mu')$ defined by

$$W(f) := (\psi \circ p_H)^{\frac{1}{p}} f \text{ for } f \in L_p(\pi, \mu).$$

Now

$$\begin{aligned} \|(\psi \circ p_H)^{\frac{1}{p}}(f)\|_p &= \left(\int_{\frac{G}{H}} (\psi(p_H(x))\|f(x)\|^p d\mu'\right)^{\frac{1}{p}}, \\ &= \left(\int_{\frac{G}{H}} \|f(x)\|^p d\mu\right)^{\frac{1}{p}}. \end{aligned}$$

Also, every $g \in L_p(\pi, \mu')$ is of the form $(\psi \circ p)^{\frac{1}{p}} f$ for some $f \in L_p(\pi, \mu)$. Hence W is an isometry from $L_p(\pi, \mu)$ to $L_p(\pi, \mu')$. For $x, y \in G$, let $\psi_y(x) = \psi(xy)$. Since

$$\mu_y(E) = \mu(E.y) = \int_{E.y} \psi d\mu' = \int_E \psi_y d\mu'_y(x),$$

where E is a Borel subset of X, we note that $d\mu_y / d\mu'_y = \psi_y$. Then, for $f \in L_p(\pi, \mu)$, $x \in G$ and for a fixed $y \in G$,

$$\begin{aligned} (W{}^{\mu}U_y^{\pi} f)(x) &= (\psi(p_H(x)))^{\frac{1}{p}}({}^{\mu}U_y^{\pi} f)(x), \\ &= (\psi(p_H(x)))^{\frac{1}{p}}\lambda(x, y)^{\frac{1}{p}} f(xy), \end{aligned} \tag{3.26}$$

by (3.25). On the other hand,

$$\begin{aligned} ({}^{\mu'} U_y^{\pi} W f)(x) &= \lambda'(x, y)^{\frac{1}{p}}(Wf)(xy), \\ &= \lambda'(x, y)^{\frac{1}{p}}(\psi(p(xy)))^{\frac{1}{p}}(f)(xy). \end{aligned} \tag{3.27}$$

In order to have equality in equations (3.26) and (3.27), we must have

$$(\psi(p(x)))^{\frac{1}{p}}\lambda(x, y)^{\frac{1}{p}} = \lambda'(x, y)^{\frac{1}{p}}(\psi(p(xy)))^{\frac{1}{p}}, \text{ for all } x \in G \text{ and a fixed } y \in G,$$

which is the same as

$$\frac{d\mu}{d\mu'} \cdot \frac{d\mu_y}{d\mu} = \frac{d\mu'_y}{d\mu'} \cdot \frac{d\mu_y}{d\mu'_y}, \text{ for a fixed } y \in G.$$

But the last equality is certainly true (see Halmos[24], Sec.32); and since $y \in G$ is arbitrary, we have the desired equality for all $y \in G$. Thus, ${}^{\mu'}U^{\pi}_y$ and ${}^{\mu}U^{\pi}_y$ are equivalent.

□

The equivalence class of ${}^{\mu}U^{\pi}$ (denoted by U^{π}) is called the **representation of G induced by the representation π of H**. The corresponding Banach space of (equivalence classes) of functions is denoted by $L_p(\pi)$. (The most appropriate notation for the p-induced representation (induced by π) would be U^{π}_p; but for simplicity of notation we use U^{π} unless the former is necessary to avoid confusion.)

Throughout our work, we assume that the Banach space $\mathcal{H}(\pi)$ of a representation π of a subgroup H of a group G stays within the class of spaces satisfying the Radon-Nikodym property (cf.Sec.2.5).

Assume now that the Banach space $\mathcal{H}(\pi)$ is reflexive so that π^* is a representation of H. Let us consider the Banach space $L_{p'}(\pi^*)$ and the induced representation $U^{\pi^*}_{p'}$ of G. The dual pairing between $L_p(\pi)$ and $L_{p'}(\pi^*)$ is given by

$$\langle f, g\rangle = \int_{\frac{G}{H}} \langle f(x), g(x)\rangle d\mu(x), \text{ for } f \in L_p(\pi) \text{ and } g \in L_{p'}(\pi^*).$$

The above integral is well defined since, for any $h \in H$ and $x \in G$,

$$\begin{aligned} \langle f(hx), g(hx)\rangle &= \langle \pi(h)f(x), \pi^*(h)g(x)\rangle, \\ &= \langle \pi(h)f(x), (\pi(h^{-1}))^* g(x)\rangle, \\ &= \langle f(x), g(x)\rangle. \end{aligned}$$

Also, for any $y \in G$,

$$\begin{aligned} \langle U^{\pi}_p(y)f, U^{\pi^*}_{p'}(y)g\rangle &= \int_{\frac{G}{H}} \langle \lambda(x,y)^{\frac{1}{p}} f(xy), \lambda(x,y)^{\frac{1}{p'}} g(xy)\rangle d\mu(x) \\ &= \int_{\frac{G}{H}} \lambda(x,y)\langle f(xy), g(xy)\rangle d\mu(x) \\ &= \langle f, g\rangle, \end{aligned}$$

the last equality of which was obtained by changing variables $x \mapsto xy$. This implies that

$$U^{\pi^*}_{p'}(y) = (U^{\pi}_p(y^{-1}))^* \;=\; (U^{\pi}_p)^*(y), \text{ for all } y \in G. \tag{3.28}$$

Next we turn to a result on the space $L_p(\pi)$; we show that $L_p(\pi)$ is an $L_1(G)$-module for $1 \le p < \infty$.

Lemma 3.3.10 *Let $1 \leq p < \infty$. For $g \in L_p(\pi)$ and $f \in L_1(G)$, the convolution $g * f$ defined by*

$$(g * f)(x) := \int_G (\lambda_H(x, y^{-1}))^{\frac{1}{p}} g(xy^{-1}) f(y) d\nu_G(y)$$

belongs to $L_p(\pi)$.

Proof: For $g \in L_p(\pi)$ and $f \in L_1(G)$, we have

$$\begin{aligned}
\|g * f\|_p^p &= \int_{\frac{G}{H}} \|(g * f)(x)\|^p d\mu_H(x), \\
&\leq \int_{\frac{G}{H}} \Big(\int_G \|(\lambda_H(x, y^{-1})^{\frac{1}{p}} g(xy^{-1})\| |f(y)| d\nu_G(y) \Big)^p d\mu_H(x), \\
&= \int_{\frac{G}{H}} \Big(\int_G (\|(\lambda_H(x, y^{-1})^{\frac{1}{p}} g(xy^{-1})\|^p |f(y)|)^{\frac{1}{p}} |f(y)|^{1-\frac{1}{p}} d\nu_G(y) \Big)^p d\mu_H(x).
\end{aligned}$$

Using Hölder's inequality (Hewitt and Ross[26], p.137),

$$\begin{aligned}
\|g * f\|_p^p &\leq \int_{\frac{G}{H}} \Big(\int_G (\|(\lambda_H(x, y^{-1})^{\frac{1}{p}} g(xy^{-1})\|^p |f(y)| d\nu_G(y))^{\frac{1}{p}} \|f\|_1^{\frac{1}{p'}} \Big)^p d\mu_H(x), \\
&= \|f\|_1^{p-1} \int_{\frac{G}{H}} \int_G \| (\lambda_H(x, y^{-1})^{\frac{1}{p}} g(xy^{-1})\|^p |f(y)| d\nu_G(y) d\mu_H(x), \\
&= \|f\|_1^{p-1} \int_G \Big(\int_{\frac{G}{H}} \| (\lambda_H(x, y^{-1})^{\frac{1}{p}} g(xy^{-1}) \|^p d\mu_H(x) \Big) |f(y)| d\nu_G(y), \\
&= \|f\|_1^p \|g\|_p^p, \qquad (3.29) \\
&< \infty.
\end{aligned}$$

To complete the proof we note that, for $h \in H$ and $x \in G$,

$$\begin{aligned}
(g * f)(hx) &= \int_G \lambda_H(hx, y^{-1})^{\frac{1}{p}} g(hxy^{-1}) f(y) d\nu_G(y), \\
&= \pi(h) \int_G \lambda_H(x, y^{-1})^{\frac{1}{p}} g(xy^{-1}) f(y) d\nu_G(y), \\
&= \pi(h)(g * f)(x),
\end{aligned}$$

which proves that $g * f$ satisfies the covariance condition.

□

Lemma 3.3.11 *Under the convolution $*$ defined in Lemma 3.3.10, we have*

$$g * (h * f) = (g * h) * f$$

for all $g \in L_p(\pi)$ and $h, f \in L_1(G)$.

Proof: For $g \in L_p(\pi)$, $h, f \in L_1(G)$ and $x \in G$, we have

$$\begin{aligned}
&((g * h) * f)(x) \\
&= \int_G \lambda_H(x, y^{-1})^{\frac{1}{p}} (g * h)(xy^{-1}) f(y) d\nu_G(y), \\
&= \int_G \lambda_H(x, y^{-1})^{\frac{1}{p}} \int_G \lambda_H(xy^{-1}, z^{-1})^{\frac{1}{p}} g(xy^{-1}z^{-1}) h(z) f(y) d\nu_G(z) d\nu_G(y), \\
&= \int_G \int_G \lambda_H(x, y^{-1}z^{-1})^{\frac{1}{p}} g(xy^{-1}z^{-1}) h(z) f(y) d\nu_G(z) d\nu_G(y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(g * (h * f))(x) &= \int_G \lambda(x, z^{-1})^{\frac{1}{p}} g(xz^{-1})(h * f)(z) d\nu_G(z), \\
&= \int_G \int_G \lambda(x, z^{-1})^{\frac{1}{p}} g(xz^{-1}) \lambda(z, y^{-1}) h(zy^{-1}) f(y) d\nu_G(y) d\nu_G(z).
\end{aligned}$$

Changing variables $zy^{-1} \mapsto z$,

$$\begin{aligned}
(g * (h * f))(x) &= \int_G \int_G (\lambda(x, y^{-1}z^{-1}))^{\frac{1}{p}} g(xy^{-1}z^{-1}) h(z) f(y) d\nu_G(z) d\nu_G(y), \\
&= ((g * h) * f)(x),
\end{aligned}$$

for any $x \in G$. Hence the result.

□

Proposition 3.3.12 *$L_p(\pi)$ is an $L_1(G)$-module.*

Proof: For any fixed $f \in L_1(G)$, the map $T_f : L_p(\pi) \mapsto L_p(\pi)$ given by $T_f(g) = g * f$ is clearly linear, and is bounded since

$$\|T_f(g)\| = \|g * f\| \leq \|g\|_p \|f\|_1, \tag{3.30}$$

by (3.29). Also, (3.30) implies that the bilinear map $L_1(G) \times L_p(\pi) \mapsto L_p(\pi)$ defined by the convolution is continuous. Hence the result is an immediate consequence of Lemmas 3.3.10 and 3.3.11.

□

Proposition 3.3.13

$$Hom_G(L_p(\pi), L_q(\gamma)) = Int_G(U_p^{\pi}, U_q^{\gamma}). \tag{3.31}$$

Proof: Let T be any bounded linear operator from $L_p(\pi)$ to $L_q(\gamma)$ and T^* be its adjoint operator. For any $g \in L_p(\pi), f \in L_1(G)$ and $k \in L_{q'}(\gamma^*)$,

$$\begin{aligned}
\langle T(g * f), k \rangle &= \langle g * f, T^* k \rangle, \\
&= \int_{\frac{G}{H}} \langle \int_G (\lambda(x, y^{-1}))^{\frac{1}{p}} g(xy^{-1}) f(y) d\nu_G(y), T^* k(x) \rangle d\mu_H(x), \\
&= \int_G f(y) \int_{\frac{G}{H}} \langle (U_p^{\pi}(y^{-1}) g)(x), T^* k(x) \rangle d\mu_H(x) d\nu_G(y), \\
&= \int_G f(y) \langle U_p^{\pi}(y^{-1}) g, T^* k \rangle d\nu_G(y).
\end{aligned}$$

Hence,

$$\langle T(g * f), k\rangle = \int_G f(y)\langle TU_p^\pi(y^{-1})g, k\rangle d\nu_G(y). \tag{3.32}$$

On the other hand,

$$\begin{aligned}\langle T(g) * f, k\rangle &= \int_{\frac{G}{K}} \langle (T(g) * f)(x), k(x)\rangle d\mu_K(x), \\ &= \int_{\frac{G}{K}} \langle \int_G (\lambda(x, y^{-1}))^{\frac{1}{q'}} T(g)(xy^{-1})f(y), k(x)\rangle d\mu_K(x) d\nu_G(y), \\ &= \int_G f(y) \int_{\frac{G}{K}} \langle (U_q^\gamma(y^{-1})Tg)(x), k(x)\rangle d\mu_K(x) d\nu_G(y).\end{aligned}$$

Therefore,

$$\langle T(g) * f, k\rangle = \int_G f(y)\langle U_q^\gamma(y^{-1})Tg, k\rangle d\nu_G(y). \tag{3.33}$$

If $T \in Hom_G(L_p(\pi), L_q(\gamma))$, we see, by (3.32) and (3.33), that

$$TU_p^\pi(y)g = U_q^\gamma(y)Tg, \tag{3.34}$$

for almost all $y \in G$. By continuity, (3.34) is true for all $y \in G$. Hence $T \in Int_G(U_p^\pi, U_q^\gamma)$. Conversely, $T \in Int_G(U_p^\pi, U_q^\gamma)$ implies $T \in Hom_G(L_p(\pi), L_q(\gamma))$, by (3.32) and (3.33). Hence, (3.31) follows.

□

Theorem 3.3.14

$$Int_G(U_p^\pi, U_q^\gamma) \cong (L_p(\pi) \otimes_G^\alpha L_{q'}(\gamma^*))^*.$$

Proof: We will give a direct proof of this result even though it immediately follows from (2.8) and Proposition 3.3.13 above.

Under the isometric isomorphism $(L_p(\pi)\otimes^\alpha L_{q'}(\gamma^*))^* = \mathcal{L}_\alpha(L_p(\pi), L_q(\gamma))$ (see (2.4), Sec.2.6), an element $\Theta \in (L_p(\pi) \otimes^\alpha L_{q'}(\gamma^*))^*$ defines an element $A_\Theta \in \mathcal{L}_\alpha(L_p(\pi), L_q(\gamma))$ by

$$\langle A_\Theta(f), g\rangle = \Theta(f \otimes g), \quad \text{for } f \in L_p(\pi) \text{ and } g \in L_{q'}(\gamma^*).$$

We can consider Θ as an operator on $(L_p(\pi)\otimes_G^\alpha L_{q'}(\gamma^*))$ if and only if it vanishes on

$$L = \{\langle U_p^\pi(s)f \otimes g - f \otimes (U_q^\gamma)^*(s)g\rangle : \quad s \in G\}. \tag{3.35}$$

That is, if and only if

$$\Theta(U_p^\pi(s)f \otimes g) = \Theta(f \otimes (U_q^\gamma)^*(s)g),$$

for $f \in L_p(\pi)$, $g \in L_{q'}(\gamma^*)$ and for all $s \in G$. This implies that the corresponding A_Θ has the property,

$$\langle A_\Theta U_p^\pi(s)(f), g\rangle = \langle A_\Theta(f), (U_q^\gamma)^*(s)g\rangle \text{ for all } s \in G.$$

Therefore, we have for all $f \in L_p(\pi), g \in L_{q'}(\gamma^*)$ and $s \in G$,

$$\begin{aligned}\Theta \in (L_p(\pi) \otimes_G^\alpha L_{q'}(\gamma^*))^* &\iff \langle A_\Theta U_p^\pi(s)(f), g\rangle = \langle (U_q^\gamma)(s)A_\Theta(f), g\rangle \\ &\iff A_\Theta U_p^\pi(s) = U_q^\gamma(s)A_\Theta,\end{aligned}$$

as required.

□

Moore's version of the Frobenius Reciprocity Theorem for the induced representation U_1^π is restricted to the situation where the corresponding homogeneous space possesses a right invariant measure (see Moore[34]). Here, we prove the result in a more general setting.

Theorem 3.3.15 *Let G be a locally compact group and let K be a closed subgroup of G. Let γ and π be representations of K and G, on the separable Banach spaces $\mathcal{H}(\gamma)$ and $\mathcal{H}(\pi)$, respectively. If $(\pi)_K$ denotes the restriction of π to K then,*

$$Int_K(\gamma, (\pi)_K) \cong Int_G(U_1^\gamma, \pi).$$

Proof: Let $B \in Int_K(\gamma, (\pi)_K)$. We define a mapping $\psi : Int_K(\gamma, (\pi)_K) \mapsto Int_G(U_1^\gamma, \pi)$ by

$$\psi(B)f = \int_{\frac{G}{K}} \pi_s^{-1} B f(s) d\mu_K(s) \text{ for } f \in L_1(\gamma).$$

It is easy to observe that the integral is well defined since the integrand can be regarded as a function on the coset space, and $\psi(B)$ is bounded with norm $\leq \|B\|$. Now to show that $\psi(B) \in Int_G(U_1^\gamma, \pi)$, consider

$$\pi_t\psi(B)f = \int_{\frac{G}{K}} \pi_{st^{-1}}^{-1} B f(s) d\mu_K(s) \text{ for } t \in G.$$

Changing variables $s \mapsto st$, we get

$$\int_{\frac{G}{K}} \pi_s^{-1} B\lambda_K(s,t) f(st) d\mu_K(s) = \psi(B)(U_1^\gamma(t)f),$$

for all $f \in L_1(\gamma)$, proving that $\psi(B) \in Int_G(U_1^\gamma, \pi)$.

Now we need to prove that ψ is a bijection. Fix a regular Borel section S of K in G, and observe that $G/K \simeq S$. Let $f \in L_1(\gamma)$ be continuous with compact support. By Lemma 3.3.3 there exists a corresponding function $g \in L_1(S, \mathcal{H}(\gamma))$ with compact support, and by Lemma 3.3.4, every continuous function in $L_1(S, \mathcal{H}(\gamma))$ with compact support arises this way. (We view μ_K as a measure on S.) Then the operator $\psi(B)$ is represented as

$$\psi(B)f = \int_S \pi_s^{-1} Bf(s) d\mu_K(s),$$

where $f \in L_1(S, \mu_K, \mathcal{H}(\gamma))$ is continuous with compact support. Since the set of continuous functions with compact support is dense in $L_1(S, \mu_K, \mathcal{H}(\gamma))$, we have

$$\psi(B)f = \int_S \pi_s^{-1} Bf(s) d\mu_K(s),$$

for any $f \in L_1(S, \mu_K, \mathcal{H}(\gamma))$. Suppose that $\psi(B) = 0$ for some $B \in Int_K(\gamma, (\pi)_K)$. Then for $u \in \mathcal{H}(\gamma)$ and $g \in C_0(S, \mu_K)$, $g(s).u \in \mathcal{M}_K(\gamma)$ and we have

$$\int_S g(s) \langle \pi_s^{-1} Bu, v \rangle d\mu(s) = 0 \text{ for all } v \in \mathcal{H}(\pi)^*.$$

Therefore we have $\langle \pi_s^{-1} Bu, v \rangle = 0$ almost everywhere on S. Let N be a null set on which $\langle \pi_s^{-1} Bu_i, v_i \rangle = 0$ for all i, j where $\{u_i\}$ and $\{v_i\}$ are countable dense sets in $\mathcal{H}(\gamma)$ and $\mathcal{H}(\pi)^*$ respectively. By continuity of $\langle \pi_s^{-1} Bu, v \rangle$ in u and v we have that $\langle \pi_s^{-1} Bu, v \rangle = 0$ if $s \notin N$ for all u and v. Now $\langle \pi_{k^{-1}s}^{-1} Bu, v \rangle = \langle \pi_s^{-1} \pi_k Bu, v \rangle = \langle \pi_s^{-1} B \gamma_k u, v \rangle = 0$ for $k \in K$ and $s \notin N$. The complement $K.N$ of $K(S - N)$ is a Haar null set in G, and therefore $\langle \pi_s^{-1} Bu, v \rangle = 0$ almost everywhere in G. Now by continuity, it is zero everywhere on G. Therefore $B = 0$ and ψ is injective.

Let $T : L_1(S, \mathcal{H}(\gamma)) \mapsto \mathcal{H}(\pi)$ be a bounded linear operator. For $u \in \mathcal{H}(\gamma)$, we define a linear map $T_u : L_1(S, \mu_K) \mapsto \mathcal{H}(\pi)$ by $T_u(g) = T(g.u)$ where $g \in L_1(S, \mu_K)$. Since $\|T_u(g)\| = \|T\| \|g.u\| \leq \|T\| \|u\| \|g\|$, we see that T_u is bounded and its norm $\|T_u\| \leq \|T\| \|u\|$. By Dunford and Schwartz[10] Theorem 10, p.507, there exists an essentially unique Borel function $\chi_u(s)$ on S to $\mathcal{H}(\pi)$ so that

$$T_u(g) = \int_S g(s) \chi_u(s) d\mu_K(s),$$

with ess sup $\|\chi_u(s)\| \leq \|T\| \|u\|$. Let A be a countable dense subset of $\mathcal{H}(\pi)$ and a vector space over the field $\mathcal{Q}(\sqrt{-1})$ of complex numbers of the form $a + bi$ with a and b rational. We can find a null set N so that each of the countable number of relations $\chi_{au}(s) + \chi_{bu}(s) = \chi_{au+bu}(s)$ and $\|\chi_u(s)\| \leq \|u\| \|T\|$ for $u, v \in A$ and $a, b \in \mathcal{Q}(\sqrt{-1})$ hold simultaneously for $s \notin N$. Then the map $u \mapsto \chi_u(s)$ uniquely extends to a bounded linear map $\chi(s)$ from $\mathcal{H}(\gamma)$ to $\mathcal{H}(\pi)$ with $\|K(s)\| \leq \|T\|$ for $s \notin N$. If we define $\chi(s) = 0$ for $s \in N$, $\chi(s)u = \chi_u(s)$ almost everywhere on S and for each u, so that

$$T_u(g) = \int_S \chi(s) u d\mu_K(s).$$

Since finite sums of the functions of the form $g.u$ for $u \in \mathcal{H}(\gamma)$ and $g \in L_1(S, \mu_K)$ form a dense subspace of $L_1(S, \mu_K, \mathcal{H}(\gamma))$ we see that for $g \in L_1(S, \mu_K, \mathcal{H}(\gamma))$,

$$T(g) = \int_S \chi(s) g(s) d\mu_K(s), \tag{3.36}$$

where $\chi(s)$ is essentially unique and ess sup $\|\chi(s)\| \leq \|T\|$.

Let $T \in Int_G(U_1^\gamma, \pi)$. We want to show that $T = \psi(B)$ for some $B \in Int_K(\gamma, (\pi)_K)$ with $\|B\| = \|T\|$. Let Φ be the mapping from the set of continuous functions with compact support in $L_1(S, \mathcal{H}(\gamma))$ to that in $L_1(\gamma)$ and let $f = \Phi g$. Clearly, $g(s) = f(s)$ for $s \in S$. Since the set of continuous functions with compact support is dense in $L_1(S, \mathcal{H}(\gamma))$, the mapping $T\Phi$ can be extended to an operator from $L_1(S, \mathcal{H}(\gamma))$ to $\mathcal{H}(\pi)$, and hence has a representation of the form (3.36). Consequently, there exists a mapping χ from S to the set of bounded linear maps from $\mathcal{H}(\gamma)$ to $\mathcal{H}(\pi)$, so that for any continuous function $f \in L_1(\gamma)$ with compact support, we have

$$Tf = T\Phi(g) \quad = \quad \int_S \chi(s) g(s) d\mu_K(s). \tag{3.37}$$

Now using the Borel isomorphism $G \simeq K \times S$, any $t \in G$ can be written as $t = k(e,t)\ell(e,t)$ where $k(e,t) \in K$ and $\ell(e,t) \in S$. Both k and ℓ are Borel functions on $S \times G$. Define a function κ on G by $\kappa(t) := \chi(\ell(e,t))\gamma((k(e,t))^{-1})$. Using the fact that $f(s) = g(s)$ for $s \in S$, we can rewrite (3.37) as

$$Tf = \int_{\frac{G}{K}} \kappa(t) f(t) d\mu_K(t),$$

and the integral is well defined since for any $k_1 \in K$ and $t \in G/K$,

$$\begin{aligned}
\kappa(k_1 t) f(k_1 t) &= \chi(\ell(e,t))\gamma((k_1 k(e,t))^{-1})\gamma(k_1) f(t) \\
&= \chi(\ell(e,t))\gamma((k(e,t))^{-1})\gamma(k_1{}^{-1})\gamma(k_1) f(t) \\
&= \kappa(t) f(t).
\end{aligned}$$

Since the set of continuous functions with compact support is dense in $L_1(\gamma)$ we have

$$Tf = \int_{\frac{G}{K}} \kappa(t) f(t) d\mu_K(t),$$

for any $f \in L_1(\gamma)$. Let us write $\pi(t)\kappa(t) = B(t)$. Then, for any $y \in G$,

$$\pi(y)(Tf) = \int_{\frac{G}{K}} \pi^{-1}(ty^{-1}) B(t) f(t) d\mu_K(t).$$

On the other hand,

$$\begin{aligned}
T(U_1^\gamma(y) f) &= \int_{\frac{G}{K}} \pi^{-1}(t) B(t) \lambda(t,y) f(ty) d\mu_K(t), \\
&= \int_{\frac{G}{K}} \pi^{-1}(ty^{-1}) B(ty^{-1}) f(t) d\mu_K(t),
\end{aligned}$$

on changing variables $t \mapsto ty^{-1}$. Consequently, we have $B(ty^{-1}) = B(t)$ for almost all $t \in G$ and all $y \in G$. In particular, for some t_0,

$$B(t_0 y^{-1}) = B(t_0),$$

for all $y \in G$, which implies that $B(t)$ is equal to a constant B for almost all $t \in G$. Also,

$$\begin{aligned}
(Tf) &= \int_{\frac{G}{K}} \pi^{-1}(t)Bf(t)d\mu_K(t), \\
&= \int_{\frac{G}{K}} \pi^{-1}(kt)Bf(kt)d\mu_K(t), \\
&= \int_{\frac{G}{K}} \pi^{-1}(t)(\pi^{-1}(k)B\gamma(k))f(t)d\mu_K(t),
\end{aligned}$$

implying that

$$\pi^{-1}(k)B\gamma(k) = B \text{ for all } k \in K.$$

Hence $B \in Int_K(\gamma, (\pi)_K)$ so that ψ is surjective. Finally,

$$\|\psi(B)\| = \|T\| = \|C\| \geq \text{ess sup}\,\|\chi(s)\| = \text{ess sup}\,\|B(s)\| = \|B\|,$$

which, together with $\|\psi(B)\| \leq \|B\|$, implies that ψ is an isometry, which establishes the theorem.

□

Chapter 4

Projective tensor products and A_p^q spaces

The aim of this chapter is to extend the notion of A_p^q space from its historical context and to recognise such spaces as preduals of spaces of intertwining operators. Considering the theory already established along these lines, the most closely related results to our work are those of Rieffel[36] and Figá-Talamanca[17]. In both cases, the authors dealt with the intertwining operators of regular representation. The situation in [17] is even more special in that the group G is assumed to be Abelian. Here, we seek the relationship between the two spaces in a more general setting.

Throughout this chapter we shall let G be a second countable locally compact group, with closed subgroups H and K. Thus, the corresponding homogeneous spaces are Hausdorff and second countable, which in turn implies that any Borel measure on such spaces is regular. In addition, we will assume that H and K are regularly related (Sec.2.2, p.9). μ_H and μ_K will denote fixed quasi-invariant measures on G/H and G/K, respectively. We choose a family of quasi-invariant measures $\{\mu_{x,y} : x \in G/H, y \in G/K\}$, where $\mu_{x,y}$ is a measure on $G/(H^x \cap K^y)$, in such a manner that for a function f defined and integrable on $(G/H) \times (G/K)$, we have

$$\int_{G/H} \int_{G/K} f(x,y) d\mu_H(x) d\mu_K(y) = \int_{D(x,y)\in\Upsilon} \int_{t\in\frac{G}{H^x\cap K^y}} f(xt, yt) d\mu_{x,y}(t) d\mu_{H,K}(D),$$

by disintegration of measures discussed in Lemma 3.1.3. For a given $\mu_{x,y}$, $\rho_{H^x\cap K^y}$ and $\lambda_{H^x\cap K^y}$ will denote the corresponding ρ-function and the λ-function respectively. For any $x \in G$, the quasi-invariant measure μ_{H^x} on G/H^x will always considered to be $\mu_{H^x} = \mu_H \circ \phi_x$, where $\phi_x : G/H^x \mapsto G/H$ is the homeomorphism given by $\phi_x(u) = xu$ (see Lemma 3.1.2). By ρ_{H^x} we mean the corresponding ρ-function of the above μ_{H^x}.

π and γ will denote representations of H and K on Banach spaces $\mathcal{H}(\pi)$ and

$\mathcal{H}(\gamma)$, respectively.

Section 4.1 discusses the construction of the convolution formula when the corresponding tensor product spaces are endowed with the greatest cross-norm. The proofs of the existence and the finiteness of the integral mainly depend on the properties of λ-functions (see Sections 2.2 and 3.1) and the most important identity (3.9) among such functions:

$$\lambda_H(xts^{-1}, s)\lambda_K(yts^{-1}, s)\lambda_{H^x \cap K^y}(t, s^{-1}) \;=\; 1.$$

(See Chapter 3 for details).

In Sec.4.2 we discover the fact that the range space of the convolutions has a Banach (semi-)bundle structure. Most of the terminology and the results in this section are dependent on the knowledge of Banach (semi-)bundles as discussed in Sec.3.2.

The definition of the space A_p^q is given in Sec.4.3 and some of its properties are then discussed.

We consider the relationship between the A_p^q spaces and the intertwining operators in Sec.4.4. The main result in this section states that the A_p^q spaces are preduals of the corresponding spaces of intertwining operators if and only if the intertwining operators can be approximated (in the ultraweak*- operator topology) by integral operators. Here, also, we see that the simplifications of the formulas are possible because of the identity (3.9) among λ-functions.

4.1 Construction of the convolution formula

Recall (Sec.2.7) that the space A_p^q in the classical case consists of convolutions of complex-valued functions of $L_p(G)$ and $L_q(G)$. Our aim is to construct A_p^q spaces using spaces of induced representations, $L_p(\pi)$ and $L_{q'}(\gamma^*)$, which are spaces of vector-valued functions. Therefore, our task is to construct a formula (see 4.15) for a convolution of functions in $L_p(\pi)$ and $L_{q'}(\gamma^*)$. The case where G/H and G/K are not compact is similar to that in the classical case in the sense that the non-triviality of the tensor product $L_p(\pi) \otimes_G^\sigma L_{q'}(\gamma^*)$ depends on the value of $1/p + 1/q'$ as the following theorem shows.

