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C. HIGGINBOTTOM,

Non-uniform Torsion of Thin-walled Beams and Shells; application to round tubes containing axial slit of any length

by

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Thanks are given to Mr. D. Doble and Mr. G. Sved of the University of Adelaide, and Dr. R. Smyth of the South Australian Institute of Technology, for the advice they have given and for their willingness to discuss various aspects of the thesis, and to Mr. T. Kopetsky of the S.A. Institute of Technology for interpretation of some Russian scripts.

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British Tube Mills kindly supplied a quantity of steel tubing.

CONTENT

REFERENCES

List of references quoted or discussed in the thesis; listed alphabetically according to author's name.

SUMMARY

It is shown theoretically in the thesis that the classical solution for the non-uniform torsion of thin-walled open-section long-span beams may appreciably underestimate the angle of twist of short to medium length beams of high warping rigidity; the reason given is that the direct shear deformation is not taken into account.

It is also shown theoretically that the same classical theory does give a valid estimate of the stress distribution for short spans of the above type of beam, even though the direct shear deformation is ignored, and on this basis a shear correction is obtained for the classical angle of twist via the strain energy of the classical shear stresses.

This thesis result gives corrections of up to 10% of the classical angle of twist, depending on the beam configuration and the type of loading for spans approximately 20 times the depth of the cross-section, and increasing corrections for shorter spans.

Experiments carried out by the author on round thin-walled metal tubes with a narrow longitudinal slit through the wall for the full length agreed well with the above thesis solution.

SUMMARY (continued)

A semi-empirical solution is proposed for the angle of twist of metal torque tubes containing a longitudinal slit of any length up to 10 diameters, with an unslit length of at least 1 diameter at both ends.

This solution is obtained by the substitution of an experimentally determined equivalent length into a theoretical result of V. Z. Vlasov for beams with large cut-outs, with the addition of the thesis shear correction.

The same approach is expected to give design solutions for longitudinal slit-like apertures in any thin-walled box structures.

A discussion of the works of some other authors who consider the effect of shear in non-uniform torsion is given in the appendix.

SUMMARY (continued)

Assumptions made throughout the thesis are

a) small, linearly elastic deflections

b) no lateral or radial deformations, including no buckling

c) uniform thickness

d) isotropic materials

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, contains no material previously published or written by any other person except where due reference is made in the thesis.

Signed:

Geoffrey Edwin Higginbottom

SYMBOLS AND DEFINITIONS

a — mean radius - average of inside and outside radius of thin-walled tube; (a = constant).

b — distance from arc radius centre to shear centre.

i — suffix relating to unslit part of part-slit tube

j — suffix relating to the cross-section containing the end of the slit in a part-slit tube

k — dimensionless characteristic, $= \frac{L}{2}\sqrt{\frac{GJ}{EJ_w}}$

$2L_i$ — total length of unslit part of slit tube

L_s — length of slit

m — moment arm, distance from centre of twist to tangents from cross-section profile line.

p — moment arm to centre of area of cross section

s — circumferential coordinate on profile line (z = constant)

$\pm S_o$ — limits of circumferential coordinate, (symmetrical arc section)

t — wall thickness, assumed always uniform

u — axial or longitudinal displacement

v — tangential or circumferential displacement

w — $\int_0^s m\,ds$

x — section constant $1/(1 + \frac{J}{J_s}) = 1-1.28\,(\frac{t}{D})^2$ for the slit tube.

z — axial or longitudinal coordinate on generator (a = constant)

A — cross-sectional area

C_1 C_2 — constants of integration

D	mean diameter of circular arc section (i.e. 2a)
E	Young's modulus of elasticity
G	Shear modulus of elasticity
J	St. Venant or pure torsion section constant. For slit thin walled tube $\simeq 1.047\ t^3 B$
J_o	pure torsion constant for thin-walled round (closed) tube, $.785\ tD^3$
J_p	polar moment of inertia $= \int_A p^2\ dA$
J_s	section shear constant of non-uniform torsion, $= \frac{(J_w)^2}{S_{ww}}$. For slit thin-walled tube $= .817\ tD^3$
J_w	section warping constant of non-uniform torsion, $= \int_A m\ S_w\ dA$ or $\int_A w^2\ dA$. For slit thin-walled tube $= .253\ tD^5$
$K_{S,B,T}$	dimensionless characteristics which are functions of k, being the correction factors for ϕ_{SO}, ϕ_{BO} and ϕ_{TO} respectively to give the part of the angle of twist associated with flexural torsional shear, flexural torsional axial normal stresses, and pure torsion, respectively. $K_s = (\frac{1}{2k}\sinh 2k + 1)/(\cosh 2k + 1)$ $K_B = \frac{3}{k^2}\ (\frac{1}{2k}\sinh 2k - 1)/(\cosh 2k + 1)$, $K_T = K_{BT} - \frac{k^2}{3} K_B$
K_{BT} =	$1 - \frac{1}{k}\tanh k$
L	total length of beam or shell (i.e. span)
L'	equivalent length
L_e	equivalent length of part slit tube

L_i		length of unslit part of part slit tube = $\frac{L - L_s}{2}$
L_s		length of narrow axial slit, symmetrical within tube span
S_w	=	$\int_S^{So} w\, ds$, (with s = 0 opposite slit)
S_{ww}	=	$t\int_A (S_w)^2\, dA$
T		torque at any cross-section, or
T		applied end torque = "uniform" torque (where "uniform" means a constant torque along the length or $T^1 = 0$)
T^1		rate of torque loading, = $\frac{dT}{dz}$ (= torque/unit length)
T_B		that part of the total torque resisted by flexural torsion
T_T		that part of the total torque resisted by pure (i.e. St. Venant) torsion
V_s		non-uniform torsion shear constant defined by $\theta_s' = \frac{Ts}{G\, V_s}$
β		dimensionless geometrical parameter relating warping rigidity of slit part of tube to unslit part.
γ		direct, or axial, shear strain of middle surface of beam wall
ε		axial, or longitudinal strain
θ		angle of twist of any cross-section (relative to a datum section)
θ_{BT}		angle of twist at any cross-section (like θ) obtained from the classic differential equation for non-uniform torsion taking no account of shear strain
θ_S		angle of twist at any cross-section (like θ) due only to the axial-circumferential (direct) shear strain

θ' rate of twist, $\frac{d\theta}{dz}$

θ'', θ''' $\frac{d^2\theta}{dz^2}$, $\frac{d^3\theta}{dz^3}$ respectively

ν Poisson's ratio

σ axial or longitudinal stress

τ axial shear stress (i.e. average shear stress through wall thickness)

ϕ total angle of twist of complete span L

ϕ_o total angle of twist of closed (unslit) tube, TL/GJ_o.

ϕ_S that part of the angle of twist associated with the axial shear stresses of non-uniform torsion ($= K_S \phi_{SO}$)

ϕ_B that part of the angle of twist associated with the axial tensile and compressive stresses of non-uniform torsion ($= K_B \phi_{BO}$)

ϕ_T that part of the angle of twist associated with the pure torsion due to the variation of shear through the wall thickness ($= K_T \phi_{TO}$)

ϕ_{BT} $\phi_B + \phi_T$ ($= K_{BT} \phi_{TO}$)

ϕ_{SO}, ϕ_{BO} the limits approached by ϕ_S and ϕ_B respectively when $\phi_T \to 0$, that is when L, or more precisely $k, \to 0$.
($\phi_{SO} = TL/GJ_s$, $\phi_{BO} = TL^3/12EJ_w$).

ϕ_{TO} TL/GJ. This is the angle of twist of a free ended beam, or the limit approached by a fixed end beam as the span becomes very long.

φ angular profile coordinate = s/a

φ_o s_o/a

$w = \int_o^s m\,ds$ (s = 0 opposite slit)

thin-walled beam	$\frac{D}{t} > 10$
long beam	one where ϕ_S is negligible; $k > 3.3$; generally $\frac{L}{D} > 10$.
medium length beam	one where ϕ_S, ϕ_B and ϕ_T should be taken into account, $(.3 < k < 3.3)$
short beam	one where ϕ_T is negligible, that is where a membrane state of stress is approached, $(k < .3$, which for the slit tube means approximately $\frac{Lt}{D^2} < .5)$.
shell	a 'beam' of such short span that the centre of twist is determined principally by the end conditions. In the thesis a rotation about the shear centre is usually assumed, even for short spans.

INTRODUCTION

The problem is to determine the effect of a short to medium-length narrow axial slit on the torsional properties of a round thin-walled tube, for small elastic deflections.

It may be applicable to practical situations such as seam welded propeller shafting with a missed length of weld, special purpose torque tubes whose rigidity it is required to control by the use of a slit, an aircraft fuselage with a slit-like cut-out, or an aircraft wing whose flexural centre it may be possible to adjust with a slit.

The mathematical model chosen is a short to medium length thin-walled open-section torque tube (the slit length) with elastic end supports (the unslit end lengths).

It is envisaged that the cross-sections containing the ends of a slit will be strongly restrained from axial warping by the unslit end lengths. Thus the first case to be studied will be a torque tube slit full length with the end cross-sections fully constrained to remain plane.

The rate of change of angle of twist along the length of such a structure will vary, even under a torque which is uniform along the length. This is a case of "non-uniform torsion".

An explanation of non-uniform torsion of a slit tube

The well known stress pattern in a closed tube subject to a pure torque about its axis is indicated in Fig. 0(a). Cross-sections remain plane during twisting.

If an axial slit is cut through the wall of the tube for its full length, while the torque remains, the shear is removed from the slit edge faces so that one edge will spring up (i.e. axially) and the other edge will spring down. See Fig. 0(b).

The slit tube is twisting in the "St. Venant" or "pure torsional" mode - as was the closed tube - because all cross-sections of the tube are free to warp. But now the cross-sections are warped, and if the torque is uniform along the length the warping will be constant, the rate of twist will be constant, and we have a case of "uniform torsion".

The torque is no longer resisted internally by a uniform circumferential "shear flow" as in the closed tube but by a stress pattern varying from plus to minus through the wall (actually giving a zero average shear flow). This stress pattern is similar to that in any thin-walled open-section beam under uniform torsion, all such sections being formed effectively from a folded plate or thin rectangular section.

To force the slit tube, under torque, back to an unwarped geometry, axial compressive and tensile forces would be required as indicated in Fig. 0(c).

It may thus be visualised, that when a torque is applied to a slit tube whose ends are externally forced to remain plane as in Fig. 1.1 (page 3), axial tensile and compressive stresses must be induced.

But how is the torque resisted in this slit tube with constrained ends?

The answer is that on any cross-section where the warping is completely prevented the torque is resisted entirely by a circumferential shear flow of the closed tube type, but of course falling off to zero at the two slit edges.

On other cross-sections, which will in general be "partially warped", there is a superposition of this direct shear flow plus the St. Venant shear stresses of Fig. 0(b) to give the shear stress distribution shown in Fig. 0(c) inset. (If the tube is long, the St. Venant shear stress becomes predominant well away from the constrained cross-sections). It follows that the internal resisting torque is made up of two components; — the moment of the direct or circumferential shear component about the centre of twist, plus the twisting moment within the walls of the tube caused by the antisymmetrical St. Venant shear distribution through the walls.

"Non-uniform" torsion is the name given to this kind of behaviour under torque loading because the varying shear pattern which occurs along the length of the tube gives rise to a varying warping of the cross-sections and also a varying rate of change of angle of twist.

Thin-walled beams of open-section are used extensively in engineering structures and their behaviour in non-uniform torsion has been investigated by many authors, including V.Z. Vlasov [1]* and S.P. Timoshenko [1] in the 1940's and more recently by Z.P. Bazant [1] and Kuang-Hau Chu and Anatole Longinow [1]. Extensive bibliographies and literature surveys on the subject are contained in Vlasov's book and in a paper by J. Novinski [1].

The thesis problem.

Fig. 0(d) illustrates the analogy between the slit-tube with constrained ends under torque and a built-in beam subject to vertical shear.

Now the vertical shear deflection of long beams is generally negligible compared to the bending deflection. Precisely the same is true of the analogous direct or circumferential shear deflection in long torque-tubes, and the classical theory described in the above references does in fact ignore this shear deformation.

Unlike our rather precise knowledge of beam theory however the relative importance of the direct shear deformation in non-uniform torsion does not seem to be fully understood.

In the majority of cases in civil engineering it has been found in practice that the classic theory is adequate. For short torque tubes it can be expected that direct shear deformation will become more important, as in the analogous beam.

* References are listed alphabetically at the end of the thesis.

Following the preliminary studies described in Chapter 1 of the main text, it became clear that the basic problem of the thesis would be to discover the importance of the direct shear deformation in non-uniform torsion.

Previous work on this topic is rather limited.

In Russian papers, Vorobev [1] utilises Vlasov's general equations in a Castigliano energy approach; Mescheryakov [1] obtains a solution which does not apply to long beams, by following up work of Goldenveyzer and ignoring variation of shear through the wall thickness; and Marin [1] provides an approximate solution by superposing the result of a simple shell analysis which assumes no axial distortions to the classic results of Vlasov. In a Japanese paper Mizoguchi and Shiota [1] obtain solutions by using a typical shell theory but do not appear to take account of twisting about the shear centre. In the U.S.A. Reissner [1] obtains solutions for solid sections via the variational calculus, and Aggarawal and Cranch [1] include the shear effect in a dynamic situation. A total of only two single-test experimental results is given in these references.

Most of these references are discussed in the first chapter of the appendix (pp A1 to A22).

The bulk of the thesis is taken up with other theoretical methods of investigating this problem, together with the presentation and discussion of a number of experiments carried out on slit-tubes.

The assumptions made relating to deformation and stress patterns are basically the same as those of the classic theory (which are summarised below and discussed further in Chapter 1 following) except that the classic assumption, shear strain = zero, is not made.

The remainder of the thesis is concerned with an initial study into the effects of a slit of any length on the torsional properties of a round tube. Such a tube is a unique case of a thin-walled open-section beam with elastic supports.

A summary of the Classic Theory

The following analysis is a summary of the classic analysis of the non-uniform torsion of thin-walled open section beams as presented by Timoshenko [1] and Vlasov [1]. A knowledge of this theory is assumed in the text of the thesis.

The classic theory is a semi-membrane approach in the sense that lateral bending of the walls or radial deflection is not taken into account,* but in the theoretical model stress system of axial tensile and compressive stress ($\sigma_z = \sigma$, uniform through the wall thickness) and shear stresses $(\tau_{zs} = \tau)$, the shear stresses are allowed to vary through the wall thickness to take account of St. Venant or "pure" torsion.

Hence applied torques are considered to be internally resisted at any cross-section by the circumferential shear flow which is the average shear across the thickness, together with the resisting moment due to the varying shear through the thickness.

* See also Novozhilov [1] for the requirements of a "membrane".

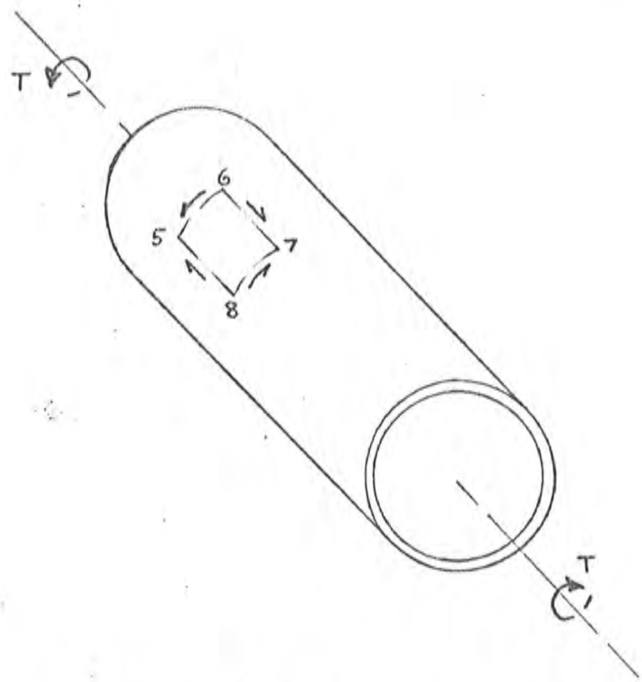


Fig. O(a) Closed tube subjected to a pure torque.

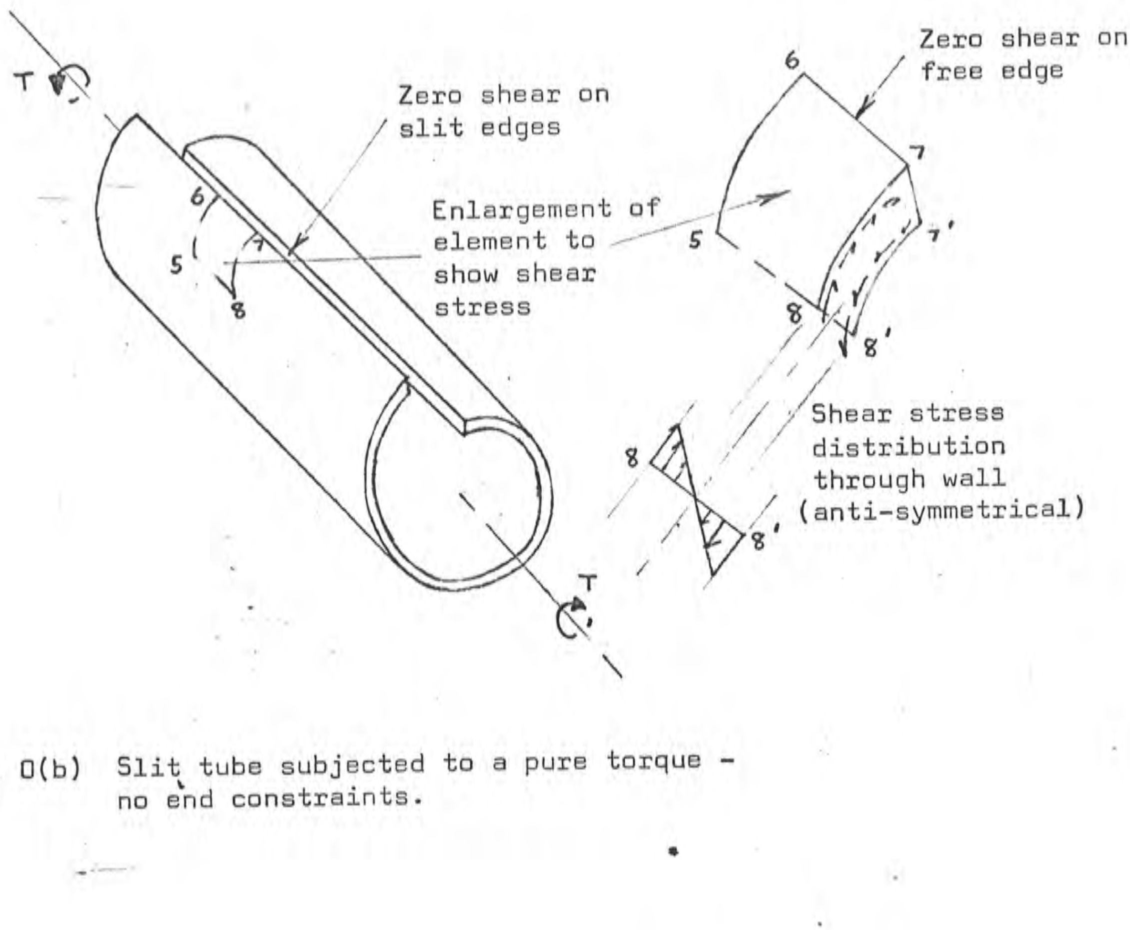


Fig. O(b) Slit tube subjected to a pure torque - no end constraints.

