XI.—The Theory of the Mechanical Analysis of Sediments by means of the Automatic Balance. By R. A. Fisher, M.A., and Professor Sven Odén. *Communicated* by The General Secretary.

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In 1916, in a paper on the size of the particles in deep-sea deposits, Odén showed that the distribution by mass of a suspension into classes of particles of different size could be inferred from a study of the course of sedimentation of such a suspension from a state of uniform dispersion in water. The experimental data required constitute the "sedimentation curve" of the suspension, showing the total weight deposited, at a fixed depth, at the end of any assigned length of time.

In a note appended to the above communication Professor Knott presented in a simplified form the mathematical theory of the derivation of the distribution, emphasising that if we use as abscissa of the distribution curve the *time* taken to fall through a given height, or the *velocity* of subsidence, then the experimental data may be interpreted without assuming the validity of Stokes' law.

The progress which has since been made in the development of the method, and the prospect that in the near future accurate sedimentation curves may be obtainable without great labour, makes it desirable to set forth, in a more complete form than has hitherto been attempted, the theory of the derivation of the distribution curve and the statistical methods appropriate for its deduction from the physical data. We shall follow Knott in using the velocity of subsidence, or rather its logarithm, as the variate of the distribution, and shall treat the use of Stokes' law and the effects of varying the viscosity and density of the fluid as an altogether separate problem.

1. The Theory of Changes during Sedimentation.

If v is the velocity of subsidence of any class of particles, then the constitution of the sediment in respect of sedimentation is specified by P, a function of v, representing the fraction by weight (in water) of the whole which has a velocity of subsidence less than v. If

$$F = v\frac{dP}{dv}$$

then

$$dP = F d(\log v),$$

so that F will represent the ordinate of the distribution curve, of which the variate log v is the abscissa. It is required to find the value of F for any value of v.

We shall suppose that the following experimental conditions have been fulfilled:—

(i) Complete dispersion of the sediment particles and prevention of any tendency to coagulation.

(ii) Uniform distribution of every class of particles at the starting time $t=0$.

(iii) Constant and uniform temperatures so that no convection currents or other disturbances occur during the sedimentation.

(iv) A sufficiently dilute concentration so that the particles may fall independently, and that the density of the fluid displaced by any particle is not appreciably different from that of water.

At time t from the commencement of sedimentation, the class of particles with velocity v will have completely disappeared from the layer of fluid to a depth vt from the surface; below that depth it will be uniformly distributed as at the start. Consequently, at any depth x the fluid will be occupied with the initial concentration of particles with velocities less than $v=x/t$, but with none with higher velocities, so that the density of the suspension at this depth will be

$$\tau = 1 + c\mathrm{P} \qquad \text{(I)}$$

where c is a constant depending upon the original concentration, and P is the fraction by weight of the particles with velocities less than v.

A. *Variation of Density with Depth at a Constant Time.*—Differentiating equation I with respect to x, we have

$$\frac{\partial\tau}{\partial x} = c\frac{dv}{dx}\cdot\frac{d\mathrm{P}}{dv} = \frac{c}{x}\mathrm{F} \qquad \text{(II)}$$

Each value of x corresponds to a distinct velocity v, and to a distinct rate of increase of density with depth. Equation II enables us to find the value of F corresponding to each value of v, and to construct the distribution curve.

B. *Rate of Change of Density at a Constant Depth.*—Differentiating equation I with respect to t, we have

$$\frac{\partial\tau}{\partial t} = c\frac{dv}{dt}\cdot\frac{d\mathrm{P}}{dv} = -\frac{c}{t}\mathrm{F} \qquad \text{(III)}$$

At a given depth, each value of t corresponds to a distinct velocity v, and to a distinct rate of decrease of the density; so that from observations of the density at a constant depth also we may construct the distribution curve.

C. *Variation of the Hydrostatic Pressure with Depth at Constant Time.*—If π is the hydrostatic pressure at any depth, then using the appropriate units, we may write

$$\frac{\partial \pi}{\partial x} = r \qquad \text{(IV)}$$

and thence find by (II)

$$\frac{\partial^2 \pi}{\partial x^2} = \frac{c}{x}\mathrm{F}\,; \qquad \text{(V)}$$

observations of the hydrostatic pressure at different depths, at a constant time, will therefore also provide the values of F corresponding to each value of v, and enable the distribution curves to be constructed.

D. *Variation of the Hydrostatic Pressure with Time at a Constant Depth.*—Since π vanishes at the surface, we can obtain, by equations IV and I, its value in the form

$$\pi = x + ct\int_0^{x/t} \mathrm{P}dv\,. \qquad \text{(VI)}$$

when differentiating with respect to t, we have

$$\frac{\partial \pi}{\partial t} = c\left\{\int_0^{x/t} \mathrm{P}dv - \mathrm{P}v\right\}, \qquad \text{(VII)}$$

and

$$\frac{\partial^2 \pi}{\partial t^2} = -\frac{cx}{t^2}\left\{\mathrm{P} - \mathrm{P} - v\frac{d\mathrm{P}}{dv}\right\} = \frac{cx}{t^2}\mathrm{F}\,; \qquad \text{(VIII)}$$

this equation enables us to determine F, for the corresponding values of v, from measurements of the hydrostatic pressure at a fixed depth.