Theorem 4.1.1 *Let $1/p + 1/q' < 1$, $1 < p, q' < \infty$. Suppose that for any given compact set F in G, there exists $x \in G$ such that $HFx \cap HF = \emptyset$ and $KFx \cap KF = \emptyset$. Then*

$$L_p(\pi) \otimes_G^\sigma L_{q'}(\gamma^*) = \{\underline{0}\}.$$

Proof:(cf.Hörmander[27]) Note that if $H = K$ and is compact in G, and G/H is not compact then there exists $x \in G$ such that $HFx \cap HF = \emptyset$; for, if $HFx \cap HF \neq \emptyset$

for all $x \in G$, we find that for each $x \in G$ there exist $h, h' \in H$ and $c, c' \in F$, such that $hcx = h'c'$. That is, $x = c^{-1}h^{-1}h'c'$ for each $x \in G$. But this means that $G = F^{-1}HF$, and hence compact; which is a contradiction.

Let $T \in Hom_G(L_p(\pi), L_q(\gamma))$ and $f \in L_p(\pi)$. Suppose that

$$\|Tf\|_q \leq C\|f\|_p, \tag{4.1}$$

where C is a constant. T, being an intertwining operator, gives us

$$\|Tf + U_x^\gamma Tf\|_q = \|T(f + U_x^\pi f)\|_q \leq C\|f + U_x^\pi f\|_p. \tag{4.2}$$

Let us write $f = \ell_1 + \ell_2$, and $Tf = \ell_1' + \ell_2'$ where ℓ_1 and ℓ_1' have compact support HD and KD' in G respectively and $\|\ell_2\|_p < \epsilon$ and $\|\ell_2'\|_q < \epsilon$ for given $\epsilon > 0$. Let $D \cup D' = F$. Under the assumptions of the theorem, there exists $x \in G$ such that $HFx \cap HF = \emptyset$ and $KFx \cap KF = \emptyset$.

Thus, for such $x \in G$,

$$\|\ell_1 + U_x^\alpha \ell_1\|_p = 2^{\frac{1}{p}}\|\ell_1\|_p, \tag{4.3}$$

$$\|\ell_1' + U_x^\alpha \ell_1'\|_q = 2^{\frac{1}{q}}\|\ell_1'\|_q. \tag{4.4}$$

Also,

$$\|\ell_2 + U_x^\pi \ell_2\|_p \leq \|\ell_2\|_p + \|U_x^\pi \ell_2\|_p < 2\epsilon, \tag{4.5}$$

and

$$\|\ell_2{}' + U_x^\pi \ell_2{}'\|_p \leq \|\ell_2{}'\|_p + \|U_x^\pi \ell_2{}'\|_p < 2\epsilon, \tag{4.6}$$

as U_x^π is an isometry. Therefore,

$$\begin{aligned} \|f + U_x^\pi f\|_p &\leq \|\ell_1 + U_x^\pi \ell_1\|_p + \|\ell_2 + U_x^\pi \ell_2\|_p, \\ &\leq 2^{\frac{1}{p}}\|\ell_1\|_p + 2\epsilon, \text{ by (4.3) and (4.5).} \end{aligned} \tag{4.7}$$

But $\ell_1 = f - \ell_2$ implies that $\|\ell_1\|_p \leq \|f\|_p + \|\ell_2\|_p < \|f\|_p + \epsilon$. This, together with (4.7), gives

$$\begin{aligned} \|f + U_x^\pi f\|_p &\leq 2^{\frac{1}{p}}(\|f\|_p + \epsilon) + 2\epsilon, \\ &< 2^{\frac{1}{p}}\|f\|_p + 3\epsilon, \text{ since } 1 < p < \infty. \end{aligned} \tag{4.8}$$

Similarly,

$$\begin{aligned} \|Tf + U_x^\gamma Tf\|_q &\geq \|\ell'_1 + U_x^\gamma \ell'_1\|_q - \|\ell'_2 + U_x^\gamma \ell'_2\|_q, \\ &> 2^{\frac{1}{q}}\|\ell'_1\|_q - 2\epsilon, \text{ by (4.4) and (4.6),} \\ &> 2^{\frac{1}{q}}\|Tf\|_q - 3\epsilon, \text{ since } 1 < q' < \infty. \end{aligned} \tag{4.9}$$

Using (4.2),(4.8) and (4.9) we have

$$2^{\frac{1}{q}}\|Tf\|_q - 3\epsilon < C(2^{\frac{1}{p}}\|f\|_p + 3\epsilon),$$

which, in turn, gives

$$2^{\frac{1}{q}}\|Tf\|_q \;<\; C(2^{\frac{1}{p}}\|f\|_p) + (3+3C)\epsilon. \tag{4.10}$$

Since (4.10) holds for any $\epsilon > 0$,

$$2^{\frac{1}{q}}\|Tf\|_q \;\leq\; C(2^{\frac{1}{p}}\|f\|_p),$$

and therefore,

$$\|Tf\|_q \;\leq\; 2^{\frac{1}{p}+\frac{1}{q'}-1}C\|f\|_p. \tag{4.11}$$

But since $1/p + 1/q' < 1$, if we choose $C = \|T\|$, eqn.(4.11) leads us to a contradiction unless $C = 0$; in which case $T = \underline{0}$. Hence the result.

□

We do not know whether Theorem 4.1.1 is true in the absence of the condition that there exists an element $x \in G$ such that $HFx \cap HF = \emptyset$ and $KFx \cap KF = \emptyset$ for a given compact set F in G.

Let us turn to the construction of the convolution formula. The following proposition states a result that equips us with the necessary ground work.

Proposition 4.1.2 *Let $1 \leq p, q' < \infty$. For $\sum_{i=1}^{\infty} f_i \otimes g_i$ in $L_p(\pi) \otimes^\sigma L_{q'}(\gamma^*)$ and for almost all $x \in G/H$ and $y \in G/K$,*

$$\sum_{i=1}^{\infty} f_i(x) \otimes g_i(y) \in \mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*).$$

Proof: We want to show that $\sum_{i=1}^{\infty} \|f_i(x)\|\|g_i(y)\| < \infty$ for almost all $x, y \in G$. For any two compact sets $K_1 \in G/H$ and $K_2 \in G/K$, and for a fixed n,

$$\begin{aligned}
\int_{K_1\times K_2} \sum_{i=1}^{n} \|f_i(x)\|\|g_i(y)\| d\mu_H(x)d\mu_K(y) \\
&= \sum_{i=1}^{n} \int_{K_1\times K_2} \|f_i(x)\|\|g_i(y)\| d\mu_H(x)d\mu_K(y), \\
&= \sum_{i=1}^{n} \int_{K_1} \|f_i(x)\| d\mu_H(x) \int_{K_1} \|g_i(y)\| d\mu_K(y).
\end{aligned} \tag{4.12}$$

If $p = q' = 1$, (4.12) implies that

$$\begin{aligned}
\int_{K_1\times K_2} \sum_{i=1}^{n} \|f_i(x)\|\|g_i(y)\| d\mu_H(x)d\mu_K(y) \\
&\leq \sum_{i=1}^{n} \|f_i\|_1\|g_i\|_1.
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\int_{K_1 \times K_2} \sum_{i=1}^{\infty} \|f_i(x)\| \|g_i(y)\| d\mu_H(x) d\mu_K(y) \leq \sum_{i=1}^{\infty} \|f_i\|_1 \|g_i\|_1,$$

the right hand side of which is finite since $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes_G^\sigma L_{q'}(\gamma^*)$.

If $1 < p, q' < \infty$, using Hölder's inequality on the right hand side of (4.12), we get

$$\begin{aligned}
&\int_{K_1 \times K_2} \sum_{i=1}^{n} \|f_i(x)\| \|g_i(y)\| d\mu_H(x) d\mu_K(y) \\
&\leq \sum_{i=1}^{n} \Big(\int_{K_1} \|f_i(x)\|^p d\mu_H(x) \Big)^{\frac{1}{p}} (\mu_H(K_1))^{\frac{1}{p'}} \Big(\int_{K_2} \|g_i(y)\|^{q'} d\mu_K(y) \Big)^{\frac{1}{q'}} (\mu_K(K_2))^{\frac{1}{q}}, \\
&\leq (\mu_H(K_1))^{\frac{1}{p'}} (\mu_K(K_2))^{\frac{1}{q}} \sum_{i=1}^{n} \|f_i\|_p \|g_i\|_{q'}. \qquad (4.13)
\end{aligned}$$

Taking the limit as $n \to \infty$ in (4.13), we get

$$\begin{aligned}
&\int_{K_1 \times K_2} \sum_{i=1}^{\infty} \|f_i(x)\| \|g_i(y)\| d\mu_H(x) d\mu_K(y) \\
&\qquad \leq (\mu_H(K_1))^{\frac{1}{p'}} (\mu_K(K_2))^{\frac{1}{q}} \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{q'}.
\end{aligned}$$

The last expression is obviously finite since $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes_G^\sigma L_{q'}(\gamma^*)$. It is clear that we can achieve the desired result in the case where $p = 1$ and $q' > 1$ (or $p > 1$ and $q' = 1$), by using (4.12) and an argument similar to that which leads to (4.13). Hence $\sum_{i=1}^{\infty} \|f_i(x)\| \|g_i(y)\| < \infty$ for almost all $x, y \in G$, which proves the Proposition.

□

Our objective is to define a mapping on $L_p(\pi) \otimes^\sigma L_{q'}(\gamma^*)$ so that its image space is a generalisation of the space of convolutions as in the classical case. Let us consider the integral

$$\int_G f(xt) \otimes^\sigma g(yt) d\nu_G(t), \qquad (4.14)$$

where $f \in L_p(\pi), g \in L_{q'}(\gamma^*)$ and $(x, y) \in G \times G$. It is easy to see that the norm of the integrand is constant on the subgroup $H^x \cap K^y$ of G; for, if $t = x^{-1}hx = y^{-1}ky$ for some $h \in H$ and $k \in K$, then $f(xt) \otimes g(yt) = \pi(h)f(x) \otimes \gamma^*(k)g(y)$. This implies that the space over which we integrate must reduce to $G/(H^x \cap K^y)$, in order to avoid the integrand becoming too large. The integrand is constant over a given coset of $G/(H^x \cap K^y)$ if

$$f(xst) \otimes g(yst) = f(xt) \otimes g(yt),$$

for all $s \in H^x \cap K^y$. But

$$f(xst) \otimes g(yst) = \pi^x(s)f(xt) \otimes \gamma^{*y}(s)g(yt).$$

This suggests that the integrand must have its value at (x, y) in the quotient space of $\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*)$, in which we have the equality

$$\pi^x(s)f(xt) \otimes \gamma^{*y}(s)g(yt) = f(xt) \otimes g(yt).$$

This calls for the following definition.

Definition 4.1.3 *For any $x, y \in G$, the* subspace *$\mathcal{H}_{x,y}$ of $\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*)$ is defined to be the closed linear span of elements of the form*

$$\pi^x(b)\xi \otimes \eta - \xi \otimes (\gamma^y(b))^*\eta,$$

where $b \in H^x \cap K^y, \xi \in \mathcal{H}(\pi)$ and $\eta \in \mathcal{H}(\gamma^*)$. The quotient Banach space $\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*)/\mathcal{H}_{x,y}$ is denoted by $\mathcal{A}_{x,y}$.

Note that, using the notation in Sec.2.6 (see (2.7)), $\mathcal{A}_{x,y}$ can be written as $\mathcal{H}(\pi^x) \otimes^\sigma_{H^x \cap K^y} \mathcal{H}(\gamma^{*y})$.

Proposition 4.1.4 *The spaces $\{\mathcal{H}_{x,y} : x, y \in G\}$, and hence the spaces $\{\mathcal{A}_{x,y} : x, y \in G\}$, satisfy the property that*

$$\mathcal{H}_{xs,ys} = \mathcal{H}_{x,y} \quad \text{and} \quad \mathcal{A}_{xs,ys} = \mathcal{A}_{x,y}, \tag{4.15}$$

for any $s \in G$.

Proof: For $s \in G$, the space $\mathcal{H}_{xs,ys}$ is the closed linear span of elements of the form

$$\pi^{xs}(b)\xi \otimes \eta - \xi \otimes (\gamma^{ys}(b))^*\eta,$$

where $b \in H^{xs} \cap K^{ys}, \xi \in H(\pi)$ and $\eta \in H(\gamma)$ with

$$\pi^{xs}(b) = \pi(xsbs^{-1}x^{-1}) = \pi^x(sbs^{-1}).$$

Since $b \in H^{xs} \cap K^{ys}$, there exist $h \in H$ and $k \in K$ such that $b = s^{-1}x^{-1}hxs = s^{-1}y^{-1}kys$. Hence $sbs^{-1} = x^{-1}hx = y^{-1}ky$, showing that $sbs^{-1} \in H^x \cap K^y$. Therefore,

$$\pi^{xs}(b)\xi \otimes \eta - \xi \otimes (\gamma^{ys}(b))^*\eta = \pi^x(sbs^{-1})\xi \otimes \eta - \xi \otimes (\gamma^y(sbs^{-1}))^*\eta,$$

with $sbs^{-1} \in H^x \cap K^y, \xi \in H(\pi)$ and $\eta \in H(\gamma)$. This implies that $\mathcal{H}_{xs,ys} \subseteq \mathcal{H}_{x,y}$, which in turn gives us that $\mathcal{H}_{x,y} = \mathcal{H}_{xss^{-1},yss^{-1}} \subseteq \mathcal{H}_{xs,ys}$, for all $s \in G$. Hence (4.15) follows.

□

For $u \otimes v \in \mathcal{H}(\pi) \otimes_\sigma \mathcal{H}(\gamma^*)$, we use the notation $u \otimes_{x,y} v$ to denote the element of $\mathcal{A}_{x,y}$ to which $u \otimes v$ belongs. Then the integral (4.14) must be written in the form

$$\int_{\frac{G}{H^x \cap K^y}} f(xt) \otimes_{x,y} g(yt) d\mu_{x,y}(t), \tag{4.16}$$

for a suitably chosen quasi-invariant measure $\mu_{x,y}$ on the homogeneous space $G/(H^x \cap K^y)$. For each $x, y \in G$, the value of the integral belongs to the quotient Banach space $\mathcal{A}_{x,y}$. The next obvious step in this construction is to check whether the integral is finite and, to this end, we see that a further modification of the integrand is necessary. Propositions (4.1.5) and (4.1.6) state the conditions under which this modified integral is well defined and finite, respectively.

Note that if we define a function $\rho_{H_{x,y}}$ on G by $\rho_{H_{x,y}} := \rho_{H^x \cap K^y}/\rho_{H^x}$, we have

$$\rho_{H_{x,y}}(sz) = \rho_{H^x \cap K^y}(sz)/\rho_{H^x}(sz) = \Delta_{H^x \cap K^y}(s)/\Delta_{H^x}(s)\rho_{H_{x,y}}(z),$$

for $s \in H^x \cap K^y$ and $z \in G$. Thus $\rho_{H_{x,y}}$, restricted to H^x, is a ρ-function for the homogeneous space $H^x/(H^x \cap K^y)$. We let $\mu_{H_{x,y}}$ be a quasi-invariant measure associated with this ρ-function and $\lambda_{H_{x,y}}$ be the corresponding λ-function. Similarly, we can define a ρ-function $\rho_{K_{x,y}}$ for the homogeneous space $K^y/(H^x \cap K^y)$ and the corresponding λ-function will be denoted by $\lambda_{K_{x,y}}$.

Proposition 4.1.5 *Let p, q and m be positive real numbers with $1 \le p, q' < \infty$. Then, for $\sum_{i=1}^\infty f_i \otimes g_i \in L_p(\pi) \otimes^\sigma L_{q'}(\gamma^*)$ and $x, y \in G$,*

$$t \mapsto \sum_{i=1}^{\infty} \frac{1}{\lambda_{H^x \cap K^y}(e,t)^{\frac{1}{m}}} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{q'}} g_i(yt) \tag{4.17}$$

is a mapping on the coset space $G/(H^x \cap K^y)$ in each of the following cases:

(a) *$p = m$ and G/K having invariant measure (or $q' = m$ and G/H having invariant measure);*

(b) *G/K and G/H both having invariant measures;*

(c) *$p = q' = m$;*

(d) *$H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ having invariant measures.*

Proof: First let us consider the expression

$$\frac{\lambda_H(x,t)^{\frac{1}{p}} \lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x \cap K^y}(e,t)^{\frac{1}{m}}},$$

under the cases (a), (b) and (c).

Consider (a). Assuming $p = m$ and using the identity (3.9), we have,

$$\Big(\frac{\lambda_H(x,t)}{\lambda_{H^x\cap K^y}(e,t)}\Big)^{\frac{1}{p}} = \lambda_K(y,t)^{-\frac{1}{p}}.$$

If the measure on G/K is invariant, then $\lambda_K(y,t) = 1$; hence

$$\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x\cap K^y}(e,t)^{\frac{1}{p}}} = \lambda_K(y,t)^{\frac{1}{q'}-\frac{1}{p}} = 1.$$

A similar argument holds in the case where $q' = m$ and G/H possesses an invariant measure. In the case of (b), $\lambda_H(x,t) = \lambda_K(y,t) = 1$, and then by identity (3.9), $\lambda_{H^x\cap K^y}(e,t) = 1$, giving

$$\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x\cap K^y}(e,t)^{\frac{1}{m}}} = 1.$$

Clearly, under condition (c),

$$\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x\cap K^y}(e,t)^{\frac{1}{m}}} = \Big(\frac{\lambda_H(x,t)\lambda_K(y,t)}{\lambda_{H^x\cap K^y}(e,t)}\Big)^{\frac{1}{p}} = 1,$$

using the identity (3.9).

Therefore, under the conditions (a), (b) or (c), (4.17) can be simplified to

$$t \mapsto \Sigma_{i=1}^{\infty} f_i(xt)\otimes_{x,y} g_i(yt),$$

which is constant on each coset of $H^x \cap K^y$ in G. Hence it is a mapping on the coset space $G/(H^x \cap K^y)$.

For the case (d), it only remains to show that

$$\frac{1}{\lambda_{H^x\cap K^y}(e,st)^{\frac{1}{m}}}\lambda_H(x,st)^{\frac{1}{p}}\lambda_K(y,st)^{\frac{1}{q'}} = \frac{1}{\lambda_{H^x\cap K^y}(e,t)^{\frac{1}{m}}}\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}},$$

for $s \in H^x \cap K^y$. Letting $s = x^{-1}hx = y^{-1}ky$, for $h \in H$ and $k \in K$, we have

$$\begin{aligned}
&\frac{1}{\lambda_{H^x\cap K^y}(e,st)^{\frac{1}{m}}}\lambda_H(x,st)^{\frac{1}{p}}\lambda_K(y,st)^{\frac{1}{q'}}\\
&= \Big(\frac{\rho_{H^x\cap K^y}(e)}{\rho_{H^x\cap K^y}(st)}\Big)^{\frac{1}{m}}\Big(\frac{\rho_H(hxt)}{\rho_H(x)}\Big)^{\frac{1}{p}}\Big(\frac{\rho_K(kyt)}{\rho_K(y)}\Big)^{\frac{1}{q'}}\\
&= \Big(\frac{\Delta_G(s)}{\Delta_{H^x\cap K^y}(s)}\frac{\rho_{H^x\cap K^y}(e)}{\rho_{H^x\cap K^y}(t)}\Big)^{\frac{1}{m}}\Big(\frac{\Delta_H(h)}{\Delta_G(h)}\frac{\rho_H(xt)}{\rho_H(x)}\Big)^{\frac{1}{p}}\Big(\frac{\Delta_K(k)}{\Delta_G(k)}\frac{\rho_K(yt)}{\rho_K(y)}\Big)^{\frac{1}{q'}}\\
&= \Big(\frac{\Delta_G(s)}{\Delta_{H^x\cap K^y}(s)}\Big)^{\frac{1}{m}}\Big(\frac{\Delta_H(h)}{\Delta_G(h)}\Big)^{\frac{1}{p}}\Big(\frac{\Delta_K(k)}{\Delta_G(k)}\Big)^{\frac{1}{q'}}\frac{1}{\lambda(e,t)^{\frac{1}{m}}}\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}
\end{aligned} \quad (4.18)$$

Since we assume that $H^x/(H^x \cap K^y)$ has invariant measure, we have (see Reiter[35] p.159),

$$\lambda_{H_{x,y}}(e,s) = \frac{\rho_{H_{x,y}}(s)}{\rho_{H_{x,y}}(e)} = \frac{\Delta_{H^x \cap K^y}(s)}{\Delta_{H^x}(s)} = 1.$$

Now H and H^x are closed conjugate subgroups of G under an inner automorphism $\tau : G \mapsto G$ given by $\tau(y) = x^{-1}yx$. Since τ is a topological isomorphism of H onto H^x we have $\Delta_H = \Delta_{H^x}\tau$. This implies that $\Delta_{H^x}(h^x) = \Delta_H(h)$. Hence we have

$$\frac{\Delta_H(h)}{\Delta_{H^x \cap K^y}(s)} = \frac{\Delta_K(k)}{\Delta_{H^x \cap K^y}(s)} = 1, \tag{4.19}$$

for $s \in H^x \cap K^y$ with $s = x^{-1}hx = y^{-1}ky$. Consequently,

$$\Big(\frac{\Delta_G(s)}{\Delta_{H^x \cap K^y}(s)}\Big)^{\frac{1}{m}} \Big(\frac{\Delta_H(h)}{\Delta_G(h)}\Big)^{\frac{1}{p}} \Big(\frac{\Delta_K(k)}{\Delta_G(k)}\Big)^{\frac{1}{q'}} = \Big(\frac{\Delta_G(s)}{\Delta_{H^x \cap K^y}(s)}\Big)^{\frac{1}{m}-\frac{1}{p}-\frac{1}{q'}}. \tag{4.20}$$

Considering the identity (3.13) and using the fact that $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ possess invariant measures, we have

$$\begin{aligned} 1 &= \frac{\Delta_G(s)\Delta_{H^x \cap K^y}(s)}{\Delta_H(h)\Delta_K(k)}, \\ &= \frac{\Delta_G(s)\Delta_{H^x \cap K^y}(s)}{\Delta_{H^x}(s)\Delta_{K^y}(s)}, \\ &= \frac{\Delta_G(s)}{\Delta_{H^x}(s)}, \\ &= \frac{\Delta_G(s)}{\Delta_{H^x \cap K^y}(s)}, \end{aligned} \tag{4.21}$$

where the last two equalities are obtained by using (4.19). By (4.20) and (4.21) we obtain

$$\Big(\frac{\Delta_G(s)}{\Delta_{H^x \cap K^y}(s)}\Big)^{\frac{1}{m}} \Big(\frac{\Delta_H(h)}{\Delta_G(h)}\Big)^{\frac{1}{p}} \Big(\frac{\Delta_K(k)}{\Delta_G(k)}\Big)^{\frac{1}{q'}} = 1.$$

Thus, (4.18) simplifies to

$$\frac{\lambda_H(x,st)^{\frac{1}{p}}\lambda_K(y,st)^{\frac{1}{q'}}}{\lambda_{H^x \cap K^y}(e,st)^{\frac{1}{m}}} = \frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x \cap K^y}(e,t)^{\frac{1}{m}}}, \tag{4.22}$$

for $s \in H^x \cap K^y$ and therefore, the integral (4.17) is well defined in case (d) as well, completing the proof of the Proposition.

□

Recall, from the discussion preceding Lemma 3.1.3, that Υ denotes the set of all double cosets $H \times K : \Delta$ of $G \times G$. For $x, y \in G$, let

$$M_{x,y}^{(\frac{q'}{q'-1})} = \int_{\frac{H^x}{H^x \cap K^y}} \lambda_{H_{x,y}}(e,\alpha)d\mu_{H_{x,y}}(\alpha) \text{ and } N_{x,y}^{(\frac{p}{p-1})} = \int_{\frac{K^y}{H^x \cap K^y}} \lambda_{K_{x,y}}(e,\xi)d\mu_{K_{x,y}}(\xi).$$

Proposition 4.1.6 *For $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes^{\sigma} L_{q'}(\gamma^*)$ the integral*

$$\int_{\frac{G}{H^x \cap K^y}} \sum_{i=1}^{\infty} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{q'}} g_i(yt) d\mu_{x,y}(t) \tag{4.23}$$

is finite for almost all $D(x,y) \in \Upsilon$ in each of the following cases:

(a) *$p = 1$ and G/K having finite invariant measure (or $q' = 1$ and G/H having finite invariant measure);*

(b) *G/K and G/H both having finite invariant measures;*

(c) *$p = q' = 1$;*

(d) *$1 < p, q' < \infty$ with $1/p + 1/q' \geq 1$ and, $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ being compact for almost all $x, y \in G$ with $(x,y) \mapsto M_{x,y}N_{x,y}$ being a bounded function from Υ to $\mathcal{R}$.*

Proof: First let us consider the cases (a), (b) and (c). Using the disintegration of measures as in Lemma 2.2.12 in the spaces involved, we get

$$\begin{aligned}\sum_{i=1}^{\infty} \int_{D(x,y)\in\Gamma} \int_{\frac{G}{H^x \cap K^y}} & \|f_i(xt)\| \|g_i(yt)\| d\mu_D(\underline{t}) d\mu_{H,K}(D) \\ &= \sum_{i=1}^{\infty} \int_{\frac{G}{H}} \int_{\frac{G}{K}} \|f_i(x)\| \|g_i(y))\| d\mu_H(x) d\mu_K(y), \\ &= \sum_{i=1}^{\infty} \|f_i\|_1 \|g_i\|_1.\end{aligned}$$

Now if

(a) $p = 1$ and G/K has finite invariant measure (or $q' = 1$ and G/H has finite invariant measure),

(b) G/K and G/H both have finite invariant measure, or

(c) $p = q' = 1$,

we know that

$$\sum_{i=1}^{\infty} \|f_i\|_1 \|g_i\|_1 \leq \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{q'}.$$

Since $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes_G^{\sigma} L_{q'}(\gamma^*)$, we see that $\sum_{i=1}^{\infty} \|f\|_p \|g\|_{q'} < \infty$. Hence we have the desired result.

Now let us consider the case (d) where $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ are compact for almost all $x, y \in G$, together with $1/p + 1/q' > 1$. In the remainder of the proof, $\lambda_{H^x \cap K^y}(\cdot,\cdot)$ will be written as $\lambda(\cdot,\cdot)$, for simplicity of notation. First, let us consider the expression $\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}/\lambda(e,t)$. Using the identity (3.9) we see that

$$\begin{aligned}
\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda(e,t)} &= \frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_H(x,t)\lambda_K(y,t)},\\
&= \frac{1}{\lambda_H(x,t)^{\frac{1}{p'}}}\frac{1}{\lambda_K(y,t)^{\frac{1}{q}}},\\
&= \left(\frac{\lambda_K(y,t)}{\lambda(e,t)}\right)^{\frac{1}{p'}}\left(\frac{\lambda_H(x,t)}{\lambda(e,t)}\right)^{\frac{1}{q}}. \qquad (4.24)
\end{aligned}$$

Let $1/p+1/q'-1=1/r$. Then $1/p'=1-1/p=1/q'-1/r=1/q'(1-q'/r)$. Similarly, $1/q=1/p(1-p/r)$. Therefore,

$$\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda(e,t)} = \left(\frac{\lambda_K(y,t)}{\lambda(e,t)}\right)^{\frac{1}{q'}(1-\frac{q'}{r})}\left(\frac{\lambda_H(x,t)}{\lambda(e,t)}\right)^{\frac{1}{p}(1-\frac{p}{r})}. \qquad (4.25)$$

Hence we have

$$\begin{aligned}
I_i(x,y) &= \int_{\frac{G}{H^x\cap K^y}} \frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x\cap K^y}(e,t)}\|f_i(xt)\|\|g_i(yt)\|d\mu_{x,y}(t)\\
&= \int_{\frac{G}{H^x\cap K^y}} (\|f_i(xt)\|^p\|g_i(yt)\|^{q'})^{\frac{1}{r}}\left(\left(\frac{\lambda_H(x,t)}{\lambda(e,t)}\right)^{\frac{1}{p}}\|f_i(xt)\|\right)^{1-\frac{p}{r}}\times\\
&\qquad \left(\left(\frac{\lambda_K(y,t)}{\lambda(e,t)}\right)^{\frac{1}{q'}}\|g_i(yt)\|\right)^{1-\frac{q'}{r}}d\mu_{x,y}(t). \qquad (4.26)
\end{aligned}$$

Using Corollary 12.5 of Hewitt and Ross[26], the above can be simplified to obtain

$$\begin{aligned}
I_i(x,y) \leq &\left(\int \|f_i(xt)\|^p\|g_i(yt)\|^{q'}d\mu_{x,y}(t)\right)^{\frac{1}{r}}\times\\
&\left(\int \frac{\lambda_H(x,t)}{\lambda(e,t)}\|f_i(xt)\|^p d\mu_{x,y}(t)\right)^{\frac{q'-1}{q'}}\left(\int \frac{\lambda_K(y,t)}{\lambda(e,t)}\|g_i(yt)\|^{q'}d\mu_{x,y}(t)\right)^{\frac{p-1}{p}} (4.27)
\end{aligned}$$

where the three integrals are over the coset space $G/H^x\cap K^y$. Let us consider $\int_{\frac{G}{H^x\cap K^y}}\left(\frac{1}{\lambda(e,t)}\lambda_H(x,t) \parallel f_i(xt) \parallel^p\right)d\mu_{(x,y)}(t)$. By Lemma 3.1.1, there exists a quasi-invariant measure $\mu_{H_{x,y}}$ on $H^x/(H^x\cap K^y)$ such that

$$\begin{aligned}
&\int_{\frac{G}{H^x\cap K^y}}\left(\frac{\lambda_H(x,t)}{\lambda(e,t)} \parallel f_i(xt) \parallel^p\right)d\mu_{(x,y)}(t)\\
&= \int_{\frac{G}{H^x}}\int_{\frac{H^x}{H^x\cap K^y}}\left(\frac{\lambda(\alpha,t)}{\lambda_{H^x}(\alpha,t)}\frac{\lambda_H(x,\alpha t)}{\lambda(e,\alpha t)} \parallel f_i(x\alpha t) \parallel^p\right)d\mu_{H_{x,y}}(\alpha)d\mu_{H^x}(t).
\end{aligned}$$

For $\alpha = x^{-1}hx$ with $h\in H$,

$$\begin{aligned}
\lambda_H(x,\alpha t) &= \lambda_H(hx,t)\lambda_H(x,\alpha),\\
&= \lambda_H(x,t)\lambda_H(x,\alpha),\\
&= \lambda_{H^x}(e,t)\lambda_H(x,\alpha), \text{ by Lemma (3.1.2)},\\
&= \lambda_{H^x}(\alpha,t)\lambda_H(x,\alpha).
\end{aligned}$$