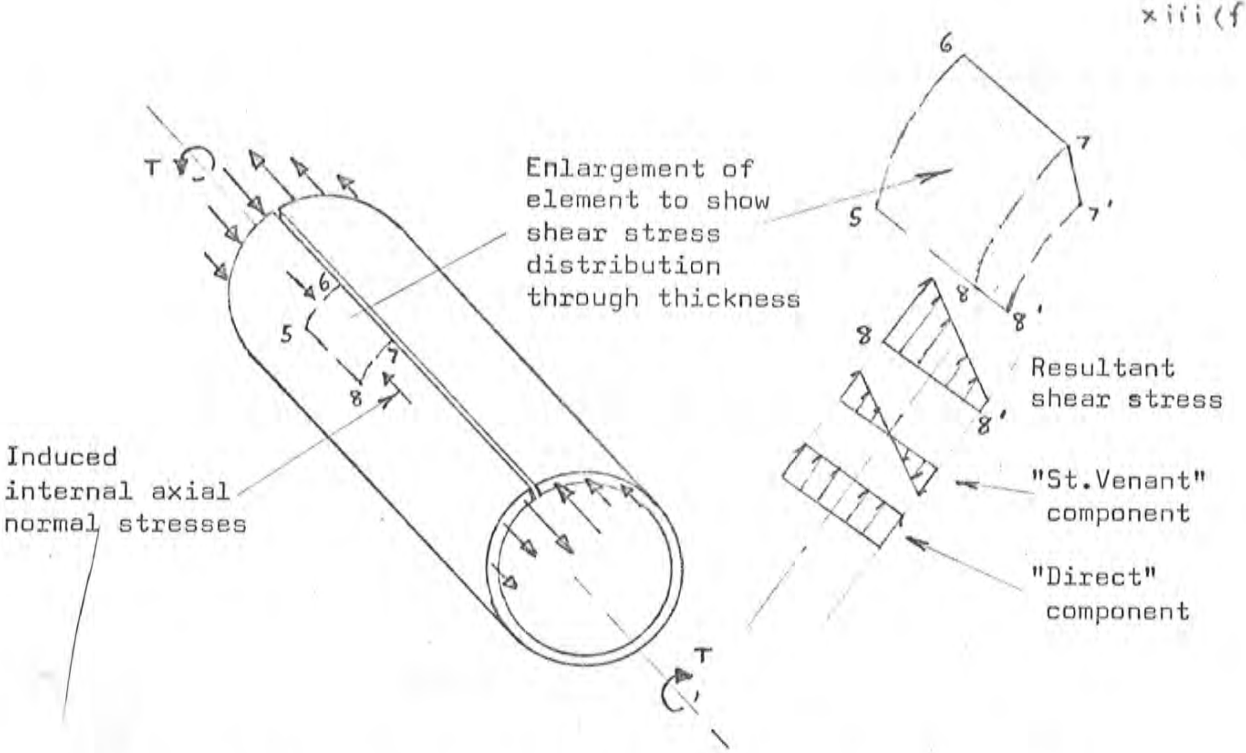


Fig. O(c) Indication of axial end loads required to force end sections of slit torque tube to become plane.

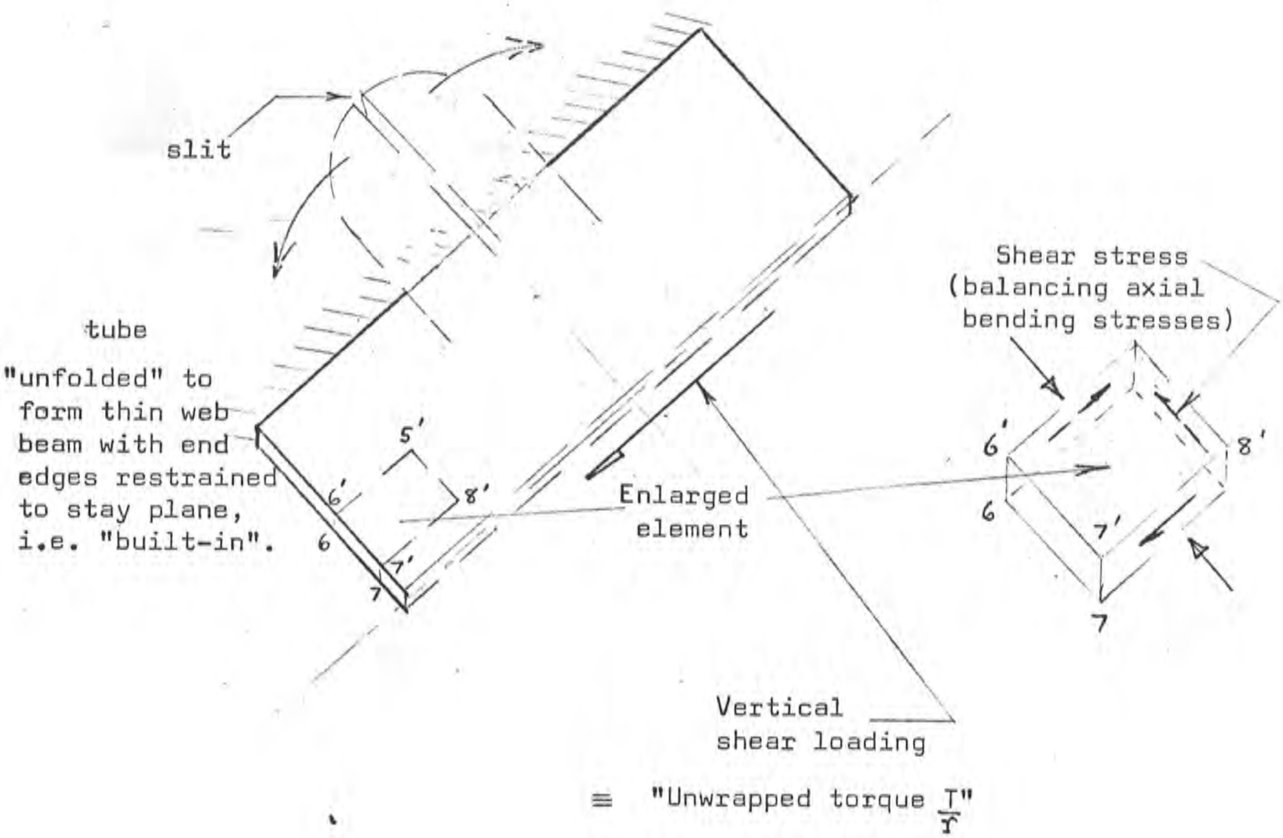
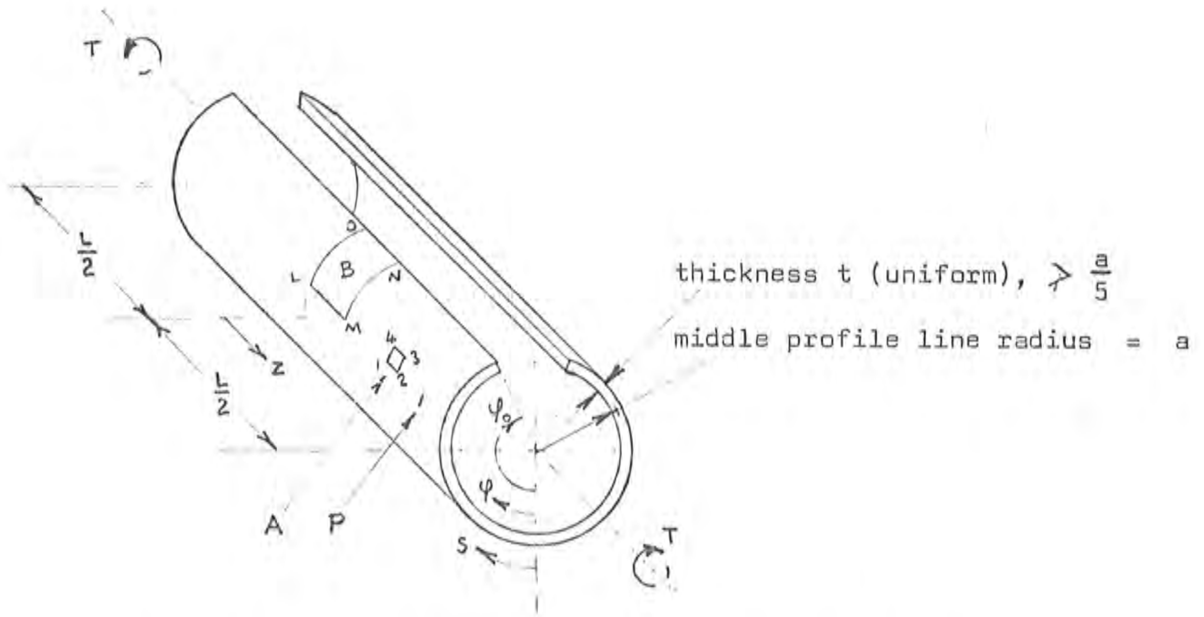
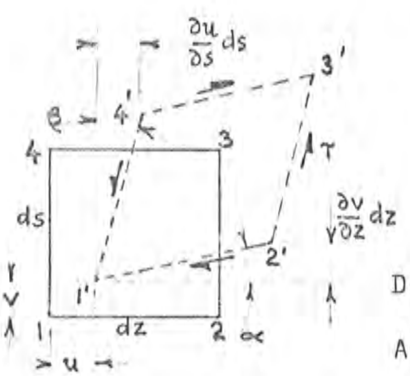


Fig. O(d) Demonstration of similarity of torsion of slit tube with constrained ends, and built-in beam with vertical shear loading.



COORDINATE SYSTEM FOR TUBE WITH FULL-LENGTH CUT-OUT & LOCATION OF ELEMENTS A, B and P

FIG. 1 (a)



$$\tau = G\,\gamma$$

where

$$\gamma = \alpha + \beta$$

Deflection of point $1 \rightarrow 1' = u, v$

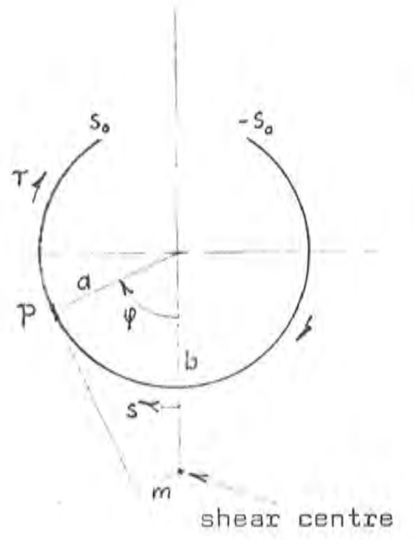
Angle $\alpha \simeq \frac{\partial v}{\partial z}\, dz/dz$

Angle $\beta \simeq \frac{\partial u}{\partial s}\, ds/ds$

ELEMENT A, dz x ds ;

ORIGINAL POSITION 1234, DEFORMED POSITION 1'2'3'4'

FIG. 1 (b)



At P,

elemental arc length = ds

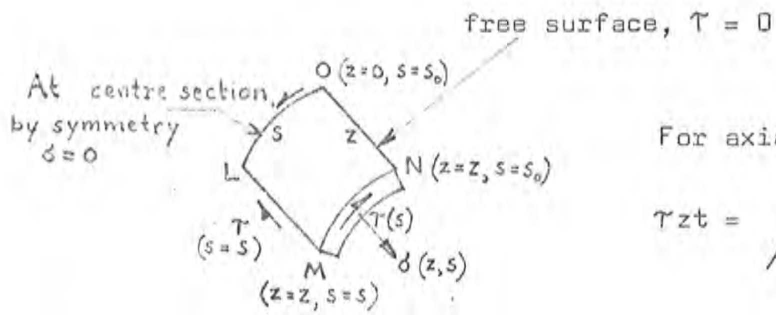
moment of shear force $\tau t\,ds$ about the shear centre

$$= \tau\, t ds.m$$

where $m = a - b \cos \varphi$

MIDDLE LINE OF CROSS-SECTION AT P

FIG. 1 (c)



For axial equilibrium

$$\tau z t = \int_{S}^{s_0} \sigma\, t\, d s$$

EQUILIBRIUM OF FINITE ELEMENT B, z × S

FIG. 1. (d)

The deformations allowed are a longitudinal warping and a rotation of the cross-section as a whole; the shape of the cross-section is assumed to be invariant. The relative movements, that is the strains, are limited in the classic theory to longitudinal tensile and compressive (warping) strains.

Consider the case of a beam of uniform thickness (t) throughout and uniform cross-section shape along the length, and consider only the affect of applied torques.

Let the axial and circumferential coordinates be z, s respectively; s being measured along the middle line of the profile; and let u (z,s), v (z,s) and θ (z) be the axial, circumferential and rotational displacements respectively. The relation between v and θ is

$$v = m\theta \qquad \text{Eq.1}$$

where m is the moment arm to the centre of twist which is assumed to be the shear centre.

The axial strain is

$$\varepsilon_z = \frac{\partial u}{\partial z} = \varepsilon \qquad \text{Eq.2}$$

The axial shear strain is

$$\gamma_{zs} = \frac{\partial v}{\partial z} + \frac{\partial u}{\partial s} = \gamma \qquad \text{Eq.3}$$

or from Eq.1

$$\gamma = m\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial s} \qquad \text{Eq.4}$$

The circumferential strain is assumed to be zero, hence the axial stress is approximately

$$\sigma_z = E\varepsilon = \sigma \qquad \text{Eq.5}$$

where the dividing factor $1 - \nu^2$ has been ignored.

The shear stress which is the average shear stress through the thickness is

$$\tau_{zs} = G\gamma = \tau \qquad \text{Eq.6}$$

but this step is not admissible in the classic theory which makes the assumption

$$\gamma = 0 \qquad \text{Eq.7}$$

[In fact it is this paradox which is the basis of the thesis study]

However Eq. 7 allows u and thence ε and σ to be obtained directly from Eqs.4, 2 and 5 to give, respectively

$$u = -w\theta' \qquad \text{Eq.8}$$

$$\varepsilon = -w\theta'' \qquad \text{Eq.9}$$

$$\sigma = -Ew\theta'' \qquad \text{Eq.10}$$

where the index dashes refer to differentiation with respect to z, and

$$w = \int_0^s m\, ds \qquad \text{Eq.11}$$

The shear stress is now obtained from the normal stress via the equation of internal equilibrium

$$0 = \frac{\partial \sigma}{\partial z} + \frac{\partial \tau}{\partial s} \qquad \text{Eq.12}$$

to give

$$\tau = E S_w \theta''' \qquad \text{Eq.13}$$

where

$$S_w = \int^{s_0} w\, ds \qquad \text{Eq.14}$$

The internal resisting torque at any cross-section is made up of that due to the average shear flow

$$T_s = \int_A (\tau t) \, m \, ds \qquad \text{Eq.15}$$

and that due to the variation (from τ) of the shear through the thickness giving rise to the pure torsional resisting moment which from St. Venant pure torsion theory is

$$T_T = GJ\theta' \qquad \text{Eq.16}$$

If the external torque per unit length at any section is $T' = dT/dz$, along the length of the beam then the change in applied torque over an elemental length dz equated to the change in total internal resisting moment gives

$$dT = \frac{\partial}{\partial z} T_s \, dz + \frac{\partial}{\partial z} T_T \, dz \qquad \text{Eq.17}$$

Substituting Eq.13 into 15 gives

$$T_s = -EJ_w\theta''' \qquad \text{Eq.18}$$

then from Eqs. 16, 17 and 18 the classic differential equation for the angle of twist is obtained

$$T' = -EJ_w\theta'''' + GJ\theta'' \qquad \text{Eq.19}$$

where

$$J_w = \int_A w^2 \, dA \qquad \text{Eq.20}$$

For a uniform torque T only acting

$$T = -EJ_w\theta''' + GJ\theta' \qquad \text{Eq.21}$$

Eq. 19 will give the classic solution for the angle of twist due to a torque caused by a load system that has a moment about the shear centre at any cross-section along the length.

Typical end conditions that might occur are:

(a) free end; $\varepsilon_o = 0$, hence from Eq.9

$$\theta_o'' = 0 \qquad \text{Eq.22}$$

(b) end free to warp, but prevented from rotating;

$$\theta_o = 0$$
$$\theta_o'' = 0 \qquad \text{Eqs.23}$$

(c) end prevented from warping; $u_o = 0$, hence from Eq.8

$$\theta_o' = 0 \qquad \text{Eq.24}$$

After solving Eq.19 the stresses may be evaluated from Eqs.10 and 13.

<u>The case of pure torsion</u>

Consider the case when the axial strain is zero (under a pure torque loading). This gives from Eq.9

$$\theta'' = 0$$

which means

$$\theta' = \text{constant}$$

A constant rate of twist is the case of pure torsion.

Note on presentation and originality

PART I of the MAIN TEXT is concerned with determining the solutions for stress and angle of twist of short to medium length torque tubes which have a full length slit and rigid end plates. (See Fig. 1.1 page 3.)

All the work is original except the analysis of Chapter 2.

Chapter 1 shows that the classical solution for angle of twist is invalid.

Chapter 2 shows that the classical solution for stresses is valid.

Chapter 3 demonstrates that the pure torsional stresses are negligible.

Chapter 4 uses the result of the preceding chapter in an analysis which confirms the conclusion of Chapter 2.

Chapter 5 presents the author's solution for angle of twist. It is a strain energy method based on the validity of the classical stresses.

Figs. 8.1 and 8.2 in PART III of the MAIN TEXT illustrate the results of the author's experiments which substantiate the above theoretical conclusions.

PART II of the MAIN TEXT (Chapter 6) deals with torque tubes which have a slit that does not extend to the ends of the tube.

Original methods are suggested for adapting an existing approximate solution for the angle of twist of closed sections which have a wide cut-out. The thesis solution for angle of twist is utilised (from PART I) and the incorporation of an "equivalent length" is suggested to take account of uncertain effects in the unslit length of tube.

This "equivalent length" is evaluated from tests described in PART III. (See Fig. 8.5). The resulting semi-empirical solution is then successfully used to predict the angle of twist of other tubes. (Fig. 8.6.)

Chapter A1 in the APPENDIX is a discussion of other authors' works concerned with the effect of shear. The most important are Reissner and Vorobev who have both obtained rather involved general solutions. They are both effectively the same solution; and they give similar (but not the same) results as the thesis method of Chapter 5.

The thesis solution is simpler, and just as valid for the cases studied.

Chapters A2 to A5 give further details of the theoretical and experimental investigations described in the MAIN TEXT.

Chapter A6 presents some original thoughts on the effect of twisting a beam about an axis other than its shear axis. It demonstrates how the principle may be applied to increase the torsional rigidity of some beams up to a hundredfold.

This last chapter need not be considered as part of the main thesis. It is included as additional material by the author within the terms of reference of his candidature.

The same is the case with the various "asides" and comments throughout the thesis.

References are quoted:

(a) where these are used as a starting point for the author's own analyses,

(b) where other authors' comments seem to be relevant to the particular discussion.

Equations and figures in the Introduction above have a single number. Those in the remainder of the thesis have a double number, the first referring to the chapter.

PART I

Tubes slit full length

Chapter 1 - theoretical considerations showing that the classic solution for angle of twist gives the wrong answer for a short slit tube, the likely reason being the neglect of the axial shear strain.

1. THE CLASSIC SOLUTION FOR ANGLE OF TWIST OF LONG OPEN-SECTION THIN-WALLED BEAMS SUBJECT TO NON-UNIFORM TORSION; ITS APPLICABILITY TO A SLIT TUBE.

A round thin-walled tube with a narrow longitudinal slit through the wall may be considered as an open-section beam. (Fig. 1(a), with $\pm \varphi_0 \equiv \pm \pi$).