If, instead of measuring the pressure at depth x, we measure the weight in the fluid of a cylindrical body immersed to a depth x in the fluid, clearly we shall have as in equation VIII

$$\frac{\partial^2 w}{\partial t^2} = -\frac{cx}{t^2}\mathrm{F} \qquad \text{(IX)}$$

Finally, we may measure the weight of sediment (in water), accumulated on a plate suspended at a depth x, the relation between the weight observed and the constitution of the sediment being again given by equation IX. Since in equation VI, $t=0$ gives $\pi = x + cx$, and $t = \infty$ gives $\pi = x$, cx is the total matter in suspension, so that if A is the matter accumulated in time t

$$\mathrm{F} = -\frac{t^2}{\mathrm{A}_\infty}\frac{\partial^2 \mathrm{A}}{\partial t^2} \qquad \text{(X)}$$

This is by far the simplest method in practice of obtaining values of the distribution function F. The weight accumulated is automatically counterpoised, and a continuous record is taken of the progress of the weight with time.

Stokes' formula

$$v=\frac{2}{9}\frac{\rho_1-\rho}{\mu}ga^2,$$

where v is the velocity of steady motion, ρ_1 the density of the falling spherical particle, and a its radius, ρ the density of the fluid and μ its viscosity, may be used to translate the velocity distribution observed into a distribution of "effective radius," the latter term being defined as a function of the velocity by the above equation.

For spherical particles, subject to the limitation that $va\rho$ is small compared to μ, we may expect the effective radius to agree with the actual radius. This limitation is equivalent to making

$$\frac{2}{9}\frac{(\rho_1-\rho)\rho}{\mu^2}a^3$$

a small quantity; if, for example, $\rho_1=2{\cdot}7$, $\rho=1$, $\mu={\cdot}0129$, $g=981$, the coefficient of a^3 is 2,227,000, and a can scarcely be larger than ·005 cm., or 50 microns. As appears below, the experimental evidence does not deal with particles coarser than this, and we cannot say if the concept of "effective radius" can be consistently used for coarser particles. An increase of fiftyfold in the viscosity would, however, make Stokes' law applicable to particles of over thirteen times the above radius.

The transformation to "effective radius" is necessary if we wish to compare the results of experiments carried out on the same soil with fluids of different viscosity; for example, with water at different temperatures. The velocity is calculated from the time by the equation, using natural logarithms

$$\log v=\log h-\log t;$$

for example, in experiment A below, $h=17{\cdot}05$ cm., and the time is given in minutes, so that v in cm. per sec. is given by

$$\log v=-1{\cdot}2583-\log t.$$

With the values given above, we have by Stokes' formula

$$\begin{aligned}\log a&=\tfrac{1}{2}\log v-5{\cdot}1328\\&=-5{\cdot}7619-\tfrac{1}{2}\log t.\end{aligned}$$

In studying a range from 14·2 minutes ($\log t=2{\cdot}65$) to 2209 minutes ($\log t=7{\cdot}70$), we have therefore $\log a$ varying from $-7{\cdot}0869$ to $-9{\cdot}6119$, and a

ranges from 8·4 to ·67 microns. The above formulæ show the advantage of using the logarithms of the quantities concerned as abscissa of the frequency curve, for the same curve may be interpreted simultaneously as a distribution in time, in velocity, and in "effective radius."

2. Schloesing's Method.

In 1903 a theory of sedimentation was put forward by M. Th. Schloesing (*Comptes Rendus*, vol. cxxxvi, pp. 1608–1613). Schloesing imagines the soil made up of parts of weights $S_1, \ldots S_n$ sinking through a given depth in times $t_1, \ldots t_n$. The successive increments accumulated in time $t_1, \ldots t_n$ will then be

$$D_1 = \left(\frac{S_1}{t_1} + \frac{S_2}{t_2} + \ldots \frac{S_n}{t_n}\right)t_1,$$

$$D_2 = \left(\frac{S_2}{t_2} + \frac{S_3}{t_3} + \ldots \frac{S_n}{t_n}\right)(t_2 - t_1),$$

and so on. From these equations the values of S may be found easily from the observed values of D. Schloesing proposes to use values of t, increasing in geometrical progression with constant ratio equal to 2; and to interpret the values of S so derived as the quantities of soil falling in times less than t_1, between t_1 and t_2, and so on.