This, together with the fact that $\lambda(e,\alpha t)=\lambda(\alpha,t)\lambda(e,\alpha)$, gives us

$$\frac{\lambda(\alpha,t)}{\lambda_{H^x}(\alpha,t)}\frac{\lambda_H(x,\alpha t)}{\lambda(e,\alpha t)}=\frac{\lambda_H(x,\alpha)}{\lambda(e,\alpha)}=\frac{\lambda_{H^x}(e,\alpha)}{\lambda(e,\alpha)}.$$

But

$$\frac{\lambda(e,\alpha)}{\lambda_{H^x}(e,\alpha)} = \lambda_{H_{x,y}}(e,\alpha), \tag{4.28}$$

$\lambda_{H_{x,y}}$ being a λ-function for $H^x/(H^x\cap K^y)$ corresponding to the measure $\mu_{H_{x,y}}$. Using the assumption that $H^x/(H^x\cap K^y)$ is compact and the fact that $\lambda_{H_{x,y}}(e,\alpha)$ is bounded on compact sets (see Lemma 2.2.7 (c)), we have

$$\int_{\frac{H^x}{H^x\cap K^y}} \lambda_{H_{x,y}}(e,\alpha)d\mu_{H_{x,y}}(\alpha) = M_{x,y}^{(\frac{q'}{q'-1})} < \infty.$$

Thus

$$\begin{aligned}\int_{\frac{G}{H^x\cap K^y}} &\Big(\frac{\lambda_H(x,t)}{\lambda(e,t)}\|f_i(xt)\|^p\Big)d\mu_{(x,y)}(t)\\ &\leq M_{x,y}^{(\frac{q'}{q'-1})}\int_{\frac{G}{H^x}}\|f_i(xt)\|^p d\mu_{H^x}(t)\\ &= M_{x,y}^{(\frac{q'}{q'-1})}\int_{\frac{G}{H}}\|f_i(t)\|^p d\mu_H(t) = M_{x,y}^{(\frac{q'}{q'-1})}\|f_i\|_p^p \end{aligned}\tag{4.29}$$

(see the discussion about μ_{H^x} at the beginning of this Chapter). Similarly, if $K^y/(H^x\cap K^y)$ is compact,

$$\int_{\frac{G}{H^x\cap K^y}}\Big(\frac{\lambda_K(y,t)}{\lambda(e,t)}\parallel g_i(yt)\parallel^{q'}\Big)d\mu_{(x,y)}(t) \leq N_{x,y}^{(\frac{p}{p-1})}\|g_i\|_{q'}^{q'}. \tag{4.30}$$

The inequalities (4.27), (4.29) and (4.30) imply that

$$\begin{aligned} I_i(x,y) &\leq \Big(\int_{\frac{G}{H^x\cap K^y}}\|f_i(xt)\|^p\|g_i(yt)\|^{q'}d\mu_{x,y}(t)\Big)^{\frac{1}{r}}\times\\ &\qquad M_{x,y}\|f\|^{p(\frac{q'-1}{q'})}N_{x,y}\|g\|^{q'(\frac{p-1}{p})}.\end{aligned}\tag{4.31}$$

Note that

$$\begin{aligned}&\Big(\int_{D(x,y)\in\Upsilon}\Big(\int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^{\infty}\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x\cap K^y}(e,t)}\|f_i(xt)\|\|g_i(yt))\|d\mu_{x,y}(t)\Big)^r d\mu_{H,K}(D)\Big)^{\frac{1}{r}}\\ &= \Big(\int_\Upsilon\Big(\sum_{i=1}^{\infty}\int_{\frac{G}{H^x\cap K^y}}\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^x\cap K^y}(e,t)}\|f_i(xt)\|\|g_i(yt))\|d\mu_{x,y}(t)\Big)^r d\mu_{H,K}(D)\Big)^{\frac{1}{r}}\\ &= \Big(\int_{D(x,y)\in\Upsilon}\Big(\sum_{i=1}^{\infty}I_i(x,y)\Big)^r d\mu_{H,K}(D)\Big)^{\frac{1}{r}}.\end{aligned}$$

Using generalised Minkowski's inequality (see Dunford and Schwartz[10], p.529) we see that

$$\left(\int_{D\in\Upsilon}\left(\sum_{i=1}^{\infty} I_i(x,y)\right)^r d\mu_{H,K}(D)\right)^{\frac{1}{r}} \leq \sum_{i=1}^{\infty}\left(\int_{D\in\Upsilon}\left(I_i(x,y)\right)^r d\mu_{H,K}(D)\right)^{\frac{1}{r}} \quad (4.32)$$

Let ess $\sup_{D(x,y)}\{(M_{x,y}N_{x,y})^r\} = S^r$. Then, by (4.31) and (4.32) we have

$$\begin{aligned}
&\left(\int_{D(x,y)\in\Upsilon}\left(\sum_{i=1}^{\infty} I_i(x,y)\right)^r d\mu_{H,K}(D)\right)^{\frac{1}{r}} \\
&\leq \sum_{i=1}^{\infty}\Bigg(\|f_i\|^{rp(\frac{q'-1}{q'})}\|g_i\|^{rq'(\frac{p-1}{p})} \times \\
&\qquad \int_{D(x,y)\in\Upsilon}(M_{x,y}N_{x,y})^r\Big(\int_{\frac{G}{H^x\cap K^y}}\|f_i(xt)\|^p\|g_i(yt)\|^{q'}d\mu_{x,y}(t)\Big)d\mu_{H,K}(D)\Bigg)^{\frac{1}{r}}, \\
&\leq \sum_{i=1}^{\infty}\Bigg(S^r\|f_i\|_p^{rp(\frac{q'-1}{q'})}\|g_i\|_{q'}^{rq'(\frac{p-1}{p})} \times \\
&\qquad \int_{\frac{G}{H}}\int_{\frac{G}{K}}\|f_i(x)\|^p\|g_i(y)\|^{q'}d\mu_H(x)d\mu_K(y)\Bigg)^{\frac{1}{r}} \qquad (4.33) \\
&= \sum_{i=1}^{\infty}\left(S^r\|f_i\|_p^{p+rp(\frac{q'-1}{q'})}\|g_i\|_{q'}^{q'+rq'(\frac{p-1}{p})}\right)^{\frac{1}{r}}. \qquad (4.34)
\end{aligned}$$

where (4.33) is obtained using disintegration of measures (see proof of Lemma 3.1.3). Since $p/r + p(q'-1)/q' = p(1/r + 1 - 1/q') = p(1/p) = 1$, and similarly $q'/r + q'(p-1)/p = 1$, we obtain

$$\left(\int_{D(x,y)\in\Upsilon}\left(\sum_{i=1}^{\infty} I_i(x,y)\right)^r d\mu_{H,K}(D)\right)^{\frac{1}{r}} \leq S\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_{q'}. \qquad (4.35)$$

This proves the finiteness of the integral (4.23) for almost all $D(x,y) \in \Upsilon$ under condition (d) together with $1/p + 1/q' > 1$.

Now let us consider the case (d) together with $1/p + 1/q' = 1$.

Using Hölder's inequality, we get

$$\begin{aligned}
&\int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^{\infty}\frac{1}{\lambda(e,t)}\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{p'}}\|f_i(xt)\|\|g_i(yt)\|d\mu_{x,y}(t) \\
&\leq \sum_{i=1}^{\infty}\left(\int_{\frac{G}{H^x\cap K^y}}\Big(\frac{\lambda_H(x,t)^{\frac{1}{p}}}{\lambda(e,t)^{\frac{1}{p}}}\parallel f_i(xt)\parallel\Big)^p d\mu_{x,y}(t)\right)^{\frac{1}{p}} \times \\
&\qquad \left(\int_{\frac{G}{H^x\cap K^y}}\Big(\frac{\lambda_K(y,t)^{\frac{1}{p'}}}{\lambda(e,t)^{\frac{1}{p'}}}\parallel g_i(yt)\parallel\Big)^{p'} d\mu_{x,y}(t)\right)^{\frac{1}{p'}},
\end{aligned}$$

$$= \sum_{i=1}^{\infty}\Big(\int_{\frac{G}{H^x\cap K^y}}\big(\frac{\lambda_H(x,t)}{\lambda(e,t)}\parallel f_i(xt)\parallel^p\big)d\mu_{(x,y)}(t)\Big)^{\frac{1}{p}}\times$$

$$\Big(\int_{\frac{G}{H^x\cap K^y}}\big(\frac{\lambda_K(y,t)}{\lambda(e,t)}\parallel g_i(yt)\parallel^{p'}\big)d\mu_{(x,y)}(t)\Big)^{\frac{1}{p'}}.$$

By (4.29) and (4.30) we have

$$\int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^{\infty}\frac{\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{p'}}}{\lambda(e,t)}\parallel f_i(xt)\parallel\parallel g_i(yt)\parallel d\mu_{(x,y)}(t)$$
$$\leq M_{x,y}N_{x,y}\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_{p'}.$$

Again, since $\sum_{i=1}^{\infty} f_i\otimes g_i \in L_p(\pi)\otimes_G^{\sigma} L_{q'}(\gamma^*)$, we see that $\sum_{i=1}^{\infty}\|f\|_p\|g\|_{q'} < \infty$. Hence the result follows.

□

In view of Propositions 4.1.5 and 4.1.6, we can formally define the convolution of functions in $L_p(\pi)$ and $L_{q'}(\gamma^*)$.

Definition 4.1.7 *Let H and K be regularly related. For each $x, y \in G$ let $\mu_{x,y}$ be a quasi-invariant measure on the homogeneous space $G/(H^x \cap K^y)$ so that the identity (3.9) holds. Let p, q be positive real numbers. The map Ψ on $L_p(\pi)\otimes_\sigma L_{q'}(\gamma^*)$ is defined by*

$$(\Psi(\Sigma_{i=1}^{\infty} f_i\otimes g_i))(x,y) := \tag{4.36}$$
$$\int_{\frac{G}{H^x\cap K^y}}\Sigma_{i=1}^{\infty}\frac{1}{\lambda_{H^x\cap K^y}(e,t)}\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt)d\mu_{(x,y)}(t)$$

whenever one of the following conditions holds:

(a) *$p = 1$ and G/K has finite invariant measure (or $q' = 1$ and G/H has finite invariant measure);*

(b) *G/K and G/H both have finite invariant measures;*

(c) *$p = q' = 1$;*

(d) *$1 < p, q' < \infty$ with $1/p + 1/q' \geq 1$, $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ are compact and possess invariant measures for almost all $x, y \in G$ and the map $(x,y) \mapsto M_{x,y}N_{x,y}$ is bounded from Υ to $\mathcal{R}$.*

It is clear that for each $(x,y) \in G \times G$, the value of $(\Psi(\Sigma_{i=1}^{\infty} f_i \otimes g_i))(x,y)$ belongs to the quotient space $\mathcal{A}_{x,y}$. We investigate the properties of the image space of Ψ in the following section.

4.2 The structure of the image space of Ψ

The image space of Ψ is contained in a space of mappings acting on $G \times G$, whose values at $(x, y) \in G \times G$ belong to a collection of Banach spaces $\{\mathcal{A}_{x,y} : (x, y) \in G \times G\}$. This suggests that the image space has the structure of the space of cross-sections of a Banach bundle or a Banach semi-bundle where the bundle space is a union of quotient spaces of a given Banach space. We considered the generalised version of such a situation in Sec.3.2 and shall adopt similar notations for the Banach bundles in this special case.

Let

$$
\begin{aligned}
\mathcal{B}_0 &= \mathcal{H}(\pi) \otimes^{\sigma} \mathcal{H}(\gamma^*) \times G \times G, \\
\mathcal{B}_0^{\Delta} &= \mathcal{H}(\pi) \otimes^{\sigma} \mathcal{H}(\gamma^*) \times (G \times G)/\Delta, \\
\mathcal{B}_1 &= \cup_{(x,y)\in G\times G}\{\mathcal{H}_{x,y} \times \{(x, y)\}\}, \\
\mathcal{B}_1^{\Delta} &= \cup_{(x,y)\in G\times G}\{\mathcal{H}_{x,y} \times \{(x, y)\Delta\}\}, \\
\mathcal{B}_2 &= \cup_{(x,y)\in G\times G}\{\mathcal{A}_{x,y} \times \{(x, y)\}\}, \text{ and} \\
\mathcal{B}_2^{\Delta} &= \cup_{(x,y)\in G\times G}\{\mathcal{A}_{x,y} \times \{(x, y)\Delta\}\}.
\end{aligned}
$$

It is clear that $\mathcal{B}_1$ is a subspace of $\mathcal{B}_0$, and $\mathcal{B}_1^{\Delta}$ is a subspace of $\mathcal{B}_0^{\Delta}$.

With j denoting any one of $\{0, 1, 2\}$, let $\theta_j : \mathcal{B}_j \mapsto G \times G$ be defined by $\theta_j(\zeta, (x, y)) = (x, y)$, and let $\theta_j^{\Delta} : \mathcal{B}_j^{\Delta} \mapsto (G \times G)/\Delta$ be defined by $\theta_j^{\Delta}(\zeta, (x, y)\Delta) = (x, y)\Delta$, where ζ belongs to the corresponding Banach space. Let $q : \mathcal{B}_0 \mapsto \mathcal{B}_2$ be the quotient map given by $q(h, x) = (\{\mathcal{H}_x + h\}, x)$. Similarly, the quotient map $q_{\Delta} : \mathcal{B}_0^{\Delta} \mapsto \mathcal{B}_2^{\Delta}$ is given by $q_{\Delta}(h, r(x)) = (\{\mathcal{H}_x + h\}, r(x))$. $\mathcal{B}_0$ has the product topology, and we topologize $\mathcal{B}_2^{\Delta}$ so that the map p_{Δ} is continuous and open.

Define $\underline{\mathcal{B}}_j := (\mathcal{B}_j, \theta_j)$ and $\underline{\mathcal{B}}_j^{\Delta} := (\mathcal{B}_j^{\Delta}, \theta_j^{\Delta})$, for $j \in \{0, 1, 2\}$.

Obviously, $\underline{\mathcal{B}}_0$ and $\underline{\mathcal{B}}_0^{\Delta}$ are trivial bundles with constant fiber $\mathcal{H}(\pi) \otimes^{\sigma} \mathcal{H}(\gamma^*)$. The space $(G \times G)/\Delta$ is Hausdorff since Δ is a closed subgroup of $G \times G$.

It is clear that this is a particular case of the general structure of the Banach (semi-)bundles discussed in Sec.3.2 (p.32). The results in Sec.2.3 lead us to the following conclusion.

Proposition 4.2.1

(i) $\underline{\mathcal{B}}_0^{\Delta}$ *and* $\underline{\mathcal{B}}_0$ *are Banach bundles over* $(G \times G)/\Delta$ *and* $G \times G$ *respectively.*

(ii) $\underline{\mathcal{B}}_2^{\Delta}$ *and* $\underline{\mathcal{B}}_2$ *are Banach semi-bundles over* $(G \times G)/\Delta$ *and* $G \times G$ *respectively.*

Proof: (i) is clearly true since $\underline{\mathcal{B}}_0^\Delta$ and $\underline{\mathcal{B}}_0$ are trivial bundles (see the discussion preceeding Lemma 3.2.3). (ii) is just a special case of Lemma 3.2.6.

□

It is possible for $\underline{\mathcal{B}}_1^\Delta$ to become a semi-subbundle of $\underline{\mathcal{B}}_0^\Delta$, when the subgroups H and K are related in a manner described in the following definition.

Definition 4.2.2 *The closed subgroups H and K of G are said to be* smoothly related *if, for a given $(x_0, y_0) \in G \times G$, an element $b_0 \in H^{x_0} \cap K^{y_0}$ and $\epsilon > 0$, there exists a neighborhood N of (x_0, y_0) and a continuous map $c : N \mapsto G$ such that $c(x,y) \in H^x \cap K^y$ and the distance $d(c(x_0, y_0), b_0) < \epsilon$.*

We turn to some examples.

(1) Let G be an Abelian group. Then for any $(x, y) \in G \times G$, $H^x \cap K^y = H \cap K$. Therefore, for any $b_0 \in H^{x_0} \cap K^{y_0}$, we can choose a neighborhood N of (x_0, y_0) and a map $c : N \mapsto G$ such that $c(x,y) = b_0$, which satisfies the conditions given in the definition.

(2) Let H be a normal subgroup of G. Then $H^x \cap K^y = (H \cap K)^y$, for any $(x,y) \in G \times G$. If $b_0 \in (H \cap K)^{y_0}$, then $b_0{}^{y_0^{-1}} \in H \cap K$, and therefore, $b_0{}^{y_0^{-1}y} \in (H \cap K)^y = H^x \cap K^y$, for any $(x,y) \in G \times G$. For any neighborhood N of (x_0, y_0) define $c : N \mapsto G$ such that $c(x,y) = b_0{}^{y_0^{-1}y}$. This has the required properties given in the definition.

(3) Let us give an example of a case where the above condition does not hold. Let H be a closed subgroup of a group G which has the property that $H^x \cap H = e$ unless $x \in H$ (see example (1) of Chapter 6). Choose $(x_0, y_0) = (e, e)$ and let b_0 be any element of H other than the identity e. Suppose there exists a neighborhood N of (e,e) and a continuous map $c : N \mapsto G$ such that $c(x,y) \in H^x \cap H = e$ and $d(c(x_0, y_0), b_0) < \epsilon$. The continuity of c implies $c(x,y) = e$ for all $(x,y) \in N$. Then by triangle inequality,

$$d(c(x,y), b_0) \;\leq\; d(c(x,y), c(x_0,y_0)) + d(c(x_0,y_0), b_0) < \epsilon.$$

This means $d(c(x,y), b_0) = d(e, b_0) < \epsilon$, which contradicts the fact that $b_0 \neq e$.

Proposition 4.2.3 *If H and K are smoothly related then $\underline{\mathcal{B}}_1^\Delta$ is a semi-subbundle of $\underline{\mathcal{B}}_0^\Delta$.*

Proof: Clearly, $\mathcal{B}_1^\Delta$ is a subset of $\mathcal{B}_0^\Delta$ and $\theta_0^{-1}((x,y)\Delta) \cap \mathcal{B}_1^\Delta = \mathcal{H}_{x,y} \times \{(x,y)\Delta\}$, is a subspace of $\mathcal{H}(\pi) \otimes_\sigma \mathcal{H}(\gamma^*) \times \{(x,y)\Delta\}$. Given $(x_0, y_0) \in G \times G$, let $\vartheta \in \mathcal{H}_{x_0,y_0}$. The Proposition is proved if we can show that for a given $\epsilon > 0$ there exists a neighborhood N of (x_0, y_0) and a continuous cross-section α of $\underline{\mathcal{B}}_0^\Delta$ and $\|\alpha(x_0, y_0) - \vartheta\| < \epsilon$ (see Definition 2.3.6 (b)).

Now $\vartheta \in \mathcal{H}_{x_0,y_0}$ can be approximated in norm by elements of the form

$$\sum_{i=1}^{n}(\pi^{x_0}(b_i)\xi_i \otimes \eta_i - \xi_i \otimes (\gamma^{y_0}(b_i))^*\eta_i),$$

where $b_i \in H^{x_0} \cap K^{y_0}, \xi_i \in H(\pi)$ and $\eta_i \in H(\gamma^*)$. Without loss of generality, we can assume that

$$\vartheta = \pi^{x_0}(b_0)\xi \otimes \eta - \xi \otimes (\gamma^{y_0}(b_0))^*\eta,$$

with $b_0 \in H^{x_0} \cap K^{y_0}$. Let $\epsilon > 0$ be given. Since $h \mapsto \pi(h)(u)$ and $k \mapsto \gamma(k)(v)$ are continuous from H to $\mathcal{H}(\pi)$ and K to $\mathcal{H}(\gamma)$ for any $u \in \mathcal{H}(\pi)$ and $v \in \mathcal{H}(\gamma)$ respectively, there exist positive numbers δ_1 and δ_2 such that

$$\|(\pi(h))(u) - (\pi(h_0))(u)\| < \frac{\epsilon}{2\|v\|} \text{ and } \|(\gamma(k))(u) - (\gamma(k_0))(v)\| < \frac{\epsilon}{2\|u\|},$$

whenever $d(h, h_0) < \delta_1$ and $d(k, k_0) < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$.

Since H and K are smoothly related, there exists a neighborhood N of (x_0, y_0) and a continuous function c from N to G such that $c(x, y) \in H^x \cap K^y$ and the distance

$$d(c(x_0, y_0), b_0) < \delta.$$

Define a mapping $\alpha : N \mapsto \mathcal{B}_0^{\Delta}$ by

$$\alpha(x, y) = \pi^x(c(x, y))\xi \otimes \eta - \xi \otimes (\gamma^y(c(x, y)))^*\eta. \tag{4.37}$$

Obviously, $\alpha(x, y) \in \mathcal{H}_{x,y}$. Since we have c continuous, $x \mapsto xtx^{-1}$ continuous for any $t \in G$ and $h \mapsto \pi(h)(u)$ continuous for any $u \in \mathcal{H}(\pi)$, we see that $(x, y) \mapsto \pi^x(c(x, y))\xi$ is continuous. Similarly, $(x, y) \mapsto \gamma^y(c(x, y))\eta$ is continuous so that α, as defined by (4.37), is continuous. Now

$$\begin{aligned}
&\|\alpha(x_0, y_0) - \vartheta\| \\
&= \|\Big(\pi^{x_0}(c(x_0, y_0))\xi \otimes \eta - \xi \otimes (\gamma^{y_0}(c(x_0, y_0)))^*\eta\Big) - \\
&\qquad \Big(\pi^{x_0}(b_0)\xi \otimes \eta - \xi \otimes (\gamma^{y_0}(b_0))^*\eta\Big)\|, \\
&= \|\Big(\pi^{x_0}(c(x_0, y_0))\xi - \pi^{x_0}(b_0)\xi\Big) \otimes \eta - \xi \otimes \Big(\gamma^{y_0}(c(x_0, y_0))\eta - \gamma^{y_0}(b_0)\eta\Big)\|, \\
&\leq \|\Big(\pi^{x_0}(c(x_0, y_0))\xi - \pi^{x_0}(b_0)\xi\Big)\|\|\eta\| + \|\xi\|\|\Big(\gamma^{y_0}(c(x_0, y_0))\eta - \gamma^{y_0}(b_0)\eta\Big)\|.
\end{aligned}$$

But since

$$d(x_0c(x_0, y_0)x_0^{-1}, x_0b_0x_0^{-1}) = d(c(x_0, y_0), b_0) = d(y_0c(x_0, y_0)y_0^{-1}, y_0b_0y_0^{-1}) < \delta,$$

we have

$$\|\Big(\pi^{x_0}(c(x_0, y_0))\xi - \pi^{x_0}(b_0)\xi\Big)\| < \frac{\epsilon}{2\|\eta\|} \text{ and } \|\Big(\gamma^{y_0}(c(x_0, y_0))\eta - \gamma^{y_0}(b_0)\eta\|\Big)\| < \frac{\epsilon}{2\|\xi\|}.$$

Hence

$$\|\alpha((x_0, y_0)) - \vartheta\| < \epsilon.$$

□

The next important result required for the study of the space of cross-sections of these Banach (semi-)bundles was dealt with, under a more general situation, in Proposition 3.2.7. We shall state the result in this special case.

Proposition 4.2.4 *Let p_Δ be the canonical mapping from $G \times G$ to $(G \times G)/\Delta$. The Banach bundle retraction*

$$\underline{\mathcal{B}}_2^{\Delta\#} := (\mathcal{B}_2^{\Delta\#}, \theta_2^{\Delta\#})$$

of $\underline{\mathcal{B}}_2^{\Delta}$ by p is topologically equivalent to $\underline{\mathcal{B}}_2$.

Proof: This is a consequence of Proposition 3.2.7.

□

4.3 The space A_p^q

First we shall show that the image space of Ψ consists of mappings which are constant on the right cosets $(G \times G)/\Delta$ under the conditions (a), (b) and (c) of Definition 4.1.7.

Proposition 4.3.1 *Let α be an element of the image space of Ψ. For any $h_0 \in H, k_0 \in K$, $x, y \in G$ and $s \in G/(H^x \cap K^y)$*

$$\alpha(h_0xs, k_0ys) = \pi(h_0) \otimes \gamma^*(k_0)\alpha(x, y)$$

under the conditions (a), (b) and (c) of Definition 4.1.7.

Proof: By Proposition 4.1.4 we have

$$\mathcal{H}_{xs,ys} = \mathcal{H}_{x,y}$$

for all $x, y, s \in G$.

Now any element $\omega \otimes_{x,y} \varrho$ of $\mathcal{A}_{x,y}$ is of the form

$$\begin{aligned} \omega \otimes_{x,y} \varrho &= \mathcal{H}_{x,y} + \omega \otimes \varrho \\ &= \langle \{\pi^x(b)\xi \otimes \eta - \xi \otimes (\gamma^y(b))^*\eta, b \in H^x \cap K^y, \xi \in H(\pi), \eta \in H(\gamma)\}\rangle + \omega \otimes \varrho. \end{aligned}$$

If this element is translated by $\pi(h_0) \otimes \gamma^*(k_0)$ from the left, we get

$\pi(h_0) \otimes \gamma^*(k_0)(\omega \otimes_{x,y} \varrho)$

$= \langle\{\pi(h_0)\pi^x(b)\xi \otimes \gamma^*(k_0)\eta - \pi(h_0)\xi \otimes \gamma^*(k_0)(\gamma^y(b))^*\eta;\ b \in H^x \cap K^y\}\rangle + \pi(h_0)\omega \otimes \gamma^*(k_0)\varrho.$

But

$$\pi(h_0)\pi^x(b)\xi = \pi^{h_0x}(b)\pi(h_0)\xi \text{ and } \gamma^*(k_0)(\gamma^y(b))^*\eta = (\gamma^{k_0y}(b))^*\gamma^*(k_0)\eta,$$

hence

$$\begin{aligned}
&\pi(h_0) \otimes \gamma^*(k_0)(\omega \otimes_{x,y} \varrho) \\
&= \langle\{\pi^{h_0x}(b)\pi(h_0)\xi \otimes \gamma^*(k_0)\eta - \pi(h_0)\xi \otimes (\gamma^{k_0y}(b))^*\gamma^*(k_0)\eta;\ b \in H^{h_0x} \cap K^{k_0y}\}\rangle \\
&\qquad + \pi(h_0)\omega \otimes \gamma^*(k_0)\varrho, \\
&= \mathcal{H}_{h_0x,k_0y} + \pi(h_0)\omega \otimes \gamma^*(k_0)\varrho, \\
&= \pi(h_0)\omega \otimes_{h_0x,k_0y} \gamma^*(k_0)\varrho. \qquad (4.38)
\end{aligned}$$

Any α in the image space of Ψ can be expressed as $\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)$. Without loss of generality, we consider an element of the form $\Psi(f \otimes g)$; the argument is then valid for any α by linearity. Consider the homeomorphism $\phi_s : G/(H^x \cap K^y) \mapsto G/(H^x \cap K^y)^s$ given by $\phi_s(v) = s^{-1}v$, (as in Lemma 3.1.2) and use the fact that $\mu_{xs,ys} = \mu_{x,y} \circ \phi_s$ to get

$$\begin{aligned}
&(\Psi(f \otimes g))(h_0xs, k_0ys) \\
&= \int_{\frac{G}{H^{xs} \cap K^{ys}}} \frac{1}{\lambda_{(H^x \cap K^y)^s}(e,t)} \lambda_H(xs,t)^{\frac{1}{p}} f(xst) \otimes_{xs,ys} \lambda_K(ys,t)^{\frac{1}{q'}} g(yst)\, d\mu_{(xs,ys)}(t) \\
&= \int_{\frac{G}{H^{xs} \cap K^{ys}}} \left(\frac{\lambda_{H^x \cap K^y}(e,s)}{\lambda_{H^x \cap K^y}(e,st)}\right)\left(\frac{\lambda_H(x,st)}{\lambda_H(x,s)}\right)^{\frac{1}{p}} f(xst) \otimes_{xs,ys} \\
&\qquad\qquad \left(\frac{\lambda_K(y,st)}{\lambda_K(y,s)}\right)^{\frac{1}{q'}} g(yst)\, d\mu_{(xs,ys)}(t), \\
&= \int_{\frac{G}{H^x \cap K^y}} \left(\frac{\lambda_{H^x \cap K^y}(e,s)}{\lambda_{H^x \cap K^y}(e,t)}\right)\left(\frac{\lambda_H(x,t)}{\lambda_H(x,s)}\right)^{\frac{1}{p}} f(xt) \otimes_{xs,ys} \left(\frac{\lambda_K(y,t)}{\lambda_K(y,s)}\right)^{\frac{1}{q'}} g(yt)\, d\mu_{(x,y)}(t), \\
&= \frac{\lambda_{H^x \cap K^y}(e,s)}{\lambda_H(x,s)^{\frac{1}{p}} \lambda_K(y,s)^{\frac{1}{q'}}} (\pi(h_0) \otimes \gamma^*(k_0))[\Psi(f \otimes g)](x,y).
\end{aligned}$$

Hence we see that

$$\alpha(h_0xs, k_0ys) = \pi(h_0) \otimes \gamma^*(k_0)\alpha(x,y)$$

only if

$$\frac{\lambda_{H^x \cap K^y}(e,s)}{\lambda_H(x,s)^{\frac{1}{p}} \lambda_K(y,s)^{\frac{1}{q'}}} = 1$$

for all $s \in G/(H^x \cap K^y)$. It is clear that this last condition is true in the cases (a), (b) and (c) given in Definition 4.1.7.

□

Lemma 4.3.2 *Consider the conditions (a), (b), (c) and (d) of Definition 4.1.7. For* $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes L_{q'}(\gamma)$, *the element* $\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)$ *is a cross-section of* $\underline{\mathcal{B}}_2^{\Delta}$, *if the integral (4.36) is constructed under one of the conditions (a), (b) or (c). It is a cross-section of* $\underline{\mathcal{B}}_2$ *if it is constructed under the condition (d).*

Proof: This is an immediate consequence of Proposition 4.3.1.

□

Definition 4.3.3 *The* **space** A_p^q *is defined to be the range of* Ψ *with the quotient norm.*

In other words, A_p^q is contained in the space of cross-sections of the Banach semi-bundle $\underline{\mathcal{B}}_2^{\Delta}$ in the cases (a), (b) and (c) of Definition 4.1.7. In the case (d), it is contained in the space of cross-sections of the Banach semi-bundle $\underline{\mathcal{B}}_2$.

By a **continuous family of functions** we mean a family of functions $\{\beta_x : x \in G\}$ such that $(x,t) \mapsto \beta_x(t)$ is a continuous map from $G \times G$ to $\mathcal{R}$.