The classic formula for the angle of twist θ of an open-section thin-walled beam with both ends restricted from warping and subject to a uniform torque T, is solving Eq.21 with the end condition of Eq.24

$$\theta = \frac{T}{GJ} \left(z \cosh k - \frac{L}{2k} \sinh \frac{2kz}{L}\right)/\cosh k \qquad - \text{Eq.1.1}$$

which for $z = \frac{L}{2}$ gives the angle of twist ϕ for the whole span.

$$\phi = \frac{TL}{GJ} \left(1 - \frac{1}{k} \tanh k\right) \qquad - \text{Eq.1.1(a)}$$

$$\text{where } k = \frac{L}{2} \sqrt{\frac{GJ}{EJ_w}} \qquad - \text{Eq.1.2}$$

EJ_w is the flexural torsional warping rigidity and GJ the St. Venant torsional rigidity; L is the span.

Vlasov [2] and Timoshenko [2] for example, obtained this result.

The general applicability of Eq.1.1 to shorter beams will be discussed later. The particular limitations of this equation when applied to a slit tube section can be determined at once by comparing numerical solutions for the slit tube with the known angles of twist of round closed tubes of identical L/D and D/t.

For the slit tube, (see Vlasov [3]),

$$J = \frac{\pi}{3} t^3 D \qquad - \text{Eq.1.3}$$

$$J_w = \frac{1}{48} tD^5 (\pi^3 - 6\pi) \qquad - \text{Eq.1.4}$$

$$= .253 \; tD^5$$

where D is the "mean diameter" of the tube (i.e. measured between the middle line of the wall thickness), and t is the uniform thickness.

Whence, for the slit tube

$$k = \frac{L}{2}\sqrt{\frac{GJ}{EJ_w}} = 1.017 \; \frac{L}{D} \cdot \frac{t}{D} \cdot \sqrt{\frac{G}{E}} \qquad - \text{Eq.1.5}$$

Use a typical value for structural metals of

$$\frac{G}{E} = .39 \qquad - \text{Eq.1.6}$$

and the previous equation becomes

$$k = .635 \; \frac{L}{D} \cdot \frac{t}{D} \qquad - \text{Eq.1.7}$$

The angle of twist of a closed round tube in uniform torsion is

$$\phi_o = \frac{TL}{GJ_o} \qquad - \text{Eq.1.8}$$

For a thin walled tube take

$$J_o = \frac{\pi}{4} tD^3 \qquad - \text{Eq.1.9}$$

The total angle of twist of the slit tube relative to the unslit or closed tube is obtained by dividing Eq.1.1(a) by Eq.1.8, thus

$$\frac{\phi}{\phi_o} = \frac{J_o}{J} \left(1 - \frac{1}{k} \tanh k\right) \qquad - \text{Eq.1.10}$$

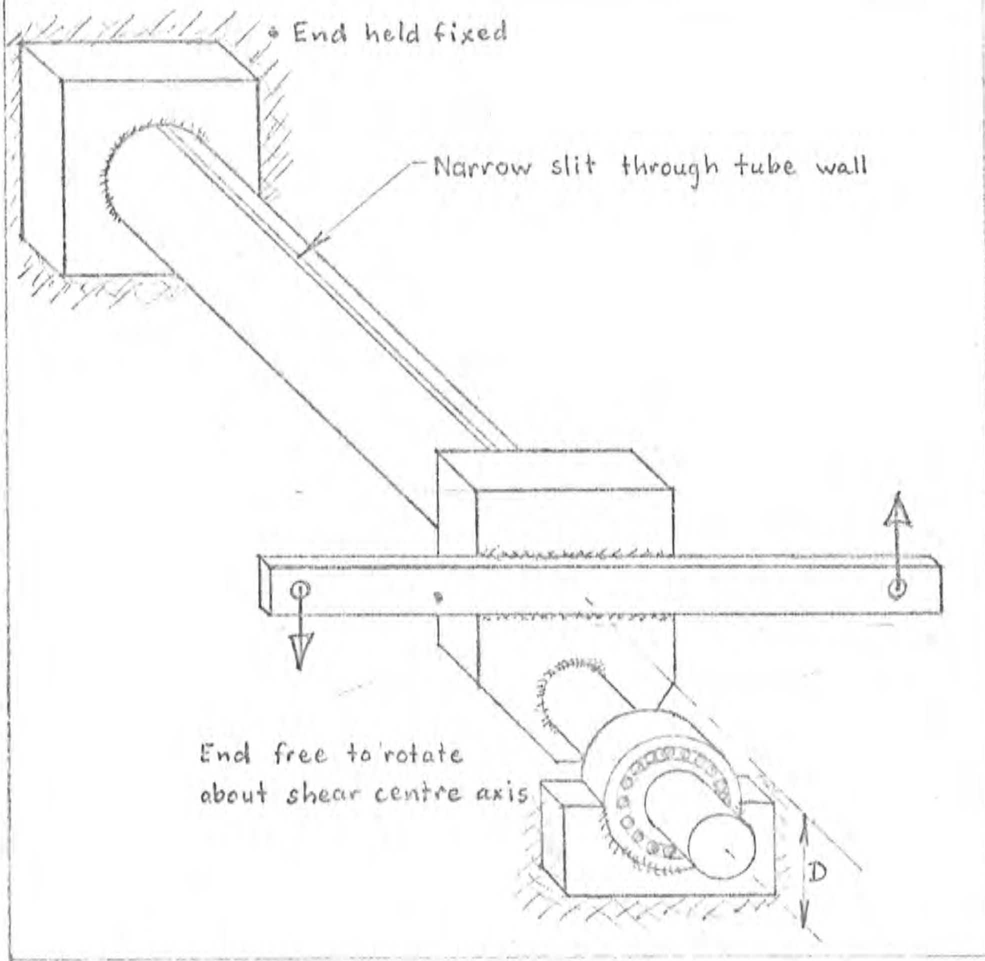
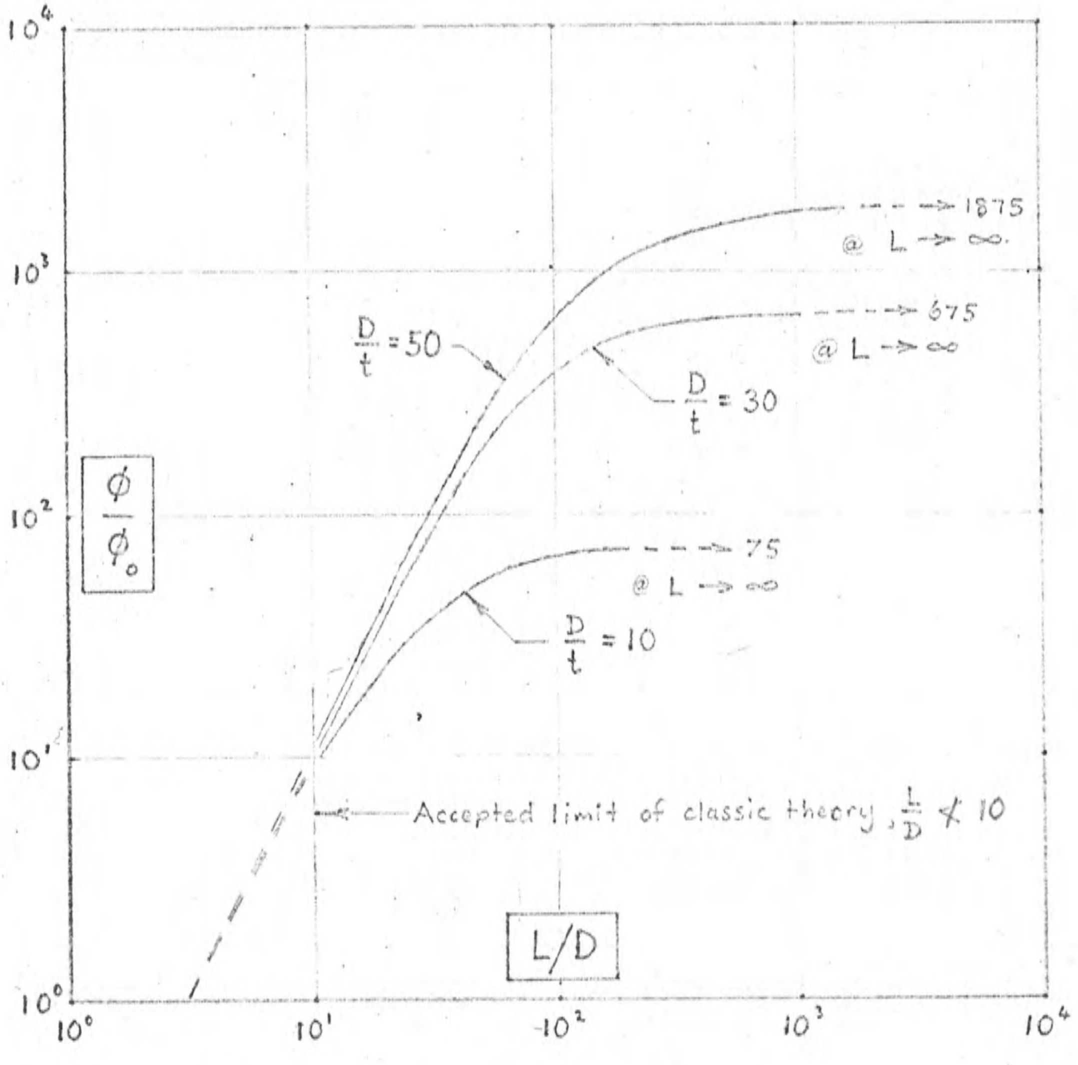


FIG. 1.1 APPLYING A UNIFORM TORQUE TO A SLIT TUBE WITH BOTH END SECTIONS CONSTRAINED TO REMAIN PLANE

(See also Figs. 7.1)



CLASSIC SOLUTION FOR ANGLE OF TWIST
of round thin-walled tubes with full-length slit, and with ends constrained to remain plane
(Eqs. 1.11 and 1.7)

FIG 1.2

Substituting for Jo and J from Eqs.1.9 and 1.3

$$\frac{\phi}{\phi_o} = \frac{3}{4}\left(\frac{D}{t}\right)^2 \left(1 - \frac{1}{k}\tanh k\right) \qquad \text{Eq.1.11}$$

The following series is useful for small values of k,

$$\frac{\tanh k}{k} \simeq 1 - \frac{k^2}{3} + \frac{2}{15}k^4 \qquad \text{Eq.1.12}$$

Eq.1.11 is evaluated for some spans up to twenty times the diameter in FIG.1.3

D/t	10		30		50	
L/D	k	ϕ/ϕ_o	k	ϕ/ϕ_o	k	ϕ/ϕ_o
1	.0635	.10	.0212	.10	.0127	.10
3	.195	.98	.0635	.91	.0381	.91
6	.381	3.84	.127	3.65	.0762	3.65
10	.635	8.6	.212	10.1	.127	10.1
20	1.27	24.7	.424	37.2	.254	41.5

Values of k from Eq.1.7, and

Values of ϕ/ϕ_o from Eq.1.11

FIG.1.3

A complete picture of Eq.1.11 is given in Fig.1.2 which covers a range of spans from L = 3D to infinity.

The table and the graph show that, for tubes of span less than approximately 3D, the classic solution says that the torsional rigidity of a slit tube is more than that of an otherwise similar closed (unslit) tube.

This is presumably incorrect.

Now Vlasov [4] states what he means by long thin-walled beams, limiting the applicability of his theory to beams with $D/t > 10$ and $L/D > 10$; nevertheless, it does appear to have been a more or less arbitrary limitation which, in any case, may not apply in the same degree, i.e. to the same accuracy, for different types of open section.

If the reason for the inapplicability of the theory to the short slit tube can be found, it can be expected that a more complete picture of the real limitations of the classic theory will follow .

Assumptions made in the classic analysis

The two basic assumptions of the classic analysis will now be considered. From Vlasov [5] -

"(a) a thin-walled beam of open section can be considered as a shell or rigid (undeformable) section;

(b) the shearing deformation of the middle surface (characterising the change in angle between the circumferential lines z = constant and the longitudinal lines s = constant) may be assumed to vanish."

The first assumption is equivalent to saying that there are no radial or circumferential normal stresses in the thin wall of the beam. This, in turn, implies a membrane state of stress with no transverse bending (or local buckling) of the tube walls.

This will be accepted as a reasonable assumption for small elastic deflections and, in any case, it can hardly be the reason for the discrepancy between two theoretical formulas, both of which contain effectively the same assumption.

The second assumption means that the direct circumferential shear deflection is negligible compared with the flexural torsional and pure torsional deformations. Practical experience has proved this to be a valid assumption for normal (open) beam sections of usual length. But, in a manner similar to simple beam theory where vertical shear is independent of span, and bending moment increases with increasing span (for a concentrated load), it may be expected that the direct shear deflection in a beam with constrained ends and subject to an end torque will increase in relative importance as the span decreases.

In fact the direct circumferential shear stress is part of the flexural torsional state due to the restraining of the end sections and provides part of the internal resistance to the applied torque; (the pure torsional moment within the tube walls provides the other part of the resisting moment). It seems unreasonable therefore to arbitrarily neglect the direct deflection due to this shear in any analysis before it is shown mathematically that it is negligible.

The bulk of the analyses in this thesis are aimed at determining the likely magnitude of these direct shear deflections in open section beams with fixed ends. For simplicity the loading is always considered to be an end torque, but the results are equally applicable to the general case of twisting of beams, except perhaps in a combined loading case where beam vertical shear is important in addition to the flexural torsional shear - see Vorobev [4].

If the second assumption (b) is not made, and an otherwise similar analysis to Vlasov-Timoshenko is carried out, integral equations which cannot be solved by this author are obtained; (see next chapter). Indirect approaches will therefore be used throughout to determine that part of the angle of twist due to the direct shear strain.

The few published works (mentioned below) that have attacked this problem have used differing approaches and arrived at solutions in various forms. These analyses are considered in the first chapter of the appendix.

Except where stated otherwise in the text, an independent approach is adopted here, and the solutions obtained are compared with the publications of E. Reissner [1], Mizoguchi and Shiota [1], Vorobev [1] and Meshcheryakov [1] (using Goldenveyzer's solution).

Chapter 2 - showing by the method of successive approximations that the classic solution for stresses, developed for long beams, is valid for the short slit tube.

This conclusion is confirmed in chapter 4.

2. THE CLASSIC SOLUTIONS FOR STRESSES DUE TO TWISTING OF A LONG THIN-WALLED BEAM OF OPEN SECTION WITH CONSTRAINED ENDS: THEIR APPLICABILITY TO SHORT BEAMS.

The solution for the longitudinal-circumferential shear stress in a thin-walled beam of open section with both ends fully constrained against warping and subject to an end torque T is, according to Vlasov [2], (or from Eqs.21, 24 and 13, together with the condition of symmetry at the centre-section which makes the strain zero, or from Eq.9, $\Theta'' = 0$ at the centre section),

$$\tau = T \; \frac{S_w}{J_w} \; \frac{\cosh\,(2\,kz/L)}{\cosh\,k} \qquad \text{Eq.2.1}$$

where z is the longitudinal co-ordinate measured from the centre of the span, and depends on the geometry of the section and the circumferential co-ordinate s . For the slit tube, (see Vlasov [6]),

$$S_w = \frac{D^3}{16}\;(5.87 - \varphi^2 - 4\cos\varphi) \qquad \text{Eq.2.2}$$

where φ is the angle between the tube radius opposite the slit and the radius to the point on the section being considered. That is,

$$s = \frac{D}{2}\,\varphi$$

where s is zero opposite the slit.

k and J_w are given by Eqs.1.2 and 1.4 respectively.

The longitudinal normal stresses obtained by Vlasov 2 are

$$\sigma = \frac{TL}{2k} \quad \frac{w}{J_w} \quad \frac{\sinh 2\,k\,z/L}{\cosh k} \qquad \text{Eq.2.3}$$

(Or use Eqs.21, 24, 13 plus $\Theta'' = 0$ at the centre-section).

For the slit tube, (see Vlasov [3]),

$$w = \frac{D^2}{4} \;(\varphi - 2 \sin\varphi) \qquad \text{Eq.2.4}$$

These results are accepted as giving good answers for the design of long beams, (see for example Vlasov [7], or Timoshenko [3]).

The distribution patterns given by Eqs.2.1 and 2.3 are shown in Figs. 2.1 (a) and 2.1 (b) for the case of the slit tube.

<u>Effect of assuming shear strain $\neq$ 0.</u>

Consider the classic derivation of the stresses, (Vlasov [8]), commencing with

$$\text{shear strain } \gamma = \frac{\partial v}{\partial z} + \frac{\partial u}{\partial s} = 0 \qquad \text{(Eqs.3,7)}$$

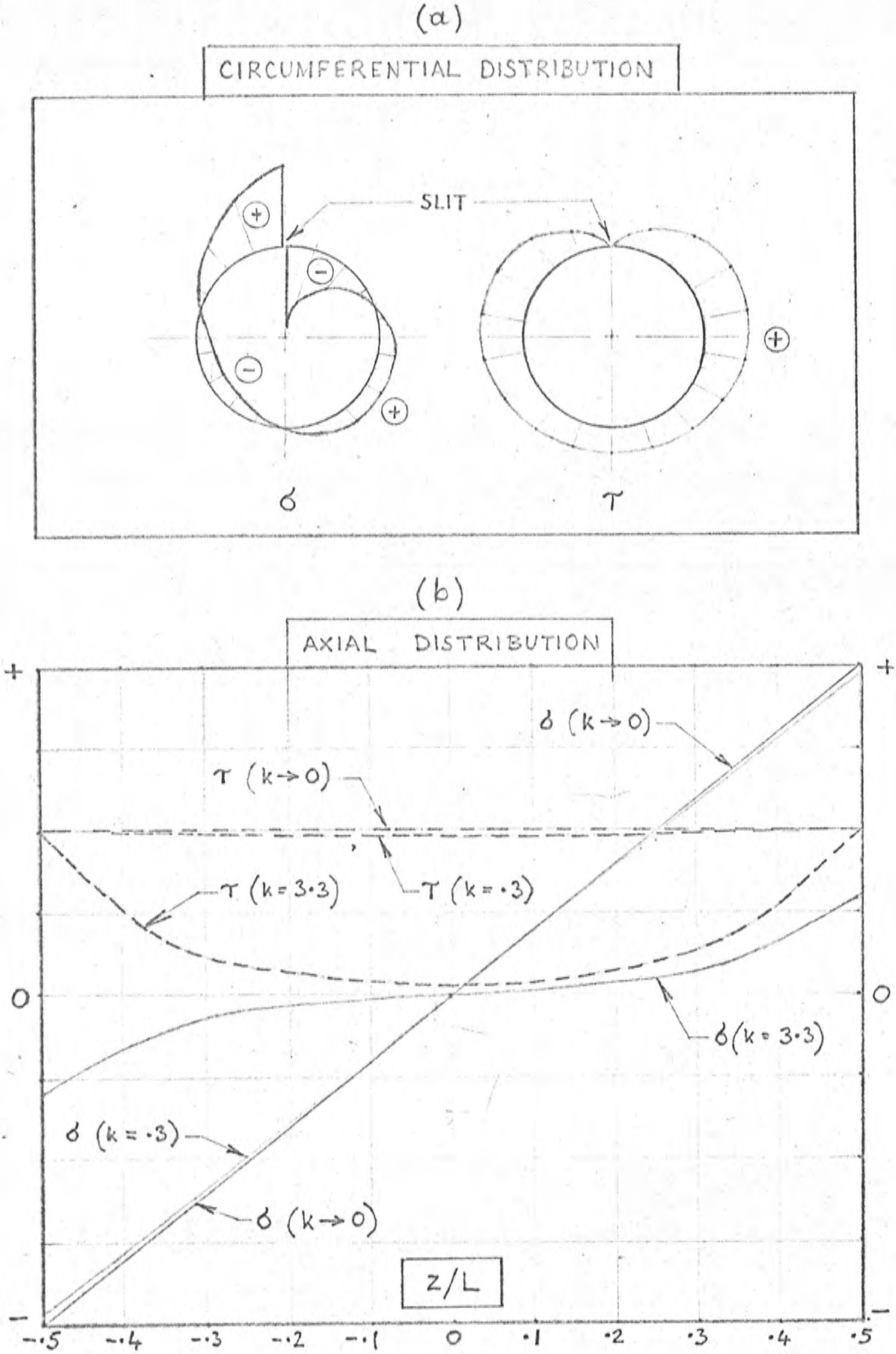
where u and v are the longitudinal and circumferential displacements; retain this expression as

$$\frac{\partial v}{\partial z} + \frac{\partial u}{\partial s} = \gamma \qquad \text{(Eq.3)}$$

so that Eq.8 for the axial displacement becomes

$$u = -\Theta' w + \eta$$

$$\text{or} \quad u = u_{\text{classic}} + \eta$$



CLASSIC STRESS SYSTEM FOR THE SLIT TUBE
WITH BOTH ENDS CONSTRAINED TO REMAIN PLANE
AND SUBJECTED TO A UNIFORM TORQUE

FIG. 2.1

where

$$u_{classic} = -\theta' w \qquad \text{(Eq.8)}$$

and

$$\eta = \int_0^s \gamma \, ds \qquad \text{Eq.2.5}$$

The new strain and stress are then

$$\varepsilon = \varepsilon_{classic} + \eta' \qquad \text{Eq.2.6}$$

(where the index dash refers to differentiation along the length z),

and

$$\sigma = \sigma_{classic} + E\eta'$$

or

$$\sigma = \sigma_{classic}\left[1 + \frac{E\ \eta'}{\sigma\ classic}\right] \qquad \text{Eq.2.7}$$

and finally, the shear stress becomes

$$\tau = \tau_{classic} - E\ s\ \eta''$$

or

$$\tau = \tau_{classic}\left[1 - \frac{E\ s\ \eta''}{\tau_{classic}}\right] \qquad \text{Eq.2.8}$$

Fig. 2.1.(b) indicates that η' and η'' are small for short spans because $\tau \rightarrow$ constant value along the length when $k \rightarrow 0$; nevertheless this may be misleading so that the analysis will be carried a step further in order that some numerical values of the above correction factors may be determined.