In view of the foregoing analysis, it would appear that Schloesing's method introduces considerable systematic errors of variable amounts, for if Sdt represent the quantity of matter whose time of fall lies in the range dt, we should have

$$D_1 = \int_0^{t_1} Sdt + \int_{t_1}^{\infty} \frac{t_1}{t} Sdt$$

$$D_2 = \int_{t_1}^{t_2}\left(1 - \frac{t_1}{t}\right)Sdt + \int_{t_2}^{\infty}(t_2 - t_1)\frac{S}{t}dt.$$

But according to Schloesing's formula

$$S_1 = D_1 - D_2\frac{t_1}{t_2 - t_1},$$

which may be written

$$\int_0^{t_1} Sdt + \int_{t_1}^{t_2} \frac{t_1}{t} \cdot \frac{t_2 - t}{t_2 - t_1} \cdot Sdt.$$

Schloesing's method assumes, in fact, that

$$\frac{t_1}{t_2 - t_1}\int_{t_1}^{t_2}\left(\frac{t_2}{t} - 1\right)Sdt$$

is negligible. If $t_2 = 2t_1$, this is far from being the case. The error of S is always positive, and depends upon the actual distribution; as an illustration we may take the case, which is approximately the condition often found over considerable ranges, where

$$S = \frac{A}{t}.$$

The value of the error is then

$$A\int_{t_1}^{2t_1}\left(\frac{2t_1}{t^2} - \frac{1}{t}\right)dt = A(1 - \log_e 2) = \cdot 3069\,A\,;$$

this may be expressed in terms of the equivalent error in the time by equating it to

$$\int_{t_1}^{t_1+\epsilon} S\,dt = A\log\left(1 + \frac{\epsilon}{t_1}\right),$$

whence

$$1 + \frac{\epsilon}{t_1} = 1\cdot 359,$$

or the equivalent error in the time is 35·9 per cent.

3. The Practical Reduction of Sedimentation Data.

In the practical reduction of the instrumental data the outstanding feature is the great range of velocities to be studied; a single observational series may deal with times and therefore with velocities with a range of over a thousandfold. Consequently, the rate of deposition changes enormously during the experiment; and, while rapidly succeeding observations are required in the early part of the experiment, far longer intervals are desirable during the greater part of the time. For the statistical reduction it is convenient that the observations should be read off at equal intervals of the logarithm of the time.

If

$$x = k\log t,$$

then

$$-t^2\frac{d^2A}{dt^2} = k\frac{dA}{dx} - k^2\frac{d^2A}{dx^2} \qquad . \quad . \quad . \quad . \quad . \quad (XI)$$

multiplication by t^2 is thus avoided; and when the appropriate formula for obtaining $\frac{dA}{dx}$ and $\frac{d^2A}{dt^2}$ from a group of successive values of A have been decided on, then, since k is constant, the values of F may be obtained by applying a single sequence of multipliers to such groups of successive values.

The particular formula which it is appropriate to apply must depend upon (i) the magnitude of the instrumental errors, (ii) the rapidity with which the true value of F varies as v is decreased. The more terms are used in the formula, and the lower the order of differences ignored, the smoother will be the resultant series of estimates of F, and the more thoroughly will instrumental errors be obliterated. On the other hand, if instrumental errors are small, and the structure of the soil in respect of velocity distribution is highly complex, with rapidly recurring maxima and minima, then high order differences should be retained and the sequence of values used should be curtailed.

To obtain information as to instrumental errors, it is evidently necessary to study the records of parallel soil samples, sedimenting preferably through different heights. The ideal method of reduction may then be regarded as the one which brings out the maximum detail common to the several series, with the minimum of discrepancies between them. In order to study the errors themselves, however, we shall not, in the present case, carry the smoothing so far.

To illustrate the principles underlying this method we may consider the possible formulæ for the first differential coefficient. If only three points were taken we should have no choice but to take $\frac{1}{2}(y_1 - y_{-1})$ as the differentiated coefficient at the central point, for this is the slope of the tangent to the parabola passing through the three points. With five points we may take either

$$\tfrac{1}{10}\{2(y_2 - y_{-2}) + (y_1 - y_{-1})\}$$

or

$$\tfrac{1}{12}\{-(y_2 - y_{-2}) + 8(y_1 - y_{-1})\},$$

the first representing the slope of the best fitting straight line (or parabola), and the second that of the best fitting cubic polynomial, or of the quartic which passes through the five points. The difference between these two formulæ is

$$\tfrac{17}{60}\{-(y_2 - y_{-2}) + 2(y_1 - y_{-1})\}$$
$$= -\tfrac{17}{60}\{(y_2 - 3y_1 + 3y_0 - y_{-1}) + (y_1 - 3y_0 + 3y_{-1} - y_{-2})\};$$

in the latter form it is seen to be proportional to the mean of the two third differences derivable from the five given points. If such third differences are not negligible, then either (i) a smooth curve of a higher degree than a parabola is required to represent the data, or (ii) the data are affected by random experimental errors, and so do not give a smooth curve. In the first case the second formula is required, for the first will suffer from a systematic error in the neglect of the third differences. In the second case both formulæ are free from systematic error, but the first formula is preferable, since it is very much the less affected by random experimental errors.