Proposition 4.3.4 *Suppose that the spaces* $G/H, G/K$ *and the numbers* p, q *satisfy one of the conditions (a), (b), (c) or (d) as described in Definition 4.1.7. Suppose further that there exists a continuous family* $\{\beta_{x,y} : (x,y) \in G \times G\}$ *of functions where* $\beta_{x,y}$ *is a Bruhat function for* $H^x \cap K^y$. *Let* f *and* g *be functions with compact support from* $L_p(\pi)$ *and* $L_{q'}(\gamma^*)$ *respectively. Then,*

$$(x,y) \quad \mapsto \quad \int_{\frac{G}{H^x \cap K^y}} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes_{x,y} \lambda_K(x,t)^{\frac{1}{q'}} g(yt) d\mu_(x,y)(t) \qquad (4.39)$$

is a continuous cross-section of the corresponding Banach semi-bundle.

Proof: For any $x \in G$ and $f \in L_p(\pi)$, the function ${}_xf$ defined by

$${}_xf(t) = f(xt)$$

is a function in $L_p(\pi^x)$ since

$${}_xf(h^x t) = f(xx^{-1}hxt) = \pi(h)_x f(t).$$

Similarly, a function $g \in L_{q'}(\gamma^*)$ gives rise to a function ${}_yg$ in $L_{q'}(\gamma^*)$.

Now suppose f and g are continuous with compact support. Then there exist compact sets G_1 and G_2 of G such that $H^x G_1$ and $K^y G_2$ are the supports of ${}_xf$ and ${}_yg$ respectively.

Recall (Lemma 2.2.6) that a Bruhat function $\beta_{x,y}$ for $H^x \cap K^y$ is a function on G satisfying the following conditions:

(i) if F is a compact set in G, then $\beta_{x,y}$ coincides on the strip $(H^x \cap K^y)F$ with a function in $C_0^+(G)$, and,

(ii) $\int_{H^x \cap K^y} \beta_{x,y}(st) d\nu_{H^x \cap K^y}(s) = 1$ for all $t \in G$.

Suppose that the integral in (4.39) is constructed under one of the conditions (a), (b), (c) or (d) of Definition 4.1.7. Consider the map

$$(x, y, t) \mapsto \beta_{x,y}(t) \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes \lambda_K(y,t)^{\frac{1}{q'}} g(yt)$$

from $(G \times G \times G)$ to $\mathcal{B}_0$. This is a cross-section of the Banach bundle retraction of $\underline{\mathcal{B}}_0$ by $p : G \times G \times G \mapsto G \times G$. It is a continuous cross-section since, under the assumptions, $\{\beta_{x,y} : (x,y) \in G \times G\}$ is a continuous family of Bruhat functions. Therefore, we can form the integral

$$\widetilde{\Gamma}(x,y) := \int_G \beta_{x,y}(t) \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes \lambda_K(y,t)^{\frac{1}{q'}} g(yt) d\nu_G(t)$$

(see the discussion on page 36) and by Lemma 3.2.11, we see that $\widetilde{\Gamma}$ is a continuous cross-section of $\underline{\mathcal{B}}_0$.

Considering the diagram

$$\begin{array}{ccc} & \mathcal{B}_0 \overset{q}{\mapsto} \mathcal{B}_2 & \\ \widetilde{\Gamma} \uparrow & & \nearrow \Gamma, \\ & G \times G & \end{array}$$

where $q(\xi, x) = (\{\mathcal{H}_x + \xi\}, x)$, we find that

$$\Gamma(x,y) := \int_G \beta_{x,y}(t) \lambda_H(x,t)^{\frac{1}{q}} f(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{p'}} g(yt) d\nu_G(t)$$

is a continuous cross-section of $\underline{\mathcal{B}}_2$. Note that the Theorem 2.2.9 (a) implies that we can assume $\rho_{H^x \cap K^y}(e) = 1$, for all $x, y \in G$. Using Corollary 2.2.11, we get

$$\begin{aligned} \Gamma(x,y) &= \int_{\frac{G}{H^x \cap K^y}} \int_{H^x \cap K^y} \beta_{x,y}(st) \frac{1}{\lambda_{H^x \cap K^y}(e, st)} \lambda_H(x, st)^{\frac{1}{p}} f(xst) \otimes_{x,y} \\ & \qquad \lambda_K(y, st)^{\frac{1}{q'}} g(yst) d\nu_H(s) d\mu_{x,y}(t). \qquad (4.40) \end{aligned}$$

But under the conditions (a), (b), (c) or (d) in Definition 4.1.7, (4.22) implies that, for $s \in H^x \cap K^y$,

$$\frac{\lambda_H(x, st)^{\frac{1}{p}} \lambda_K(y, st)^{\frac{1}{q'}}}{\lambda_{H^x \cap K^y}(e, st)} = \frac{\lambda_H(x, t)^{\frac{1}{p}} \lambda_K(y, t)^{\frac{1}{q'}}}{\lambda_{H^x \cap K^y}(e, t)}.$$

Therefore the integral (4.40) can be simplified to give

$$\begin{aligned} \Gamma(x,y) &= \int_{\frac{G}{H^x \cap K^y}} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes_{x,y} \lambda_K(x,t)^{\frac{1}{q'}} g(yt) \times \\ & \qquad \left(\int_{H^x \cap K^y} \beta_{x,y}(st) d\nu_{H^x \cap K^y}(s) \right) d\mu_{x,y}(t), \\ &= \int_{\frac{G}{H^x \cap K^y}} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes_{x,y} \lambda_K(x,t)^{\frac{1}{q'}} g(yt) d\mu_{x,y}(t). \end{aligned}$$

Hence the mapping given by (4.39) is continuous in the Banach semi-bundle $\underline{\mathcal{B}}_2$. In the case of (a), (b) or (c) in Definition 4.1.7, we can consider the mapping (4.39) as a cross-section of the Banach semi-bundle retraction $\underline{\mathcal{B}}_2^{\Delta\#}$ of $\underline{\mathcal{B}}_2^{\Delta}$ by the cannonical mapping $p : G \times G \to (G \times G)/\Delta$. By Proposition 3.2.8, this cross-section gives rise to the continuous cross-section in $\underline{\mathcal{B}}_2^{\Delta}$ given by (4.39), as required. In the case (d), the mapping given by (4.39) is continuous in the Banach semi-bundle $\underline{\mathcal{B}}_2$, as required.

□

Proposition 4.3.5

(1) *If A_p^q is constructed under the conditions (a), (b) or (c) of Definition 4.1.7, then $A_p^q \subseteq L_1(\underline{\mathcal{B}}_2^{\Delta}; \mu_{H,K})$. In particular, if G/H and G/K possess finite invariant measure and $1/p + 1/q' > 1$, then $A_p^q \subseteq L_r(\underline{\mathcal{B}}_2^{\Delta}; \mu_{H,K})$ where $1/r = 1/p + 1/q' - 1$.*

(2) *If A_p^q is constructed under the condition (d) and if $1/p + 1/q' > 1$, then $A_p^q \subseteq L_r(\underline{\mathcal{B}}_2^{\Delta}; \mu_{H,K})$ where $1/r = 1/p + 1/q' - 1$.*

(3) *If A_p^q is constructed under the condition (d) and if $1/p + 1/q' = 1$, then $A_p^q \subseteq L_\infty(\underline{\mathcal{B}}_2; \mu_{H\times K})$.*

Proof: (1) Consider the space A_p^q under any of the conditions (a) to (c) given in Definition 4.1.7. According to the calculations in Proposition 4.1.6, we see that

$$\int_{D(x,y)\in\Gamma} \|\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)(x,y)\| d\mu_{H,K}(x,y) \leq \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{q'},$$

for any $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes L_{q'}(\gamma)$, showing that $A_p^q \subseteq L_1(\underline{\mathcal{B}}_2^{\Delta}; \mu_{H,K})$.

If G/H and G/K have finite invariant measure μ_H and μ_K respectively, we see that μ_{H^x} and μ_{K^y} are finite invariant measures on G/H^x and G/K^y for $x, y \in G$. Hence $\lambda_{H^x}(z,t) = 1 = \lambda_{K^y}(w,t)$ for $z \in G/H^x$, $w \in G/K^y$ and $t \in G$. Using the identity (3.9), we see that $\lambda_{H^x\cap K^y} = 1$, for almost all $(x,y) \in (G\times G)/(H\times K)$. Therefore, for $f \in L_p(\pi)$,

$$\int_{\frac{G}{H^x\cap K^y}} \|f(xt)\|^p d\mu_{x,y}(t) = \int_{\frac{G}{H^x}} \int_{\frac{H^x}{H^x\cap K^y}} \frac{\lambda_{H^x\cap K^y}(\alpha,t)}{\lambda_{H^x}(\alpha,t)} \|f(x\alpha t)\|^p d\mu_{H_{x,y}}(\alpha) d\mu_{H^x}(t),$$

where $\mu_{H_{x,y}}$ is the measure on the coset space $H^x/(H^x \cap K^y)$ as defined in Lemma 3.1.1 which is finite and invariant (see also the proof of Proposition 4.1.6). Simplifying,

$$\begin{aligned}\int_{\frac{G}{H^x\cap K^y}} \|f(xt)\|^p d\mu_{x,y}(t) &= \int_{\frac{G}{H^x}} \int_{\frac{H^x}{H^x\cap K^y}} \|f(xt)\|^p d\mu_{H_{x,y}}(\alpha) d\mu_{H^x}(t) \\ &= \int_{\frac{G}{H^x}} \|f(xt)\|^p d\mu_{H^x}(t) \\ &= \|f\|_p^p.\end{aligned}$$

Similarly,

$$\int_{\frac{G}{H^x\cap K^y}} \|g(yt)\|^{q'} d\mu_{x,y}(t) = \|g\|_{q'}^{q'},$$

for $g \in L_{q'}$. Therefore, using Corollary 12.5 of Hewitt and Ross [26] we obtain

$$\begin{aligned}
&\int_{\frac{G}{H^x\cap K^y}} \|f(xt)\|\|g(yt)\| d\mu_{x,y}(t) \\
&= \int_{\frac{G}{H^x\cap K^y}} (\|f(xt)\|^p \|g(yt)\|^{q'})^{\frac{1}{r}} \|f(xt)\|^{1-\frac{p}{r}} \|g(yt)\|^{1-\frac{q'}{r}} d\mu_{x,y}(t), \\
&\leq \left(\int_{\frac{G}{H^x\cap K^y}} \|f(xt)\|^p \|g(yt)\|^{q'} d\mu_{x,y}(t)\right)^{\frac{1}{r}} \left(\int_{\frac{G}{H^x\cap K^y}} \|f(xt)\|^p d\mu_{x,y}(t)\right)^{\frac{q'-1}{q'}} \times \\
&\qquad \left(\int_{\frac{G}{H^x\cap K^y}} \|g(yt)\|^{q'} d\mu_{x,y}(t)\right)^{\frac{p-1}{p}}, \\
&= \left(\int_{\frac{G}{H^x\cap K^y}} \|f(xt)\|^p \|g(yt)\|^{q'} d\mu_{x,y}(t)\right)^{\frac{1}{r}} \|f\|_p^{p(\frac{q'-1}{q'})} \|g\|_{q'}^{q'(\frac{p-1}{p})},
\end{aligned}$$

which is similar to the right hand side of (4.31) (Proposition 4.1.6). Note that

$$\begin{aligned}
&\left(\int_{D(x,y)\in\Upsilon} \|\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)(x,y)\|^r d\mu_{H,K}(x,y)\right)^{\frac{1}{r}} \\
&\leq \left(\int_{D(x,y)\in\Upsilon} \left(\int_{\frac{G}{H^x\cap K^y}} \sum_{i=1}^{\infty} \|f_i(xt)\otimes g_i(yt)\| d\mu_{x,y}(t)\right)^r d\mu_{H,K}(D)\right)^{\frac{1}{r}} \\
&\leq \left(\int_{D(x,y)\in\Upsilon} \left(\sum_{i=1}^{\infty} \int_{\frac{G}{H^x\cap K^y}} \|f_i(xt)\|\|g_i(yt)\| d\mu_{x,y}(t)\right)^r d\mu_{H,K}(D)\right)^{\frac{1}{r}}
\end{aligned}$$

Using the same notation as in (4.31), by generalised Minkowski's inequality (see Dunford and Schwartz[10] p.529) we obtain

$$\begin{aligned}
&\left(\int_{D(x,y)\in\Upsilon} \|\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)(x,y)\|^r d\mu_{H,K}(x,y)\right)^{\frac{1}{r}} \\
&\leq \sum_{i=1}^{\infty} \left(\int_{D(x,y)\in\Upsilon} (I_i(x,y))^r d\mu_{H,K}(D)\right)^{\frac{1}{r}}
\end{aligned}$$

Hence using the same calculations which follow inequality (4.32), we achieve the required result

$$\|\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)\|_r \leq \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{q'}.$$

(2) This is evident from (4.35) (Proposition 4.1.6).

(3). Suppose that $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ are compact for almost all $(x, y) \in G \times G$, and $p = q$. Consider the supremum norm on $\mathcal{B}(\underline{\mathcal{B}})$. For any $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes L_{p'}(\gamma^*)$,

$\|\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)\|_\infty =$

ess $\sup_{(x,y)\in G\times G}\{\| \sum_{i=1}^{\infty} \int_{\frac{G}{H^x\cap K^y}} \frac{1}{\lambda(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{p'}} g_i(yt) d\mu_{x,y}(t)\|\}.$

Now, following the argument in Proposition 4.1.6, we see that

$$\|\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)\|_\infty \le \text{ess } \sup_{(x,y)}\{M_{x,y}N_{x,y}\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_{p'}\} \le S\sum_{i=1}^{\infty}\|f_i\|_p\|g_i\|_{p'},$$

where $S = \text{ess } \sup_{(x,y)} M_{x,y}N_{x,y}$ is a constant, as required.

□

4.4 Induced representations and Integral Intertwining Operators

In this section we shall investigate the possibility of generalising Rieffel's result (see Rieffel[36] Theorem 5.5) on classical A_p^q spaces which asserts that such a space is the predual of the space of intertwining operators if and only if those operators can be approximated, in the ultraweak*-operator topology, by integral operators. To begin, we shall give the definition of an integral operator from $L_p(\pi)$ to $L_q(\gamma)$, and discuss some of its properties.

Definition 4.4.1 *Let T be a bounded linear operator from $L_p(\pi)$ into $L_q(\gamma)$. T is called an* integral operator *if there exists a $\mu_H \times \mu_K$ measurable function Φ, called the* kernel *of T, from $G/H \times G/K$ to $\mathcal{L}(H(\pi), H(\gamma))$ such that for a given $f \in L_p(\pi)$,*

(1) *the function $x \mapsto \Phi(y,x)f(x)$ is integrable for almost all $y \in G/K$,*

(2) *$y \mapsto \int_{\frac{G}{H}} \Phi(y,x)f(x)d\mu_H(x)$ belongs to $L_q(\gamma)$ and*

(3) *$(Tf)(y) = \int_{\frac{G}{H}} \Phi(y,x)f(x)d\mu_H(x)$, for almost all $y \in G/K$.*

The next result describes the properties of the kernel of an intertwining integral operator. The existence of such operators will be discussed in Proposition 4.4.3 and in Section 4.4.1.

Proposition 4.4.2 *Let Φ be the kernel of a given integral intertwining operator for induced representations U_p^π and U_q^γ. Then Φ satisfies the following properties.*

(1) *For almost all $x \in G/H$, $y \in G/K$ and for all $s \in G$,*

$$\lambda_H(x,s^{-1})^{\frac{1}{p'}}\Phi(y,xs^{-1}) = \lambda_K(y,s)^{\frac{1}{q}}\Phi(ys,x). \tag{4.41}$$

(2) *For all $h \in H$, $k \in K$, and for almost all $x \in G/H$, $y \in G/K$,*

$$\Phi(ky, hx)\pi_h \quad = \quad \gamma_k \Phi(y, x). \tag{4.42}$$

(3) *Under the conditions given in Definition 4.1.7, $\Phi(y, x)$ is an intertwining operator of the representations π^x and γ^y of the subgroup $H^x \cap K^y$ of G for almost all $x \in G/H$ and $y \in G/K$.*

Proof : (1) Suppose that T is an integral operator from $L_p(\pi)$ to $L_q(\gamma)$ with the kernel Φ. Then for $f \in L_p(\pi)$ and $y \in G$,

$$(Tf)(y) = \int_{\frac{G}{H}} \Phi(y, x) f(x) d\mu_H(x).$$

In addition, if $T \in Hom_G(L_p(\pi), L_q(\gamma))$ then

$$(TU_s^\pi f)(y) = (U_s^\gamma Tf)(y) \text{ for almost all } y \in G/K \text{ and for } s \in G.$$

Now

$$\begin{aligned} (TU_s^\pi f)(y) &= \int_{\frac{G}{H}} \Phi(y, x)(U_s^\pi f) d\mu_H(x), \\ &= \int_{\frac{G}{H}} \Phi(y, x) \lambda_H(x, s)^{\frac{1}{p}} f(xs) d\mu_H(x). \end{aligned}$$

Changing variables $xs \mapsto x$, we find

$$(TU_s^\pi f)(y) = \int_{\frac{G}{H}} \Phi(y, xs^{-1}) \lambda_H(x, s^{-1}) \lambda_H(xs^{-1}, s)^{\frac{1}{p}} f(x) d\mu_H(x).$$

Since, by Lemma 2.2.7(a), $\lambda_H(x, s^{-1})\lambda_H(xs^{-1}, s) = 1$, the above integral simplifies to

$$(TU_s^\pi f)(y) \quad = \quad \int_{\frac{G}{H}} \Phi(y, xs^{-1}) \lambda_H(x, s^{-1})^{\frac{1}{p'}} f(x) d\mu_H(x). \tag{4.43}$$

On the other hand,

$$\begin{aligned} (U_s^\gamma Tf)(y) &= \lambda_K(y, s)^{\frac{1}{q}} (Tf)(ys), \\ &= \lambda_K(y, s)^{\frac{1}{q}} \int_{\frac{G}{H}} \Phi(ys, x) f(x) d\mu_H(x). \end{aligned} \tag{4.44}$$

Therefore, by (4.43) and (4.44), property (1) follows.

(2) For $k \in K$ and $y \in G$,

$$\begin{aligned} \gamma_k(Tf)(y) = (Tf)(ky) &= \int_{\frac{G}{H}} \Phi(ky, x) f(x) d\mu_H(x), \\ &= \int_{\frac{G}{H}} \Phi(ky, hx) \pi_h f(x) d\mu_H(x), \end{aligned} \tag{4.45}$$

for $h \in H$. On the other hand,

$$\gamma_k(Tf)(y) = \gamma_k \int_{\frac{G}{H}} \Phi(y,x)f(x)d\mu_H(x). \tag{4.46}$$

It is clear that property (2) follows from (4.45) and (4.46).

(3) We want to show that

$$\gamma_b^y \Phi(y,x) = \Phi(y,x)\pi_b^x, \tag{4.47}$$

for all $b \in H^x \cap K^y$ and for almost all $x \in G/H$ and $y \in G/K$. For any $b \in H^x \cap K^y$ we have $b = y^{-1}ky = x^{-1}hx$ for some $h \in H$ and $k \in K$. Using (4.42),

$$\gamma_{yby^{-1}}\Phi(y,x) = \Phi(yby^{-1}y, xbx^{-1}x)\pi_{xbx^{-1}},$$

which implies

$$\begin{aligned}
\gamma_b^y \Phi(y,x) &= \Phi(yb,xb)\pi_b^x, \\
&= \frac{\lambda_H(xb,b^{-1})^{\frac{1}{p'}}}{\lambda_K(y,b)^{\frac{1}{q}}}\Phi(y,x)\pi_b^x, \text{ by (4.41) }, \\
&= \frac{1}{\lambda_H(x,b)^{\frac{1}{p'}}\lambda_K(y,b)^{\frac{1}{q}}}\Phi(y,x)\pi_b^x, \text{ by Lemma 2.2.7 (a)} \qquad (4.48)
\end{aligned}$$

Under conditions (a), (b) or (c) of Def.4.1.7, (4.48) simplifies to (4.47), as required.

Now suppose that the condition given in (d) of Definition 4.1.7 applies. Consider the right hand side of (4.48). We see that

$$\begin{aligned}
\frac{1}{\lambda_H(x,b)^{\frac{1}{p'}}\lambda_K(y,b)^{\frac{1}{q}}} &= \left(\frac{\lambda_{H^x\cap K^y}(e,b)}{\lambda_K(y,b)}\right)^{\frac{1}{p'}}\left(\frac{\lambda_{H^x\cap K^y}(e,b)}{\lambda_H(x,b)}\right)^{\frac{1}{q}}, \text{ by (3.9)}, \\
&= \left(\frac{\lambda_{H^x\cap K^y}(e,b)}{\lambda_{K^y}(e,b)}\right)^{\frac{1}{p'}}\left(\frac{\lambda_{H^x\cap K^y}(e,b)}{\lambda_{H^x}(e,b)}\right)^{\frac{1}{q}}, \text{ by Lemma 3.1.2.}
\end{aligned}$$

Under the condition that $H^x/(H^x\cap K^y)$ and $K^y/(H^x\cap K^y)$ have invariant measure, we have

$$\frac{\lambda_{H^x\cap K^y}(e,b)}{\lambda_{H^x}(e,b)} = \frac{\lambda_{H^x\cap K^y}(e,b)}{\lambda_{K^y}(e,b)} = 1$$

(see (4.19) in the proof of Proposition 4.1.5). Therefore,

$$\gamma_b^y \Phi(y,x) = \Phi(y,x)\pi_b^x, \tag{4.49}$$

for all $b \in H^x \cap K^y$ and for almost all $x \in G/H$ and $y \in G/K$. Hence the result.

□

Following an argument similar to that of Moore[34], we shall obtain a result for intertwining operators between $L_1(\pi)$ and $L_q(\gamma), q > 1$.

Proposition 4.4.3 *Let U_1^π and U_q^γ be induced representations of the locally compact group G with the corresponding Banach spaces of functions $L_1(\pi)$ and $L_q(\gamma)$ $(q > 1)$, respectively. Then, if the Banach space $\mathcal{H}(\pi)$ is separable, the intertwining operators T for these representations are integral operators with the corresponding kernel Φ satisfying*

$$ess\ sup_{x\in\frac{G}{H}}(\int_{\frac{G}{K}} \|\Phi(y,x)\|^q d\mu_K(y))^{\frac{1}{q}} \leq \|T\|.$$

Proof : The proof is in two parts:

(1). Let S and R be fixed Borel cross-sections of H and K in G. Then $G/H \simeq S$, $G/K \simeq R$ and we regard μ_H and μ_K as measures on S and R . Let C be a continuous linear map of $L_1(S, \mathcal{H}(\pi), \mu_{\mathcal{H}})$ into $L_q(R, \mathcal{H}(\gamma), \mu_K)$. Firstly we prove that C is an integral operator. For $u \in H(\pi)$ define $C_u : L_1(S, \mu_H) \to L_q(R, \mathcal{H}(\gamma), \mu_K)$ by

$$C_u(g) \quad = \quad C(gu), \tag{4.50}$$

for $g \in L_1(S, \mu_H)$. $C_u(g)$ is bounded since $\|C_u\| \leq \|C\|.\|u\|$. Then by Dunford and Schwartz[10], Theorem 10, p.507, there exists a μ_H-essentially unique bounded measurable function $\chi_u(.)$ on S to a weakly compact subset of $L_q(R, \mathcal{H}(\gamma), \mu_K)$ such that

$$C_u(g) = \int_S \chi_u(s) g(s) d\mu_H(s)$$

and $\|C_u\| = \text{ess sup}\ \|\chi_u(s)\|$. Let $K_u(., s) = \chi_u(s)$ so that $K_u : R\times S \mapsto H(\gamma)$. Then K_u is $\mu_H \times \mu_K$ measurable (see Dunford and Schwartz[10], Theorem 17 p.198), and we have

$$(C_u(g))(t) \quad = \quad \int_S g(s) K_u(t, s) d\mu(s) \tag{4.51}$$

with ess $\sup_{s\in S}(\int_R \|K_u(t, s)\|^q d\mu_K(t))^{\frac{1}{q}} \leq \|C\|\|u\|$. Now let Λ_0 be a countable dense subset of the scalar field Λ of $\mathcal{H}(\pi)$ and let A be a countable dense subset of $\mathcal{H}(\pi)$. Then, since the countable number of relations

$$K_{\alpha u}(t, s) + K_{\beta w}(t, s) = K_{\alpha u+\beta w}(t, s) \text{ and } \| K_u(., s) \|\leq\| C \| \cdot \| u \|$$

for $\alpha, \beta \in \Lambda_0$ and $u, w \in A_0$ fails to hold only on a $\mu_K \times \mu_H$ null set, it is possible to find a null set N in $R\times S$ so that they all simultaneously subtend for $(t, s) \notin N$. Then the map $u \mapsto K_u(t, s)$ uniquely extends to a bounded linear map $K(t, s)$ of $\mathcal{H}(\pi)$ to $\mathcal{H}(\gamma)$ with $\| K(s, t) \|\leq\| C \|$. Now define the map K on $R \times S$ such that $K(t, s) = 0$ for $(t, s) \in N$ and $K(t, s)u = K_u(t, s)$ otherwise, for each $u \in \mathcal{H}(\pi)$. Then, by (4.50) and (4.51), we have

$$C(gu)(t) = \int_S K(t, s) g(s) u d\mu(s)$$

and ess $\sup_{s\in S}(\int_R \frac{\|K(t,s)u\|}{\|u\|}^q d\mu_K(t))^{\frac{1}{q}} \leq \|C\|$ for any $u \in H(\pi)$, which implies that ess $\sup_{s\in S}(\int_R \|K(t, s)\|^q d\mu_K(t))^{\frac{1}{q}} \leq \|C\|$. Hence for $g \in L_1(S, \mathcal{H}(\pi), \mu_{\mathcal{H}})$ we have

$$(Cg)(t) = \int_S K(t, s) g(s) d\mu_H(s). \tag{4.52}$$

(2). Secondly, we prove that the intertwining operators T from $L_1(\pi)$ to $L_q(\gamma)$ are integral operators.

Observing that $G \simeq H \times S$, for a given continuous function $f' \in L_1(S, \mathcal{H}(\pi), \mu_H)$ we can define a function $(\Phi_1 f') \in L_1(\pi)$ by

$$(\Phi_1 f')(y) = \pi(h) f'(s),$$

where $y \in G$ with $y = hs$ for $h \in H$ and $s \in S$. Similarly, since $G \simeq K \times R$, for a given continuous function $g \in L_q(\gamma)$, we define the function $(\Phi_q g) \in L_q(R, \mathcal{H}(\gamma), \mu_K)$ by

$$(\Phi_q g)(r) = g(r)$$

for $r \in R$. Clearly, $\|f'\|_1 = \|(\Phi_1 f')\|_1$ and $\|g\|_q = \|(\Phi_q g)\|$.

For a given intertwining operator T from $L_1(\pi)$ to $L_q(\gamma)$ we define an operator $\tilde{T}$ on the space of continuous functions in $L_1(S, \mathcal{H}(\pi), \mu_H)$ to $L_q(R, \mathcal{H}(\gamma), \mu_K)$ by

$$\tilde{T} := \Phi_q T \Phi_1.$$

Since the space of continuous functions in $L_1(S, \mathcal{H}(\pi), \mu_H)$ is dense in $L_1(S, \mathcal{H}(\pi), \mu_H)$, we have the following commutative diagram:

$$L_1(S, \mathcal{H}(\pi), \mu_H) \overset{\tilde{T}}{\mapsto} L_q(R, \mathcal{H}(\gamma), \mu_K)$$

$$\Phi_1 \downarrow \qquad\qquad \uparrow \Phi_q$$

$$L_1(\pi) \overset{T}{\mapsto} L_q(\gamma)$$

with $\tilde{T}(f') = \Phi_q T \Phi_1(f')$ for $f' \in L_1(S, \mathcal{H}(\pi), \mu_H)$. Clearly, $\|T\| = \|\tilde{T}\|$. Using the result in part (1), we see that there exists a map K from $S \times R$ to the set of bounded linear maps from $\mathcal{H}(\pi)$ to $\mathcal{H}(\gamma)$ such that

$$(\tilde{T} f')(t) = \int_S K(t, s) f'(s) d\mu_H(s),$$

for $f' \in L^1(S, \mathcal{H}(\pi), \mu_{\mathcal{H}})$ and $t \in R$. Using the Borel isomorphism $G \simeq K \times R$ any $y \in G$ can be written as $y = k(e, y)\ell(e, y)$ where $k(e, y) \in K$ and $\ell(e, y) \in R$. Both k and ℓ are Borel functions on $R \times G$. Then, for $f \in L_1(\pi)$

$$\begin{aligned}(Tf)(y) = (\Phi_q{}^{-1} \tilde{T} \Phi_1{}^{-1}(f))(y) &= \gamma(k(e, y))((\tilde{T}\Phi_1{}^{-1})f)(\ell(e, y)), \\ &= \gamma(k(e, y)) \int_S K(\ell(e, y), s)((\Phi_1{}^{-1})f)(s) d\mu_H(s).\end{aligned}$$

But since $(\Phi_1{}^{-1} f)(s) = f(s)$, we have

$$(Tf)(y) \;=\; \gamma(k(e, y)) \int_S K(\ell(e, y), s) f(s) d\mu_H(s).$$

Now the Borel isomorphism $G \simeq H \times S$, allows us to express any $x \in G$ in the form $x = h(e, x)m(e, x)$, where h and m are Borel functions on $H \times S$, $h(e, x) \in H$ and

$m(e,x) \in S$. If we define $\Phi(y,x) = \gamma(k(e,y))K(\ell(e,y), m(e,x))\pi(h(e,x))^*$, then we have $\|\Phi(y,x)\| = \|K(\ell(e,y), m(e,x))\|$ and

$$(Tf)(y) = \int_S \Phi(y,s)f(s)ds \;=\; \int_{\frac{G}{H}} \Phi(y,x)f(x)d\mu_H(x), \tag{4.53}$$

with ess $\sup_{x\in G/H}(\int_{G/K} \|\Phi(y,x)\|^q d\mu_K(y))^{\frac{1}{q}} \leq \|T\|$. Therefore T is an integral operator.

□

4.4.1 Group theoretic conditions for integral intertwining operators

In the case where both G/H and G/K possess finite invariant measures, we can show that the intertwining operators can be approximated by integral operators at least under the conditions given in the Lemmas 4.4.4 and 4.4.5.

A set U in G is said to be **invariant under the action of** H (or H-**invariant**) if for each $u \in U$, $h^{-1}uh = u$ for all $h \in H$.