The procedure to be followed is that of Inan [1] who used the method of successive approximations to show that the correction factors in square brackets in Eqs.2.9 and 2.10 are of the order $(\frac{t}{D})^2$ from which he deduced that they are always small.

Following the procedure of Inan [2], substitute the general expressions for $\delta_{classic}$ and $\tau_{classic}$ which are Eqs.10 and 13 respectively, to give

$$\delta = \delta_{classic}\left[1 - \frac{\eta'}{\theta'' w}\right] \qquad \text{Eq.2.9}$$

and

$$\tau = \tau_{classic}\left[1 + \frac{s\,\eta''}{\theta''' S_w}\right] \qquad \text{Eq.2.10}$$

where

$$\eta = \frac{1}{G}\int_0^s \tau\, ds \qquad \text{(from Eq.2.6)}$$

$$\eta' = \frac{1}{G}\int_0^s \tau'\, ds \qquad \text{Eqs.2.11}$$

$$\eta'' = \frac{1}{G}\int \tau''\, ds$$

For example consider the case of a uniform torque on a beam with both ends built-in, and evaluate Eq.2.1 for shear stress to determine approximate values for η, η' and η'' from Eqs.2.11.

$$\eta = \int_0^s \frac{\tau}{G}\, ds = \frac{T}{GJ_w} \; \frac{\cosh 2kz/L}{\cosh k} \int_0^s S_w \, ds$$

$$\eta' = \frac{T}{GJ_w} \; \frac{2k}{L} \; \frac{\sinh 2kz/L}{\cosh k} \int_0^s S_w \, ds$$

$$\eta'' = \frac{T}{GJ_w} \; \frac{4k^2}{L^2} \; \frac{\cosh 2kz/L}{\cosh k} \int_0^s S_w \, ds$$

Substitute these values of η' and η'' in Eqs. 2.7 and 2.8 together with the values for $\sigma_{classic}$ and $\tau_{classic}$ from Eqs.2.1 and 2.3

$$\sigma = \sigma_{classic}\left[1 + 4k^2 \; \frac{E}{G} \; \frac{\int_0^s S_w \, ds}{L^2 w}\right]$$

Substituting

$$k^2 = \frac{L^2}{4}\sqrt{\frac{GJ}{EJ_w}} \qquad \text{(from Eq. 1.2)}$$

$$\sigma = \sigma_{classic}\left[1 + \frac{J}{J_w} \; \frac{\int_0^s S_w \, ds}{w}\right] \qquad \text{Eq.2.12}$$

$$\tau = \tau_{classic}\left[1 + 4k^2 \; \frac{E}{G} \; \frac{s\int_0^s S_w \, ds}{L^2 S_w}\right]$$

or

$$\tau = \tau_{classic}\left[1 + \frac{J}{J_w} \; \frac{s\int_0^s S_w \, ds}{S_w}\right] \qquad \text{Eq.2.13}$$

The terms in square brackets in Eqs.2.12 and 2.13 are the respective approximate correction factors for the stresses, if they are small. Both factors are independent of the span, and are constant along a generator. These are the Inan correction factors and they are of the order $(\frac{t}{D})^2$.

However the numerical constants in $\frac{J}{J_w}$ are so large for the more open sections like a semi-circular arc that the correction factors are not small for these sections except for very thin walls; the validity of the Inan conclusions is therefore doubted when applied to such sections. Nevertheless it will be shown in chapter 4 that the classic stresses do approach the same values for short beams as the solutions to an analysis which ignores the pure torsional shear but takes into account the direct shear, whence it is tentatively concluded that the classic stress system can be assumed to be approximately correct for all long and short thin-walled open-section beams and shells when twisting occurs about the shear centres.

Effect of assuming shear strain = constant

It can be seen from Eqs.2.7 and 2.8 that the classic stress system of σ and τ (which are derived from the assumption $\gamma = 0$) remain unaltered if in fact γ = constant; that is, Eqs.10 and 13 remain unaltered.

Chapter 4 will examine in detail the effect of assuming γ = constant, while assuming that the pure torsion is negligible; the latter is shown to be a reasonable assumption in the next chapter.

Chapter 3 - showing that the St. Venant or pure torsion is negligible in short slit tubes; that is, the shear stress through the wall thickness is effectively uniform.

3. AN ESTIMATION OF THE RELATIVE IMPORTANCE OF THE ST. VENANT PURE TORSIONAL SHEAR STRESSES IN A SHORT SLIT TUBE

The differential equation for long thin-walled beams from which Eq.1.1 is derived is

$$GJ\theta' - EJ_w\theta''' = T \qquad \text{(Eq.21)} \qquad \text{Eq.3.1}$$

where the first term represents that part of the torque resisted by the St. Venant pure torsional shear stresses (which are the difference between the actual circumferential shear at any point within the thickness and the average stress across the thickness), and the second term represents the resistance due to the flexural torsion system of axial normal and shear stresses caused by the ends of the tube being forced to remain plane.

The index dashes refer to differentation with respect to the longitudinal z axis.

If the pure torsional shear stresses are small the deflection due to flexural torsion alone may be estimated by ignoring the first term of the above equation to give the simple differential equation

$$-EJ_w\theta''' = T \qquad \text{Eq.3.2}$$

At this stage it cannot be said that the St. Venant shear stresses are small since no attempt has been made to evaluate them. But it will now be shown that Eq.3.2 does give similar results to Eq. 3.1 for short thin-walled beams so that the pure torsional shear stresses are indeed small for such beams.

From Eq.3.2

$$\theta''' = -\frac{T}{EJ_w}$$

Integrating

$$\theta'' = -\frac{T}{EJ_w} z + C_1$$

At the centre of the span by virtue of symmetry

$$\theta'' = 0$$

so if z is measured from the centre of the span, at z = 0,

$$C_1 = 0$$

Integrating again

$$\theta' = -\frac{T}{EJ_w} \frac{z^2}{2} + C_2$$

The rate of twist at the ends of the span must now be assumed. To be consistent with the Vlasov-Timoshenko assumption of vanishing shear strain which was used to obtain equation 1.1, (with which the following result will be compared), take

$$\theta' = 0 \text{ at } z = \pm \frac{L}{2}$$

This gives

$$C_2 = \frac{T}{EJ_w} \frac{L^2}{8}$$

Integrating again,

$$\theta = \frac{T}{EJ_w} \frac{z^3}{6} + \frac{T}{EJ_w} \frac{L^2 z}{8} + C_3$$

Take $\theta = 0$ at $z = 0$, so that $C_3 = 0$, then

$$\theta = \frac{T}{EJ_w}\left(\frac{L^2 z}{8} - \frac{z^3}{6}\right) \qquad \text{Eq.3.3}$$

The total angle of twist is twice that for half the span with $z = \frac{L}{2}$, thus

$$\phi = \frac{T L^3}{12\ EJ_w} \qquad \text{Eq.3.4}$$

Substituting for J_w for the slit tube from Eq.1.4 the twist given by Eq.3.4 relative to the twist of a closed tube ϕ_o becomes

$$\frac{\phi}{\phi_o} = \frac{T L^3}{12\ EJ_w} \div \frac{TL}{GJ_o}$$

$$= \frac{L^2}{12}\ \frac{GJ_o}{EJ_w}$$

$$= \frac{L^2}{12}\ \frac{G}{E}\ \frac{\frac{\pi}{4}\ tD^3}{.253\ tD^5}$$

$$\frac{\phi}{\phi_o} = .26\ \frac{G}{E}\ \left(\frac{L}{D}\right)^2 \qquad \text{Eq.3.5}$$

This is the angle of twist due to flexural torsion (apart from the direct shear deflection). This equation is the first approximation of Eq.1.11 obtained by substituting $1 - \frac{k^2}{3}$ for $\frac{1}{k}\tanh k$, that is when k^4 is negligible; i.e. Eq.1.1 (a) approaches Eq.3.4 when k (or L) approaches zero.

This proves that the torsional shear stress is negligible when, approximately

$$\frac{L}{D} < .3 \ \frac{D}{t}\sqrt{\frac{E}{G}}$$

for the slit tube. This finding will be utilised in the next chapter.

The limitations of Eq.3.5 as a solution in itself have already been indicated in Chapter 1.

Note that it has not been implied here that θ' itself is small, but the complete term $GJ\theta'$ relative to $EJ\theta'''$, that is, $GJ\theta'/EJ_w\theta'''$ which is like k^2.

Chapter 4 - showing that a membrane theory, which assumes uniform shear stresses through the wall thickness and takes account of the axial shear strain as constant along the length, gives solutions for the stresses and differential change in rate of twist to which the classic solutions approach as the span decreases.

4. BASIC DIFFERENTIAL EQUATIONS FOR STRESS AND ANGLE OF TWIST OF A SHORT TO MEDIUM LENGTH THIN-WALLED OPEN SECTION CIRCULAR CYLINDER CONSIDERED AS A MEMBRANE, SUBJECTED TO AN END TORQUE ONLY: SOLUTIONS.

When the span of an open-section beam with constrained ends is short enough to give a value of $k < .3$ the pure torsion will be assumed to be negligible and the beam will be assumed to have the requirements of a membrane as explained by Novozhilov [1].*

An analysis like the one which follows is presented by Beskin [1].

Figure 1 (a) shows a cylinder of constant thickness t, radius a, to centre of wall thickness, ($t < \frac{2a}{10}$), length L, subjected to end torques T and with both ends constrained to remain plane. (See also Fig. 1.1).

Some assumptions to be made initially are that

1. Only two stresses are important - the "membrane" stresses σ and τ, both uniform through the thickness, where σ can be tension or compression.

2. Only two forms of deformation are important - axial displacements and a rotation of the cross-section.

This means there are no radial displacements (i.e. no transverse bending), and no circumferential strain. Thus all cross-sections retain their size and shape.

* See also Gol'denveizer [1].

3. The deflections are small.

4. Hooke's Law is everywhere always valid.

5. The shear strain and hence shear stress is constant along a generator.

6. The axis of rotation (centre of twist) of the cylinder under the effect of the torque is the line of the shear centres.

7. The reduced modulus of elasticity $E_1 = E/_{1-\gamma^2}$ is assumed approximately equal to E. This is the same assumption as made in the classic analysis.

Coordinate system

It has been found convenient to choose the two symmetrical cutting planes of the cylinder as the zero (reference) planes. See fig. 1 (a).

The longitudinal coordinate z is measured plus to the right of the central cross section and minus to the left. Considerations of symmetry of the model indicate that the longitudinal normal stress (σ) is zero at this centre section, for a uniform torque.

The circumferential coordinate s is measured positive clockwise from the central point of the arc, viewed from the right hand end, and negative anticlockwise from this point. The coordinate s will generally be converted to the angular coordinate $\varphi = \frac{s}{a}$ where a is the (constant) arc radius of the middle line of the cross sectional thickness.

The middle surface of the shell is used throughout to describe the shell.

Displacements, strains, and stresses

(a) A longitudinal displacement in the positive direction (left to right) is called + u : see Fig. 1 (a).

A circumferential or tangential displacement in the positive s or φ direction (clockwise looking at a section from the right hand end) is called + v .

(b) The longitudinal strain is then

$$\varepsilon_z = \frac{\partial u}{\partial z} \qquad \text{Eq.4.1}$$

$$\varepsilon_z = \frac{\sigma}{E} - \nu \frac{\sigma_s}{E} \qquad \text{(i)}$$

The circumferential strain is

$$\varepsilon_s = \frac{\sigma_s}{E} - \nu \frac{\sigma}{E} \qquad \text{(ii)}$$

which is assumed to vanish as in the classic analysis, (see Introduction).

Substituting for σ_s from (i) into (ii) and equating ε_s to zero gives

$$\sigma = \frac{E}{1 - \nu^2} \varepsilon_z \qquad \text{(iii)}$$

Again making the same approximation as the classic analysis, ν^2 is ignored to give

$$\sigma = E \, \varepsilon_z = E \frac{\partial u}{\partial z} \qquad \text{Eq.4.2}$$

Note this result is the same as assuming $\sigma_s = 0$, in eq. (i) above - cf. a membrane analysis as in Den Hartog [1].

The shear strain is (from fig. 1 (b))

$$\gamma_{zs} = \frac{\partial v}{\partial z} + \frac{\partial u}{\partial s} = \gamma \qquad \text{Eq.4.3}$$

taking the deformation shown in the figure as positive. It follows that the shear stress (τ) condition shown in Fig. 1 (b) be positive, as is a clockwise torque (T) and clockwise angle of twist (θ) - viewed from the right hand end.

The circumferential displacement v is related to the (small) angle of twist about the shear centre by the equation

$$v = m\,\theta \qquad \text{Eq.4.4}$$

where m is the perpendicular distance from the tangent at the circumferential element being considered to the shear centre; that is, m is the moment arm, which is different for each element around the circumference. A positive shear flow has a negative moment about the shear centre when $b \cos \varphi > a$; see fig. 1 (c). The relation between m and φ is therefore

$$m = a - b \cos \varphi \qquad \text{Eq.4.5}$$

where b is the distance between the tube centre and the shear centre; (see Vlasov [15] or Timoshenko [6] for determination of shear centre).

For the slit tube, b = 2a (from Vlasov above), so

$$m = a\,(1 - 2 \cos \varphi) \qquad \text{Eq.4.6}$$

Thus m is negative in the central 120° opposite the slit, and positive for the two 120° sections bounding the slit.

The value for v may now be substituted in the equation for γ_{zs}, remembering that the beam is of constant cross-section so that m is a function of s only:

$$\gamma_{zs} = m \frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial s} \qquad \text{Eq.4.7}$$

It is at this point that the analysis becomes fundamentally different from the Vlasov [8] and Timoshenko [4] approach in that this shear strain is not assumed to vanish, but is used to obtain the shear stress directly via the shear modulus of elasticity. Thus

$$\tau = G \left(m \frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial s} \right) \qquad \text{Eq.4.8}$$

Equations of equilibrium

(a) For the finite element of fig. 1 (d) bordered on the left by an edge of zero normal stress, and at the top by an edge of zero shear stress,

$$\tau z t = \int_{S}^{S_o} \sigma \, t \, ds \qquad \text{Eq.4.9}$$

where $S = S_o$ is the free edge, and $S = S$ is the particular generator along which a constant value of τ is acting from $z = 0$ to $z = z$. On the cross section $z = 0$, $\sigma = 0$; on the cross section $z = z$, σ varies around the circumference.

(b) For equilibrium at any cross section (Fig. 1 (e)) the resisting torque equals the applied torque.

Force on element = τ t ds

Moment of force about shear centre = τ t m ds

The total resisting moment is the integral around the whole circumference,

$$T = \int_{-s_0}^{s_0} \tau t \, m \, ds \qquad \text{Eq.4.10}$$

where t is constant throughout.

This means the same as Eq.3.2, that is, the torque is equated to the flexural torsion effect only and the variation of the shear stress through the thickness which produces the pure torsional moment is assumed negligible.

Stresses, in terms of the angle of twist

(a) From Eq.4.8

$$\frac{\partial u}{\partial s} = \frac{\tau}{G} - m \frac{\partial \theta}{\partial z}$$

Integrate with respect to s and, for any point s = s on the section

$$u = \int_0^s \frac{\tau}{G} \, ds - \frac{\partial \theta}{\partial z} \int_0^s m ds + C_1 (s)$$

wherein the inflexibility of the cross-section shape implies a single value of θ for all points on a section.

Differentiate with respect to z

$$\frac{\partial u}{\partial z} = - \frac{\partial^2 \theta}{\partial z^2} \int_0^s m \, ds$$

the τ term disappearing because this is constant along the length, as is C_1 (s).

Multiplying both sides by E, and using Eq.4.2,

$$\sigma = - E \; \frac{\partial^2 \theta}{\partial z^2} \int_0^s m \; ds$$

For convenience, differentiations with respect to z will be denoted by the appropriate number of index dashes, and the integral will be called

$$\int_0^s m \; ds = w \qquad \text{Eq.4.11}$$

so that the above value for the normal stress may be written

$$\sigma = - E \, w \, \theta'' \qquad \text{Eq.4.12}$$

where for the general circular arc section, substituting for m from Eq.4.5 into Eq.4.11, and using $s = a\varphi$

$$w = \int_0^{\varphi} (a - b \cos \varphi) \, a \, d\varphi$$

$$w = a^2 \varphi - ab \sin \varphi \qquad \text{Eq.4.13}$$

which for the slit tube gives

$$w = a^2 \, (\varphi - 2 \sin \varphi) \qquad \text{Eq.4.14}$$

cf. Eq.2.4

w is negative between $\varphi \pm 110^\circ$ and positive for the rest of the section.

(b) From Eq.4.9

$$\tau = \frac{1}{z} \int_s^{s_o} \sigma \; ds$$

Substitute for σ from Eq.4.12

$$\tau = - \frac{E}{z} \, \theta'' \int_s^{s_o} w \; ds$$

Call the integral

$$\int_{s}^{s_0} w \, ds = S_w \qquad \text{Eq.4.15}$$

then

$$\tau = - E \, S_w \frac{\theta''}{z} \qquad \text{Eq.4.16}$$

For the slit tube where $\varphi_0 = \pi$, substituting for w from Eq.4.14 into Eq.4.15, and using $s = a\varphi$

$$S_w = \int_{\varphi}^{\pi} a^2 \, (\varphi - 2 \sin \varphi) \, a \, d\varphi$$

$$S_w = a^3 \left(\frac{\pi^2}{2} - 2 - \frac{\varphi^2}{2} - 2 \cos \varphi \right) \qquad \text{Eq.4.17}$$

cf. Eq.2.2.

This expression is always positive, except at the free edges $\varphi = \pm \pi$ where it vanishes as expected.