The possible formulæ for the first differential coefficient for seven and nine points are given below.

TABLE I.

Degree of Fitted Curve.	7.	9.
I. or II.	$\frac{1}{28}\{(y_1-y_{-1})+2(y_2-y_{-2})+3(y_3-y_{-3})\}$	$\frac{1}{60}\{(y_1-y_{-1})+2(y_2-y_{-2})+3(y_3-y_{-3})+4(y_4-y_{-4})\}$
III. or IV.	$\frac{1}{252}\{58(y_1-y_{-1})+67(y_2-y_{-2})-22(y_3-y_{-3})\}$	$\frac{1}{1188}\{126(y_1-y_{-1})+193(y_2-y_{-2})+142(y_3-y_{-3})-86(y_4-y_{-4})\}$
V. or VI.	$\frac{1}{60}\{45(y_1-y_{-1})-9(y_2-y_{-2})+(y_3-y_{-3})\}$	$\frac{1}{77220}\{25911(y_1-y_{-1})+20421(y_2-y_{-2})-12429(y_3-y_{-3})+2286(y_4-y_{-4})\}$
VII. or VIII.		$\frac{1}{840}\{672(y_1-y_{-1})-168(y_2-y_{-2})+32(y_3-y_{-3})-3(y_4-y_{-4})\}$

The influence of random errors in the case of each of these formulæ may be ascertained by expressing the variance (the mean of the square errors) due to this cause, in terms of that of the original series, as in the following table:—

DEGREE OF CURVE FITTED.

Number of Points.	I. or II.	III. or IV.	V. or VI.	VII. or VIII.
3	·5000			
5	·1000	·9028		
7	·0357	·2626	1·1706	
9	·0167	·1143	·4186	1·3626

It is evident that there is a very considerable advantage in using formulæ based upon a sufficiently large number of points. With nine points we may take account of changes in value requiring a quartic curve with nearly the same accuracy as in assuming that a parabola gives a sufficient fit for a 5-point run.

The important term in the sedimentation formula is that for the second differential coefficient. The following are two 9-point formulæ:

Degree of Curve.	y_{-4}.	y_{-3}.	y_{-2}.	y_{-1}.	y_0.	y_1.	y_2.	y_3.	y_4.	
II. or III.	+28	+7	−8	−17	−20	−17	−8	+7	+28	÷462
IV. or V.	−126	+371	+151	−211	−370	−211	+151	+371	−126	÷1716

These may be combined with those for the first differential coefficient to give separate formula appropriate to curves of the second, third, fourth, and fifth degrees, using the appropriate value of k in equation XI. For example, when the times of the successive values increase in geometrical progression, so that the difference of the natural logarithms is ·05, then $k=20$; and

$$-t^2\frac{d^2A}{dt^2}=20\frac{dA}{dx}-400\frac{d^2A}{dt^2}.$$

To obtain a formula appropriate for a fitted curve of the third degree, we insert in the above the second of the formulæ given for the first differential coefficient and the first of those given for the second differential coefficient. The resulting coefficients of the nine observed values are given below, with sufficient accuracy for the purpose for which they were calculated.

y_{-4}.	y_{-3}.	y_{-2}.	y_{-1}.	y_0.	y_1.	y_2.	y_3.	y_4.
−22·795	−8·451	+3·677	+12·597	+17·316	+16·840	+10·176	−3·670	−25·690

Any consecutive set of nine observations of the total weight accumulated, multiplied by these factors, will give a value of the required quantity, which is proportional to the frequency function. Equally by dividing the above multipliers by A_∞, we may obtain the frequency function directly in each case.

The computations, and, as it happens, the physical measurements, are somewhat simplified by using the first differences of the series of observed weights. Since the sum of the above factors is zero, the formula will apply equally to the first differences, and the multipliers then become

$y_{-3}-y_{-4}$.	$y_{-2}-y_{-3}$.	$y_{-1}-y_{-2}$.	y_0-y_{-1}.	y_1-y_0.	y_2-y_1.	y_3-y_2.	y_4-y_3.
+22·795	+31·246	+27·569	+14·972	−2·344	−19·184	−29·360	−25·690

There are now only eight multipliers instead of nine, and the differences in the weights are three figure, in place of five figure numbers.

Tables II and III show the results of two experiments recently carried out in the Physical Laboratory, Rothamsted Experimental Station, with the same soil—a clay from Rothamsted, from which the coarser fractions had been previously removed. These experiments are only of a preliminary character, carried out as part of the work of developing a sufficiently accurate technique. They serve to illustrate the methods of statistical reduction and to show how the results of the latter may be used to gauge the accuracy attained at any stage. With more complete temperature control we may anticipate a considerable advance in precision. The frequency function is expressed in percentages of the total possible deposit, A_∞, the values of which in the two cases were 11·82 and 11·21 grms. The course of the desired frequency function is shown in figs. A and B; the above 9-point formula was used in both cases, since it enables both the general course of the curves and the experimental errors to be clearly seen.