Lemma 4.4.4 *Suppose G/K and G/H have finite invariant measures. Suppose further that every neighborhood of e contains a neighborhood which is invariant under the action of H. Then, it is possible to select an approximate identity $\{J_N : N \in \mathcal{G}\}$, of norm 1 in $L_1(G)$, where $\mathcal{G}$ is a directed set of H-invariant neighborhoods of the identity e of G, so that the operator defined by $T_{J_N}f := J_N * f$ is an intertwining operator from $L_1(\pi)$ to $L_p(\pi)$. Moreover, $J_N * f \to f$ for $f \in L_p(\pi)$.*

Proof : We show that we can select an approximate identity $\{J_N\}$ of norm 1 in $L_1(G)$ so that

(a) the convolution $(J_N * f)$ is well defined in the sense that $\int_G \|J_N(xy^{-1})f(y)\|d\nu_G(y) < \infty$,

(b) $T_{J_N}f \in L_p(\pi)$ for all $f \in L_1(\pi)$,

(c) T_{J_N} is an intertwining operator for the two spaces involved, and

(d) $J_N * f \to f$ for $f \in L_p(\pi)$.

Let J be an element of $L_1(G)$ with compact support.

(a) For $f \in L_1(\pi)$,

$$(T_J f)(x) = (J * f)(x) = \int_G J(xy^{-1})f(y)d\nu_G(y).$$

As G/H has invariant measure, by Corollary 2.2.11, we get

$$\begin{aligned}
&\int_G \|J(xy^{-1})f(y)\|d\nu_G(y) \\
&= \int_{\frac{G}{H}}\int_H \|J(xy^{-1}h^{-1})f(hy)\|d\nu_H(h)d\mu_H(y), \\
&\leq \int_{\frac{G}{H}} \|f(y)\| \int_H |J(xy^{-1}h^{-1})|d\nu_H(h)d\mu_H(y).
\end{aligned} \tag{4.54}$$

Let $F(u) = \int_H |J(uh^{-1})|d\nu_H(h)$. Since F has compact support and is bounded, there exists a constant C such that

$$\int_{\frac{G}{H}} \|f(y)\|F(xy^{-1})d\mu_H(y) \leq C\int_{\frac{G}{H}} \|f(y)\|d\mu_H(y) = C\|f\|_1,$$

which implies that $\int_G \|J(xy^{-1})f(y)\|d\nu_G(y) < \infty$, as required.

(b) Let $f \in L_1(\pi)$ and J be an element of $L_1(G)$ with compact support and F be defined as in (a) above. Then,

$$\begin{aligned}
&\left(\int_{\frac{G}{H}} \|(J*f)(x)\|^p d\mu_H(x)\right)^{\frac{1}{p}} \\
&\leq \left(\int_{\frac{G}{H}}\left(\int_{\frac{G}{H}} \|f(y)\||F(xy^{-1})|d\mu_H(y)\right)^p d\mu_H(x)\right)^{\frac{1}{p}}, \text{ using (4.54)}, \qquad (4.55)\\
&\leq \int_{\frac{G}{H}}\left(\int_{\frac{G}{H}} \|f(y)\|^p|F(xy^{-1})|^p d\mu_H(x)\right)^{\frac{1}{p}} d\mu_H(y), \text{ by Minkowski's inequality ,} \\
&= \int_{\frac{G}{H}} \|f(y)\|\left(\int_{\frac{G}{H}} |F(xy^{-1})|^p d\mu_H(x)\right)^{\frac{1}{p}} d\mu_H(y).
\end{aligned}$$

Since $F(u)$ is bounded and has compact support, there exists a constant D such that

$$\int_{\frac{G}{H}} \|f(y)\|\left(\int_{\frac{G}{H}} |F(xy^{-1})|^p d\mu_H(x)\right)^{\frac{1}{p}} d\mu_H(y) \leq D\int_{\frac{G}{H}} \|f(y)\|d\mu_H(y) = D\|f\|_1,$$

which is finite as $f \in L_1(\pi)$. To complete the proof of (b), we need to show that $J*f$ satisfies the covariance condition $(J*f)(hx) = \pi(h)(J*f)(x)$, for all $h \in H$ and $x \in G$. Assume that we have selected a directed set $\mathcal{G}$ of H-invariant neighborhoods of the identity e of G, which is appropriate to obtain the limit in (d). For such a neighborhood $N \in \mathcal{G}$, let J_N be a function in $L_1(G)$ which is a positive scalar multiple of the characteristic function of N and which has $L_1(G)$-norm 1. We have

$$(J_N * f)(hx) = \int_G J_N(hxy^{-1})f(y)d\nu_G(y).$$

Changing variables $y \mapsto hy$, we find,

$$(J_N * f)(hx) = \int_G J_N(hxy^{-1}h^{-1})f(hy)d\nu_G(y).$$

But, under our assumption on N, $J_N(hxy^{-1}h^{-1}) = J_N(xy^{-1})$ for all $x, y \in G$. Therefore,

$$\begin{aligned}(J_N * f)(hx) &= \pi(h)\int_G J_N(xy^{-1})f(y)d\nu_G(y)\\ &= \pi(h)(J_N * f)(x).\end{aligned}$$

Thus, $T_{J_N}f \in L_p(\pi)$ for all $f \in L_1(\pi)$.

(c) Let J be of the form J_N as defined in part (b). We show that T_J is an intertwining operator. For $f \in L_1(\pi)$ and $s \in G$,

$$(U_x^\pi T_J f)(s) = (T_J f)(sx) = \int_G J(sxy^{-1}).f(y)d\nu_G(y)$$

Changing variables $y \mapsto yx$,

$$\begin{aligned}(U_x^\pi T_J f)(s) &= \int_G J(sy^{-1})f(yx)d\nu_G(y),\\ &= \int_G J(sy^{-1})(U_x^\pi f)(y)d\nu_G(y),\\ &= (T_J(U_x^\pi f))(s).\end{aligned}$$

Hence T_J is an intertwining operator for the spaces $L_1(\pi)$ and $L_p(\pi)$.

(d) Let $f \in L_p(\pi)$ with compact support. Then,

$$\begin{aligned}\|J * f - f\|_p &= \left(\int_{\frac{G}{H}} \|J * f(x) - f(x)\|^p d\mu_H(x)\right)^{\frac{1}{p}},\\ &= \left(\int_{\frac{G}{H}} \|\int_G J(y^{-1})f(yx)d\nu_G(y) - f(x)\|^p d\mu_H(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $m = \int_G J(y^{-1})d\nu_G(y)$, we can now write the above in the form

$$\begin{aligned}&\left(\int_{\frac{G}{H}} \|\int_G J(y^{-1})f(yx)d\nu_G(y) - \frac{1}{m}\int_G J(y^{-1})f(x)d\nu_G(y)\|^p d\mu_H(x)\right)^{\frac{1}{p}}\\ &\leq \left(\int_{\frac{G}{H}}\left(\int_G \|\frac{J(y^{-1})}{m}(mf(yx) - f(x))\|d\nu_G(y)\right)^p d\mu_H(x)\right)^{\frac{1}{p}},\\ &\leq \int_G\left(\int_{\frac{G}{H}} \|\frac{J(y^{-1})}{m}(mf(yx) - f(x))\|^p d\mu_H(x)\right)^{\frac{1}{p}} d\nu_G(y),\end{aligned}$$

by Minkowski's inequality (see, for example, Hewitt and Ross[26], p.138). Hence

$$\|J * f - f\|_p \leq \int_G \frac{J(y^{-1})}{m}\|m({}_yf) - f\|_p d\nu_G(y). \tag{4.56}$$

Since $f \in L_p(\pi)$ has compact support, then by Lemma 3.3.4, f is of the form θ_ℓ for a suitable $\ell \in C_0(G, \mathcal{H}(\pi))$. This means that for $x \in G$, we have

$$f(x) = \theta_\ell(x) = \int_H \pi(h^{-1})\ell(hx)d\nu_H(h).$$

Now

$$\begin{aligned}\|_yf - f\|_p &= \Big(\int_{\frac{G}{H}} \|\theta_\ell(yx) - \theta_\ell(x)\|^p d\mu_H(x)\Big)^{\frac{1}{p}}, \\ &\leq \Big(\int_{\frac{G}{H}} \Big(\int_H \|\ell(hyx) - \ell(hx)\| d\nu_H(h)\Big)^p d\mu_H(x)\Big)^{\frac{1}{p}}.\end{aligned}$$

In order to estimate the measure of the set where the integrand does not vanish we proceed as follows. Let S be the compact support of the continuous function ℓ. If $a \in G$ but $a \notin HS$ then $a^{-1} \notin S^{-1}H$ and so $Sa^{-1} \cap H = \emptyset$. Hence if $a \notin HS$ then $\nu_H(Sa^{-1} \cap H) = 0$. If $a \in HS$ then $a^{-1} \in S^{-1}h$ for some $h \in H$ and so $Sa^{-1} \subseteq SS^{-1}h$. Hence

$$\nu_H(Sa^{-1} \cap H) \leq \nu_H(SS^{-1}h \cap H) = \nu_H(SS^{-1} \cap H).$$

We have proved that for every $a \in G$ we have $\nu_H(Sa^{-1} \cap H) < \infty$. Since ℓ has compact support it is uniformly continuous from the right (see Lemma 3.3.7). Therefore, given $\epsilon > 0$, there is a neighborhood N of e in G such that $\|\ell(yhx) - \ell(hx)\| < \epsilon$ for every $h \in H$ provided $y \in N$. Let N_ϵ be a neighborhood of the identity contained in N which is invariant under the action of H. If we choose J_ϵ such that $\int_G J_\epsilon(y) d\nu_G(y) = 1$ and $J_\epsilon(y) = 0$ for $y \notin N_\epsilon$ then

$$\|\ell(hyx) - \ell(hx)\| < \|\ell(yhx) - \ell(hx)\| < \epsilon$$

for every $h \in H$ and $y \in N_\epsilon$. Hence

$$\|_yf - f\|_p \leq \epsilon \nu_H(SS^{-1} \cap H)\mu_H(G/H).$$

However, since Δ is continuous, if N_ϵ is sufficiently small we can approximate 1 by $\Delta(y^{-1})$ in N_ϵ and then m is arbitrarily close to 1 since

$$\int_G J_\epsilon(y^{-1})\Delta(y^{-1}) d\nu_G(y) \approx \int_G J_\epsilon(y) d\nu_G(y) = 1.$$

Therefore, (4.56) gives,

$$\|J * f - f\|_p \leq \epsilon \nu_H(SS^{-1} \cap H)\mu_H(G/H).$$

Now let $\mathcal{G}$ denote the set of all neighborhoods N of the identity of G, symmetric and invariant under H. Letting $N_1 \leq N_2$ if $N_1 \supseteq N_2$ we obtain a directed set. For a given N, choosing a non-negative function J_N from $L_1(G)$ such that $\int_G J_N(x) d\nu_G(x) = 1$ and $J_N(x) = 0$ for $x \notin N$, we can construct a net $J_N (N \in \mathcal{G})$. From above we see that $J_N * f \to f$ for every $f \in L_p(\pi)$ as required in (d).

□

Lemma 4.4.5 *Suppose G/K and G/H have finite invariant measures. Suppose further that $H \lhd G$ and there is a continuous map $x \mapsto A_x$ from G to the set of isometries from $\mathcal{H}(\pi)$ to itself such that*

$$\pi(n^x) = A_x^{-1}\pi(n)A_x,$$

for $n \in H$. Then, it is possible to select an approximate identity $\{J_N : N \in \mathcal{G}\}$ of norm 1 in $L_1(G)$, where $\mathcal{G}$ is a directed set of neighborhoods of the identity e of G, such that the operator $\widetilde{T}_{J_N}$ defined by

$$(\widetilde{T}_{J_N} f)(x) := (J_N \star f)(x) = \int_G J_N(xy^{-1})A_{xy^{-1}}f(y)d\nu_G(y)$$

is an intertwining operator from $L_1(\pi)$ to $L_p(\pi)$. Moreover, $J_N \star f \to f$ for $f \in L_p(\pi)$.

(Note the different notations $J_N * f$ and $J_N \star f$ of convolutions defined in Lemmas 4.4.4 and 4.4.5, respectively.)

Proof: Let $J \in L_1(G)$ be with compact support and $\int_G J(y)d\nu_G(y) = 1$. We follow the same steps as in lemma 4.4.4.
(a) First we shall show that

$$(\widetilde{T}_J f)(x) := (J \star f)(x) = \int_G J(xy^{-1})A_{xy^{-1}}f(y)d\nu_G(y)$$

is finite, for $f \in L_1(\pi)$. Using Corollary 2.2.11, and the fact that G/H has invariant measure, we have

$$\begin{aligned}
&\int_G \|J(xy^{-1})A_{xy^{-1}}f(y)\|d\nu_G(y) \\
&\quad = \int_{\frac{G}{H}}\int_H \|J(xy^{-1}h^{-1})A_{xy^{-1}}f(hy)\ \|\ d\nu_H(h)d\mu_H(y), \\
&\quad \leq \int_{\frac{G}{H}}\int_H |J(xy^{-1}h^{-1})|\|A_{xy^{-1}}f(hy)\|d\nu_H(h)d\mu_H(y), \\
&\quad = \int_{\frac{G}{H}} \|f(y)\| \int_H |J(xy^{-1}h^{-1})|d\nu_H(h)d\mu_H(y), \qquad (4.57)
\end{aligned}$$

where the last equality is obtained by using the fact that A_x is an isometry for $x \in G$. This is the same as the eqn (4.54) in lemma 4.4.4; and therefore, following the same reasoning, we see that

$$\int_G \|J(xy^{-1})A_{xy^{-1}}f(y)\|d\nu_G(y) < \infty.$$

(b) We shall show that $(J \star f) \in L_p(\pi)$ for $f \in L_1(\pi)$.

$$\left(\int_{\frac{G}{H}} \|(J \star f)(x)\|^p d\mu_H(x)\right)^{\frac{1}{p}}$$

$$= \left(\int_{\frac{G}{H}} \| \int_G J(xy^{-1}) A_{xy^{-1}} f(y) d\mu_H(y) \|^p d\mu_H(x) \right)^{\frac{1}{p}},$$
$$\leq \left(\int_{\frac{G}{H}} \left(\int_{\frac{G}{H}} \|f(y)\| |F(xy^{-1})| d\mu_H(y) \right)^p d\mu_H(x) \right)^{\frac{1}{p}},$$

by (4.56), where $F(u) = \int_H |J(uh^{-1})| d\nu_H(h)$ as in the proof of Lemma 4.4.4 (a). The last inequality above is the same as (4.56) in part (b) of previous lemma.

We shall now show that $(J \star f)$ satisfies the covariance condition. For $x \in G$ and $h \in H$,

$$(J \star f)(hx) = \int_G J(y^{-1}) A_{y^{-1}} f(yhx) d\nu_G(y).$$

Now $f(yhx) = f(yhy^{-1}yx) = \pi(yhy^{-1}) f(yx)$ since $H \lhd G$, and π satisfies the condition $\pi(yhy^{-1}) = \pi(h^{y^{-1}}) = A_{y^{-1}}^{-1} \pi(h) A_{y^{-1}}$ for $y \in G$ and $h \in H$. Thus

$$\begin{aligned} (J \star f)(hx) &= \pi(h) \int_G J(y^{-1}) A_{y^{-1}} f(yx) d\nu_G(y), \\ &= \pi(h)(J \star f)(x). \end{aligned}$$

for $h \in H$ and $x \in G$.

(c) We shall show that $\widetilde{T}_J$ is an intertwining operator. For $x, s \in G$,

$$(U_x^\pi \widetilde{T}_J f)(s) = \int_G J(sxy^{-1}) A_{sxy^{-1}} f(y) d\nu_G(y).$$

Changing variables $y \mapsto yx$,

$$\begin{aligned} (U_x^\pi \widetilde{T}_J f)(s) &= \int_G J(sy^{-1}) A_{sy^{-1}} f(yx) d\nu_G(y), \\ &= \int_G J(sy^{-1}) A_{sy^{-1}} (U_x^\pi f)(y) d\nu_G(y), \\ &= (T_J(U_x^\pi f))(s), \end{aligned}$$

as required.

(d) Let $f \in L_p(\pi)$ with compact support. Then,

$$\begin{aligned} \|J \star f - f\|_p &= \left(\int_{\frac{G}{H}} \|J \star f(x) - f(x)\|^p d\mu_H(x) \right)^{\frac{1}{p}} \\ &= \left(\int_{\frac{G}{H}} \| \int_G J(y^{-1}) A_{y^{-1}} f(yx) d\nu_G(y) - f(x) \|^p d\mu_H(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Letting $m = \int_G J(y^{-1}) d\nu_G(y)$, we can now write the above in the form

$$\left(\int_{\frac{G}{H}} \| \int_G J(y^{-1}) A_{y^{-1}} f(yx) d\nu_G(y) - \frac{1}{m} \int_G J(y^{-1}) f(x) d\nu_G(y) \|^p d\mu_H(x) \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\frac{G}{H}}\left(\int_G \|\frac{J(y^{-1})}{m}(mA_{y^{-1}}f(yx) - f(x))\|d\nu_G(y)\right)^p d\mu_H(x)\right)^{\frac{1}{p}},$$
$$\leq \int_G\left(\int_{\frac{G}{H}} \|\frac{J(y^{-1})}{m}(mA_{y^{-1}}f(yx) - f(x))\|^p d\mu_H(x)\right)^{\frac{1}{p}} d\nu_G(y),$$

by Minkowski's inequality. Hence

$$\|J \star f - f\|_p \leq \int_G \frac{J(y^{-1})}{m}\|m(A_{y^{-1}})_y f - f\|_p d\nu_G(y).$$

But

$$\begin{aligned}\|m(A_{y^{-1}})_y f - f\|_p &\leq \|A_{y^{-1}}(m({}_y f) - f)\|_p + \|(A_{y^{-1}}f - f)\|, \\ &\leq \|(m({}_y f) - f)\|_p + \|(A_{y^{-1}}f - f)\|.\end{aligned}$$

Now $y \mapsto A_{y^{-1}}$ being continuous, for a given ϵ it is possible to find a neighborhood U of the identity e such that $\|(A_{y^{-1}}f - f)\| \leq \epsilon/2$ for all $y \in U$. We could choose μ_H to be the Haar measure on G/H since H is a normal subgroup of G. Then we see that the map $y \mapsto {}_y f$ of G/H into $L_p(\pi)$ is uniformly continuous from the right. Hence, there exists a neighborhood N of the identity e in G (considering the canonical mapping $G \mapsto G/H$,) such that $\|{}_y f - f\|_p < \epsilon/2$. Let $N_\epsilon = N \cap U$. Now we choose J_ϵ such that $\int_G J_\epsilon(y)d\nu_G = 1$ and $J(y) = 0$ for $y \notin N_\epsilon$. Since Δ is continuous, if N_ϵ is sufficiently small we can approximate 1 by $\Delta(y^{-1})$ in N_ϵ and then m is arbitrarily close to 1 since

$$\int_G J_\epsilon(y^{-1})\Delta(y^{-1})d\nu_G(y) \approx \int_G J_\epsilon(y)d\nu_G(y) = 1.$$

Then we have

$$\|J \star f - f\|_p \leq \int_G \frac{J(y^{-1})}{m}\|m_y f - f\|_p d\nu_G(y) + \epsilon/2 < \epsilon.$$

Therefore, using the same argument as in the last paragraph of the proof of the previous Lemma, we conclude that we can construct a net J_N for $N \in \mathcal{G}$, where $\mathcal{G}$ denotes a directed set of symmetric neighborhoods of the identity e, such that $J_N \star f \to f$ for all $f \in L_p(\pi)$.

□

Lemma 4.4.6 *Suppose G/K and G/H have finite invariant measures. If either*

(a) *every neighborhood of e contains a neighborhood which is invariant under the action of H,*

or

(b) *$H \lhd G$ and there is a continuous map $x \mapsto A_x$ from G to the set of isometries from $\mathcal{H}(\pi)$ to itself such that*

$$\pi(n^x) = A_x^{-1}\pi(n)A_x, \text{ for every } n \in H,$$

then the intertwining operators of the induced representations U_p^π and U_q^γ can be approximated in the strong operator topology by integral operators.

Proof: Let $T : L_p(\pi) \mapsto L_q(\gamma)$ be an intertwining operator.

(a) Let T_{J_N} be defined as in Lemma 4.4.4. Then $T \circ T_{J_N} : L_1(\pi) \mapsto L_q(\gamma)$ is an intertwining operator. By Proposition 4.4.3, $T \circ T_{J_N}$ is an integral operator.

Now

$$\|T \circ T_{J_N}(f) - T(f)\| \quad = \quad \|T(J_N * f - f)\|. \tag{4.58}$$

Lemma 4.4.4 asserts that the expression on the right hand side of (4.58) can be made as small as we wish. Hence the intertwining operator $T : L_p(\pi) \mapsto L_q(\gamma)$ can be approximated by the integral operators $T \circ T_{J_N}$ in the strong operator topology.

(b) Let $\widetilde{T}_{J_N}$ be defined as in Lemma 4.4.5. Then, $T \circ \widetilde{T}_{J_N}$ is an integral intertwining operator from $L_1(\pi)$ to $L_q(\gamma)$. Moreover,

$$\|T \circ \widetilde{T}_{J_N}(f) - T(f)\| \quad = \quad \|T(J_N \star f - f)\|. \tag{4.59}$$

By Lemma 4.4.5, the right hand side of (4.59) can be made arbitrarily small, hence we have the required result.

□

4.4.2 The space A_p^q as the predual of the space of intertwining operators

We are now in a position to state the main result of this section, which is a generalisation of Rieffel's result([36] Theorem 5.5) on classical A_p^q spaces.

Theorem 4.4.7 *Suppose that the space A_p^q, $(q' > 1,)$ is constructed under one of the conditions given in Definition 4.1.7. Then the following statements are equivalent.*

(a) $L_p(\pi) \otimes_G^\sigma L_{q'}(\gamma^*) \simeq A_p^q$.

(b) *Every element of $Int_G(U_p^\pi, U_q^\gamma)$ can be approximated in the ultraweak*-operator topology by integral operators.*

Proof:(b)⇒(a) Suppose that every element of $Int_G(U_p^\pi, U_q^\gamma)$ can be approximated in the ultraweak*-operator topology by integral operators. First we show that the kernel of Ψ contains the subspace L of $L_p(\pi)\otimes L_{q'}(\gamma^*)$ (cf. Sec.2.6, p.22 and eqn.(3.35)) given by

$$L = \langle\{U_p^\pi(s)f\otimes g - f\otimes (U_q^\gamma)^*(s)g : s\in G\}\rangle.$$

That is,

$$\Psi(\Sigma_{i=1}^\infty U^\pi(s)f_i\otimes g_i) = \Psi(\Sigma_{i=1}^\infty f_i\otimes (U^\gamma)^*(s)g_i)$$

for $s\in G$. In the following we write $\lambda(\cdot,\cdot)$ for $\lambda_{H^x\cap K^y}(\cdot,\cdot)$. Now

$$\begin{aligned}
&\Psi(\sum_{i=1}^\infty U^\pi(s)f_i\otimes g_i)(x,y)\\
&= \int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^\infty \frac{1}{\lambda(e,t)}\lambda_H(x,t)^{\frac{1}{p}}\lambda_H(xt,s)^{\frac{1}{p}}f_i(xts)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt)d\mu_{x,y}(t),\\
&= \int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^\infty \frac{1}{\lambda(e,t)}\lambda_H(x,ts)^{\frac{1}{p}}f_i(xts)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt)d\mu_{x,y}(t),\\
&= \int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^\infty \frac{\lambda(t,s^{-1})}{\lambda(e,ts^{-1})}\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}\lambda_K(y,ts^{-1})^{\frac{1}{q'}}g_i(yts^{-1})d\mu_{x,y}(t),
\end{aligned}$$

on changing variables $ts\mapsto t$. Since $\lambda(t,s^{-1})/\lambda(e,ts^{-1}) = 1/\lambda(e,t)$, and $\lambda_K(y,ts^{-1}) = \lambda_K(yt,s^{-1})\lambda_K(y,t)$(see Lemma 2.2.7, (a)),

$$\begin{aligned}
&\Psi(\sum_{i=1}^\infty U^\pi(s)f_i\otimes g_i)(x,y)\\
&= \int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^\infty \frac{1}{\lambda(e,t)}\lambda_H(x,t)^{\frac{1}{p}}f_i(xts)\otimes_{x,y}\\
&\qquad\qquad \lambda_K(y,t)^{\frac{1}{q'}}\lambda_K(yt,s^{-1})^{\frac{1}{q'}}g_i(yts^{-1})d\mu_{x,y}(t),\\
&= \int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^\infty \frac{1}{\lambda(e,t)}\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}(U^\gamma)^*(s)g_i(yt)d\mu_{x,y}(t),\\
&= \Psi(\sum_{i=1}^\infty f_i\otimes (U^\gamma)^*(s)g_i).
\end{aligned}$$

Now it only requires to prove that the kernel of Ψ is contained in L. To achieve this, it suffices to show that any bounded linear functional F on $L_p(\pi)\otimes_G^\sigma L_q(\gamma)$ which annihilates L also annihilates the kernel of Ψ. Since F annihilates L, there exists $T\in Int_G(U_p^\pi, U_q^\gamma)$ such that

$$\langle r, F\rangle = \sum_{i=1}^\infty \langle g_i, Tf_i\rangle, \tag{4.60}$$

for any $r\in L_p(\pi)\otimes_G^\sigma L_{q'}(\gamma^*)$ with the expansion

$$r = \sum_{i=1}^\infty f_i\otimes g_i.$$

Suppose now that r is in the kernel of Ψ. Then,

$$\sum_{i=1}^{\infty} \int_{\frac{G}{H^x \cap K^y}} \frac{1}{\lambda(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{q'}} g_i(yt) \mu_{x,y}(t) = 0. \tag{4.61}$$

By (4.60), it suffices to show that

$$\sum_{i=1}^{\infty} \langle g_i, T f_i \rangle = 0.$$

Under the assumption that the operator T can be approximated by the integral operators $\{T_j : j \in I\}$ in the ultraweak*-operator topology, we have

$$\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle \rightarrow \sum_{i=1}^{\infty} \langle g_i, T f_i \rangle.$$

Hence in order to prove $\sum_{i=1}^{\infty} \langle g_i, T f_i \rangle = 0$, it is sufficient to prove

$$\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle = 0,$$

for each T_j. Since T_j is an integral operator, we have

$$(T_j f_i)(y) = \int_{\frac{G}{H}} \Phi_j(y,x) f_i(x) d\mu_H(x),$$

where Φ_j is the kernel of T_j as described in Definition 4.4.1. Thus,

$$\begin{aligned}
&\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle \\
&= \sum_{i=1}^{\infty} \int_{\frac{G}{K}} \langle g_i(y), (T_j f_i)(y) \rangle d\mu_K(y), \\
&= \sum_{i=1}^{\infty} \int_{\frac{G}{K}} \int_{\frac{G}{H}} \langle g_i(y), \Phi_j(y,x) f_i(x) \rangle d\mu_H(x) d\mu_K(y), \\
&= \sum_{i=1}^{\infty} \int_{\frac{G \times G}{H \times K}} \langle g_i(y), \Phi_j(y,x) f_i(x) \rangle d\mu_{H \times K}(x,y), \\
&= \sum_{i=1}^{\infty} \int_{D \in \Upsilon} \int_{\frac{G}{H^x \cap K^y}} \langle g_i(yt), \Phi_j(yt, xt) f_i(xt) \rangle d\mu_{x,y}(t) d\mu_{(H,K)}(D),
\end{aligned}$$

using disintegration of measures as explained in Lemma 3.1.3. (Also, see the discussion preceeding the Lemma). By Proposition 4.4.2 (1), $\lambda_H(xt, t^{-1})^{\frac{1}{p'}} \Phi_j(y,x) = \lambda_K(y,t)^{\frac{1}{q}} \Phi_j(yt, xt)$ for almost all $x \in G/H$.

Therefore,

$$\sum_{i=1}^{\infty}\langle g_i, T_j f_i\rangle$$
$$= \sum_{i=1}^{\infty}\int_{D\in\Upsilon}\int_{\frac{G}{H^x\cap K^y}}\langle g_i(yt), \frac{\lambda_H(xt,t^{-1})^{\frac{1}{p'}}}{\lambda_K(y,t)^{\frac{1}{q}}}\Phi_j(y,x)f_i(xt)\rangle d\mu_{x,y}(t)d\mu_{(H,K)}(D).$$

From the identity (3.9), we see that

$$\frac{\lambda_H(xt,t^{-1})^{\frac{1}{p'}}}{\lambda_K(y,t)^{\frac{1}{q}}} = \frac{1}{\lambda_H(x,t)^{\frac{1}{p'}}\lambda_K(y,t)^{\frac{1}{q}}} = \frac{1}{\lambda(e,t)}\lambda_H(x,t)^{\frac{1}{p}}\lambda_K(y,t)^{\frac{1}{q'}}.$$

Consequently,

$$\sum_{i=1}^{\infty}\langle g_i, T_j f_i\rangle$$
$$= \sum_{i=1}^{\infty}\int_{D\in\Upsilon}\int_{\frac{G}{H^x\cap K^y}}\frac{1}{\lambda(e,t)}\langle\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt), \Phi_j(y,x)\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\rangle$$
$$d\mu_{x,y}(t)d\mu_{(H,K)}(D). \tag{4.62}$$

By Proposition 4.4.2 (3), $\Phi_j(y,x) \in Int_{H^x\cap K^y}(H(\pi^x), H(\gamma^y))$ under the conditions given in Definition 4.1.7. Hence there exists
$\Theta_j(y,x) \in (H(\pi^x)\otimes_{H^x\cap K^y} H((\gamma^y)^*))^*$ such that

$$\sum_{i=1}^{\infty}\langle\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt), \Phi_j(y,x)\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\rangle$$
$$= \sum_{i=1}^{\infty}\langle\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt), \Theta_j(x,y)\rangle,$$

(see (2.7)). Therefore we have,

$$\sum_{i=1}^{\infty}\langle g_i, T_j f_i\rangle =$$
$$\sum_{i=1}^{\infty}\int_{D\in\Upsilon}\int_{\frac{\Delta}{(H\times K)^{(x,y)}\cap\Delta}}\langle\frac{1}{\lambda(e,t)}\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{q'}}g_i(yt), \Theta_j(x,y)\rangle$$
$$d\mu_{x,y}(t)d\mu_{(H,K)}(D). \tag{4.63}$$

Hence, by (4.61),

$$\sum_{i=1}^{\infty}\langle g_i, T_j f_i\rangle = 0,$$

as required.

(a)⇒ (b) Now suppose that the kernel of Ψ is L. We want to show that the integral operators of the form $T_\phi f(y) = \int_{G/H}\phi(y,x)f(x)d\mu_H(x)$ form a dense set in $Hom_G(L_p(\pi), L_q(\gamma))$ in the ultraweak*-operator topology; or equivalently, the corresponding linear functionals are dense in $(L_p(\pi) \otimes_G L_{q'}(\gamma^*))^*$ in the weak*-topology. Hence, we only need to show that the annihilator of these functionals, regarded as functionals on $(L_p(\pi) \otimes_\sigma L_{q'}(\gamma^*))^*$, is L. But by (4.63) we see that the annihilator of these linear functional is the kernel of Ψ which is equal to L under our assumption. This concludes the proof of the Theorem.