Differential Equation for Angle of Twist

Substituting from Eq.4.16 into Eq.4.10

$$T = - E \frac{\theta''}{z} \, t \int_{-s_0}^{s_0} m \, S_w \, ds$$

Call the quantity

$$t \int_{-s_0}^{s_0} m \, S_w \, ds = J_w \qquad \text{Eq.4.18}$$

then

$$T = - E J_w \frac{\theta''}{z} \qquad \text{Eq.4.19}$$

which is Eq.3.2 integrated once, as expected.

Evaluate J_w for the slit tube by substituting Eq.4.4 and Eq.4.17 in Eq.4.18, with $s = a\varphi$ and $s_o = \pi a$

$$J_w = ta^4 \int_{-\pi}^{\pi} (1 - 2\cos\varphi)\left(\frac{\varphi^2}{2} - 2 - \frac{\varphi^2}{2} - 2\cos\varphi\right) a \, d\varphi$$

$$J_w = ta^5 \left(\frac{2}{3}\pi^3 - 4\pi\right) \qquad \text{Eq.4.20}$$

cf. Eq.1.4.

This is the particular value of J_w for the slit tube, and works out at $8.10\ ta^5$, or $.253\ tD^5$ where D is the mean diameter of the tube.

Summary of results so far

$$\sigma = -Ew\theta'' \qquad \text{(Eq.4.12)}$$

$$\tau = -E S_w \frac{\theta''}{z} \qquad \text{(Eq.4.16)}$$

$$\theta'' = -\frac{Tz}{EJ_w} \qquad \text{from (Eq.4.19)}$$

m = moment arm to centre of twist

$$w = \int_0^s m\,ds \qquad \text{(Eq.4.11)}$$

$$S_w = \int_s^{s_o} w\,ds \qquad \text{(Eq.4.15)}$$

$$J_w = t\int_{-s_o}^{s_o} m\,S_w\,ds \qquad \text{(Eq.4.18)}$$

w and S_w are effectively the same geometrical constants derived by Vlasov [9], and already used in previous chapters of this thesis; similarly EJ_w is the same flexural warping rigidity as derived by Vlasov [10] and the C_1 of Timoshenko [5], already used, although expressed in a slightly different form here.

Comparing the above three equations for σ, τ and θ'' with Vlasov's results [11] it is seen they will be identical if the St. Venant torsional effect is negligible, which assumption is in any case inherent in this membrane approach.

It should still be remembered however that in both the classic solutions and in the above solutions, E is strictly $E/1-\nu^2$.

Solution for normal stress

From Eqs.4.12 and 4.19

$$\sigma = \frac{T}{J_w} w z \qquad \text{Eq.4.21}$$

The limitations of this equation to short beams is indicated by the linear increase of σ with z which is typical of membrane solutions. (See Gol'denveizer [2].)

For the slit tube, substituting for w and J_w from Eqs.4.14 and 4.20,

$$\sigma = \frac{T}{8.10\ ta^3}(\varphi - 2\sin\varphi)\ z$$

or

$$\sigma = .988\frac{T}{tD^3}\ (\varphi - 2\sin\varphi)\ z \qquad \text{Eq.4.22}$$

which, relative to the shear stress (τ_o) in the closed tube is

$$\frac{\sigma}{\tau_o} = 1.55 \frac{z}{D} \left(\varphi - 2 \sin \varphi \right) \qquad \text{Eq.4.23}$$

where

$$\tau_o = \frac{2T}{\pi D^2 t} \qquad \text{Eq.4.24}$$

σ is completely anti-symmetrical, changing signs at $\varphi = \pm 110^o$ around the circumference as well as at $\varphi = 0$, opposite the slit. The maximum values are at the fixed ends and along the slit, with

$$\sigma_{max} = \pm 1.55 \frac{TL}{tD^3}$$

which relative to the shear stress τ_o in a similar closed tube is

$$\frac{\sigma_{max}}{\tau_o} = 2.44 \ L/D$$

The classic solution is

$$\sigma = \frac{T}{J_w} \quad w \ z \left[\frac{L \sinh 2 \ kz/L}{2 \ kz \cosh k} \right] \qquad \text{(Eq.2.3)}$$

which is Eq.4.21 multiplied by the factor in square brackets which is close to unity for values of $k < .3$.

The membrane stress distribution in slit tubes is that illustrated in Fig. 2.1 (a), and in Fig.2.1 (b) for $k \to 0$.

Solution for shear stress

From Eqs. 4.16 and 4.19

$$\tau = \frac{T}{J_w} \quad S_w \qquad \text{Eq.4.25}$$

For the slit tube, substituting for [illegible] and J[illegible] from Eqs.4.17 and 4.20,

$$\tau = \frac{T}{8.10\ ta^2}\ \left(\frac{\pi^2}{2} - 2 - \frac{\varphi^2}{2} - 2\cos\varphi \right)$$

or

$$\tau = .247\ \frac{T}{tD^2}\ (5.87 - \varphi^2 - 4\cos\varphi) \qquad \text{Eq.4.26}$$

which, relative to the stress (τ_o) in the unslit tube is

$$\frac{\tau}{\tau_o} = .388\ (5.87 - \varphi^2 - 4\cos\varphi) \qquad \text{Eq.4.27}$$

τ is constant along a generator and symmetrical around the circumference; it is a zero at the slit and equals $.461\ T/tD^2$ opposite the slit, with the maximum value at $\varphi = \pm 110^\circ$ of

$$\tau_{max} = .875\ \frac{T}{tD^2}$$

which relative to the shear stress in the closed tube is

$$\frac{\tau_{max}}{\tau} = 1.38 \qquad \text{Eq.4.28}$$

The classic solution is

$$\tau = \frac{T}{J_w}\ S_w \left[\frac{\cosh 2\ k\ z/L}{\cosh k} \right] \qquad \text{(Eq.2.1)}$$

which is Eq.4.25 multiplied by the factor in square brackets which is close to unity when $k < .3$

At the centre of the span where $z = 0$ this factor represents more than a 10% difference when $\cosh k > 1.1$, or $k > .442$, which for the slit tube, with $\frac{G}{E} = .39$, means approximately

$$\frac{L}{D} \cdot \frac{t}{D} > .72$$

In practice this difference is probably not important since the maximum shear stress in both equations is the same (at $z = \pm \frac{L}{2}$).

Note that the factor in square brackets above may also be taken as an indication of the presence of the St. Venant torsional shear stresses away from the fixed ends which must be present to make up the resisting torque.

Graphical illustrations of the membrane stress distribution in slit tubes are given in Fig.2.1 (a), and Fig. 2.1 (b) with $k \to 0$.

As mentioned in chapter 2, it is seen that Eqs.4.21 and 4.25 are the limits approached by the classic Eqs.2.1 and 2.3 when $k \to 0$ indicating that the classical solutions are valid for all lengths of thin-walled open-section beam, when twisting occurs about the shear-centre.

<u>Solution of the membrane equations for angle of twist including a finite value for rate of twist at the fixed ends.</u>

From Eq.4.19

$$\theta'' = -\frac{Tz}{EJ_w}$$

Integrating,

$$\theta' = -\frac{Tz^2}{2EJ_w} + C_2 \qquad \text{(i)}$$

The classic solution to this equation naturally takes the assumption of vanishing shear strain to imply a zero rate of twist at the constrained end sections ($z = \pm \frac{L}{2}$); but this can hardly be assumed in the present approach which is aware of a finite value for shear strain, (see the comments of E. Reissner [2] and Novozhilov [2]).

So the determination of a real value of θ' at the constrained ends becomes an important feature of this membrane approach.

(a) <u>Determination of end condition for rate of twist, and hence the shear deflection, using longitudinal deflection = 0 at fixed ends.</u>

At the fixed ends u = constant around the circumference, therefore

$$\frac{\partial u_o}{\partial s} = 0$$

Substitute in Eq.4.8 which is

$$\tau = G\ (m\,\theta' + \frac{\partial u}{\partial s})$$

to give

$$\tau(z = \pm L/2) = Gm\,\theta_o'$$

where θ_o' represents the rate of twist at the end section. Substitute this in Eq.4.10, which is

$$T = \int_{-S_o}^{S_o} \tau\ t\ m\ ds$$

to give

$$T = \int_{-s_0}^{s_0} G\, m\, \Theta_0'\, t\, m\, ds$$

or

$$T = G\, \Theta_0'\, t\, a \int_{-\varphi_0}^{\varphi_0} m^2\, d\varphi \qquad \text{(ii)}$$

call

$$ta \int_{-\varphi_0}^{\varphi_0} m^2\, d\varphi = J_m \qquad \text{Eq.4.29}$$

Then

$$\Theta_0' = \frac{T}{GJ_m} \qquad \text{Eq.4.30}$$

For a circular arc section,

$$m = a - b\cos\varphi \qquad \text{(Eq.4.5)}$$

hence

$$\frac{J_m}{ta} = \int_{-\varphi_0}^{\varphi_0} m^2\, d\varphi$$

$$J_m = ta\left[(2a^2 + b^2)\,\varphi_0 - 4\,ab\sin\varphi_0 + \frac{b^2}{2}\sin(2\,\varphi_0)\right] \qquad \text{Eq.6.3}$$

For the slit tube, with $b = 2a$ and $\varphi_0 = \pi$,

$$J_m = 6\pi\, t\, a^3$$

so

$$\Theta_0' = \frac{T}{6\pi G\, ta^3} \equiv \frac{T}{3\, GJ_o} \qquad \text{Eq.4.31}$$

where GJ_o is the modulus of rigidity of the closed tube.

Continuing now with the integration by substituting Eq.4.20

in eq. (i) above, for $z = \pm \frac{L}{2}$

$$\frac{T}{GJ_m} = -\frac{TL^2}{8EJ_w} + C_2$$

Substituting this value of the constant in eq. (i)

$$\theta' = \frac{T}{GJ_m} + \frac{TL^2}{8EJ_w} - \frac{Tz^2}{2EJ_w} \qquad \text{Eq.4.32}$$

Integrate again

$$\theta = \frac{Tz}{GJ_m} + \frac{Tz}{24EJ_w}(3L^2 - 4z^2) \qquad \text{Eq.4.33}$$

where the constant of integration is zero if $\theta = 0$ at $z = 0$

The total angle of twist is twice the value for $z = \frac{L}{2}$, thus

$$\phi = \frac{TL}{GJ_m} + \frac{TL^3}{12EJ_w} \qquad \text{Eq.4.34}$$

The second term is the classic solution when the pure torsional stresses are very small or ignored (see Eq.3.4); the first term, representing the direct shear deflection, is now added to this.

It remains to discuss the validity of this result, in particular Eq.4.30.

It is noted that Eq.4.31 for the slit tube does not line up with the thoughts contained in chapter A.2 (appendix) and expressed in the semi-intuitive Eq.A.2.6 wherein the twist due to shear is guessed as TL/GJ_δ.

The discrepancy seems to lie in the disregarding of any notion of twisting about the shear centre in chapter A2 whilst this assumption is used in the derivation of Eq.4.30. Whereas the pure torsion plus axial normal stress system combine to cause a rotation about the shear centre, axial shear strain causes a displacement of the cross-section about a different axis, which for a slit tube section would be the arc radius centre. And at the fixed end the rate of twist is due entirely to shear so that the moment arm m in eq.(ii) above for θ_o' should be replaced by a, to give

$$T = G\,\theta_o' \;\; ta \int_{-\varphi_o}^{\varphi_o} a^2 \, d\varphi$$

Call

$$ta \int_{-\varphi_o}^{\varphi_o} a^2 \, d\varphi = J_a \qquad \text{Eq.4.35}$$

Then

$$\theta_o' = \frac{T}{GJ_a} \qquad \text{Eq.4.36}$$

and

$$\phi = \frac{TL}{GJ_a} + \frac{TL^3}{12\;EJ_w} \qquad \text{Eq.4.37}$$

which relative to a closed round tube of the same length is

$$\frac{\phi}{\phi_o} = \frac{J_o}{J_a} + \frac{L^2}{12}\;\frac{G}{E}\;\frac{J_o}{J_w} \qquad \text{Eq.4.38}$$

For the slit tube with $\varphi_o = \pi$, J_a becomes the polar moment of inertia J_p, (as in the Reissner solution [1] mentioned in chapter A1), hence

$$GJ_a = GJ_p = G\,2\pi\,ta^3 = GJ_o$$

where GJ_o is the modulus of rigidity of the closed tube, implying that the rate of twist at the fixed end for the slit tube is the same as that for a closed tube. Then Eq.4.36 becomes, for the slit tube

$$\phi = \frac{TL}{GJ_o} + \frac{TL^3}{12EJ_w}$$

or

$$\frac{\phi}{\phi_o} = 1 + \frac{L^2}{12}\,\frac{G}{E}\,\frac{J_o}{J_w}$$

where ϕ_o is the angle of tiwst of the similar closed tube; and substituting for J_o and J_w for the slit tube, gives

$$\frac{\phi}{\phi_o} = 1 + .26\,\frac{G}{E}\,\left(\frac{L}{D}\right)^2 \qquad \text{Eq.4.39}$$

which is the same as Eq.A.2.6.

It is now necessary to verify that with the new value of θ', $GJ\theta'$ is still negligible relative to $EJ_w\,\theta'''$ in that the assumption of negligible pure torsional effects remains valid, (see chapter 3).

From Eq.4.32, replacing J_m by J_d

$$GJ\,\theta' = \left[\frac{T}{GJ} + \frac{TL^2}{8EJ_w}\left(1 - 4\left(\frac{z}{L}\right)^2\right)\right] GJ$$

Also, from Eq.4.19,

$$EJ_w\,\theta''' = T$$

Therefore

$$\frac{GJ\theta'}{EJ_w\theta'''} = -\left(\frac{J}{J_a} + \frac{1}{2}\,\frac{L^2}{4}\,\frac{GJ}{EJ_w}\,\left(1 - 4\,\frac{z^2}{L^2}\right)\right)$$

$$= -\left(\frac{J}{J_a} + \frac{k^2}{2}\,\left(1 - 4\,\frac{z^2}{L^2}\right)\right)$$

So k^2 must be small; and also $\frac{J}{J_a}$ must be small.

For the slit tube, using Eqs.1.3 and $J_p = J_o$,

$$\frac{J}{J_a} = \frac{\frac{2}{3}\pi a\,t^3}{2\pi a^3 t} = \frac{1}{3}\,\left(\frac{t}{a}\right)^2$$

which is small for the thinner sections.

For the semi-circle, (like structural beam open sections), with $p = a$ instead of being measured to the C. of G.,

$$\frac{J}{J_a} = \frac{\frac{1}{3}\pi a\,t^3}{\pi a^3 t} = \frac{1}{3}\,\left(\frac{t}{a}\right)^2$$

which is small for the thinner sections.

Eq.4.39 is a suitable equation to be used for the short slit tube with ends prevented from warping. Since Eq.A.4.37 gives a different result from the methods following for more open sections further consideration, and experiments need to be carried out to determine its general suitability.

(b) Shear deflection of circular arc section from shear strain

Assume that the angle of twist at any section, due to the shear strain only, is given by the average shear strain around the section. The shear is constant over the length, so that the total shear deflection from one end of a generator to the other end is

$$\delta_s = L\,\gamma$$

where δ_s varies around the circumference because of $\gamma(s)$.

The angle of rotation of one end of a generator relative to the other is, due to the shear strain only,

$$\frac{\delta_s}{a} = L\,\frac{\gamma}{a}$$

for a given generator of the beam, provided the beam maintains its circular shape.

An average value of $\frac{\delta_s}{a}$ is required around the whole circumference to give an approximation to the angle of twist due to shear. Thus

$$\phi_s = \frac{1}{2\,S_o}\int_{-S_o}^{S_o} L\,\frac{\gamma}{a}\,ds$$

where $\gamma = \frac{\tau}{G}$, and τ is given by Eq.4.25

Substituting gives

$$\phi_s = \frac{TL}{GJ\,S_o\,J_w\,a}\int_{-S_o}^{S_o} S_w\,ds \qquad \text{Eq.4.40}$$

For the slit tube, substitute for m, S_w and J_w from Eqs.4.6, 4.17 and 4.20 respectively, and with $S_o = \pi a$

$$\phi_s = \frac{TL}{2 G \pi a J_w} \int_{-\pi}^{\pi} a^3 \left(\frac{\pi^2}{2} - 2 - \frac{\varphi^2}{2} - 2 \cos \varphi \right) a \, d\varphi$$

$$= \frac{TL}{2 \pi G a^3 t}$$

$$= \phi_o$$

which is the angle of twist of the closed tube, as before.

Remark

While agreeing with the intuitive result expected for the slit tube, the above two methods ((a) and (b)) for finding the shear deflection were regarded as unsatisfactory. The method described in the next chapter is thought to be more reasonable.

Chapter 5 - showing that the classic solution for angle of twist may be corrected for shear by adding to it the shear deflection obtained from the strain energy of the classic shear stresses.

5. ANGLE OF TWIST DUE TO AXIAL SHEAR STRAIN FOR ANY THIN-WALLED OPEN-SECTION SECTION BEAM, BY STRAIN ENERGY FROM THE CLASSIC STRESS SYSTEM

Consider a beam subject to a uniform torque T and with both ends constrained to remain plane.

The external work done by the torque in twisting the beam through a small angle $d\theta$ over an elemented length dz is $\frac{1}{2} T d\theta$ for the elastic case, and this equals the internal strain energy imparted to the system within the length dz. The angle of twist θ over a finite length is then given by equating $\frac{1}{2} T \theta$ to the total strain energy within the system over the same length. Ref.Williams [2] .

Similarly, let $\frac{1}{2} T\, d\theta_s$ be the external work associated with that part of the angle of twist (θ_s) which is due solely to the axial or direct shear strain of the middle surface of the beam wall. The internal strain energy associated with the shear angle is $\frac{1}{2}\tau\gamma$, or $\frac{\tau^2}{2G}$, per unit volume for an element t x ds x dz. For a finite length z therefore,

$$\int_0^z \frac{1}{2} T\, d\theta_s = \int_{\text{volume } (0 \rightarrow z)} \frac{\tau^2}{2G}\, d\,(\text{volume})$$

which for the coordinate system of Fig. 1 (a) is

$$\frac{1}{2} T \theta_s = \frac{t}{2G} \int_0^z \int_{-s_0}^{s_0} \tau^2\, ds\, dz \qquad \text{Eq.5.1}$$

The accuracy of Eq.5.1 as a means for determining Θ_s depends primarily on the accuracy of the expression substituted for the shear stresses τ. Chapters 2, and 3 with 4, have indicated that the classic solution for shear stresses (Eq.2.1) is valid for all lengths of thin-walled open-section beam or shell that have an unchanging shape of cross-section when under load, and which twist about the shear centres of each cross-section.

Therefore substitute Eq.2.1 in Eq.5.1, to give

$$\tfrac{1}{2} T \; \Theta_s = \frac{t}{2G} \int_0^z \int_{-s_0}^{s_0} \left(\frac{T}{J_w} \; S_w \; \frac{\cosh 2kz/L}{\cosh k}\right)^2 ds \, dz$$

$$\therefore \quad \Theta_s = \frac{T \; t}{G \, J_w^{\,2} \cosh^2 k} \int_{-s_0}^{s_0} (S_w)^2 \, ds \int_0^z (\cosh^2 2 \, kz/L) \, dz$$

Performing the integration with respect to z,

$$\Theta_s = \frac{T \; t}{G J_w^{\,2} \cosh^2 k} \int_{-s_0}^{s_0} (S_w)^2 ds \; \left(\frac{z}{2} + \frac{L}{8k} \sinh 4kz/L\right) \qquad \text{Eq.5.2}$$

and this is the angle of twist due to shear only measured from the centre section.

Eq.5.2 may be simplified by defining a "basic shear rigidity, GJ_s, in non-uniform torsion", where

$$J_s = \frac{J_w^{\,2}}{t \int_{-s_0}^{s_0} (S_w)^2 \, ds} = \frac{J_w^{\,2}}{S_{ww}} \quad \text{say} \qquad \text{Eq.5.3}$$

J_w and S_w being given by Eqs.4.18 and 4.15 respectively.