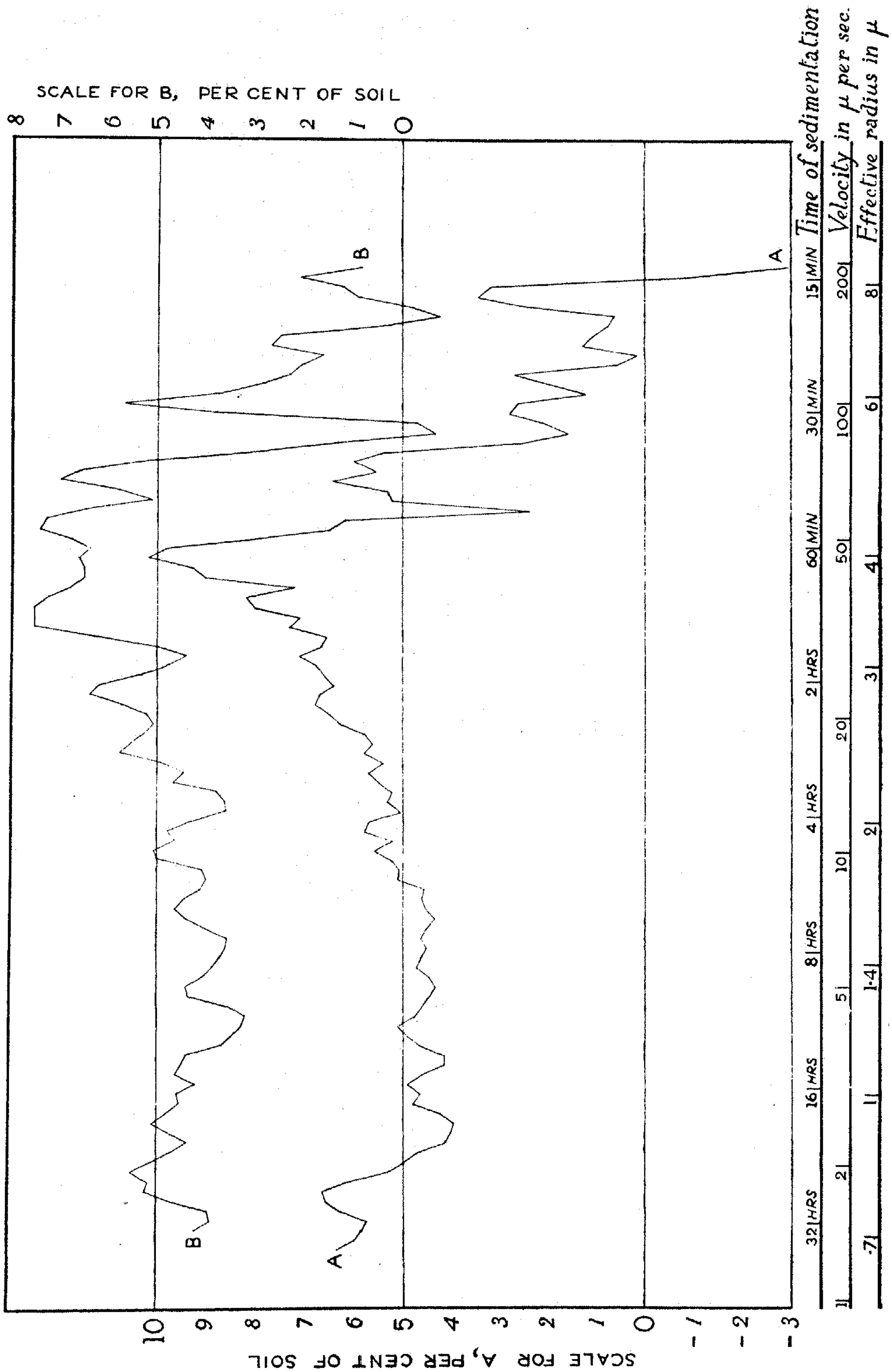
TABLE II.

	·00.	·05.	·10.	·15.	·20.	·25.	·30.	·35.	·40.	·45.
2·5	...	...	...	−2·94	−·91	3·16	3·41	2·48	·59	·74
3·0	1·01	1·24	·15	·54	2·66	2·00	1·18	2·60	2·78	2·05
3·5	1·55	2·50	5·35	6·00	5·56	6·82	5·48	5·42	2·71	6·20
4·0	6·51	8·04	9·77	10·18	9·26	9·13	7·18	8·21	8·07	7·12
4·5	7·28	6·57	6·70	7·12	6·78	6·62	6·43	6·73	6·79	6·50
5·0	6·27	5·80	5·63	5·83	5·40	5·74	5·46	5·21	5·28	5·05
5·5	5·69	5·80	5·20	5·57	5·23	5·07	5·07	4·54	4·59	4·47
6·0	4·30	4·46	4·61	4·48	4·64	4·72	4·44	4·31	4·42	4·59
6·5	4·77	5·08	4·86	4·64	4·07	4·06	4·57	4·91	4·64	4·76
7·0	4·16	3·91	3·98	4·12	4·67	4·95	5·31	6·18	6·63	6·58
7·5	6·25	5·75	5·85	5·99	6·36	...	...	...	...	...