$\square$

Corollary 4.4.8 *Suppose that every element of $Int_G(U_p^\pi, U_q^\gamma)$ can be approximated in the ultraweak*-operator topology by integral operators. Then the intertwining number $\partial(U_p^\pi, U_q^\gamma)$ is equal to the dimension of the space of all functions Φ given in Definition 4.4.1. Moreover, if H and K are discretely related,*

$$\partial(U_p^\pi, U_q^\gamma) = \sum_{\vartheta \in \Upsilon} d_\vartheta,$$

where d_ϑ is the dimension of the set of all functions Φ which vanish outside the double coset ϑ.

Proof: Let $T \in Int_G(U_p^\pi, U_q^\gamma)$. By (4.63) we have

$$\sum_{i=1}^{\infty} \langle g_i, Tf_i \rangle = \int_{D \in \Upsilon} \langle \Psi(x,y), \Theta(x,y) \rangle d\mu_{(H,K)}(D) = \langle \Psi, \Theta \rangle. \tag{4.64}$$

Now using Theorems 3.3.14 and 4.4.7,

$$(A_p^q)^* \simeq Hom_G(L_p(\pi), L_q(\gamma)).$$

By (4.64), the intertwining number $\partial(U_p^\pi, U_q^\gamma)$ is equal to the dimension of the space of all functions Θ which, in turn is equal to the dimension of the space of all functions Φ.

If H and K are discretely related, G is a union of a null set and a countable collection of double cosets. By Proposition 4.4.2 (2), the value of Φ on ϑ is uniquely determined by its value $\Phi(x_0, y_0)$ at (x_0, y_0) where $(x_0, y_0) \in \vartheta$.

Hence

$$\partial(U_p^\pi, U_q^\gamma) = \sum_{\vartheta \in D} d_\vartheta.$$

$\square$

Chapter 5

p-nuclear norms and Hilbert-Schmidt norms

In the construction of A_p^q spaces in Chapter 4, the corresponding tensor products of L_p spaces were endowed with the greatest cross-norm. Here, we investigate the possibility of using the p-nuclear norm and the Hilbert-Schmidt norm in this regard.

We let G be a second countable locally compact group with closed subgroups H and K. π and γ will denote representations of H and K on Banach spaces $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$, respectively.

5.1 p-nuclear norms

Recall, from Definition 2.6.5 (3), that for Banach spaces S and R and $z = \sum_{i=1}^{n} x_i \otimes y_i \in S \otimes R$, the p-nuclear norm $\alpha_p(z)$, $(1 \leq p \leq \infty,)$ of z is defined by

$$\alpha_p(z) := \inf\Big\{ \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}} \mu_{p'}(y_1, \ldots.., y_n) : z = \sum_{i=1}^{n} x_i \otimes y_i \Big\},$$

where

$$\mu_{p'}(y_1, \ldots.., y_n) = \sup_{\psi}\Big\{ \Big(\sum_{i=1}^{n} |\langle \psi, y_i\rangle|^{p'}\Big)^{\frac{1}{p'}} : \psi \in Y^*, \|\psi\| = 1\Big\},$$

for $1 \leq p' < \infty$ and

$$\mu_{\infty}(y_1, \ldots.., y_n) = \sup_{\psi}\Big\{ \max_{1\leq i\leq n} |\langle \psi, y_i\rangle| : \psi \in Y^*, \|\psi\| = 1\Big\}.$$

The infimum is taken with respect to all representations of z and $\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}}$ is understood to mean $\max_i \|x_i\|$ when $p = \infty$.

The first problem that arises is to show that

$$\sum_{i=1}^{n} f_i(x) \otimes g_i(y) \in \mathcal{H}(\pi) \otimes^{\alpha_p} \mathcal{H}(\gamma),$$

for $\sum_{i=1}^{n} f_i \otimes g_i$ in $L_p(\pi) \otimes^{\alpha_p} L_p(\gamma)$ and $x, y \in G$. We see that this is true at least in the case where $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$ are L_p spaces of finite measure. To achieve this, we use the following result.

Proposition 5.1.1 *Let $S, T, X,$ and Y be finite measure spaces and $1 \leq p < \infty$. Then*

$$L_p(X, L_p(S)) \otimes^{\alpha_p} L_p(Y, L_p(T)) \cong L_p(X \times Y, L_p(S) \otimes^{\alpha_p} L_p(T)).$$

Proof: Let Γ_1 be the natural embedding of $L_p(X) \otimes^{\alpha_p} L_p(S)$ to $L_p(X, L_p(S))$ given by

$$(\Gamma(\sum_{i=1}^{n} f_i \otimes g_i))(x) = \sum_{i=1}^{n} f_i(x) g_i.$$

By Theorem 2.6.7 (a), Γ_1 is an isometry. Therefore,

$$L_p(X) \otimes^{\alpha_p} L_p(S) \cong L_p(X, L_p(S)) \tag{5.1}$$

Similarly,

$$L_p(Y) \otimes^{\alpha_p} L_p(T) \cong L_p(Y, L_p(T)). \tag{5.2}$$

By Theorem 2.6.7 (b) (1), the natural embedding Γ_1 of $L_p(X) \otimes^{\alpha_p} L_p(S)$ into $L_p(X \times S)$ given by

$$(\Gamma_1(\Sigma_{i=1}^{n} f_i \otimes g_i))(x, s) = \Sigma_{i=1}^{n} f_i(x) g_i(s)$$

is an isometry. Hence, using (5.1) and (5.2), we get

$$L_p(X, L_p(S)) \cong L_p(X \times S) \quad \text{and} \quad L_p(Y, L_p(T)) \cong L_p(Y \times T). \tag{5.3}$$

Therefore,

$$\begin{aligned} & L_p(X, L_p(S)) \otimes^{\alpha_p} L_p(Y, L_p(T)) \\ \cong\ & L_p(X \times S) \otimes^{\alpha_p} L_p(Y \times T), \text{ by (5.3)}, \\ \cong\ & L_p(X \times S \times Y \times T), \text{ by Theorem 2.6.7 (b) (2)}, \\ \cong\ & L_p(X \times Y \times S \times T), \\ \cong\ & L_p(X \times Y, L_p(S \times T)), \text{ by (5.3)}, \\ \cong\ & L_p(X \times Y, L_p(S) \otimes^{\alpha_p} L_p(T)), \text{ by Theorem 2.6.7 (b) (2)}, \end{aligned}$$

as required.

□

The above result shows that, in the case where G/H and G/K possess finite measures and $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$ are L_p spaces, we have a corresponding result to that of Proposition 4.1.2, in terms of α_p norms.

Proposition 5.1.2 *Let $\mathcal{H}(\pi) = L_p(S)$ and $\mathcal{H}(\gamma) = L_p(T)$ where S and T are finite measure spaces. For $\Sigma_{i=1}^n f_i \otimes g_i$ in $L_p(\pi) \otimes^{\alpha_p} L_p(\gamma)$ and $x, y \in G$,*

$$\sum_{i=1}^n f_i(x) \otimes g_i(y) \in L_p(S) \otimes^{\alpha_p} L_p(T),$$

for almost all $x \in G/H$ and $y \in G/K$.

Proof: This is an immediate consequence of Proposition 5.1.1.

□

Throughout the remainder of this section, we assume that $\mathcal{H}(\pi) = L_p(S)$ and $\mathcal{H}(\gamma) = L_p(T)$ where S and T are finite measure spaces.

Theorem 5.1.3 *Suppose that the homogeneous spaces G/H and G/K are finite measure spaces. Let p be a real number with $1 \leq p < \infty$. Then*

$$U_p^\pi \otimes^{\alpha_p} U_p^\gamma \approx U_p^{\pi \otimes^{\alpha_p} \gamma},$$

where $\pi \otimes^{\alpha_p} \gamma : H \times K \mapsto U(\mathcal{H}(\pi) \otimes^{\alpha_p} \mathcal{H}(\gamma))$ is the representation given by $(\pi \otimes^{\alpha_p} \gamma)(h, k) = \pi(h) \otimes^{\alpha_p} \gamma(k)$.

Proof: For simplicity of notation we will write U^π for U_p^π. By Proposition 5.1.1 there exists an isometry Γ from $L_p(\pi) \otimes^{\alpha_p} L_p(\gamma)$ to $L_p(\pi \otimes^{\alpha_p} \gamma)$. For $z = \sum_{i=1}^\infty f_i \otimes g_i \in L_p(\pi) \otimes^{\alpha_p} L_p(\gamma)$, let F be the image of z in $L_p(\pi \otimes^{\alpha_p} \gamma)$ under Γ so that

$$(\Gamma z)(x, y) = F(x, y) = \sum_{i=1}^n f_i(x) \otimes g_i(y).$$

We have to show that F satisfies the covariance condition

$$F(hx, ky) = \pi(h) \otimes \gamma(k) F(x, y) \tag{5.4}$$

for $h \in H$ and $k \in K$ (see Sec.2.5). Without loss of generality, we let $F(x, y) = f(x) \otimes g(y)$, and the result follows for any F by linearity. Now

$$\begin{aligned} F(hx, ky) &= f(hx) \otimes g(ky), \\ &= \pi(h) \otimes \gamma(k)(f(x) \otimes g(y)), \\ &= \pi(h) \otimes \gamma(k) F(x, y), \end{aligned}$$

as required. Also, we see that Γ is an intertwining operator for $U^{\pi \otimes \gamma}$ and $U^\pi \otimes U^\gamma$. For $t_1, t_2 \in G$,

$$\begin{aligned} (U_{t_1, t_2}^{\pi \otimes \gamma} \Gamma z)(x, y) &= F(xt_1, yt_2), \\ &= f(xt_1) \otimes g(yt_2), \\ &= (U_{t_1}^\pi f)(x) \otimes (U_{t_1}^\gamma g)(y), \\ &= \left(\Gamma\left(U_{t_1}^\pi f \otimes U_{t_1}^\gamma g\right)\right)(x, y), \\ &= \left(\Gamma((U_{t_1}^\pi \otimes U_{t_1}^\gamma)(f \otimes g))\right)(x, y), \\ &= (\Gamma(U_{t_1}^\pi \otimes U_{t_1}^\gamma) z)(x, y). \end{aligned}$$

Hence,

$$U^{\pi\otimes\gamma}\Gamma = \Gamma(U^{\pi}\otimes U^{\gamma}),$$

as required.

□

For $x, y \in G$, let ${}_p\mathcal{H}_{x,y}$ be the subspace of $\mathcal{H}(\pi)\otimes^{\alpha_p}\mathcal{H}(\gamma)$ which is the closed linear span of elements of the form

$$\pi^x(b)\xi\otimes\eta - \xi\otimes(\gamma^y(b))^{-1}\eta,$$

for $b \in H^x \cap K^y$, $\xi \in \mathcal{H}(\pi)$ and $\eta \in \mathcal{H}(\gamma)$ (cf. Definition 4.1.3). Let ${}_p\mathcal{A}_{x,y}$ be the quotient Banach space $\mathcal{H}(\pi)\otimes^{\alpha_p}\mathcal{H}(\gamma)/{}_p\mathcal{H}_{x,y}$. Clearly, (as shown in Proposition 4.1.4,) ${}_p\mathcal{H}_{xs,ys} = {}_p\mathcal{H}_{x,y}$ and ${}_p\mathcal{A}_{xs,ys} = {}_p\mathcal{A}_{x,y}$, for any $s \in G$. Here again, for $u\otimes v \in \mathcal{H}(\pi)\otimes^{\alpha_p}\mathcal{H}(\gamma^*)$, we use the notation $u\otimes_{x,y} v$ to denote the element of ${}_p\mathcal{A}_{x,y}$ to which $u\otimes v$ belongs. We have the following analogous results to those of Propositions 4.1.5 and 4.1.6.

Proposition 5.1.4 *Let p and m be real numbers with $1 \le p < \infty$. Let H and K be regularly related, closed subgroups of the locally compact group G. Suppose G/H and G/K are finite measure spaces. Then, for $\sum_{i=1}^n f_i\otimes g_i \in L_p(\pi)\otimes^{\alpha_p} L_p(\gamma^*)$ and $x, y \in G$,*

$$t \;\mapsto\; \sum_{i=1}^{n}\frac{1}{\lambda_{H^x\cap K^y}(e,t)^{\frac{1}{m}}}\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{p}}g_i(yt) \tag{5.5}$$

is a mapping on the coset space $G/(H^x\cap K^y)$ in each of the following cases:

(a) *$p = m$;*

(b) *G/K and G/H both having invariant measures;*

(c) *$H^x/(H^x\cap K^y)$ and $K^y/(H^x\cap K^y)$ having invariant measures.*

Proof: The fact that the mapping (5.5) is a well defined function on the space $G/(H^x\cap K^y)$ is clearly true just as it is in Proposition 4.1.5, since the norm of the underlying tensor product space plays no role in the calculations.

□

Using the same notations as in Chapter 4, p.62, let

$$M_{x,y}^{(\frac{q'}{q'-1})} = \int_{\frac{H^x}{H^x\cap K^y}}\lambda_{H_{x,y}}(e,\alpha)d\mu_{H_{x,y}}(\alpha) \text{ and } N_{x,y}^{(\frac{p}{p-1})} = \int_{\frac{K^y}{H^x\cap K^y}}\lambda_{K_{x,y}}(e,\xi)d\mu_{K_{x,y}}(\xi).$$

Proposition 5.1.5 *For $\sum_{i=1}^n f_i\otimes g_i \in L_p(\pi)\otimes^{\alpha_p} L_p(\gamma^*)$ the integral*

$$\int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^{n}\frac{1}{\lambda_{H^x\cap K^y}(e,t)}\lambda_H(x,t)^{\frac{1}{p}}f_i(xt)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{p}}g_i(yt)d\mu_{x,y}(t) \tag{5.6}$$

is finite for almost all $D(x,y) \in \Upsilon$ in each of the following cases:

(a) *$p = 1$;*

(b) *G/K and G/H both having finite invariant measures;*

(c) *$1 < p \leq 2$ and $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ being compact for almost all $x, y \in G$ with $(x, y) \mapsto M_{x,y}N_{x,y}$ being a bounded function from Υ to $\mathcal{R}$.*

Proof: Let

$$I(x,y) = \int_{\frac{G}{H^x \cap K^y}} \sum_{i=1}^{n} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{p}} g_i(yt) d\mu_{x,y}(t).$$

Following the same argument as in Proposition 4.1.6, we have, under the conditions (a) or (b) above,

$$\|I\| \leq \sum_{i=1}^{n} \|f_i\|_p \|g_i\|_p;$$

under the condition (c) with $1 < p < 2$ and $2/p - 1 = 1/r$,

$$\|I\|_r \leq S \sum_{i=1}^{n} \|f_i\|_p \|g_i\|_p$$

where $S = \text{ ess sup}_{(x,y)}\{M_{x,y}N_{x,y}\}$; and under the condition (c) with $p = 2$,

$$\|I(x,y)\| \leq M_{x,y}N_{x,y} \sum_{i=1}^{n} \|f_i\|_p \|g_i\|_p.$$

Since, in each of the above cases, the right hand side is a finite sum of finite terms (as $f_i \in L_p(\pi)$ and $g_i \in L_p(\gamma^*)$ for each i), we have the required result.

Note that in the case of either (a) or (b), we can achieve the result as follows.

$$\begin{aligned}
&\int_{D(x,y)\in\Upsilon} \|I(x,y)\| d\mu_{H,K} D(x,y) \\
&\leq \ (\int_{\Upsilon} \int_{\frac{G}{H^x \cap K^y}} \|\sum_{i=1}^{n} f_i(xt) \otimes_{x,y} g_i(yt)\|_{\alpha_p} d\mu_{x,y}(t) d\mu_{H,K}(D), \\
&\leq \ \int_{\Upsilon} \int_{\frac{G}{H^x \cap K^y}} \|\sum_{i=1}^{n} f_i(xt) \otimes g_i(yt)\|_{\alpha_p} d\mu_{x,y}(t) d\mu_{H,K}(D), \\
&= \ \int_{\frac{G}{H}} \int_{\frac{G}{K}} \|\sum_{i=1}^{n} f_i(x) \otimes g_i(y)\|_{\alpha_p} d\mu_H(x) d\mu_K(y),
\end{aligned}$$

where the last equality is obtained by disintegration of measures as in Lemma 2.2.12 (see also, Lemma 3.1.3). But under the assumption that G/H and G/K are finite measure spaces, we have

$$\begin{aligned}
&\int_{\frac{G}{H}} \int_{\frac{G}{K}} \|\sum_{i=1}^{n} f_i(x) \otimes g_i(y)\|_{\alpha_p} d\mu_H(x) d\mu_K(y) \\
&\leq \ \Big(\int_{\frac{G}{H}} \int_{\frac{G}{K}} \|\sum_{i=1}^{n} f_i(x) \otimes g_i(y)\|_{\alpha_p}^{p} d\mu_H(x) d\mu_K(y)\Big)^{\frac{1}{p}}. \qquad (5.7)
\end{aligned}$$

By Theorem 5.1.3, $\sum_{i=1}^{n} f_i \otimes g_i \in L_p(\pi \otimes^{\alpha_p} \gamma^*)$. Hence we see that the right hand side of (5.7) is finite, which leads us to the required result.

□

We are now in a position to define the map ${}_p\Psi^{\alpha_p}$ as a mapping on $L_p(\pi) \otimes^{\alpha_p} L_p(\gamma^*)$ using the integral (5.5).

Definition 5.1.6 *Let H and K be regularly related, closed subgroups of the locally compact group G. Suppose G/H and G/K are finite measure spaces. The* map *${}_p\Psi^{\alpha_p}$ on $L_p(\pi)_G \otimes^{\alpha_p} L_p(\gamma^*)$ is defined by*

$$[{}_p\Psi^{\alpha_p}(\sum_{i=1}^{n} f_i \otimes g_i)](x,y) \tag{5.8}$$
$$:= \int_{\frac{G}{H^x \cap K^y}} \sum_{i=1}^{n} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{p}} g_i(yt) d\mu_{(x,y)}(t)$$

whenever one of the following conditions holds:

(a) $p = 1$;

(b) *G/K and G/H both have finite invariant measures;*

(c) *$1 < p \leq 2$, $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ are compact and possess invariant measures for almost all $x, y \in G$ and the map $(x,y) \mapsto M_{x,y}N_{x,y}$ is bounded from Υ to $\mathcal{R}$.*

The image space of ${}_p\Psi^{\alpha_p}$ has a similar structure to that of Ψ in Chapter 2; it has the Banach bundle structure where the tensor products of Banach space are endowed with the α_p norm. We use the following notation (cf. p.65).

Let

$$\begin{aligned}
{}_p\mathcal{B}_0 &= \mathcal{H}(\pi) \otimes^{\alpha_p} \mathcal{H}(\gamma^*) \times G \times G, \\
{}_p\mathcal{B}_0^\Delta &= \mathcal{H}(\pi) \otimes^{\alpha_p} \mathcal{H}(\gamma^*) \times ((G \times G)/\Delta), \\
{}_p\mathcal{B}_1 &= \cup_{(x,y)\in G\times G}\{\mathcal{H}_{x,y} \times \{(x,y)\}\}, \\
{}_p\mathcal{B}_1^\Delta &= \cup_{(x,y)\in G\times G}\{\mathcal{H}_{x,y} \times \{(x,y)\Delta\}\}, \\
{}_p\mathcal{B}_2 &= \cup_{(x,y)\in G\times G}\{\mathcal{A}_{x,y} \times \{(x,y)\}\}, \text{ and} \\
{}_p\mathcal{B}_2^\Delta &= \cup_{(x,y)\in G\times G}\{\mathcal{A}_{x,y} \times \{(x,y)\Delta\}\}.
\end{aligned}$$

It is clear that ${}_p\mathcal{B}_1$ is a subspace of ${}_p\mathcal{B}_0$, and ${}_p\mathcal{B}_1^\Delta$ is a subspace of ${}_p\mathcal{B}_0^\Delta$. With j denoting any one of $\{0,1,2\}$, let ${}_p\theta_j : {}_p\mathcal{B}_j \mapsto G \times G$ be defined by ${}_p\theta_j(\zeta,(x,y)) = (x,y)$, and let ${}_p\theta_j^\Delta : {}_p\mathcal{B}_j^\Delta \mapsto (G\times G)/\Delta$ be defined by ${}_p\theta_j^\Delta(\zeta,(x,y)\Delta) = (x,y)\Delta$, where ζ belongs to the corresponding Banach space. We define the quotient maps $q : {}_p\mathcal{B}_0 \mapsto {}_p\mathcal{B}_2$ and $q_\Delta : {}_p\mathcal{B}_0^\Delta \mapsto {}_p\mathcal{B}_2^\Delta$ in a similar manner to those in Sec.3.2 (p.32) and topologize ${}_p\mathcal{B}_2$ and ${}_p\mathcal{B}_2^\Delta$ so that the maps q and $q_\Delta$are continuous and open.

Define ${}_p\underline{\mathcal{B}}_j := ({}_p\mathcal{B}_j, {}_p\theta_j)$ and ${}_p\underline{\mathcal{B}}_j^{\Delta} := ({}_p\mathcal{B}_j^{\Delta}, {}_p\theta_j^{\Delta})$.

Since the nature of the norm of the underlying Banach space did not play a role in our arguments in Chapter 3, we see that Propositions 3.2.1, 3.2.2, 3.2.3, 3.2.4 and 3.2.5 hold in the present situation, leading us to the fact that ${}_p\mathcal{B}_2$ and ${}_p\mathcal{B}_2^{\Delta}$ are Banach semi-bundles over $G \times G$ and $(G \times G)/\Delta$. The analogous result to that of Lemma 4.3.2 is as follows.

Lemma 5.1.7 *For $\sum_{i=1}^n f_i \otimes g_i \in L_p(\pi) \otimes L_p(\gamma)$, the element ${}_p\Psi(\Sigma_{i=1}^n f_i \otimes g_i)$ is a cross-section of ${}_p\underline{\mathcal{B}}_2^{\Delta}$, if the integral (5.8) is constructed under one of the conditions (a) or (b). It is a cross-section of ${}_p\underline{\mathcal{B}}_2$ if it is constructed under the condition (c).*

Proof: This follows immediately from the analogous result (in α_p norms) to that of Proposition 4.3.1.

□

The preceding discussion and results lead us to the following definition.

Definition 5.1.8 *The* space ${}_pA_p^p$ *is defined to be the range of ${}_p\Psi^{\alpha_p}$ with the quotient norm.*

In other words, ${}_pA_p^p$ is contained in the space of cross-sections of the Banach semi-bundle ${}_p\underline{\mathcal{B}}_2^{\Delta}$ for cases (a) and (b) of Definition 5.1.6.

We have the following analogous results to those of Propositions 4.3.4, 4.3.5 and Theorem 4.4.7.

Proposition 5.1.9 *Suppose that the spaces $G/H, G/K$ and the real number p satisfy one of the conditions (a), (b) or (c) given in Definition 5.1.6. Suppose further that there exists a continuous family of Bruhat functions $\beta_{x,y}$ on $H^x \cap K^y$ for $(x, y) \in G \times G$. Let f and g be functions with compact support from $L_p(\pi)$ and $L_p(\gamma)$ respectively. Then,*

$$(x,y) \mapsto \int_{G/H^x \cap K^y} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes_{x,y} \lambda_K(x,t)^{\frac{1}{p}} g(yt) d\mu_{x,y}(t)$$

is a continuous cross-section of the corresponding Banach semi-bundle.

Proof: The result does not depend on the norm of the underlying Banach space $L_p(\pi) \otimes^{\alpha_p} L_p(\gamma)$. Hence the proof is exactly the same as that of Proposition 4.3.4.

□

Proposition 5.1.10

(1) *Let* ${}_pA_p^p$ *be constructed under condition (a) or (b) of Definition 5.1.6. Then* ${}_pA_p^p \subseteq L_1(\underline{\mathcal{B}}_2^\Delta; \mu_{H,K})$. *In particular, if* G/H *and* G/K *possess finite invariant measure and* $2/p > 1$, *then* ${}_pA_p^p \subsetneq L_r(\underline{\mathcal{B}}_2^\Delta; \mu_{H,K})$ *where* $1/r = 2/p - 1$.

(2) *If* ${}_pA_p^p$ *is constructed under the condition (c) together with* $1 < p < 2$, *then* ${}_pA_p^p \subseteq L_r(\underline{\mathcal{B}}_2^\Delta; \mu_{H,K})$ *where*
$1/r = 2/p - 1$.

(3) *If* ${}_pA_p^p$ *is constructed under the condition (c) together with* $p = 2$, *then* ${}_pA_p^p \subseteq L_\infty(\underline{\mathcal{B}}_2; \mu_{H\times K})$.

Proof: These results were established in the proof of Proposition 5.1.5.

□

Proposition 5.1.11 *Suppose the representation spaces* $\mathcal{H}(\pi)$ *and* $\mathcal{H}(\gamma)$ *are* L_p *spaces of finite measure. Suppose further, that the space* ${}_pA_p^p$, $(p > 1,)$ *is constructed under one of the conditions (b) or (c) given in Definition 5.1.6. Then the following statements are equivalent.*

(a) $L_p(\pi)\otimes_G^{\alpha_p} L_p(\gamma) \cong {}_pA_p^p$.

(b) *Every element of* $Int_G(U_p^\pi, U_{p'}^{\gamma^*})$ *can be approximated in the weak*-operator topology by integral operators.*

Proof: This is similar to that of Proposition 4.4.7 (a), and will be omitted.

□

5.2 Hilbert-Schmidt Norms

Now we turn to the construction of A_p^q spaces where the spaces involved are Hilbert spaces. Recall, from Definition 2.6.5 (4), that for Hilbert spaces V and W and $z = \sum_{i=1}^n x_i \otimes y_i \in V \otimes W$, the Hilbert-Schmidt norm on $V \otimes W$ is defined by

$$\beta(\sum_{i=1}^n x_i \otimes y_i) = (\sum_{j=1}^n \sum_{i=1}^n \langle x_i, x_j\rangle \langle y_i, y_j\rangle)^{\frac{1}{2}}.$$

We shall now consider another definition of the Hilbert-Schmidt norm given in several places in the literature, which will be shown to be equivalent to Definition 2.6.5 (4). Theorems 5.2.1 and 5.2.2 explain this new definition.

If V and W are complex Hilbert spaces then by an **antilinear map** $T : V \mapsto W$ we mean a continuous additive operator such that $T(\alpha x) = \bar{\alpha}T(x)$, for all $x \in V$ and $\alpha \in \mathcal{C}$. If the range of the linear map T is finite-dimensional then it is called an **operator of finite rank**.

Theorem 5.2.1 *An algebraic tensor product of the Hilbert spaces V, W is the vector space $V \otimes W$ of all antilinear maps $T : W \mapsto V$ of finite rank together with the operation $(x, y) \mapsto x \otimes y$ where $x \otimes y$ maps z into $(y, z)x$.*

Proof: See Gaal[19], Theorem 1, Sec.3, Chapter VI.

□

Theorem 5.2.2 *The topological tensor product $V \otimes W$ can be interpreted as the space of antilinear Hilbert-Schmidt operators T from W to V with the inner product $\langle S, T\rangle = \sum_{i=1}^{\infty}\langle S(y_i), T(y_i)\rangle$, where $\{y_i : i \in \mathcal{N}\}$, is a maximal orthonormal set in W.*

Proof: See Gaal[19], Theorem 2, Sec.3, Chapter VI.

□

Theorem 5.2.3 *The two norms on $V \otimes W$ defined in Definition 2.6.5 (4) and Theorem 5.2.2 are equivalent.*

Proof: By Theorem 5.2.1, for every antilinear map $T : W \mapsto V$ of finite rank, there exist $\{x_i : i = 1, ..n\} \subset V$ and $\{y_i : i = 1, ..n\} \subset W$ such that

$$T = \sum_{i=1}^{n} x_i \otimes y_i.$$

Let $\{z_k : k \in \mathcal{N}\}$ be a maximal orthonormal set in W. Then,

$$\begin{aligned}
||T||^2 &= \sum_{k=1}^{\infty} ||T(z_k)||^2, \\
&= \sum_{k=1}^{\infty} ||(\sum_{i=1}^{n} x_i \otimes y_i)(z_i)||^2, \\
&= \sum_{k=1}^{\infty} || \sum_{i=1}^{n}\langle y_i, z_k\rangle x_i||^2, \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\langle y_i, z_k\rangle\overline{\langle y_j, z_k\rangle}\langle x_i, x_j\rangle, \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n}\langle x_i, x_j\rangle\sum_{k=1}^{\infty}\langle y_i, z_k\rangle\overline{\langle y_j, z_k\rangle},
\end{aligned}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n}\langle x_i, x_j\rangle\langle y_i, y_j\rangle,$$
$$= \|\sum_{i=1}^{n} x_i \otimes y_i\|_\beta^2,$$

as required.

□

It is easy to see that β is a cross-norm (see, for example, Light and Cheney[29], Lemma 1.34). The result stated in Theorem 5.1.3 is well known in the present context of Hilbert spaces, but will be stated here as it is necessary for our purposes.

Theorem 5.2.4

$$L_2(\pi) \otimes^\beta L_2(\rho) \cong L_2(\pi \otimes^\beta \rho),$$

where $\pi \otimes^\beta \rho$ is the representation on $\mathcal{H}(\pi) \otimes^\beta \mathcal{H}(\rho)$.

Proof: See Gaal[19], Theorem 1, Sec.8, Chapter VI.

□

The 2-nuclear norm and the Hilbert-Schmidt norm on $L_2(\pi) \otimes L_2(\rho)$ are equivalent at least under special conditions, as the next Proposition shows.

Proposition 5.2.5 *Let $\mathcal{H}(\pi) = L_2(S)$ and $\mathcal{H}(\gamma) = L_2(T)$ where S and T are finite measure spaces. Then*

$$L_2(\pi) \otimes^\beta L_2(\rho) \cong L_2(\pi) \otimes^{\alpha_2} L_2(\rho).$$

Proof: By Theorem 5.2.4, we have

$$\begin{aligned} L_2(\pi) \otimes^\beta L_2(\rho) &\cong L_2(\pi \otimes^\beta \rho), \\ &= L_2((G/H) \times (G/K), L_2(S) \otimes^\beta L_2(T)). \qquad (5.9) \end{aligned}$$

By Theorems 5.2.4 and 2.6.7 (b) (2), we find that

$$L_2(S) \otimes^\beta L_2(T) \cong L_2(S \times T) \cong L_2(S) \otimes^{\alpha_2} L_2(T). \qquad (5.10)$$

(see also, Light and Cheney[29], Theorem 1.39). Therefore by (5.9) and (5.10), we obtain

$$\begin{aligned} L_2(\pi) \otimes^\beta L_2(\rho) &\cong L_2((G/H) \times (G/K), L_2(S) \otimes^{\alpha_2} L_2(T)), \\ &\cong L_2(G/H, L_2(S)) \otimes^{\alpha_2} L_2(G/K, L_2(T)), \text{by Proposition 5.1.2}, \\ &= L_2(\pi) \otimes^{\alpha_2} L_2(\rho), \end{aligned}$$

as claimed in the Proposition.