Substituting Eq.5.3 in 5.2, gives

$$\theta_s = \frac{T}{GJ_s}\left[\frac{z + \frac{L}{4k}\sinh\frac{4kz}{L}}{2\cosh^2 k}\right] \qquad \text{Eq.5.4}$$

The rate of twist due to shear only $\left(\frac{d\,\theta_s}{dz}\right)$ is

$$\theta_s' = \frac{T}{GJ_s}\left[\frac{\cosh^2\frac{2kz}{L}}{\cosh^2 k}\right] \qquad \text{Eq.5.5}$$

The effect of z and k on the rate of twist due to shear only is as follows:

$$z = 0,\ \theta_s' = \frac{T}{GJ_s\cosh^2 k};\quad z = \frac{L}{2} \text{ and } k \to 0,\quad \theta_s' \Rightarrow \frac{T}{GJ_s}$$

The last case, that of $k \to 0$ represents the case of negligible pure torsion as illustrated in chapter 3.

Fig. 5.1 gives some values of the flexural torsion section constant for shear, J_s, together with values for other torsion constants.

$J_o = .785D^3t$	J $(\frac{1}{3}t^3 \times \text{profile length})$	J_w (Eq.4.18)	J_s (Eq.5.3)	J_p
SLIT TUBE	$1.048\ Dt^3$	$.253\ D^5t$	$.817\ D^3t$	$.785\ D^3t$
¾ CIRCULAR ARC	$.785\ Dt^3$	$.0266D^5t$	$.151\ D^3t$	$.589\ D^3t$
½ CIRCULAR ARC	$.525\ Dt^3$	$.00117D^5t$	$.0128D^3t$	$.393\ D^3t$

TORSION CONSTANTS FOR 3 DIFFERENT THIN-WALL OPEN SECTIONS

FIG. 5.1

<u>Angle of twist including shear</u> (Unwarped ends, uniform torque)

The classic solution for the angle of twist is

$$\theta_{BT} = \frac{T}{GJ}\left[z + \frac{L \sinh \frac{2\,kz}{L}}{2k \cosh k}\right] \qquad \text{(Eq.1.1)}$$

and, for the full span, taking $2 \times \theta_{BT}$ for $z = \frac{L}{2}$,

$$\phi_{BT} = \frac{TL}{GJ}\left(1 - \frac{\tanh k}{k}\right) \qquad \text{(Eq.1.1a)}$$

where the suffix $_{BT}$ has been introduced to indicate that this is the angle of twist taking into account the axial warping due to the axial tensile and compressive stresses (or "bending-like" stresses) plus the pure torsion, but not the axial (or direct) shear strain.

The angle of twist which includes all three types of strain is Eq.5.4 plus Eq.1.1.

viz.,

$$\theta = \theta_s + \theta_{BT}$$

or

$$\theta = \frac{T}{GJ_s}\left[\frac{z + \frac{L}{4k} \sinh \frac{4kz}{L}}{2 \cosh^2 k}\right] + \frac{T}{GJ}\left[z + \frac{L \sinh \frac{2kz}{L}}{2k \cosh k}\right]$$

Eq.5.6.

Where z = 0 is at the centre section of the beam which is subjected to a uniform torque T along the length and has both end sections held plane.

(See Fig. 1 (a) in the introduction)

The angle of twist ϕ for the whole span is given by doubling Eq. 5.6 for $z = \frac{L}{2}$; and putting symbols for the hyperbolic terms the formula for ϕ may be simplified thus:

$$\phi = \phi_s + \phi_{BT}$$

or

$$\phi = K_s \frac{TL}{GJ_s} + K_{BT} \frac{TL}{GJ} \qquad \text{Eq.5.7}$$

where

$$K_s = \frac{\frac{1}{2k}(\sinh 2k) + 1}{(\cosh 2k) + 1} \qquad \text{Eq.5.8}$$

and

$$K_{BT} = 1 - \frac{1}{k}\tanh k \qquad \text{Eq.5.9}$$

The formulas for k and J_s are given by Eqs.1.2 and 5.3 respectively.

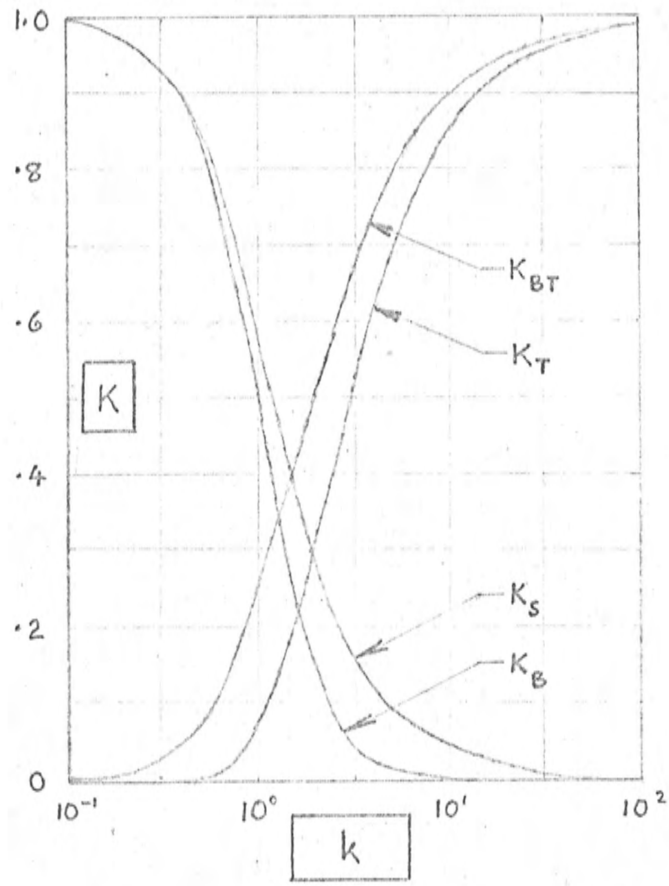
Fig. 5.2 shows how K_S and K_{BT} vary with k. This shows that K_S is small for large spans and K_{BT} is small for short spans.

The limits approached by the two terms of Eq.5.7 are of interest. As k or L $\rightarrow \infty$, $K_s \rightarrow 0$ and $K_{BT} \rightarrow 1$, hence

$$\phi_{k\rightarrow\infty} \rightarrow \frac{TL}{GJ} = \phi_{TO} \text{ say} \qquad \text{Eq.5.10}$$

And when k or L $\rightarrow 0$, $K_s \rightarrow 1$ and

$$K_{BT} \frac{TL}{GJ} \rightarrow \frac{k^2}{3} \frac{TL}{GJ} = \frac{L^2}{12} \frac{GJ}{EJ_w} \frac{TL}{GJ} = \frac{TL^3}{12\, EJ_w}$$



K FACTORS
FOR THIN-WALLED OPEN-SECTION BEAMS
WITH ENDS CONSTRAINED TO STAY PLANE,
AND SUBJECTED TO A UNIFORM TORQUE.

FIG. 5.2

Let

$$\phi_{BT_{(k \to 0)}} = \phi_{B0} = \frac{TL^3}{12\ EJ_w} \qquad \text{Eq.5.11}$$

and let

$$\phi_{S_{(k \to 0)}} = \phi_{S0} = \frac{TL}{GJ_s} \qquad \text{Eq.5.12}$$

Then

$$\phi_{(k \to 0)} = \phi_{S0} + \phi_{B0}$$

that is,

$$\phi_{(k \to 0)} = \frac{TL}{GJ_s} + \frac{TL^3}{12\ EJ_w} \qquad \text{Eq.5.13}$$

This particular result is obtained by Mescheryakov [1] and is further discussed in chapter A.1 in the appendix.

Relative Importance of Shear (Unwarped Ends, Uniform Torque)

The importance of the shear in general terms is given by the ratio of the two terms of Eq.5.7

$$\frac{\phi_s}{\phi_{BT}} = \frac{K_s}{K_{BT}} \ \frac{J}{J_s} \qquad \text{Eq.5.14}$$

The following Fig. 5.3 evaluates this ratio for the slit tube and for a semi-circular arc section at $\frac{L}{D} = 10$ for $\frac{D}{t} = 10$ and $\frac{D}{t} = 50$, and with $G/E = .39$.

L = 10D G = .39E	J_s	J	$^D/_t$	K_{BT}	K_s	$^{\phi_s}/_{\phi_{BT}}$
SLIT TUBE	$.817D^3t$	$1.048dt^3$	10	.115	.78	8.7%
			50	.00980	.99	5.2%
SEMI-CIRCLE	$.0128D^3t$	$.524dt^3$	10	.85	.0755	3.6%
			50	.345	.46	2.4%

RELATIVE IMPORTANCE OF SHEAR, Eq.5.14

FIG. 5.3

Thus it is seen that the shear term can be ignored for "long" beams of the more open section if an underestimation of the order 3% is acceptable.

For shorter beams, the shear deflection should always be considered; that is the whole of Eq.5.4.

When k is small or large the hyperbolic functions need not be used directly and the following forms of Eq.5.7 are suggested,

$k < .3$

$$\phi = (1 - \frac{2}{3} k^2) \frac{TL}{GJ_s} + \frac{k^2}{3} (1 - \frac{2}{5} k^2) \frac{TL}{GJ} \qquad \text{Eq.5.15}$$

$k > 3.3$

$$\phi = (1 - \frac{1}{k}) \frac{TL}{GJ} \qquad \text{Eq.5.16}$$

The shear term which has been dropped from Eq.5.16 because it is relatively small is $\frac{1}{2k} \frac{TL}{GJ_s}$.

For the in-between values of k, Eqs.5.8 and 5.9 are convenient to use for K_s and K_{BT}.

<u>Angle of Twist of Slit-Tube</u> (Unwarped Ends, Uniform Torque)

Relative to the angle of twist ($\phi_o = \frac{TL}{GJ_o}$) of a closed round tube of the same span and subject to the same torque Eq.5.7 becomes

$$\phi/\phi_o = \phi_s/\phi_o + \phi_{BT}/\phi_o$$

or

$$\phi/\phi_o = K_s \frac{J_o}{J_s} + K_{BT} \frac{J_o}{J} \qquad \text{Eq.5.17}$$

This equation has been evaluated in the following table 5.4 for a slit tube and for a semi-circular section of the same radius and same uniform thickness as the closed tube.

For the slit-tube, using values of the section constants from Fig.5.1, Eq.5.17 is

$$\frac{\phi}{\phi_o} = .96\ K_s + .75\ \left(\frac{D}{t}\right)^2\ K_{BT} \qquad \text{Eq.5.18}$$

which for short beams (k^4 negligible) is approximately

$$\frac{\phi}{\phi_o} = .96\ \left(1 - \frac{2}{3}\,k^2\right) + .26\ \frac{G}{E}\ \left(\frac{L}{D}\right)^2\ \left(1 - \frac{2}{5}\,k^2\right) \qquad \text{Eq.5.19}$$

cf. Eq.5.15.

This is approximately Eq.3.5 (the classic solution for $k \rightarrow 0$) plus the angle of twist of the original unslit tube. This very much simplified solution for the slit tube with ends constrained to remain plane and subject to a uniform torque applies when the pure torsion is negligible, that is when the shear stress is effectively uniform through the wall thickness.

Numerical solutions of Eq.5.19 (ignoring k^2) are also given in Fig.5.4 (illustrated in Fig.5.5) and comparing these with the solutions of Eq.5.17 it is seen that the simpler equation is valid for short slit tubes up to a span/depth ratio

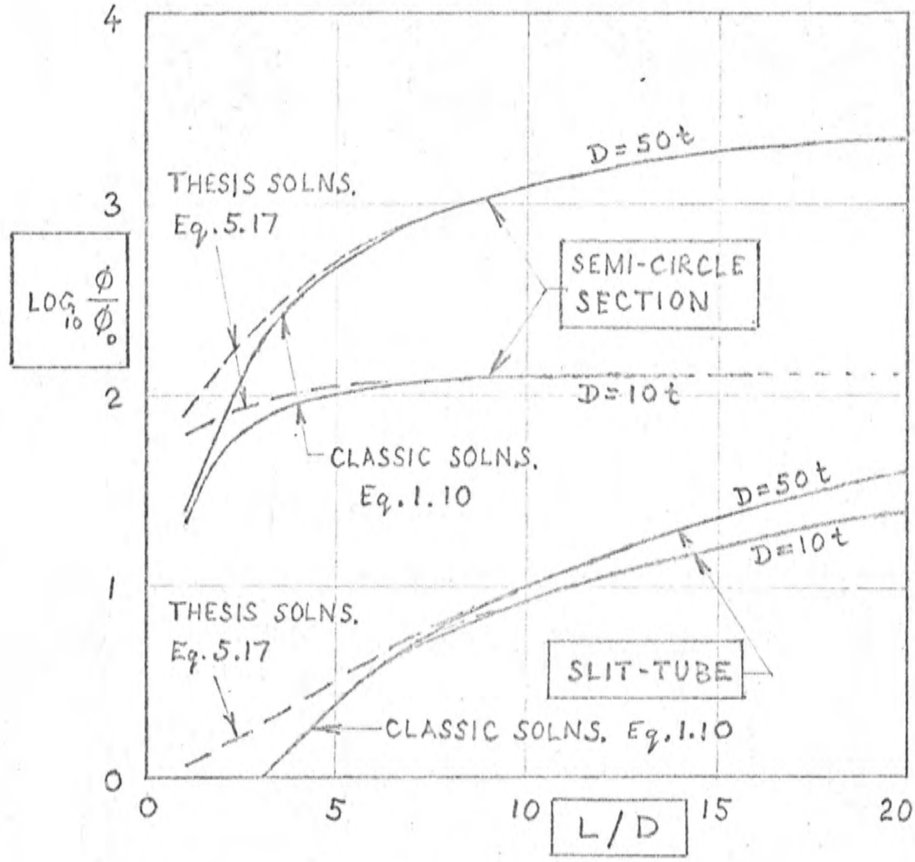
$\frac{L}{D} < .3\ \frac{D}{t}\sqrt{\frac{E}{G}}$.

Again, Eq.5.19 could be legitimately modified by replacing E by the reduced modulus $E/1 - \nu^2$, thus decreasing the value of the last term by 8 to 9%.

SLIT TUBE			$\phi \times \dfrac{.785\ t\ D^3\ G}{TL}$			$(\frac{G}{E} = .39)$
$\frac{L}{D}$	$\frac{D}{t}$	k 1.27	Eq. 5.19	Eq. 5.17	Eq. A.1.15	Eq. 1.10
20	10	-	-	25.2	25.0	24.7
	30	.423	-	38.1	38.0	37.2
	50	.254	41	42.4	42.4	41.5
10	10	.634	-	9.4	9.5	8.6
	30	.211	11.1	11.0	11.0	10.1
	50	.127	11.1	11.0	11.0	10.1
6	10	.380	-	4.71	4.27	3.84
	30	.127	4.6	4.60	4.59	3.65
	50	.0761	4.6	4.61	4.61	3.65
3	10	.190	1.9	1.94	1.90	.98
	30	.0634	1.9	1.87	1.87	.91
	50	.0380	1.9	1.87	1.87	.91
1	10	.0634	1.06	1.06	1.04	.10
	30	.0211	1.06	1.06	1.06	. 10
	50	.0127	1.06	1.06	1.06	.10
Using end condition Eq.4.36 add 1.0 to Eq.1.10						
SEMI-CIRCLE						
20	10	13.24		141	141	139
	20	4.41		1007	1011	1000
	30	2.64		2352	2360	2340
10	10	6.62		1.32	131	127.5
	30	2.21		742	740	726
	50	1.32		1308	1320	1290
6	10	3.97		120	118.5	112
	30	1.32		474	471	446
	50	.792		685	671	645
3	10	1.99		94	91	77.5
	30	.662		209	202.5	165
	50	.396		246	244	191
1	10	.662		63.2	53.3	19.1
	30	.221		80.0	74.1	20.6
	50	.132		82.2	82.5	21.6
Using end condition Eq.4.36 add 2.0 to Eq.1.10						

ANGLES OF TWIST FROM FOUR THESIS SOLUTIONS TOGETHER WITH THE CLASSIC SOLUTION

FIG. 5.4



ANGLE OF TWIST
OF THIN-WALLED OPEN-SECTION BEAMS
WITH ENDS CONSTRAINED TO STAY PLANE
AND SUBJECTED TO A UNIFORM TORQUE

FIG. 5.5

Account of shear in the general case

So far in this chapter, and indeed in most of the thesis, the only beam condition considered is a uniform torque acting along a beam with both ends constrained to remain plane. The main reason for this restriction is that this condition applies most closely to the tube with a slit part length that is considered in chapter 6. However, all methods are easily adaptable to other torque loadings and end conditions, and in addition all solutions due to the torque loadings may be superposed on other load effects provided all strains remain elastic and provided twisting occurs about the shear centre.

For example, the angle of twist due to shear was found in this chapter by equating the external work done by the torque, in turning through the shear angle only, to the internal strain energy of the classic shear stresses. In the general case this means, for an elemental length dz,

$$\tfrac{1}{2}\, T\, d\,\theta_s = \left(\int_A \frac{\tau^2}{2G}\, dA \right) dz \qquad \text{Eq.5.20}$$

where T is the (average) torque between the cross-sections distance apart dz, (allowing for a torque which varies along the length of the beam), and the integration of the shear stress τ is taken over the whole cross-section.

The rate of twist due to shear $\frac{d\,\theta_s}{dz}$ is then, from Eq.5.20

$$\theta_s' = \frac{1}{T} \int_A \frac{\tau^2}{G}\, dA \qquad \text{Eq.5.21}$$

The total rate of twist at any section is then Eq.5.21 plus the classic solution (called Θ'_{BT}).

This general result will now be applied to three other beam conditions.

(1) Ends constrained from warping, uniformly varying torque (T = hz)

The classic solution for the shear stress τ is, from Vlasov [12] , with h = constant (lbf.ins./in.),

$$\tau_s = \frac{hL}{2} \cdot \frac{S_w}{J_w} \cdot \frac{\sinh 2\,kz/L}{\sinh k} \qquad \text{Eq.5.22}$$

Substituting in Eq.5.21, performing the circumferential integration and transforming with Eq.5.3 gives

$$\theta'_s = \frac{\left(\frac{hL}{2}\right)\left(\frac{L}{2z}\right)}{GJ_s} \frac{\sinh^2 2kz/L}{\sinh^2 k} \qquad \text{Eq.5.23}$$

Thus for $z = 0, \frac{L}{2}$, $\theta'_s = 0, \frac{hL}{2}/GJ_s$ respectively.

Consider only the case of the short beam where $k < .3$ which is when the shear is important, then with k^3 negligible Eq.5.23 simplifies to

$$\theta'_s = \frac{hz}{GJ_s} \qquad \text{Eq.5.24}$$

so that the total angle of twist due to shear is

$$\phi_s = 2\int_0^{L/2} \frac{hz}{GJ_s}\, dz$$

$$\phi_s = \frac{hL^2}{4GJ_s} \qquad \text{Eq.5.25}$$

The angle of twist of a round closed tube of the same span and subject to the same end condition and torque is

$$\phi_o = \frac{hL^2}{4GJ_o}$$

Again for the slit tube it is seen that the additional twist due to shear is approximately the same as the original angle of twist of the closed tube, (since $J_s \simeq J_o$).