TABLE III.

	·00.	·05.	·10.	·15.	·20.	·25.	·30.	·35.	·40.	·45.
2·4715	...	...	...	·84	2·06	1·11	·91	−·16	−·81	·26
2·9715	2·48	2·66	1·67	2·05	2·31	2·84	3·70	5·68	3·82	−·32
3·4715	−·70	1·35	2·92	5·22	6·56	6·99	5·79	5·14	6·47	7·27
3·9715	7·41	6·80	6·43	6·61	6·54	6·54	6·82	7·15	7·55	7·54
4·4715	7·55	6·34	5·01	4·44	4·83	5·35	6·20	6·40	5·77	5·25
4·9715	5·10	5·29	5·55	5·79	4·97	4·47	4·70	3·81	3·65	3·63
5·4715	4·37	4·81	4·68	5·07	5·01	4·12	4·05	4·17	4·47	4·66
5·9715	4·41	4·05	3·60	3·63	3·75	3·91	4·11	4·43	4·36	3·58
6·4715	3·22	3·31	3·52	3·70	4·42	4·49	4·64	4·12	4·60	4·56
6·9715	4·81	5·09	4·79	4·39	4·69	5·08	5·50	5·18	5·22	4·71
7·4715	3·99	3·94	4·23	...	...	...	...	...	...	...

Note to Tables II and III.—The natural logarithm of the time corresponding to any entry is found by adding the number heading the column to the number to the left of the same row. Thus the first entry of Table III, ·84, corresponds to $\log t = 2·6215$. Both tables proceed by intervals of ·05.


SCALE FOR B, PER CENT OF SOIL
SCALE FOR A, PER CENT OF SOIL
A
B
Time of sedimentation
Velocity in μ per sec.
Effective radius in μ
15 MIN
30 MIN
60 MIN
2 HRS
4 HRS
8 HRS
16 HRS
32 HRS

In examining the extent of experimental errors we may distinguish the internal evidence of error provided by the irregularities of a single curve, assuming the true curve to be such that the square of the difference between successive values is negligible. This evidence necessarily ignores causes of error which affect a number of consecutive values in the same manner; such causes being only exposed to detection by the comparison of parallel experiments, that is, by studying the external evidence of error.

The range of values available was divided into five groups, each of twenty consecutive values. For each set was calculated the mean square difference between consecutive terms (the unit being 1 per cent.); these values show a general agreement between the two experiments. From the means of the two sets of values an estimate was obtained of the mean square error of an individual value; if consecutive values had been independent, this would be obtained by multiplying by ·5, but since the formula used introduces a correlation of adjacent values, the appropriate factor is ·90156. Thence, by taking the square root, an estimate is obtained of the standard error, based on internal evidence only.

Table IV.

Section.	Approximate Time in Minutes.	Mean Square Difference.		Mean.	Mean Square Error.	Standard Error.
		A.	B.			
I . . .	14–37	2·2142	2·3190	2·2666	2·0435	1·43
II . . .	37–100	1·6639	·6603	1·1621	1·0477	1·02
III . . .	100–280	·1068	·2413	·1740	·1569	·40
IV . . .	280–750	·0485	·1203	·0844	·0761	·28
V . . .	750–2100	·1590	·1458	·1524	·1374	·37

The obvious feature, which is well shown in the diagrams, is the relatively high values of the errors in sections I and II, especially in the former. It would appear that during the first hundred minutes of the experiments the sedimentation curve was very much less smooth than it was later, and that, if errors of any magnitude affect the later values, they must be of a kind to act in a similar direction for considerable periods, up to hours, at a time.

To utilise parallel experiments for comparison, it is necessary to fix corresponding points on the logarithmic scale. The different heights through which the soil settles may be allowed for by converting the time scale into a velocity scale, introducing merely an additive constant to bring the two experiments into correspondence. A similar additive constant will serve to allow for viscosity, when it is desired to compare two experiments carried out at different temperatures. The velocity distribution will be changed by the change in viscosity, so that the comparison has to be based on the "effective radius" calculated from Stokes' formula. We should expect this distribution to be the same in two experiments poured from the same suspension. The abscissa of the two experimental distributions may thus be made to correspond; the ordinates also will need adjusting since A_∞ is for a similar suspension proportional to the height of subsidence.

When the corresponding percentage values are compared from the two experiments it appears that there is a systematic excess of A over B, except in the first section. The mean square difference for each section may be therefore calculated with or without allowance for the systematic error.