□

As a consequence of the above result, we see that the above theory on ${}_pA_p^p$ spaces provides us with a theory for ${}_2A_2^2$ under the condition that the Hilbert spaces $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$ are L_2 spaces of finite measure. This condition is not necessary in the case of ${}_2A_2^2$ spaces, as Proposition 5.2.6, 5.2.7 and 5.2.8 show.

Proposition 5.2.6 *For $\sum_{i=1}^n f_i \otimes g_i$ in $L_p(\pi) \otimes^\beta L_p(\gamma)$ and $x \in G/H, y \in G/K$,*

$$\sum_{i=1}^n f_i(x) \otimes g_i(y) \in \mathcal{H}(\pi) \otimes^\beta \mathcal{H}(\gamma),$$

for almost all x, y.

Proof: We want to show that

$$\beta(\sum_{i=1}^n f_i \otimes g_i) = \Big(\sum_{i=1}^n \sum_{j=1}^n \langle f_i(x), f_j(x)\rangle \langle g_i(y), g_j(y)\rangle\Big)^{\frac{1}{2}} < \infty.$$

Since $\sum_{i=1}^\infty f_i \otimes g_i$ in $L_p(\pi) \otimes^\beta L_p(\gamma)$, we have

$$\sum_{i=1}^n \sum_{j=1}^n \langle f_i, f_j\rangle \langle g_i, g_j\rangle < \infty.$$

Therefore, for any compact sets C_1 and C_2 in G/H and G/K respectively,

$$\begin{aligned} &\int_{C_1}\int_{C_2} \sum_{i=1}^n \sum_{j=1}^n \langle f_i(x), f_j(x)\rangle \langle g_i(y), g_j(y)\rangle d\mu_H(x) d\mu_K(y) \\ \leq\ & \int_{G/H}\int_{G/K} \sum_{i=1}^n \sum_{j=1}^n \langle f_i(x), f_j(x)\rangle \langle g_i(y), g_j(y)\rangle d\mu_H(x) d\mu_K(y), \\ <\ & \infty. \end{aligned}$$

Hence

$$\sum_{i=1}^n f_i(x) \otimes g_i(y) \in \mathcal{H}(\pi) \otimes^\beta \mathcal{H}(\gamma),$$

for almost all x, y, as required.

□

Proposition 5.2.7 *Let H and K be regularly related, closed subgroups of the locally compact group G. Then, for $\sum_{i=1}^n f_i \otimes g_i \in L_2(\pi) \otimes_G^\beta L_2(\gamma)$ and $x, y \in G$,*

$$t \mapsto \sum_{i=1}^n \frac{1}{\lambda_{H^x \cap K^y}(e,t)^{\frac{1}{m}}} \lambda_H(x,t)^{\frac{1}{2}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{2}} g_i(yt) \tag{5.11}$$

is a mapping on the coset space in each of the following cases:

(a) $m = 2$;

(b) *G/K and G/H both having invariant measures;*

(c) *$H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ having invariant measures.*

Proof: The proofs of (a), (b) and (c) are identical to those of Proposition 4.1.5 (a),(b) and (d) respectively.

□

Proposition 5.2.8 *For $\sum_{i=1}^{n} f_i \otimes g_i \in L_2(\pi) \otimes_G^\beta L_2(\gamma)$, the integral*

$$\int_{\frac{G}{H^x \cap K^y}} \sum_{i=1}^{n} \frac{1}{\lambda_{H^x \cap K^y}(e,t)^{\frac{1}{m}}} \lambda_H(x,t)^{\frac{1}{2}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{2}} g_i(yt) d\mu_{(x,y)}(t) \quad (5.12)$$

is finite for almost all $D(x,y) \in \Upsilon$ in each of the following cases:

(a) *G/K and G/H both having finite invariant measures.*

(b) *$m = 1$, $H^x/(H^x \cap K^y)$ and $K^y/(H^x \cap K^y)$ being compact for almost all $x, y \in G$ and $(x,y) \mapsto M_{x,y} N_{x,y}$ being a bounded map from Υ to $\mathcal{R}$.*

Proof: The proof of (a) is similar to that of (b) and the proof of (b) is exactly the same as that of (d) in Proposition 5.1.5. As we saw in the proof of Proposition 5.1.5, we can achieve the result in the case of (a), also using the following method. Let

$$I(x,y) = \int_{\frac{G}{H^x \cap K^y}} \sum_{i=1}^{n} \frac{1}{\lambda_{H^x \cap K^y}(e,t)^{\frac{1}{m}}} \lambda_H(x,t)^{\frac{1}{2}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{2}} g_i(yt) d\mu_{(x,y)}(t).$$

Then

$$\begin{aligned}
&\int_{D(x,y)\in\Upsilon} \|I(x,y)\| d\mu_{H,K} D(x,y) \\
&\leq (\int_{\Upsilon} \int_{\frac{G}{H^x \cap K^y}} \| \sum_{i=1}^{n} f_i(xt) \otimes_{x,y} g_i(yt) \|_\beta d\mu_{x,y}(t) d\mu_{H,K}(D), \\
&\leq \int_{\Upsilon} \int_{\frac{G}{H^x \cap K^y}} \| \sum_{i=1}^{n} f_i(xt) \otimes g_i(yt) \|_\beta d\mu_{x,y}(t) d\mu_{H,K}(D), \\
&= \int_{\frac{G}{H}} \int_{\frac{G}{K}} \| \sum_{i=1}^{n} f_i(x) \otimes g_i(y) \|_\beta d\mu_H(x) d\mu_K(y),
\end{aligned}$$

where the last equality is obtained by disintegration of measures as in Lemma 2.2.12 (see also, Lemma 3.1.3). But under the assumption that G/H and G/K are finite measure spaces, we have

$$\int_{\frac{G}{H}}\int_{\frac{G}{K}}\|\sum_{i=1}^{n}f_i(x)\otimes g_i(y))\|_{\beta}d\mu_H(x)d\mu_K(y)$$
$$\leq \left(\int_{\frac{G}{H}}\int_{\frac{G}{K}}\|\sum_{i=1}^{n}f_i(x)\otimes g_i(y))\|_{\beta}^2 d\mu_H(x)d\mu_K(y)\right)^{\frac{1}{2}}. \tag{5.13}$$

By Theorem 5.2.4, $\sum_{i=1}^{n} f_i\otimes g_i \in L_p(\pi\otimes^{\alpha_p}\gamma)$. Hence we see that the right hand side of (5.13) is finite, which leads us to the required result.

□

The above results lead us to the following definition of the map ${}_2\Psi^{\beta}$:

Definition 5.2.9 *Let H and K be regularly related, closed subgroups of the locally compact group G. For $x,y\in G$ let us choose a family of quasi-invariant measures $\mu_{x,y}$ on the homogeneous spaces $G/(H^x\cap K^y)$ so that the identity (3.9) holds.* The map ${}_2\Psi^{\beta}$ *on $L_2(\pi)\otimes^{\beta}L_2(\gamma)$ is defined by*

$$({}_2\Psi^{\beta}(\sum_{i=1}^{n}f_i\otimes g_i))(x,y) \tag{5.14}$$
$$:= \int_{\frac{G}{H^x\cap K^y}}\sum_{i=1}^{n}\frac{1}{\lambda_{H^x\cap K^y}(e,t)^{\frac{1}{m}}}\lambda_H(x,t)^{\frac{1}{2}}f_i(xt)\otimes_{x,y}\lambda_K(y,t)^{\frac{1}{2}}g_i(yt)d\mu_{x,y}(t)$$

whenever one of the following conditions holds:

(a) *G/K and G/H both have finite invariant measures;*

(b) *$m=1$, $H^x/(H^x\cap K^y)$ and $K^y/(H^x\cap K^y)$ are compact and possess invariant measure for almost all $x,y\in G$ and $(x,y)\mapsto M_{x,y}N_{x,y}$ is a bounded map from Υ to $\mathcal{R}$.*

The discussion on Banach semi-bundles leading to the definition of the space A_q^p in terms of Hilbert-Schmidt norm and the material discussed in Lemma 5.1.7, Propositions 5.1.9, 5.1.10 and 5.1.11, including proofs, is word by word applicable to the present situation, and will be omitted. The corresponding definition of A_q^p and the analogous result to that of Theorem 4.4.7 (a) will be given below since they are necessary to discuss the main result in this section.

Definition 5.2.10 *Let ${}_2\Psi^{\beta}$ be defined as in (5.14). Then the* space $A_2(\pi,\gamma)$ *is defined to be the range of ${}_2\Psi^{\beta}$ with the quotient norm.*

We shall state the following definitions and well known results regarding 2-induced representations, which will be used in the proof of the main result.

Definition 5.2.11 *(Gaal[19], p.377.) The* space $\mathcal{L}(\pi, \rho)$ *is defined to be the space of all such functions* $\chi : G \times G \mapsto HS(\mathcal{H}(\rho)^*, \mathcal{H}(\pi))$ *which satisfy the follwing properties:*

(1) *For every* $x, y \in G$ *and* $h \in H, k \in K$,

$$\chi(hx, ky) = \pi \otimes \rho(h, k)\chi(x, y).$$

(2) *For every fixed pair of vectors* $u \in \mathcal{H}(\pi), v \in \mathcal{H}(\rho)$,

$$(x, y) \mapsto \langle \chi(x, y)u, v\rangle$$

is measurable on $G \times G$.

Since quasi-invariant measures on each of G/H and G/K form a single equivalance class the following result is independent of the choice of μ_H and μ_K.

Proposition 5.2.12 *If* $\chi_1, \chi_2 \in \mathcal{L}(\pi, \rho)$ *and* $\|\chi_k(x, y)\|(k = 1, 2)$ *is finite for almost all* $(\xi, \eta) \in (G/H) \times (G/K)$ *then*

$$(x, y) \mapsto \langle \chi_1(x, y), \chi_2(x, y)\rangle$$

is a function defined on $(G/H) \times (G/K)$ *and is measurable. Furthermore,*

$$|\langle \chi_1(x, y), \chi_2(x, y)\rangle| \leq \|\chi_1(x, y)\|.\|\chi_2(x, y)\|$$

for all $x \in \xi, y \in \eta$ *and almost all* $(\xi, \eta) \in (G/H) \times (G/K)$.

Proof: See Gaal[19], Proposition 3, Sec.7, Chapter VI.

□

Using the quasi-invariant measures μ_H and μ_K, we can define $\|\chi\|$ for $\chi \in \mathcal{L}(\pi, \rho)$ by

$$\|\chi\|^2 = \int_{\frac{G}{H}} \int_{\frac{G}{K}} \|\chi(x, y)\|^2 d\mu_H(\xi) d\mu_K(\eta).$$

If $\chi_1, \chi_2 \in \mathcal{L}(\pi, \rho)$ and $\|\chi_1\|, \|\chi_2\|$ are finite then we can introduce the inner product

$$\langle \chi_1, \chi_2\rangle = \int_{\frac{G}{H}} \int_{\frac{G}{K}} \langle \chi_1(x, y), \chi_2(x, y)\rangle d\mu_H(\xi) d\mu_K(\eta).$$

Definition 5.2.13 *(Gaal[19], p.378.) Let* $\mathcal{L}^2(\pi, \rho)$ *be the inner product space of equivalence classes of functions* χ *in* $\mathcal{L}(\pi, \rho)$ *for which* $\|\chi\|$ *is finite.*

Theorem 5.2.14 *The inner product space $\mathcal{L}^2(\pi,\rho)$ is a complex Hilbert space.*

Proof: See Gaal[19], Theorem 5, Sec.7, Chapter VI.

□

The following result describes the relationship between the spaces $HS(L_2(\rho)^*, L_2(\pi))$ and $\mathcal{L}_2(\pi,\rho)$.

Theorem 5.2.15 *A linear operator $T : L_2(\rho)^* \mapsto L_2(\pi)$ belongs to $HS(L_2(\rho)^*, L_2(\pi))$ if and only if there is a χ in $\mathcal{L}^2(\pi,\rho)$ such that*

$$\langle Tf, g\rangle = \int_{\frac{G}{H}}\int_{\frac{G}{K}} \langle \chi(x,y)f(y), g(x)\rangle d\mu_H(x)d\mu_K(y)$$

for every $g \in L_2(\pi)$ and $f \in L_2(\rho)^*$. The kernel χ is uniquely determined by T and the map $T \mapsto \chi$ is a norm preserving isomorphism of $HS(L_2(\rho)^*, L_2(\pi))$ onto $\mathcal{L}^2(\pi,\rho)$.

Proof: Gaal[19], Theorem 11, Sec.7, Chapter VI.

□

Let us denote the set of intertwining operators with Hilbert-Schmidt norm by $Int_\beta(L_2(\rho)^*, L_2(\pi))$.

Proposition 5.2.16 *For a given intertwining operator $T \in Int_\beta(L_2(\rho)^*, L_2(\pi))$, the corresponding kernel $\chi : G\times G \mapsto HS(\mathcal{H}(\rho)^*, \mathcal{H}(\pi))$ satisfies the condition*

$$\lambda_H(x,s)^{\frac{1}{2}}\lambda_K(y,s)^{\frac{1}{2}}\chi(xs,ys) \;=\; \chi(x,y), \tag{5.15}$$

for all $s \in G$, for almost all $x \in G/H$ and $y \in G/K$, in addition to the two conditions mentioned in Definition 5.2.12.

Proof: By Theorem 5.2.15, for every $g \in L_2(\pi)$, $f \in L_2(\rho)^*$ and $s \in G$, we find that

$$\begin{aligned}\langle TU_s^{\rho^*}f, g\rangle &= \int_{\frac{G}{H}}\int_{\frac{G}{K}} \langle \chi(x,y)\lambda_K(y,s)^{\frac{1}{2}}f(ys), g(x)\rangle d\mu_H(x)d\mu_K(y),\\ &= \int_{\frac{G}{H}}\int_{\frac{G}{K}} \langle \chi(x,ys)\lambda_K(y,s^{-1})^{\frac{1}{2}}f(y), g(x)\rangle d\mu_H(x)d\mu_K(y), \quad (5.16)\end{aligned}$$

by changing variables $y \mapsto ys^{-1}$ and using Lemma 2.2.7 (a). T being an intertwining operator for the representations U^{ρ^*} and U^π, we have

$$\langle TU_s^{\rho^*}f, g\rangle \;=\; \langle U_s^\pi Tf, g\rangle = \langle Tf, U_s^{\pi^*}g\rangle. \tag{5.17}$$

Using Theorem 5.2.15, we see that

$$\begin{aligned}\langle Tf, U_s^{\pi^*} g\rangle &= \int_{\frac{G}{H}}\int_{\frac{G}{K}} \langle \chi(x,y)f(y), \lambda_H(x,s^{-1})^{\frac{1}{2}} g(xs^{-1})\rangle d\mu_H(x) d\mu_K(y), \\ &= \int_{\frac{G}{H}}\int_{\frac{G}{K}} \langle \chi(xs,y)f(y), \lambda_H(x,s)^{\frac{1}{2}} g(x)\rangle d\mu_H(x) d\mu_K(y), \qquad (5.18)\end{aligned}$$

by changing variables $x \mapsto xs$ and using Lemma 2.2.7 (a). Now using (5.16), (5.17) and (5.18), we obtain

$$\lambda_K(y,s^{-1})^{\frac{1}{2}}\chi(x,ys^{-1}) = \lambda_H(x,s)^{\frac{1}{2}}\chi(xs,y), \qquad (5.19)$$

for almost all $x \in G/H$ and $y \in G/K$. Replacing y by ys, we find that

$$\lambda_K(ys,s^{-1})^{\frac{1}{2}}\chi(x,y) = \lambda_H(x,s)^{\frac{1}{2}}\chi(xs,ys). \qquad (5.20)$$

Now Lemma 2.2.7 (a) implies that $\lambda_K(ys,s^{-1})\lambda_K(y,s) = \lambda_K(y,e) = 1$, hence, $\lambda_H(x,s)^{\frac{1}{2}}\lambda_K(y,s)^{\frac{1}{2}}\chi(xs,ys) = \chi(x,y)$, for $s \in G$, as claimed.

□

The set of all functions χ with the three properties (a), (b) in Definition 5.2.12 and (5.15) above will be denoted by $\mathcal{L}^2_G(\pi,\rho)$.

Proposition 5.2.17 *Suppose that the space $A_2(\pi,\gamma)$ is constructed under one of the conditions (a), (b) or (c) given in Definition 5.2.9. Then*

$$L_2(\pi)\otimes_G^\beta L_2(\gamma) \cong A_2(\pi,\gamma).$$

Proof: The result can easily seen to be true by using Theorem 5.2.15 and an argument similar to that of Theorem 4.4.7 (a).

□

Corollary 5.2.18

$$(L_2(\pi^*)\otimes^\beta L_2(\rho^*))^* \cong (L_2(\pi)\otimes^\beta L_2(\rho)).$$

Proof:We see that the space $\mathcal{L}^2(\pi,\rho)$ is isometrically isomorphic to the Hilbert space of the induced representation $U^{\pi\otimes_\beta\rho}$ of $G\times G$. Hence we have

$$L_2(\pi\otimes^\beta\rho) \cong \mathcal{L}^2(\pi,\rho). \qquad (5.21)$$

By Theorem 5.2.16,

$$HS(L_2(\rho)^*, L_2(\pi)) \cong \mathcal{L}_2(\pi,\rho). \qquad (5.22)$$

Now (2.6) (see p.19) implies that

$$HS(L_2(\rho)^*, L_2(\pi)) \cong (L_2(\pi^*)\otimes^\beta L_2(\rho^*))^*. \qquad (5.23)$$

From the isometries (5.21), (5.22) and (5.23), it is clear that

$$(L_2(\pi^*) \otimes^\beta L_2(\rho^*))^* \;\cong\; L_2(\pi \otimes^\beta \rho). \tag{5.24}$$

Therefore, $(L_2(\pi^*) \otimes^\beta L_2(\rho^*))^*$ is reflexive, hence by Schattes[40], p.141,

$$(L_2(\pi^*) \otimes^\beta L_2(\rho^*))^* \cong (L_2(\pi) \otimes^\beta L_2(\rho)),$$

as claimed.

□

Recall, from Sec. 2.6, that $(L_2(\pi) \otimes_G^\beta L_2(\rho))$ is the quotient space of $(L_2(\pi) \otimes^\beta L_2(\rho))$ and the closed linear subspace L which is spanned by all the elements of the form

$$U_s^\pi f \otimes g - f \otimes (U_s^\rho)g, \qquad s \in G.$$

In view of the isometry given in Theorem 5.2.4, let $\tilde{L}$ be the closed subspace of $L_2(\pi \otimes^\beta \rho)$ corresponding to L in $(L_2(\pi) \otimes^\beta L_2(\rho))$. It is clear that $\tilde{L}$ is the linear span of the elements of the form

$$\lambda_H(\cdot, s)^{\frac{1}{2}} \lambda_K(\cdot, s)^{\frac{1}{2}} F((\cdot)s, (\cdot)s) - F(\cdot, \cdot),$$

where $F \in L_2(\pi \otimes^\beta \rho)$ and $s \in G$. The quotient space $L_2(\pi \otimes^\beta \rho)/\tilde{L}$ will be denoted by $L_2(\pi \otimes_G^\beta \rho)$. Therefore we have

$$L_2(\pi \otimes_G^\beta \rho) \;\cong\; (L_2(\pi) \otimes_G^\beta L_2(\rho)). \tag{5.25}$$

Obviously, $\mathcal{L}_G^2(\pi, \rho)$ and $L_2(\pi \otimes_G \rho)$ are identical. We now come to the main result of this section.

Theorem 5.2.19 *Let π, ρ, ϱ and γ be representations of the closed subgroups H, K, M and N of G. Suppose that the spaces $A_2(\pi, \rho), A_2(\varrho, \gamma)$ and $A_2(\pi \otimes_G \rho, \varrho \otimes_G \gamma)$ are constructed under conditions (a), (b) or (c) given in Definition 5.1.9. Then there exists an isomorphism*

$$A_2(\pi, \rho) \otimes_G A_2(\varrho, \gamma) \mapsto A_2(\pi \otimes_G \rho, \varrho \otimes_G \gamma).$$

Proof: Hence, using (5.25) and Proposition 5.2.17 under the conditions given in Theorem 5.2.18, we see that

$$A_2(\pi, \rho) \simeq L_2(\pi) \otimes_G^\beta L_2(\rho) \simeq L_2(\pi \otimes_G^\beta \rho).$$

This gives us the diagram

$$\begin{array}{ccc} (L_2(\pi) \otimes_G L_2(\rho)) \otimes_G (L_2(\varrho) \otimes_G L_2(\gamma)) & \cong & L_2(\pi \otimes_G \rho) \otimes_G L_2(\varrho \otimes_G \gamma) \\ \downarrow\uparrow \quad\quad \downarrow\uparrow & & \downarrow\uparrow \\ A_2(\pi, \rho) \otimes_G A_2(\varrho, \gamma) & \mapsto & A_2(\pi \otimes_G \rho, \varrho \otimes_G \gamma) \end{array}$$

where $\downarrow\uparrow$ represents an isometry, and the theorem is established.

□

Chapter 6

Examples

In the following examples, the identity element of the group is denoted by e. For simplicity of notation $(\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i))(x,y)$ will be written as $\Psi_{f_i,g_i}(x,y)$.

Example (1) Let $G = SO(3)$ be the group of rotations of the 3-dimensional euclidean space around the origin. Let $H = K \subset G$ be the subgroup of rotations h_ϕ about the north pole:

$$h_\phi = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $0 \leq \phi < 2\pi$. H is isomorphic to the group $SO(2)$ of plane rotations. Any element $t \in SO(3)$ is given by three Euler angles ϕ, θ, ψ such that $t(\phi,\theta,\psi) = h_\phi^3 h_\theta^1 h_\psi^3$ where h_α^i is a rotation about the ith coordinate axis through an angle α. It is known (see Vilenkin[44], p.106) that the rotation $t(\phi,\theta,\psi)$ has the matrix form

$$t(\phi,\theta,\psi) \quad = \quad t(\phi,0,0)t(0,\theta,0)t(\psi,0,0) \tag{6.1}$$

with

$$t(\phi,0,0) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$t(0,\theta,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix},$$

giving

$$t(\phi,\theta,\psi) = \begin{pmatrix} cos\phi cos\psi - sin\phi sin\psi cos\theta & -cos\phi sin\psi - sin\phi cos\psi cos\theta & sin\phi sin\theta \\ sin\phi cos\psi + cos\phi sin\psi cos\theta & -sin\phi sin\psi - cos\phi cos\psi cos\theta & -cos\phi sin\theta \\ sin\psi sin\theta & cos\psi sin\theta & cos\theta \end{pmatrix}.$$

The coset space G/H can be identified with the sphere

$$S^2 = \{\underline{x} \in R^3 : |\underline{x}| = 1\}$$

and the coordinates of any point $\underline{x}$ on S^2 can be expressed in the form

$$\underline{x} = (0,0,1)t_{\underline{x}}(\phi,\theta,\psi) = (sin\psi sin\theta, cos\psi sin\theta, cos\theta),$$

where $\underline{x} \mapsto t_{\underline{x}}$ is the cross-sectional function from G/H to G. Since the coordinates of $\underline{x}$ are independent of ϕ we can write

$$\underline{x} = (0,0,1)t_{\underline{x}}(0,\theta,\psi)$$

with $0 \le \theta \le \pi$ and $0 \le \psi < 2\pi$. Let $\pi_n, n \in \mathcal{N}$, be a representation of H defined by $\pi_n(h_\phi) = e^{in\phi}$. In order to consider integration on G, consider its volume element $dt = d\nu_H d\mu_H$, where $d\nu_H = (2\pi)^{-1}d\phi$ is an invariant volume element of H and $d\mu_H = (4\pi)^{-1}d\theta sin\phi d\phi$ is the element of surface area of S^2. Then the space $L_p(\pi_n)$ is defined by

$$\begin{aligned} L_p(\pi_n) \; &:= \; \{f : G \mapsto \mathcal{C} : f(h_\phi g) = e^{in\phi}f(g), h_\phi \in H, g \in G; \\ & \qquad \|f\|_p = (\int_{\frac{G}{H}} \|f(y)\|^p d\mu_H(y))^{\frac{1}{p}} < \infty\}. \end{aligned}$$

Given $f \in L_p(\pi_n)$ we can define $\tilde{f} : S^2 \to \mathcal{H}(\pi)$ by

$$\tilde{f}(\underline{x}) = f(t_{\underline{x}})$$

and the induced representation U^π is given by

$$(U_s^\pi \tilde{f})(\underline{x}) = f(t_{\underline{x}}s).$$

But if $t_{\underline{x}}s = h_\psi t_{\underline{y}}$, then $\underline{y} = (0,0,1)t_{\underline{x}}s = \underline{x}s$. Hence $t_{\underline{x}}st_{\underline{x}s}^{-1} = h_\psi$. Therefore,

$$(U_s^\pi \tilde{f})(\underline{x}) = e^{in\psi}\tilde{f}(\underline{x}s),$$

with $h_\psi = t_{\underline{x}}st_{\underline{x}s}^{-1}$. Also,

$$\|\tilde{f}\|_p = \Big(\int_{S^2} |\tilde{f}(\underline{x})|^p d\mu_H(\underline{x})\Big)^{\frac{1}{p}} = \Big(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} |\tilde{f}(\theta,\psi)|^p sin\theta d\theta d\psi\Big)^{\frac{1}{p}}.$$

Consider the space $A_p^p, p > 1$ formed by $L_p(\pi_n)$ and $L_{p'}(\pi_m^*)$ together with the projective tensor product.

Case 1: $n = m$.

We see that

$$\begin{aligned} \mathcal{H}_{x,y} \; &= \; \{\langle \pi_n^x(b)\xi \otimes \eta - \xi \otimes \pi_n^{*y}(b)\eta \rangle, b \in H^x \cap H^y, \xi, \eta \in \mathcal{C}\}, \\ &= \; 0. \end{aligned}$$

Therefore $\mathcal{A}_{x,y} = \mathcal{C}$. Since G/H has finite invariant measure (cf. Lemma 4.3.2 (b)), the corresponding Banach bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ is given by

$$\mathcal{B} = \mathcal{C} \times (G \times G)/\Delta$$

and

$$\theta(z, (x, y)) = (x, y).$$

By Proposition 4.3.1, the elements in A_p^p are cross-sections of the form

$$\Psi_{f_i,g_i}(x, y) \;=\; \Psi_{f_i,g_i}(xy^{-1}, e),$$

and hence can be thought of as a mapping of one variable. We have

$$\Psi_{f_i,g_i}(u) = \int_{\frac{G}{H^u \cap H}} \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\mu_u(t).$$

If $u \notin H$, then $H^u \cap H = \{e\}$, giving

$$\Psi(u) = \int_G \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\nu_G(t).$$

If $u \in H$, then

$$\begin{aligned}
\Psi_{f_i,g_i}(u) &= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\mu_H(t), \\
&= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(uvt)g_i(vt)d\mu_H(t), \text{ for } v \in H, \\
&= \int_{\frac{G}{H}} \int_H \sum_{i=1}^{\infty} f_i(uvt)g_i(vt)d\nu_H(v)d\mu_H(t),
\end{aligned}$$

since H is compact. Therefore for any $u \in G$,

$$\Psi_{f_i,g_i}(u) = \int_G \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\nu_G(t).$$

But for $u \in H$,

$$\begin{aligned}
\Psi_{f_i,g_i}(u) &= \int_G \sum_{i=1}^{\infty} f_j(ut)g_j(t)d\nu_G(t), \\
&= e^{in\phi} \int_G \Sigma_j f_j(t)g_j(t)d\nu_G(t). \qquad (6.2)
\end{aligned}$$

On the other hand, for $u \in H$,

$$\begin{aligned}
\Psi_{f_i,g_i}(u) &= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\mu_H(t), \\
&= \int_{S^2} \sum_{i=1}^{\infty} \tilde{f}_i((0,0,1)ut)\tilde{g}_i((0,0,1)t)d\mu_H(t), \\
&= \int_{S^2} \sum_{i=1}^{\infty} \tilde{f}_i((0,0,1)t)\tilde{g}_i((0,0,1)t)d\mu_H(t), \\
&= \int_G \sum_{i=1}^{\infty} f_i(t)g_i(t)d\nu_G(t). \qquad (6.3)
\end{aligned}$$

By equations (6.2) and (6.3), it is clear that $\Psi_{f_i,g_i}(u) = 0$ for $u \in H$. Therefore, letting $\breve{g}(t) = g(t^{-1})$, the elements Ψ_{f_i,g_i} of the space A_p^p are of the form

$$\Psi_{f_i,g_i}(u) = \begin{cases} \sum_{i=1}^{\infty} f_i * \breve{g}_i(u), & \text{if } u \notin H, \\ 0, & \text{if } u \in H, \end{cases}$$

Case 2: $n \neq m$. In this case,

$$\mathcal{H}_{x,y} = \begin{cases} 0, & xy^{-1} \notin H, \\ \mathcal{C}, & xy^{-1} \in H, \end{cases}$$

and therefore,

$$\mathcal{A}_{x,y} = \begin{cases} \mathcal{C}, & xy^{-1} \notin H, \\ 0, & xy^{-1} \in H. \end{cases}$$

The Banach bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ is given by

$$\mathcal{B} = \{(\mathcal{C}, (x,y)\Delta) : xy^{-1} \notin H\} \cup \{(0, (x,y)\Delta) : xy^{-1} \in H\},$$

and

$$\theta(z, (x,y)) = (x,y).$$

Now let us consider the elements of the space A_p^p. When $u \in H$,

$$\begin{aligned} \Psi_{f_i,g_i}(u) &= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(ut) g_i(t) d\mu_H(t), & (6.4) \\ &= \pi_n(u) \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(t) g_i(t) d\mu_H(t). & (6.5) \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(ut) g_i(t) d\mu_H(t) &= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(t) g_i(u^{-1}t) d\mu_H(t), & (6.6) \\ &= \pi_m(u) \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(t) g_i(t) d\mu_H(t). & (6.7) \end{aligned}$$

By (6.5) and (6.7), we see that

$$\int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(t) g_i(t) d\mu_H(t) = 0.$$

Therefore,

$$\Psi_{f_i,g_i}(u) = 0, \text{ for } u \in H.$$

Now for $xy^{-1} \notin H$, $H^{xy^{-1}} \cap H = \{0\}$. Therefore,

$$\Psi_{f_i,g_i}(u) = \int_G \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\nu_G(t) \text{ for } u \notin H.$$

Hence, in this case too, the elements Ψ_{f_i,g_i} of the space A_p^q can be described are of the form

$$\Psi_{f_i,g_i}(u) = \begin{cases} \sum_{i=1}^{\infty} f_i * \check{g}_i(u), & \text{if } u \notin H, \\ 0, & \text{if } u \in H, \end{cases}$$

For the remaining part of this example, let m and n be any two positive integers. Since any $u \in G$ can be expressed in matrix form as in (6.1), we see that $\Psi_{f_i,g_i}(h_\phi^3 h_\theta^1 h_\psi^3) = e^{in\phi}e^{im\psi}\Psi_{f_i,g_i}(h_\theta^1)$. Therefore $\|\Psi_{f_i,g_i}(u)\|$ depends only on θ and we can regard Ψ as a function on the double coset space $H : H \cong [0, \pi]$ (cf.Proposition 4.3.1). Suppose f_i and g_i are continuous for each i and let $u \to h_\phi$ where $u \in G$ and $h_\phi \in H$. In oder to show Ψ_{f_i,g_i} is continuous, it is sufficient to consider the elements of A_p^p which can be expressed in the form of a finite sum $\sum_{i=1}^{n} f_i * \check{g}_i(u)$, since an infinite sum of the form $\sum_{i=1}^{\infty} f_i * \check{g}_i(u)$ can be approximated by finite sums. Letting $u = h_\phi v$ we have $v \to e$, and

$$\begin{aligned} &\|\Psi_{f_i,g_i}(u) - \Psi_{f_i,g_i}(h_\phi)\| \\ &\quad = \|e^{in\phi}\int_G \sum_{i=1}^{n} f_i(vt)g_i(t)d\nu_G(t) - e^{in\phi}\int_G \sum_{i=1}^{n} f_i(t)g_i(t)d\nu_G(t)\|, \\ &\quad \leq \int_G \sum_{i=1}^{n} \|f_i(vt) - f_i(t)\|\|g_i(t)\|d\nu_G(t), \\ &\quad \leq \sum_{i=1}^{n} (\int_G \|f_i(vt) - f_i(t)\|^p d\nu_G(t))^{\frac{1}{p}} (\int_G \|g_i(t)\|^{p'} d\nu_G(t))^{\frac{1}{p'}}. \end{aligned}$$

Now $(\int_G \|g_i(t)\|^{p'} d\nu_G(t))^{\frac{1}{p'}} = (\int_{\frac{G}{H}} \int_H \|g_i(ht)\|^{p'} d\nu_H(t) d\mu_H(t))^{\frac{1}{p'}} = \|g_i\|_{p'}$. Since f_i is continuous, for a given $\epsilon > 0$ there exists a neighborhood U_i of e such that $\|f_i(vt) - f_i(t)\| < \epsilon/(2^i\|g_i\|_{p'})$ for $v \in U_i$. Hence,

$$\|\Psi_{f_i,g_i}(u) - \Psi_{f_i,g_i}(h_\phi)\| < \sum_{i=1}^{n} \frac{\epsilon}{2^i} < \epsilon.$$

This is a consequence of the general result given in Proposition 4.3.4; it is trivial that there exists a continuous family of Bruhat functions β_u, for $u \in G$, namely, $\beta_u(t) = 1$ for all $t \in G$. Hence by Proposition 4.3.4, the cross sections are continuous when the corresponding f_i and g_j are continuous with compact support.