The classic solution for angle of twist is, from Vlasov [12],

$$\Theta_{BT} = \frac{(\frac{hL}{2})\,L}{GJ}\left[\left(\frac{1}{4} - \frac{z^2}{L^2}\right) - \frac{\sinh \frac{k}{L}\left(\frac{L}{2} - z\right)\,\sinh \frac{k}{L}\left(z + \frac{L}{2}\right)}{k \sinh k}\right]$$

Eq.5.26

which for the whole span is

$$\phi_{BT} = \frac{hL^2}{4GJ}\left[1 - \frac{4\sinh^2 \frac{k}{2}}{k \sinh k}\right] \qquad \text{Eq.5.27}$$

For short beams (k^4 small) this becomes

$$\phi_{BT} = \frac{k^2}{48}\;\frac{hL^2}{GJ}\;\left(1 - \frac{k^2}{10}\right) \qquad \text{Eq.5.28}$$

and it is immediately seen that for this type of loading shear is even more important than for the uniform torque case, since the classic solution approaches zero as $L \rightarrow 0$ while the shear twist is virtually constant at Eq.5.25.

The corrected angle of twist (for a short beam) is Eq.5.25 + Eq.5.28, viz.,

$$\phi = \frac{hL^2}{4GJ_s} + \frac{k^2}{48}\,\frac{hL^2}{GJ}\,\left(1 - \frac{k^2}{10}\right) \qquad \text{Eq.5.29}$$

or relative to the closed tube (Eq. (i) above)

$$\frac{\phi}{\phi_o} = \frac{J_o}{J_s} + \frac{k^2}{12}\,\frac{J_o}{J}\,\left(1 - \frac{k^2}{10}\right) \qquad \text{Eq.5.30}$$

For the slit tube this is, (using TABLE 5.1 and substituting for k from Eq.1.5,

$$\frac{\phi}{\phi_o} = .96 + .065\,\frac{G}{E}\left(\frac{L}{D}\right)^2 \left(1 - \frac{k^2}{10}\right) \qquad \text{Eq.5.31}$$

(2) <u>Ends held from twisting but free to warp, uniformly varying torque T = hz</u>

Following the same procedure as above, i.e. utilising the classic solutions of Vlasov [13] for angle of twist and shear stress together with Eq.5.21, gives

$$\theta_s' = \frac{\left(\frac{hL}{2}\right)\left(\frac{L}{2z}\right)}{GJ_s}\,\frac{\sinh^2 \frac{2kz}{L}}{k^2 \cosh^2 k} \qquad \text{Eq.5.32}$$

or, for $k < .3$

$$\phi_s = \frac{hz}{GJ_s}\,(1 - k^2) \qquad \text{Eq.5.33}$$

and

$$\phi_{BT} = \frac{hL^2}{4GJ}\left[1 - \frac{2\,(\cosh k - 1)}{k^2 \cosh k}\right] \qquad \text{Eq.5.34}$$

or, for $k < .3$

$$\phi_{BT} = \frac{5\,k^2}{48}\,\frac{h\,L^2}{GJ}\,\left(1 - \frac{k^2}{2}\right) \qquad \text{Eq.5.35}$$

so that the total angle of twist for the short beam is Eq.5.33 plus 5.35, which for the slit-tube relative to the closed tube is

$$\frac{\phi}{\phi_o} = .96\ (1 - k^2) + .325\ \frac{G}{E}\ (\frac{L}{D})^2\ (1 - \frac{k^2}{2}) \qquad \text{Eq.5.36}$$

(3) Cantilever subject to uniform torque T

This case represents a half-span of the original beam with both ends constrained. Therefore Eqs.1.1 for θ_{BT}, 1.2 for k, 2.1 for τ, 2.3 for σ and 5.4 for θ_s apply to the cantilever if z = 0 at the free end and the full span is $\frac{L}{2}$.

If the cantilever span is made L, Eq.5.19 for the corrected angle of twist becomes for the slit tube,

$k_c\ (= L \sqrt{\frac{GJ}{EJ_w}}\ ) < .3,$

$$\frac{\phi}{\phi_o} = .96\ (1 - \frac{2}{3}\,k_c^{\ 2}) + 1.04\ \frac{G}{E}\ (\frac{L}{D})^2\ (1 - \frac{k_c^{\ 2}}{10}) \qquad \text{Eq.5.37}$$

Separate angles of twist due to shear, normal stresses and pure torsion along length of beam

To complete the picture of non-uniform torsion of thin-walled open-section beams with end sections constrained to remain plane (within the limits of the assumptions of elastic strains and undeformable cross-section) it remains to determine the separate components of ϕ_{BT}.

The second term in Eq.4.2 may be thought of as the torsional bending twist plus the pure torsional twist, and the separate parts of this may be found because this bending twist may be determined by energy means from the Vlasov normal stress system,

$$\sigma = \frac{TL}{2k\ J_w \cosh k} \quad w \sinh(2\ kz/L) \qquad \text{(Eq.2.3)}$$

The bending twist ϕ_B is given by

$$\tfrac{1}{2}\, T\, \phi_B = \int_{\text{Volume}} \frac{\sigma^2}{2E}\ d\ (\text{VOLUME})$$

or

$$\phi_B = \frac{TL^2}{4k^2 J_w^{\ 2} \cosh^2 k} \quad t \int_{-S_o}^{S_o} w^2\ ds \int_{-\frac{L}{2}}^{\frac{L}{2}} \sinh^2(2\ kz/L)\, dz$$

Now

$$t \int_{-S_o}^{S_o} w^2\ ds \equiv t \int_{-S_o}^{S_o} m\ S_w\ ds = J_w \qquad (\text{Eq. } 4.18)$$

therefore

$$\phi_B = \frac{TL^2}{4k^2\ J_w \cosh^2 k} \int_{-\frac{L}{2}}^{\frac{L}{2}} \tfrac{1}{2}\left(\cosh(4kz/L) - 1\right)\ dz$$

where the square sinh is replaced by the double angle term.

Evaluate the integral, to give

$$\phi_B = \frac{TL^3}{4k^2 EJ_w} \; \frac{\left(\frac{1}{2k} \sinh 2k - 1\right)}{\cosh 2k + 1}$$

(cf Eq.3.4)

or

$$\phi_B = K_B \; \phi_{BO} \qquad \text{Eq.5.38}$$

where

$$\phi_{BO} = \frac{TL^3}{12\, EJ_w} \qquad (\text{Eq.3.4}) \qquad \text{Eq.5.39}$$

and

$$K_B = \frac{\left(\frac{1}{2k} \sinh 2k - 1\right)}{\cosh 2k + 1} \; \frac{3}{k^2} \qquad \text{Eq.5.40}$$

The pure torsional twist ϕ_T is then ϕ_{BT} (Eq.1.1(a)) minus ϕ_B above, thus

$$\phi_T = K_{BT} \frac{TL}{GJ} - K_B \; \frac{TL^3}{12\, EJ_w}$$

$$= \frac{TL}{GJ} \; \left(K_{BT} - K_B \; \frac{L^2}{12} \; \frac{GJ}{EJ_w}\right)$$

But

$$k^2 = \frac{L^2}{4} \; \frac{GJ}{EJ_w} \qquad (\text{Eq.1.2})$$

$$\therefore \phi_T = \frac{TL}{GJ} \left(K_{BT} - K_B \; \frac{k^2}{3} \right)$$

or

$$\phi_T = K_T \; \frac{TL}{GJ} \qquad \text{Eq.5.41}$$

where

$$K_T = 1 - \frac{1}{k} \tanh k - \frac{\frac{1}{2k} \sinh 2k - 1}{\cosh 2k + 1} \qquad \text{Eq.5.42}$$

The complete equation for the total angle of twist of a thin-walled open-section beam with end sections constrained to remain plane and subject to an end torque T will now be written in terms of the three separate components:

$$\phi = K_S \, \phi_{SO} + K_B \, \phi_{BO} + K_T \, \phi_{TO} \qquad \text{Eq.5.43}$$

where K_S, K_B and K_T are given respectively by Eqs.5.8, 5.40 and 5.42 and the "limiting" values of the separate twists are respectively

$\phi_{SO} = TL/GJ_s$, $\phi_{BO} = TL^3/12\, EJ_w$ and $\phi_{TO} = TL/GJ$.

When $k \; (= \frac{L}{2} \sqrt{GJ/_{EJ_w}})$ is very small (say $k < .3$ for an error of less than 1%,) Eq.5.43 may be used in the following form,

$$\phi = (1 - \frac{2k^2}{3}) \; \frac{TL}{GJ_s} + (1 - k^2) \; \frac{TL^3}{12\, EJ_w} \qquad \text{Eq.5.44}$$

where $\phi_T = (1 - k^2) \; \frac{2}{15\, k^4} \; \frac{TL}{GJ}$ has been discarded.

when $k > 3.3$ say, Eq.5.43 may be used in the form

$$\phi = \frac{3}{2k^3} \; \frac{TL^3}{12\, EJ_w} + (1 - \frac{3}{2k}) \, \frac{TL}{GJ} \qquad \text{Eq.5.45}$$

where $\phi_s = \frac{1}{2k} \; \frac{TL}{GJ_s}$ has been discarded.

Relative to the angle of twist of a closed tube which is

$$\phi_o = \frac{TL}{GJ_o}$$

Eq.5.43 becomes

$$\frac{\phi}{\phi_o} = K_s \frac{J_o}{J_s} + K_B \frac{L^2}{12} \frac{GJ_o}{EJ_w} + K_T \frac{J_o}{J} \qquad \text{Eq.5.46}$$

which is of the form

$$\frac{\phi}{\phi_o} = K_S C_1 + K_B C_2 \left(\frac{L}{D}\right)^2 + K_T C_3 \left(\frac{D}{t}\right)^2 \qquad \text{Eq.5.47}$$

where C_1, C_2 and C_3 are constants for a given section and material. K_S and K_B decrease with increasing span while K_T increases, as indicated by Fig. 5.2.

To summarise the solutions of this chapter, the following equations may be used according to the particular value of k.

$k < \cdot 3$

$$\emptyset = (1 - 2/3\, k^2) \frac{TL}{GJ_s} + (1 - k^2) \frac{TL^3}{12\, EJ_w} \qquad \text{(Eq.5.44)}$$

$$= (1 - \frac{2}{3} k^2) \frac{TL}{GJ_s} + \frac{k^2}{3} (1 - \frac{2}{5} k^2) \frac{TL}{GJ} \qquad \text{(Eq.5.15)}$$

$.3 < k < 3.3$

$$\phi = \frac{1}{\frac{(2k \quad \sinh 2k - 1)}{\cosh 2k + 1}} \frac{TL}{GJ_s} + (1 - \frac{\tanh k}{k}) \frac{TL}{GJ} \qquad \text{(Eq.5.7)}$$

$k > 3.3$

$$\phi = (1 - \frac{1}{k}) \frac{TL}{GJ} \qquad \text{(Eq.5.16)}$$

These equations are for any thin-walled open-section with ends constrained to remain plane and subjected to an end torque T.

Summary of chapter

A simple correction to the classic solution for the angle of twist of a thin-walled open-section beam has been given to take into account shear.

It has been shown that shear should be taken into account when, in general, the span is less than 10 times the major depth of the cross-section; and in particular Eqs.5.19, 31, 36 and 37 for the slit tube demonstrate its importance in relatively stiff open-sections.

The most striking case is that of the slit-tube built-in at both ends and subject to a uniformly varying torque, (like that produced by a transverse load uniformly distributed along the span and with a line of action parallel to but not through the line of shear centres of the cross-section) because of the higher torque near the ends. For this case the angle of twist due to axial shear strain only is more than 50% of the total angle of twist (Eq.5.31) when the span is less than 6 times the mean diameter; and still approximately 25% for a span 10 times the mean diameter.

PART II

Tubes slit part length

Chapter 6 - the development of theoretical equations requiring experimental constants

6. AN INITIAL APPRAISAL OF THE CASE OF THE TUBE WITH SLIT NOT EXTENDING FULL LENGTH.

A thin-walled round tube containing a slit within the length of the tube may be considered as a thin-walled open section shell, (which is the slit length) with elastic ends, (which are the unslit lengths). The wall thickness will be taken as uniform throughout.

Such a problem is given in Vlasov [14] where the angle of twist of a tube containing a long wide cut-out is considered. The deflections at the junction of the two parts are equated, and the normal stresses in the uncut part are assumed to be constant along the length having a circumferential distribution as dictated by the central open section along the junction line.

The Vlasov procedure will now be applied to the slit tube, but a correction factor will be introduced to take into account the reduced relaxation of normal stresses near the slit. Since this is where the normal stresses are a maximum in the slit tube the prevention of their full relaxation due to their opposite nature each side of the slit can be expected to reduce appreciably the angle of twist compared to the Vlasov solution which assumes the circumferential normal stress distribution pattern in the slit length to be the same as with rigid ends but the magnitude everywhere reduced in direct proportion to the lengths of unslit tubing.

The Vlasov solution itself will not be considered because it considerably overestimates deflections due to the above reasons, and secondly because it does not take into account shear deflections as explained previously. Nevertheless the following analysis is based on the Vlasov procedure in an attempt to obtain a basis for a semi-empirical solution.

Consider the complete tube to be of length L and have a slit length L_s symmetrical within the tube span which has rigid end plates. See Fig. 6.1, (p. 73).

When a uniform torque is applied, axial stresses are transmitted into the unslit lengths from the slit part which is undergoing flexural torsion, and these stresses will be assumed to be constant along a generator and vary circumferentially with the slit part.

The (uniform) axial strain along any generator in the unslit part is

$$\varepsilon_i = \frac{\sigma_j}{E} \qquad \text{Eq.6.1}$$

The subscripts i, j, s refer to the unslit part of the tube, the junction section of the slit and unslit parts, and the slit part, respectively.

The total axial deflection in the unslit part (occurring at the junction) is then

$$u_i = u_j = - \varepsilon_i L_i$$

substituting from Eq.6.1

$$u_j = - \frac{\sigma_j}{E} L_i \qquad \text{Eq.6.2}$$

where L_i = length of unslit part ($= \frac{L - L_s}{2}$), and the minus sign means that a (positive) tensile stress or strain produces a deflection in the negative z direction measured from the centre of the span.

Consider now the junction deflection and stress from the slit side. The stress might be approximated from Eq.4.21, with $z = \frac{L_s}{2}$

$$\sigma_j = \frac{T}{J_w} \; w_s \; \frac{L_s}{2}$$

And since

$$\sigma = E \; \frac{\partial u}{\partial z}$$

and

$$u_j = - w_s \quad \theta_j^{1} \qquad\qquad \text{(from Eq.4.12)}$$

substituting for u_j and σ_j in Eq.6.2

$$\theta_j^{1} = \frac{T}{2 \; EJ_w} \; L_i \; L_s \qquad\qquad \text{Eq.6.3}$$

or, introducing a stiffening number β as explained above,

$$\theta_j^{1} = \beta \frac{T}{2 \; EJ_w} \; L_i \; L_s \qquad\qquad \text{Eq.6.4}$$

where $\beta < 1$.

Eq.6.4 gives the rate of twist at the junction due to the relaxation of the normal stresses in the slit part. It does not include the shear deflection which has not yet been added.

The angle of twist of the slit portion due to flexural torsion bending only is, using Eq.3.4 and Eq.6.4

$$\phi_{BU} + L_s\ \Theta_j^1 = \frac{TL_s^3}{12\ EJ_w} + \beta\ \frac{TL_s^2}{2\ EJ_w}\ L_i$$

where the pure torsional effect is neglected.

This is the increase in the angle of twist due to the slit, or

$$\Delta\phi = \phi_{increase} = \frac{TL_s^3}{12\ EJ_w}\left[1 + 6\beta\frac{L_i}{L_s}\right] \qquad \text{Eq.6.5}$$

Add the shear deflection of the complete span L, to give the total angle of twist ϕ for the slit plus unslit parts,

$$\phi = \frac{TL}{GJ_o} + \frac{TL_s^3}{12\ EJ_w}\left[1 + 6\ \beta\ \frac{L_i}{L_s}\right] \qquad \text{Eq.6.6}$$

Substitute $J_w = .253\ t\ D^5$, then,

relative to the angle of twist of the original unslit tube TL/GJ_o, (span L, $J_o = .785\ t\ D^3$)

$$\frac{\phi}{\phi_o} = 1 + .26\ \frac{G}{E}\ \left(\frac{L_s}{D}\right)^2 \left(\frac{L_s}{L}\right)\left[1 + 6\ \beta\left(\frac{L_i}{L_s}\right)\right] \qquad \text{Eq.6.7}$$

Substitute $\dfrac{L - L_s}{2}$ for L_i, then

$$\frac{\phi}{\phi_o} = 1 + .26\ \frac{G}{E}\ \left(\frac{L_s}{D}\right)^2 \left[3\beta + \frac{L_s}{L}\ (1 - 3\ \beta\ )\right] \qquad \text{Eq.6.7 (a)}$$

Comparing this with Eq.5.19 for a short tube with a full length slit it is seen that they are the same for $L_s = L$ ($k \to 0$) regardless of the value of β, as expected.

Note that the Vlasov method would have given, with $\beta = 1$, but with full account for shear,

$$\frac{\phi}{\phi_o} = 1 + .26\,\frac{G}{E}\left(\frac{L_s}{D}\right)^2\left(\frac{L_s}{L}\right)\left[3\,\frac{L}{L_s} - 2\right] \qquad \text{Eq.6.8}$$

where the quantity in square brackets represents the account for the elastic end effect on the slit part.

Approximate estimation of β

The actual deformations (u_{j_a}) around the junction cross-section will not be proportional to the stresses σ on the slit side of the junction section, but will become zero at the end of the slit, perhaps in the manner of the dotted line in Fig. 6.2.

An idea of β is then given by comparing the actual work done by the axial stresses at the junction with the work done for the case of a wide slit i.e. when $\beta = 1$ thus

$$\beta \simeq \frac{\int_{-S_o}^{S_o} \sigma u_j \, t \, ds}{\int_{-S_o}^{S_o} \sigma u_{j_a} \, t \, ds} \simeq \frac{\sum(\sigma u_j)}{\sum(\sigma u_{j_a})}$$

Graphically integrating using Fig. 6.2 (while assuming that σ on the junction section is the same in each case) gives

$$\beta \simeq \frac{1}{3}$$

which when substituted in Eq.6.7 (a) would give

$$\frac{\phi}{\phi_o} = 1 + .26\,\frac{G}{E}\left(\frac{L_s}{D}\right)^2 \qquad \text{- Eq.6.9}$$

indicating an appreciable stiffening compared with Eq.6.8.