TABLE V.

Section.	Mean Difference (A-B).	Mean Square (A-B).	Mean Square Adjusted.	Standard Error Unadjusted.	Standard Error Adjusted.
I	−·126	4·3545	4·5873	1·47	1·51
II	+·524	3·1065	2·9811	1·25	1·22
III	+·014	1·4501	·4451	·85	·47
IV	+·639	·7139	·3250	·60	·40
V	+·392	1·0978	·9938	·74	·70

The systematic error is doubtless due to two sets of causes: (i) errors of the abscissæ of the two diagrams, due probably to insufficient temperature readings; (ii) to sustained causes of disturbance in the sedimentation process. The existence of such causes is evidenced by the fact that even after removing the systematic contribution the standard error in sections III, IV, and V, as revealed by a comparison of parallel experiments, is considerably greater than that indicated by the internal evidence of a single experiment. If the whole systematic error were due to this cause, we should prefer the unadjusted values; but since errors of abscissa-adjustment have no doubt contributed, the better values for comparison will lie between the adjusted and the unadjusted figures.

Comparing these with the values obtained from internal evidence, we at once confirm the relative inaccuracy of the values in sections I and II: for these, indeed, the internal evidence reveals practically the whole of the causes of error. In the remaining sections, the internal evidence reveals a decreasing proportion of the actual errors present, which must therefore be largely due to sustained causes.

4. Motion of Fluid.

The main features of the above statistical examination appear to be explicable in terms of the probable movements of the fluid in which the particles are suspended. These may be considered under two heads: (i) Initial disturbance; (ii) Convection currents. In both cases we shall be concerned with exceedingly slow movements, for a movement of the fluid becomes important when it is of the order of only a small percentage of the velocity of sedimentation, which is itself very small.

(i) *Initial Disturbance.*—According to the simple theory, the fluid, at the moment when settling commences, should be both in a state of uniform mixture, and *at rest*. Actually at the last moment at which the mixture is uniform it is in a state of violent motion, as it is poured into the sedimentation vessel. To ascertain what consequences must follow from the state of initial disturbance, we must know how rapidly the initial motion dies away. We may suppose that this initial motion is compounded of the various periodic motions consistent with the form of the vessel and of the various types of steady circulation of which the fluid is capable, and that after a short time the motion will consist solely of those motions which die away least rapidly. Among these, and of chief importance for our purpose, is the motion of steady circulation down the centre of the tall cylindrical vessel, returning upwards in a zone nearer the circumference. Such a motion, whether the central column move downward or upward, would profoundly modify the process of sedimentation upon a plate occupying the central portion of the bottom of the jar, so long as the velocity of movement was appreciable compared with the velocity of sedimentation.

To calculate how rapidly a slow circulation of this type would die away, owing to the viscosity of the fluid, we may compare its motion with that in the capillary tubes used by Poisseuille. If a be the radius of such a tube, and

$$ve^{-t/\mathrm{T}}$$

be the velocity after time t, at a distance r from the centre, then equating the retardation of each zone of fluid to that caused by the viscous forces, we have

$$\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial r}\right) = -\frac{r\rho}{\mathrm{T}}v,$$

or, v satisfies the differential equation

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\rho}{\mu \mathrm{T}}v=0.$$

This equation is satisfied by the Bessel function of zero order, J_0; so that we may write

$$v=\mathrm{A}J_0\left(\sqrt{\frac{\rho}{\mu \mathrm{T}}}\cdot r\right),$$

and the condition that the velocity should vanish when $r=a$ gives us

$$a\sqrt{\frac{\rho}{\mu \mathrm{T}}}=2{\cdot}4048,$$

being the first zero of J_0. Hence

$$\mathrm{T}=\frac{a^2\rho}{(2{\cdot}4048)^2\mu}$$

gives the time required for the velocity at any point to fall through the ratio 1 : e.

It will be noted that the velocity distribution found above for steadily decaying motion differs slightly from that in the Poisseuille tube for steady maintained motions, in that the Bessel function

$$J_0\left(2{\cdot}4048\cdot\frac{r}{a}\right)$$

is substituted for

$$1-\frac{r^2}{a^2},$$

this substitution leading to a slightly reduced dissipation of energy.

In applying this theory to the motion in a jar of fluid of radius b, we may obtain a rough approximation by equating a^2 to $\frac{1}{2}b^2$; thus, for example, with $\rho=1$, $b=6{\cdot}7$ cm., and $\mu={\cdot}0129$, the value for water at 11° C., we have $a^2=22{\cdot}45$ and $\mathrm{T}=301$ secs. This rough procedure may well exaggerate the time required to the extent of doubling it, but in such a case we can only arrive at a very broad approximation.