By Proposition 4.3.5 (1), $A_p^p \subseteq L_1(H : H)$, where $L_1(H : H)$ can be identified with $L_1([0, \pi], \frac{1}{2}\sin\theta d\theta)$. Note that $L_1(H : H)$ is a commutative algebra under the convolution.

Example (2) We will construct A_p^q space (where the norm on the underlying tensor product space is the greatest cross-norm) in the case where $G = \mathcal{R}$,$H = \mathcal{Z}$ and $K = 3\mathcal{Z}$. Let the representations π of H and γ of K be given by

$$\pi_\alpha(h) = e^{ih\alpha}, \text{ for } h \in H,$$

and

$$\gamma_\beta(k) = e^{ik\beta}, \text{ for } k \in K,$$

where $\alpha, \beta \in \mathcal{R}$. The homogeneous spaces are $G/H = [0,1)$ and $G/K = [0,3)$. Hence, for $1 < p < \infty$,
$L_p(\pi) = \{f : G \mapsto \mathcal{C} : f(hx) = e^{ih\alpha} f(x), h \in H, x \in G, \|f\|_p = (\int_0^1 |f(x)|^p dx)^{\frac{1}{p}} < \infty\}$,
$L_{q'}(\gamma^*) = \{g : G \mapsto \mathcal{C} : f(kx) = e^{-ik\beta} f(x), k \in K, x \in G, \|g\|_{q'} = (\int_0^3 |g(x)|^{q'} dx)^{\frac{1}{q'}} < \infty\}$
The corresponding induced representations are defined by

$$(U_t^\pi f)(x) = f(x+t), (U_t^{\gamma^*} g)(x) = g(x+t) \text{for } f \in L_p(\pi), g \in L_{q'}(\gamma^*) \text{ and } t, x \in G.$$

It is clear that $H^x = H$, $K^y = K$, $H^x \cap K^y = K = 3Z$ for all $x, y \in G$, $(G \times G)/\Delta \simeq R$, $G/(H \cap K) = [0,3)$ and the double coset space $H : K \simeq [0,1)$. Therefore

$$\mathcal{A}_{x,y} = \langle \{\xi\eta(e^{i3n\alpha} - e^{-i3n\beta}) : \xi, \eta \in C\} \rangle^\perp.$$

Hence,

$$\mathcal{A}_{x,y} = \begin{cases} \mathcal{C}, & \text{if } \alpha = -\beta, \\ 0, & \text{if } \alpha \neq -\beta. \end{cases}$$

This implies that

$$A_p^q = \underline{0} \text{ if } \alpha \neq \beta.$$

Let us consider the case where $\alpha = -\beta$. Now since G/H and G/K have finite invariant measure, by Proposition 4.1.5 and 4.1.6 the integral (4.19) is well defined and finite. Hence we can consider the space A_p^p $(p > 1)$ whose elements are cross-sections of the trivial bundle $\underline{\mathcal{B}} = (\mathcal{B}, \theta)$ where $\mathcal{B} = \mathcal{C} \times R$, and $\theta(z, x) = x$, by Proposition 4.3.2. Hence we can regard the elements of A_p^p as functions $\psi_{f_i, g_i} : [0,1) \mapsto \mathcal{C}$ and the equation 4.19 can be simplified to give

$$\psi_{f_i,g_i}(x) = (\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i))(x, 0) = \frac{1}{3}\int_0^3 \sum_{i=1}^{\infty} f_i(x+t) g_i(t) dt,$$

where $f_i \in L_p(\pi), g_i \in L_{p'}(\gamma^*)$. Letting $\check{g}_i(t) = g_i(t^{-1})$, the above can be written as

$$\psi_{f_i,g_i}(x) = \sum_{i=1}^{\infty} f_i * \check{g}_i(x).$$

Proposition 4.3.4 states that the cross-sections ψ_{f_i,g_i} are continuous for continuous $f_i \in L_p(\pi)$ and $g_i \in L_{p'}(\gamma)$. We will show this directly. It is sufficient to consider elements of A_p^p which can be expressed in terms of finite sums since the infinite sums can be approximated by finite sums.

$$\begin{aligned}
\|\psi_{f_i,g_i}(z) - \psi_{f_i,g_i}(0)\| &= \frac{1}{3}\|\int_0^3 \sum_{i=1}^n f_i(z+t)g_i(t) - f_i(t)g_i(t)dt\|, \\
&\leq \frac{1}{3}\sum_{i=1}^n \int_0^3 \|f_i(z+t)g_i(t) - f_i(t)g_i(t)\|dt, \\
&\leq \frac{1}{3}\sum_{i=1}^n \int_0^3 \|g(t)\|\|f_i(z+t) - f_i(t)\|dt, \\
&\leq \frac{1}{3}\sum_{i=1}^n \Big(\int_0^3 \|g_i(t)\|^{p'}dt\Big)^{\frac{1}{p'}} \Big(\int_0^3 \|f_i(z+t) - f_i(t)\|^p dt\Big)^{\frac{1}{p}}, \\
&= \frac{1}{3}\sum_{i=1}^n \|g_i\|_{p'} \Big(\int_0^3 \|f_i(z+t) - f_i(t)\|^p dt\Big)^{\frac{1}{p}}.
\end{aligned}$$

Since f_i is continuous for each i, for a given $\epsilon > 0$ there exists a neighbourhood $N_i(0)$ of 0 such that $\|(f_i(z+t) - f_i(t)\|^{p'} < \epsilon/2^i\|g_i\|_{p'}$ for all $z \in N_i(0)$. Therefore we have for $z \in \cap_{i=1}^n N_i(0)$,

$$\begin{aligned}
\|\psi_{f_i,g_i}(z) - \psi_{f_i,g_i}(0)\| &\leq \sum_{i=1}^n (\|g_i\|_{p'} \frac{\epsilon}{2^i\|g_i\|_{p'}}) \\
&\leq \sum_{i=1}^n \frac{\epsilon}{2^i} = \epsilon,
\end{aligned}$$

as required.

We will show that Proposition 4.3.5 is true in this case. This now states that $A_p^p \subseteq L_1(\mathcal{R})$. For $f_i \in L_p(\pi), g_i \in L_{p'}(\gamma^*)$,

$$\begin{aligned}
\|\psi_{f_i,g_i}\| &= \int_0^1 \|\frac{1}{3}\int_0^3 \sum_{i=1}^n f_i(x+t)g_i(t)dt\|dx, \\
&\leq \sum_{i=1}^n \frac{1}{3}\int_0^1 \| \int_0^3 \|f_i(x+t)\|\|g_i(t)\|dtdx, \\
&\leq \sum_{i=1}^n \frac{1}{3}\int_0^3 \|g_i(t)\|dt(\int_0^1 \|f_i(x+t)\|dx)dt, \\
&\leq \sum_{i=1}^n \frac{1}{3}\|g_i\|_{p'}\|f_i\|_p, \\
&< \infty,
\end{aligned}$$

showing that $A_p^p \subseteq L_1([0,1))$. By Hewitt and Ross[26], Theorem 20.18, we see that

$$\psi_{f_i,g_i}(x) \in L_m[0,1),$$

where $1/m = 1/p + 1/q' - 1$ which confirms Proposition 4.3.5 (1).

It is clear that every neighborhood of 0 is invariant under the action of H. Therefore, by Lemma 4.4.6 and Theorem 4.4.7, we have for $1 < p < \infty$,

$$L_p(\pi) \otimes_G L_{p'}(\gamma^*) \cong A_p^p.$$

Example (3) Consider the group $G = \{x_{\theta,z,\phi} = \begin{pmatrix} e^{i2\pi\theta} & z \\ 0 & e^{i2\pi\phi} \end{pmatrix} : z \in \mathcal{C}, 0 \leq \theta, \phi \leq 1\}$, with subgroups $K = \{k_{\theta,z} = \begin{pmatrix} e^{i2\pi\theta} & z \\ 0 & 1 \end{pmatrix} : z \in \mathcal{C}, 0 \leq \theta \leq 1\}$,
$H = \{h_{z,\phi} = \begin{pmatrix} 1 & z \\ 0 & e^{i2\pi\phi} \end{pmatrix} : z \in \mathcal{C}, 0 \leq \phi \leq 1\}$, $N = \{n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathcal{C}\}$,
$A = \{a_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi\phi} \end{pmatrix} : 0 \leq \phi \leq 1\}$ and $B = \{b_\theta = \begin{pmatrix} e^{i2\pi\theta} & 0 \\ 0 & 1 \end{pmatrix} : 0 \leq \theta \leq 1\}$.
Note that $G/H \simeq B$, $G/K \simeq A$, $K = N.B$, $H = N.A$, N is abelian and
$A \cap N = B \cap N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also, $x_{\theta,z,\phi} = h_{z,\phi} b_\theta = k_{\theta, ze^{-i2\pi\phi}} a_\phi$. We shall use the method of Little Groups (see Serre[41]) to construct irreducible representations of H and K. Let $\imath$ be the trivial representation of N. Then the stabilizer of $\imath$ under the action of K is K itself. Now $\tilde{\gamma}_n : B \mapsto \mathcal{C}$, $n \in \mathcal{N}$, defined by $\tilde{\gamma}_n(b_\theta) = e^{i2\pi n\theta}$, is a one-dimensional representation of B. Then

$$\gamma_n = \tilde{\gamma}_n \circ p_K,$$

where $p_K : K \mapsto K/N$ is the canonical mapping, defines an irreducible representation of K, by the method of Little Groups.

Similarly, for $m \in \mathcal{N}$, we can define an irreducible representation π_m of H by letting

$$\pi_m = \tilde{\pi}_m \circ p_H$$

where $\tilde{\pi}_m : A \mapsto \mathcal{C}$ is a representation of A defined by $\tilde{\pi}_m(a_\phi) = e^{i2\pi m\phi}$ and $p_H : H \mapsto H/N$ is the canonical mapping. The induced representation spaces $L_p(\pi_n)$ and $L_{q'}(\gamma_n^*)$ are given by

$$L_p(\pi_m) = \{f : G \mapsto \mathcal{C} : f(x_{\theta,z,\phi}) = e^{i2\pi m\phi} f(b_\theta), (\int_B \|f(b_\theta)\|^p d\theta)^{\frac{1}{p}} < \infty\},$$

$$L_{q'}(\gamma_n^*) = \{g : G \mapsto \mathcal{C} : g(x_{\theta,z,\phi}) = e^{i2\pi n\theta} g(a_\phi), (\int_A \|g(a_\phi)\|^{q'} d\phi)^{\frac{1}{q'}} < \infty\}.$$

The corresponding induced representations $U_p^{\pi_m}$ and $U_{q'}^{\gamma_n^*}$ of G are given by

$$(U_p^{\pi_m}(x_{\theta,z,\phi})f)(g_{\theta',z',\phi'}) = f(x_{\theta+\theta', ze^{i2\pi\phi'} + z'e^{i2\pi\theta}, \phi+\phi'}),$$

for $f \in L_p(\pi_m)$ and

$$(U_{q'}^{\gamma_n^*}(x_{\theta,z,\phi})g)(x_{\theta',z',\phi'}) = g(x_{\theta+\theta', ze^{i2\pi\phi'} + z'e^{i2\pi\theta}, \phi+\phi'}),$$

for $g \in L_{q'}(\gamma_n^*)$. Now $H^x = H, K^y = K$ and $H^x \cap K^y = N$ for all $x, y \in G$. For $s \in H^x \cap K^y$, $\pi^x(s) = \pi(xsx^{-1}) = (\tilde{\pi} \circ p_H)\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = 1$. Similarly, $\gamma^y(s) = 1$.

Since G/H and G/K have finite invariant measure, we can construct A_p^q space for $1 < p, q < \infty$, formed by $f \in L_p(\pi_m)$ and $L_{q'}(\gamma_n^*)$ together with the projective tensor product (see Propositions 4.1.5 and 4.1.6).

Therefore

$$\begin{aligned} \mathcal{A}_{x,y} &= \{\langle \overline{\pi}^x(s)\xi \otimes \eta - \xi \otimes (\overline{\gamma}^y(s))^*\eta \rangle : s \in N, \xi, \eta \in \mathcal{C}\}^{\perp}, \\ &= \{\langle \xi \otimes \eta - \xi \otimes \eta \rangle : \xi, \eta \in \mathcal{C}\}^{\perp}, \\ &= \mathcal{C}. \end{aligned}$$

Note that the double coset space $H : K = \{e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$, hence by Proposition 4.3.1, for $h_{z,\phi} \in H, k_{\theta,z} \in K$ and $s \in G$ we have

$$\begin{aligned} \Psi_{f_i,g_i}(h_{z,\phi}s, k_{\theta,z}s) &= \pi(h_{z,\phi})\gamma(k_{\theta,z})\Psi_{f_i,g_i}(e,e), \\ &= e^{i2\pi n\theta}e^{i2\pi m\phi}\Psi_{f_i,g_i}(e,e). \end{aligned}$$

Since $G/(H^x \cap K^y) = \{\begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{i\pi\phi} \end{pmatrix}, \quad 0 < \theta, \phi \leq 1\}$ for any $x, y \in G$,

$$\begin{aligned} \Psi_{f_i,g_i}(e,e) &= \int_{\frac{G}{H\cap K}} \sum_{i=1}^{\infty} f_i(t) \otimes_{e,e} g_i(t)dt, \\ &= \int_0^1 \int_0^1 \sum_{i=1}^{\infty} e^{i2\pi m\phi} f_i(b_\theta).e^{i2\pi m\theta} g_i(a_\phi)d\theta d\phi \\ &= \sum_{i=1}^{\infty} \int_0^1 e^{i2\pi n\phi} g_i(a_\phi)d\phi. \int_0^1 e^{i2\pi n\theta} f_i(b_\theta)d\theta. \end{aligned} \tag{6.8}$$

In the case of $p = q' = 2$, (6.8) can be written as

$$\Psi_{f_i,g_i}(e,e) = \sum_{i=1}^{\infty} \hat{f}_i(n)\hat{g}_i(m), \tag{6.9}$$

where $\hat{f}_i$ and $\hat{g}_i$ represent the Fourier transforms of f_i and g_i respectively. Therefore $\Psi_{f_i,g_i}(hxs, kys) = e^{i2\pi n\theta}e^{i2\pi m\phi}\sum_{i=1}^{\infty}\hat{f}_i(n)\hat{g}_i(m)$. In this case, it is clear that Ψ_{f_i,g_i} is continuous. This is to be expected by Proposition 4.3.4, since $\beta\begin{pmatrix} e^{i2\pi\theta} & z \\ 0 & e^{i2\pi\phi} \end{pmatrix} = \frac{1}{2\pi}e^{-|z|^2}$ for $z \in \mathcal{C}, 0 \leq \theta, \phi \leq 1$ defines a Bruhat function of N. In the case $n = m$, (6.9) can be further simplified to

$$\Psi_{f_i,g_i}(e,e) = \sum_{i=1}^{\infty}(f_i * g_i)\hat{}(n) \tag{6.10}$$

Example (4) Let $G = \Pi \times \mathcal{C}$, the semi-direct product of the circle group Π and the set of complex numbers $\mathcal{C}$. The binary operation in G is given by

$$(e^{i\theta}, z)(e^{i\phi}, w) = (e^{i(\theta+\phi)}, e^{i\phi}z + w),$$

where $0 \leq \theta, \phi < 2\pi$ and $z, w \in \mathcal{C}$. The identity $e = (1, 0)$. Let $H = K = \Pi$. Then $G/H = G/K \simeq \mathcal{C} \simeq H : H$.

Let $\pi_n, n \in \mathcal{N}$ be a representation of H defined by $\pi_n(e^{i\theta}, 0) = e^{in\theta}$. Then, the space $L_p(\pi_n)$ is defined by

$$L_p(\pi_n) = \{f : G \mapsto \mathcal{C} : f((e^{i\theta}, w) = f((e^{i\theta}, 0)(1, w)) = e^{in\theta} f(1, w), (\int_{\mathcal{C}} \|f(1, w)\|^p dw)^{\frac{1}{p}} < \infty\}.$$

The corresponding induced representation $U_p^{\pi_n}$ on G is defined by

$$(U_p^{\pi}(e^{i\phi}, w)f)(e^{i\theta}, z) = f(e^{i(\theta+\phi)}, e^{i\phi}z + w).$$

For $x = (e^{i\phi}, w)$,

$$H^x = \{(e^{-i\phi}, -e^{-i\phi}w)(e^{i\theta}, 0)(e^{i\phi}, w) : 0 \leq \theta < 2\pi\} = \{(e^{i\theta}, w - e^{i\theta}w) : 0 \leq \theta < 2\pi\}.$$

Since $(e^{i\theta}, w - e^{i\theta}w) \in H$ only if $w = 0$ or $e^{i\theta} = 1$, we have

$$H \cap H^x = \begin{cases} \{(1,0)\} & \text{for } x \notin H, \\ H & \text{for } x \in H, \end{cases}$$

which then implies

$$H/(H \cap H^x) \simeq \begin{cases} H & \text{for } x \notin H, \\ \{(1,0)\} & \text{for } x \in H. \end{cases}$$

Clearly, $H/(H \cap H^x)$ has finite invariant measure for all $x \in G$, hence by Propositions 4.1.5 and 4.1.6, Ψ is well defined on $L_p(\pi_n) \otimes_G^{\sigma} L_{p'}(\pi_m^*)$, for $n, m \in \mathcal{N}$, and the elements of the space A_p^p are defined by

$$\Psi_{f_i,g_i}(x, y) = \int_{\frac{G}{H^x \cap H^y}} \sum_{i=1}^{\infty} f_i(xt) g_i(yt) d\mu_{x,y}(t).$$

If $xy^{-1} = h = (e^{i\phi}, 0) \in H$, then $H^x \cap H^y = H^y$. This gives

$$\begin{aligned} \Psi_{f_i,g_i}(x, y) &= \int_{\frac{G}{H^y}} \sum_{i=1}^{\infty} f_i(hyt) g_i(yt) d\mu_{H^y}(t), \\ &= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(ht) g_i(t) d\mu_H(t), \text{ (see Lemma 3.1.2,)} \\ &= e^{in\theta} \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(t) g_i(t) d\mu_H(t), \\ &= \Psi_{f_i,g_i}(xy^{-1}, e). \end{aligned}$$

If $xy^{-1} \notin H$, then $H^x \cap H^y = \{(1,0)\}$ and hence

$$\Psi_{f_i,g_i}(x,y) = \int_G \sum_{i=1}^{\infty} f_i(xt)g_i(yt)d\nu_G(t).$$

But since the Haar measure on G is left- and right-invariant, we have

$$\Psi_{f_i,g_i}(x,y) = \int_G \sum_{i=1}^{\infty} f_i(xy^{-1}t)g_i(t)d\nu_G(t) = \Psi_{f_i,g_i}(xy^{-1},e).$$

Consequently we have, on writing u for xy^{-1},

$$\begin{aligned}
\Psi_{f_i,g_i}(u) &= \int_{\frac{G}{H}} \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\mu_H(t), \text{ for } u \in H, \\
&= \int_G \sum_{i=1}^{\infty} f_i(ut)g_i(t)d\nu_G(t), \text{ for } u \notin H.
\end{aligned}$$

Ψ_{f_i,g_i} can be shown to be continuous for continuous f_i, g_i, using a similar argument to that of Example 1, p.114.

Now let us calculate $\Psi_{f_i,g_i}(u)$. Consider the case where $n = m$. Letting $u = (e^{i\theta}, z)$ we get

$$\begin{aligned}
(\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i))(e^{i\theta}, z) &= \int_G \sum_{i=1}^{\infty} f_i((e^{i\theta}, z)(e^{i\phi}, w))g_i((e^{i\phi}, w))d\phi dw, \\
&= \int_G \sum_{i=1}^{\infty} f_i((e^{i(\theta+\phi)}, e^{i\phi}z + w))g_i((e^{i\phi}, w))d\phi dw.
\end{aligned}$$

For either $f \in L_p(\pi)$ or $f \in L_{p'}(\pi^*)$ define $\tilde{f}(w) = f(1,w)$. Since $(e^{i(\theta+\phi)}, e^{i\phi}z + w) = (e^{i(\theta+\phi)}, 0)(1, e^{i\phi}z + w)$,

$$\begin{aligned}
\Psi_{f_i,g_i}(e^{i\theta}, z) &= \int_G \sum_{i=1}^{\infty} e^{in(\theta+\phi)} \tilde{f}_i(e^{i\phi}z + w)e^{-in\phi}\tilde{g}_i(w)d\phi dw, \\
&= e^{in\theta} \int_0^{2\pi} \int_{w\in\mathcal{C}} \sum_{i=1}^{\infty} \tilde{f}_i(e^{i\phi}z + w)\tilde{g}_i(w)d\phi dw.
\end{aligned}$$

Letting $\breve{g}_i(w) = \tilde{g}_i(-w)$, we see that the above can be written as

$$\Psi_{f_i,g_i}(e^{i\theta}, z) = e^{in\theta} \int_0^{2\pi} \sum_{i=1}^{\infty} \tilde{f}_i * \breve{g}_i(e^{i\phi}z)d\phi.$$

Suppose $p = 2$. Then, for each i, $\tilde{f}_i$ and $\breve{g}_i$ can be regarded as functions in $L_2(\mathcal{C})$. Hence $\tilde{f}_i * \breve{g}_i = \hat{h}$, for some $h \in L_1(\mathcal{C})$. Therefore, we have

$$\Psi_{f_i,g_i}(e^{i\theta}, z) = e^{in\theta} \int_0^{2\pi} \hat{h}(e^{i\phi}z)d\phi. \tag{6.11}$$

For $z = x + iy$, let $e^{i\phi}z = (x\cos\phi - y\sin\phi) + i(x\sin\phi + y\cos\phi = R_\phi(x,y)$. Then

$$\begin{aligned}\int_0^{2\pi} \hat{h}(e^{i\phi}z)d\phi &= \int_0^{2\pi}\int_{\mathcal{R}^2} h(u,v)e^{2\pi i(u,v).R_\phi(x,y)}dudvd\phi,\\ &= \int_0^{2\pi}\int_{\mathcal{R}^2} h(u,v)e^{2\pi i R_{-\phi}(u,v).(x,y)}dudvd\phi.\end{aligned}$$

Changing variables $(u,v) \mapsto R_\phi(u,v)$, we obtain

$$\int_0^{2\pi}\int_{\mathcal{R}^2} h(u,v)e^{2\pi i R_\phi(u,v).(x,y)}dudvd\phi = \int_{\mathcal{R}^2} e^{2\pi i(u,v).(x,y)}(\int_0^{2\pi} h(R_\phi(u,v))d\phi)dudv.$$

Change variables (u,v) to polar cordinates (r,θ) and assume that $\int_0^{2\pi} h(R_\phi(u,v))d\phi$ is independent of θ. Letting $k(r) = \int_0^{2\pi} h(R_\phi(u,v))d\phi$, we obtain

$$\begin{aligned}&\int_{\mathcal{R}^2} e^{2\pi i(u,v).(x,y)}(\int_0^{2\pi} h(R_\phi(u,v))d\phi)dudv\\ &\qquad = \int_0^{\infty} k(r)\int_0^{2\pi} e^{2\pi i(r\cos\theta, r\sin\theta).(x,y)}rdrd\theta,\\ &\qquad = \int_0^{\infty} k(r)\int_0^{2\pi} e^{2\pi i(xr\cos\theta + yr\sin\theta)}d\theta rdr.\end{aligned}$$

Letting $w = e^{i\theta}$, we see that

$$\begin{aligned}\int_0^{2\pi} e^{2\pi i(xr\cos\theta+yr\sin\theta)}d\theta &= \int_{|w|=1} \frac{e^{2\pi i r w\frac{(x-iy)}{2}}e^{2\pi i\frac{r}{w}\frac{(x+iy)}{2}}}{iw}dw,\\ &= \int_{|w|=1}\frac{1}{iw}e^{\pi i r(w\bar{z}+\frac{z}{w})}dw.\end{aligned}$$

Letting $iw = v$, the above gives

$$\int_0^{2\pi} e^{2\pi i(xr\cos\theta+yr\sin\theta)}d\theta = \int_{|v|=1}\frac{1}{iv}e^{\pi r(v\bar{z}-\frac{z}{v})}dv.$$

Let $t = v\bar{z}/|z|$. Then the above simplifies to

$$\int_0^{2\pi} e^{2\pi i(xr\cos\theta+yr\sin\theta)}d\theta = \int_{|t|=1}\frac{1}{it}e^{\pi r|z|(t-\frac{1}{t})}dt.$$

But

$$\int_{|t|=1}\frac{1}{2\pi i t}e^{\pi r|z|(t-\frac{1}{t})}dt = J_0(\pi r|z|)$$

where J_0 is the Bessel function of order 0 (see Whittaker and Watson [47], p.355). Hence

$$\Psi_{f_i,g_i}(e^{i\theta},z) = 2\pi e^{in\theta}\int_0^{\infty} k(r)J_0(\pi r|z|)rdr.$$

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Index of Notations and Symbols

Notations

$S.Int_G(\pi, \gamma)$	the space of intertwining operators for π and γ which are in the Schmidt class, p.16
$U(X)$	the group of all isometries of X onto itself, p.16.
V^*	the dual space of V, p.15.

Special Symbols

$\mathcal{R}, \mathcal{C}, \mathcal{Q}, \mathcal{N}, \mathcal{Z}$	p.5.
$\mathcal{Q}(\sqrt{-1})$	p.50.
$\pi \approx \gamma$	equivalence of π and γ, p.17.
$\mu \succ \lambda$	$\lambda(\cdot, y)$ is a Radon-Nikodym derivative of the measure μ_y with respect to μ, p.8.
$\cong$	isometric isomorphism, p.5.
$\simeq$	topological equivalence, p.5.
$\Longleftrightarrow$	if and only if
$x \longmapsto y$	the value of a function at x is y; used to define functions by their values.
$A \longmapsto B$	a mapping from space A into space B
$\longrightarrow$	converges to
$f * g$	convolution product, p.23.
$u \otimes v$	p.20.
$u \otimes_{x,y} v$	p.59.
$X \otimes^{\alpha} Y$	p.21.
π^*	the adjoint of the representation π, p.17.
π^x	p.16.
$\lambda_H(x, y)$	p.8.
G/H	the homogeneous space of the set of all right-cosets of H in G, p.5.
$H : K$	the double coset space of a group G corresponding to the subgroups H and K, p.6.
H^x	p.16.
Δ	the diagonal subgroup of $G \times G$, p.28.
Δ_G	p.6.
p_H	the canonical mapping from a group G to the space of right-cosets G/H, p.5.
p'	the conjugate exponent of a number p, $1 \leq p \leq \infty$, p.5.
q	p.32.
q_R	p.32.
θ_j	p.32.
θ_j^R	p.32.
$\theta^\#$	p.11.
$i^\#$	p.11.

$j^{\#}$	p.12.
T^*	the adjoint of the operator T, p.47.
I	directed set, p.5.
Υ	set of all double-cosets $H \times K : \Delta$ of $G \times G$, p.28
$\sigma(z)$	the greatest cross-norm σ of z, p.20.
$\alpha_p(z)$	the p-nuclear norm α_p of z, p.20.
$\beta(z)$	the Hilbert-Schmidt norm β of z, p.21.
$\gamma(z)$	the least cross-norm γ of z, p.20.
$\langle , \rangle$	the dual pairing between a space V and its dual V^*, p.4.
$\langle \{u_\alpha : \alpha \in J\} \rangle$	the linear span of vectors $\{u_\alpha : \alpha \in J\}$ in any given vector space, where J is a set of indices, p.4.
$\langle \{u_\alpha : \alpha \in J\} \rangle^{\perp}$	the orthogonal complement of $\langle \{u_\alpha : \alpha \in J\} \rangle$, p.115
$H \lhd G$	H is a normal subgroup in G, p.84.
$\partial(\pi, \gamma)$	the intertwining number of the representations π and γ, p.16.
$f \circ g$	composition of the mappings f and g. p.4.

Subject Index