To ascertain for how long such a circulation will continue to be of importance relative to the rate of sedimentation, a similar approximation must suffice; for the actual initial velocity is unknown. After time t, soil will be sinking through the liquid at all velocities up to h/t, where h is the height of liquid above the plate. Taking $h/2t$ as a standard, the velocity of the liquid will fall below 1 per cent. of this, when $v=h/200t$. If, then,

$$v=v_0e^{-t/\mathrm{T}}$$

we shall have

$$\frac{t}{\mathrm{T}}e^{-t/\mathrm{T}}=\frac{h}{200v_0\mathrm{T}}={\cdot}000283,$$

putting $h=17$ cm., $T=300$ secs., $v_0=10$ cm. per sec. The latter value is very arbitrary, but it may be altered considerably without much affecting the value of t/T obtained. With the above values we have $t=10{\cdot}5T=53$ min.

After a lapse of 53 minutes, therefore, there is reason to think that the initial disturbance will have subsided to an extent to render the direct effect of fluid motion entirely negligible. Indeed, for some time before the rate of flow falls to 1 per cent. of the rate of sedimentation the direct effect will be small compared to the actual variations observed. There is, however, a secondary effect of such fluid motion, which will probably continue to disturb the sedimentation curve for a longer period. While there is any circulation comparable with the rate of sedimentation, the aggregate of particles of a given size, which on the simple theory should be uniformly distributed over a portion of the fluid, ceasing abruptly at a given height, will be displaced from this simple distribution, and will extend to different heights in different parts of the fluid. In consequence, it is probable that they will not be deposited with perfect regularity on the suspended plate, but, being distributed in clouds or layers, will be deposited to some extent intermittently, so disturbing the second differential of the amount deposited, which quantity is very sensitive to any disturbance of this type. Such an irregular distribution will not be induced in the finer particles, which will still be uniformly distributed over almost the whole fluid, at the time when the initial motion has subsided; but the coarser of those particles which have not settled at this time will be so affected, and the disturbance may be expected to continue until these also have settled out.

The above considerations probably afford an explanation of the irregularities observable in the first two sections of the record, which extend to about 100 minutes from the commencement. It is probable that these irregularities could not be wholly removed by any improvement of the recording mechanism. In so far as they are due to initial disturbance, a complete remedy would seem to lie in the use of fluids of higher viscosity, for in this way the initial disturbance may be very quickly damped out, and, at the same time, the time of sedimentation of the coarser particles is much increased.

(ii) *Convection Currents.*—Little need be said in addition to the above in reference to convection currents. Currents of this kind will undoubtedly be set up in the jar, if, owing to gradual rise in temperature, the walls of the jar are maintained at a higher temperature than the interior. During the latter days of the experiment, especially, very minute currents will serve to produce relatively great disturbances in the rate of sedimentation. Evidently for long-period experiments increasingly accurate temperature control will be required. The efficacy of temperature control would be much increased by fixing the temperature at 4° C., at which point the density is stationary; a reduction in the diameter of the jar should also do much to cut down such disturbances.

One direct effect of changing temperature may be noted, namely, that which is due to the change of viscosity with temperature. A slight rise in temperature is accompanied, apart from convection effects, by a speeding up of the process of sedimentation; in the neighbourhood of 11° C., the rate of deposition will be increased by 2·83 per cent. for a rise of 1° C., in the temperature. With falling temperature the apparent value of F will thus contain a positive error, and with rising temperature a negative error. Sensible errors in the apparent velocity distribution will in this way be introduced by small temperature changes during the experiment; as in the case of convection currents, the need for accurate temperature control increases with the length of the experiment.

Summary.

(1) A simplified mathematical statement of the theory of sedimentation through a stationary fluid leads to the formula originally indicated by Odén, and shows that the characteristic distribution of the sediment may be obtained—

(*a*) from the variation of density with depth,
(*b*) from the rate of change of density at a given depth,
(*c*) from the variation of hydrostatic pressure with depth,
(*d*) from the rate of change of hydrostatic pressure at a given depth;

the last relationship affording the theoretical basis of the sedimentation method.

(2) Schloesing's sedimentation theory is incomplete, and leads to somewhat large errors in the interpretation of the observations.

(3) A discussion is given of the statistical problems arising in the reduction of sedimentation data derived from the automatic balance, and examples of such data from two duplicate experiments are utilised to examine into the experimental errors actually present.

(4) Two types of fluid motion appear to influence the results:

(*a*) A vertical circulation set up by the initial disturbance of the fluid, the theoretical effects of which agree with errors

indicated by the experiments as occurring during the first 100 minutes of sedimentation. This disturbance may be remedied by using fluids of higher viscosity, especially as this procedure increases the time taken by the coarser particles to settle, in addition to reducing the time necessary to obliterate the initial disturbance.

(*b*) Convection currents of unspecified type will become important in prolonged experiments, where the finer particles are being studied. Great experimental refinement may be necessary to avoid these: their effect should be much reduced by maintaining the temperature of the water as close as possible to its temperature of maximum density.