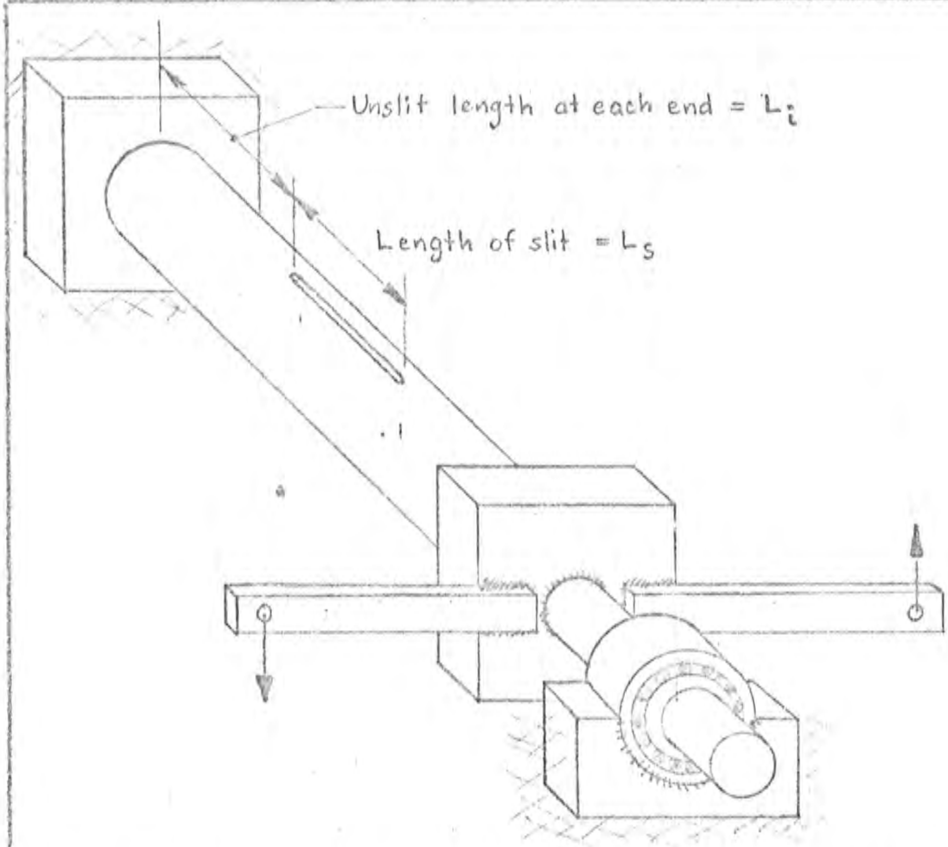


FIG. 6.1 APPLYING A UNIFORM TORQUE TO A TUBE SLIT PART LENGTH

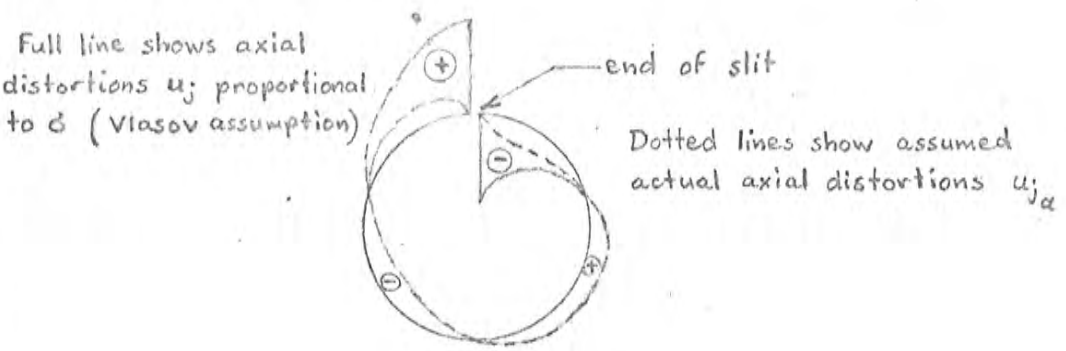


FIG. 6.2 AXIAL DISTORTIONS ON A CROSS-SECTION CONTAINING ONE END OF THE SLIT

Elastic effect of real end plates

The actual non-rigidity of real end plates could be taken into account by the use of an equivalent length of unslit tube, L_i', where $L_i' = \dfrac{L' - L_s}{2}$, and where L' ($> L$) is an equivalent total span.

Substituting this in Eq.6.6 in place of L_i,

$$\phi = \frac{TL}{GJ_o} + \frac{TL_s^3}{12\, EJ_w}\left[1 + 3\,\beta\,\frac{L' - L_s}{2}\right]$$

or

$$\frac{\phi}{\phi_o} = 1 + .26\,\frac{G}{E}\left(\frac{L_s}{D}\right)^2\left[\frac{L_s}{L} + 3\,\beta\left(\frac{L'}{L} - \frac{L_s}{L}\right)\right] \qquad \text{Eq.6.10}$$

Equivalent elastic lengths of unslit parts

Alternatively, β and L' above may be combined into a single equivalent length representing an effective L_i in Eq.6.6 or 6.7 which with $\beta = 1$ would give the same as the measured increase in angle of twist.

Thus substitute $L_i = (L_e - L_s)/2$ and $\beta = 1$ in Eq.6.7 (or rewrite Eq.6.8 with the L in square brackets $= L_e$),

$$\frac{\phi}{\phi_o} = 1 + .26\,\frac{G}{E}\left(\frac{L_s}{D}\right)^2\left(\frac{L_s}{L}\right)\left[\frac{3\,L_e}{L_s} - 2\right]$$

Taking $\frac{G}{E} = .39$, as previously, this gives

$$\frac{\phi}{\phi_o} = 1 + .101\left(\frac{L_s}{D}\right)^2\left(\frac{L_s}{L}\right)\left[3\,\frac{L_e}{L_s} - 2\right] \qquad \text{Eq.6.11}$$

alternatively, the increase in angle of twist due to the slit may be written,

$$\Delta\phi = .1\left(\frac{L_s}{D}\right)^2\left(\frac{L_s}{L}\right)\left[3\,\frac{L_e}{L_s} - 2\right]\phi_o$$

For the ideal case of a tube with ends fully constrained to remain plane, L_e will always be less than L for a tube slit only part length. If a certain amount of elasticity is present in the real end fixing, L_e will exceed L as L_s approaches L.

Solution for a slit in an infinite plate subject to pure shear

From the work of Mushkelishvili, Koiter [1] obtains a particular solution for the increase in elastic strain energy due to a crack in an infinite rectangular plate subject to pure shear.

This solution might be expected to give a satisfactory answer for very short slits in round tubes subject to torsion, but not for the longer slits.

Assume that the increase in strain energy due to a very short slit in a thin-walled round tube subject to a uniform and constant torque T is given by the Koiter solution,

$$\Delta U = \frac{\pi}{8} (\varkappa + 1) \frac{\tau^2 c^2 t}{G} \qquad \text{Eq.6.12}$$

where

$$\varkappa + 1 = \frac{3 - \nu}{1 + \nu} + 1$$

$$= 3.08 \text{ if Poisson's Ratio } \nu = .3$$

$2c$ = slit length = L_s

τ = original shear stress = $\tau_o = \dfrac{2T}{\pi D^2 t}$

Substituting the above values, Eq.6.12 becomes,

$$\Delta U = \frac{\pi \times 3.08}{8} \times \frac{4T^2}{\pi^2 D^4 t^2} \times \frac{L_s^2 t}{4G}$$

$$\Delta U = \frac{.385}{\pi} \frac{T^2 L_s^2}{D^4 t G} \qquad \text{Eq.6.13}$$

Or, equating this to the increase in external work done,

$$\frac{1}{2} T \phi_{increase} = \frac{.385}{\pi} \frac{T^2 L_s^{\,2}}{D^4 \; t \; G}$$

giving

$$\phi_{increase} = .245 \; \frac{TL_s^{\,2}}{GD^4 t} \qquad \text{Eq.6.14}$$

(cf. Eq.6.5, with $G = .385E$, and $3 \beta = 1$).

The total angle of twist, including the slit effect (relative to the angle of twist of the original unslit tube), is then

$$\frac{\phi}{\phi_o} = 1 + \frac{\phi_{increase}}{\phi_o}$$

$$= 1 + \frac{.245 \; TL_s^{\,2}/GD^4 t}{TL/.785 \; GD^3 t}$$

$$\frac{\phi}{\phi_o} = 1 + .193 \; \frac{L_s^{\,2}}{LD}$$

or

$$\frac{\phi}{\phi_o} = 1 + .193 \; \left(\frac{L_s}{D}\right)^2 \frac{D}{L} \qquad \text{Eq.6.15}$$

Compare this with Eq.6.9 and it is seen that both solutions are parabolic in $L_s^{\,2}$ and give the same result when

$$\frac{L}{D} \simeq 2$$

If $\frac{L}{D} < 2$, Eq.6.15 is larger than 6.8 and smaller if $\frac{L}{D} > 2$.

It might be that a shell $\frac{L}{D} \simeq 2$ with a very short slit comes closest to the assumption of an infinite plate, being least subject to end effects or curvature effects.

However it is realised that both equations are very approximate, being obtained simply as a basis to fit to experimental results and that the sameness for L/D = 2 is probably coincidental.

Further discussion on the stresses in the part-slit tube

Because the warping restriction in the vicinity of the ends of a slit is high, due to the opposite nature of the stresses either side of the slit, the magnitude of these stresses might be approximately those given by the classic Eqs.2.1 and 2.3 or Eq.4.22 and 4.26 for slits of length L_s less than 10 diameters. That is, the axial stresses along the edges of the slit (which are the maximum around the circumference) might still be given by - substituting $\varphi = \pm 180^\circ$ in Eq.4.22 -

$$\sigma = \pm .988\,\pi \frac{Tz}{D^3 t} \qquad \text{(from Eq.4.22)}$$

or, relative to the shear stress (τ_o) in the closed tube

$$\frac{\sigma}{\tau_o} = \pm 1.55\,\pi\; z/D \qquad \text{Eq.6.16}$$

where z is measured axially for the centre of the slit length.

However towards the ends of the slit stress-concentration effects can be expected so that the maximum stresses will be greater than those forecast by Eq.6.16 above, i.e. numerically,

$$\sigma_{max} > .988\,\pi \frac{TL_s}{2\,D^3 t}$$

or

$$\frac{\delta_{max}}{\tau_o} > 2.44\ L_s/D \qquad \text{Eq.6.17}$$

In order to obtain limiting torques for experiments, the author (arbitrarily) assumed a stress concentration factor of 2 to give an estimated maximum principal stress ($\delta_{1,2}$) near the end of a slit

$$\frac{\delta_{1,2}}{\tau_o} \simeq 5 \quad \frac{L_s}{D} \qquad \text{Eq.6.17 (a)}$$

Now according to L.H. Donnell (1) a crack in a thin sheet subjected to pure shear might produce shear stress concentrations about 5 times the original shear, suggesting for the part-length slit tube,

$$\frac{\tau_{max}}{\tau_o} \simeq 5 \qquad \text{Eq.6.18}$$

So the shear stressus are the limiting factor for short slits, and the axial stresses for the longer slits. And if stress concentrations must be taken into account, a slit tube has a strength $\frac{1}{5}$ of the unslit tube for short slits, and much less still for long slits.

In addition to this experiments by the author have shown that local buckling near the ends of the slit, on the "compressive" side of the slit is an important and likely mode of failure of very thin tubes or tubes of low modulus of elasticity, although the buckling may not be sudden nd in the initial stages may not even measurably effect the angle of twist. However, neither local buckling nor "large deflections" will be discussed further in this thesis.

Away from the immediate vicinity of the slit, the stress distribution (in the slit length of the tube) may be like that of the classic Eqs. 2.1 and 2.3, or the membrane Eqs. 4.26 and 4.22, but with a general reduction in magnitude because of the elastic end effect of the closed part of the tube, although the shear stress at $\varphi = \pm 110^{\circ}$ may still be approximately 1.38 τ_o (Eq.4.28).

PART III

Experiments

Chapter 7 - brief description of experimental apparatus and test procedures; statements of accuracy. (Specimen calculations of theoretical and experimental accuracies are in the appendix chapter A3).

7. EXPERIMENTAL APPARATUS AND TEST PROCEDURE

Torsion machine and specimens

Figs. 7.1 (a) and (b) show the experimental set-up built by the author to apply simple torque loadings from 100 to 15 000 inch pounds to the two main tubes which were $3\frac{1}{2}$" diameter, $\frac{1}{10}$" wall thickness, mild steel tubes of spans 32" and 22".

Each tube was fillet welded at each end, (by Mr. Edward Cam of the South Australian Institute of Technology) to a very heavy solid steel 3 " long $5\frac{1}{2}$" diameter end-block which in turn was welded to a large $\frac{1}{2}$" thick flange plate.

The flange plate at one end was bolted to a rigid foundation.

The tubes were supported at the other end in a $2\frac{3}{4}$" roller bearing via a 5" long $2\frac{1}{4}$" diameter solid steel round bar welded to the flange plate in line with the tube structural axis; in the case of the full-length slit tube span 22" a test was also made with the end bar and bearing in line with the shear centre axis directly opposite the slit.

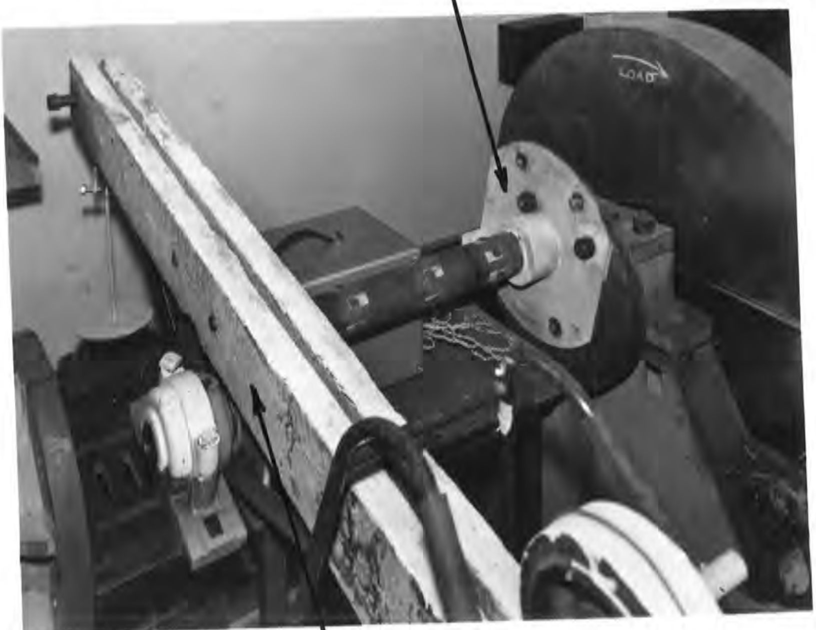
The bearing was in a plummer block fixed to a rigid foundation.

Under all possible experimental loadings the end bearing was pre-tested and found to be virtually frictionless and without hysteresis.

A 60" wooden cross-beam with diagonal steel struts was

Fig. 7.1 (a) Test Apparatus
with specimen 2.2 in position

This end of tube welded to massive end plate which is bolted to rigid foundation.

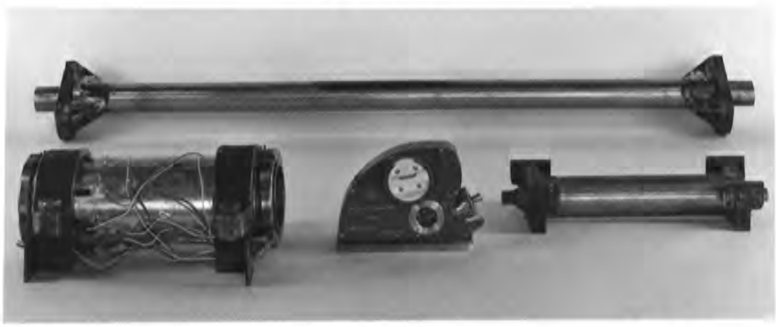


This end of tube welded to similar end plate (carrying wooden cross-beam) with stub axle welded to other side.

The axle can rotate in bearing in plummer block which is bolted to rigid foundation.

Fig. 7.1 (b) Test Apparatus with specimen 2.2 in position

Tube 5



Tube 6 Tube 4

Clinometer

Fig. 7. 2 Other Test Specimens and Clinometer

(For details of all specimens see Appendix chapter A5) bolted symmetrically to the ½" thick flange plate at the rotationally free (bearing) end. This enabled a uniform torque (constant along the length of the beam) in the form of an end couple to be applied to the experimental tube by hanging equal or nearly equal weights from 5 lbf to 250 lbf to each end of the cross-beam, one weight producing an upward load via a pulley. Pre-testing of this pulley again indicated effectively frictionless operation. The moment arm of each weight was 30" for the upward acting load and 28.5" for the other.

The self weight of the tube, end plates and cross-beam caused bending in the tube which acted as a beam with one fixed end and one pinned end. This bending was ignored. Also ignored was the bending produced whenever the loading weights were unequal. The tubes were slit on top, so that the line of action of these bending loads was through the shear-centre axis and produced no additional torsion.

Instrumentation, measuring techniques*

The torque was measured accurately (better than 1% error) by the addition of the products of each weight on the cross-beam and the perpendicular distance of its line of action from the tube structural axis or the shear centre axis as appropriate.

The angle of twist of the whole span of the test tube was measured with a Watts 90° clinometer, (Figs. 7.1 and 7.2), which read the slope of a plane surface to the nearest ½ minute by means of a bubble level.

* See Chapter A.3 in the appendix for detailed discussion of experimental errors.

This instrument was simple to use and of a convenient sensitivity, in addition to being accurate enough and having more than adequate range. It was calibrated by means of sine bars and found to give angular differences to better than $\pm \frac{1}{4}$ minute.

A short length of channel, made into a double knife edge on which the clinometer could be placed, was welded to each flange plate. Then at each load increment clinometer readings were taken at both ends of the tube; the change in angle of tiwst for a particular load increment was given by the change in the difference of the clinometer reading at both ends.

The following tables explains this procedure.

LOAD (TORQUE)	CLINOMETER READINGS		
	Left Hand	Right Hand	Difference
T_1	ϕ_{L_1}	ϕ_{R_1}	$\phi_{R_1} - \phi_{L_1} = \phi_1$
T_2	ϕ_{L_2}	ϕ_{R_2}	$\phi_{R_2} - \phi_{L_2} = \phi_2$

Change in Torque $= T_2 - T_1$

Change in Angle of Twist $= \phi_2 - \phi_1$

FIG. 7.3

The measured change in angle of twist for any torque increment was ~~accura~~ accurate within $\pm$ 1 minute, but for a typical maximum angle of twist of 30 minutes the slope of most of the T - ϕ plots is confidently estimated within 2% error.

The stresses were obtained via "Shinkoh" 120 Ω paper strain gauges, gauge length 8 mm, and a Phillips PR 9302 strain bridge. See Fig. 7.1.

On the outside surface of the tubes 7 or more rosettes (0-45°-90°) and 11 single strain gauges were used. No strains were measured on the inside surface of the tubes so that no account was taken of any lateral bending of the walls that might occur.

The axial shear stresses (τ) and the axial tensile and compressive stresses (σ) could be calculated from the measured strains (ε) using the following formulas given by Durelli [1],

$$\tau = G \left[2 \, (\varepsilon_z - \varepsilon_{45})^2 + 2 \, (\varepsilon_{45} - \varepsilon_s)^2 \right. \qquad \text{Eq.7.1}$$

where

$$\tan 2\theta = \frac{2 \, \varepsilon_{45} - \varepsilon_z - \varepsilon_s}{\varepsilon_z - \varepsilon_s} \qquad \text{Eq.7.2}$$

$$\sigma = \frac{E}{1 - \nu^2} \, (\varepsilon_z + \nu \, \varepsilon_s) \qquad \text{Eq.7.3}$$

where ε_z, ε_{45}, ε_s are the strains measured axially, (ε_{0°) at 45° to the axis, and circumferentially (ε_{90°) respectively.

Test procedure

Paragraphs 1. to 4. below concern a single piece of 3.5" diameter mild-steel seam welded tubing, wall thickness 0.1" and the same strain gauges attached throughout.

1. 32" span original (unslit) tube loaded in approximately 1500 inch pound increments up to approximately 15000 inch pound. Measured at each increment were the angle of twist ϕ_o and the diagonal strain $(\varepsilon_{45})_o$.

The shear modulus of elasticity G was derived from the angle of twist measurements enabling the shear stress τ_o to be checked from the strain measurements. See appendix chapter 15.

2. With the tube in situ, a single axial slit .040" in width was made through the wall of the tube symmetrically about the centre of the span.

This was done by first scraping through the tube wall with a hand scraper and when the gap was long enough inserting a hacksaw blade to extend the slit to the required length.

3. A complete test run was carried out for each of 18 different (increasing) lengths of slit including the final test with the slit running the full length.

The torque was applied about the tube structural axis throughout these tests, and was limited according to the following table.

Length of slit L_s (ins.)	0	3	6	10	20	32
Torque allowed (kips ins)	37	13.5	8.7	5.2	2.6	1.6

<u>No slit</u>: torque limited by $\tau_o = 20$ ksi $= 2T/\pi tD^2$

<u>3" slit</u>: torque limited (arbitrarily) by $\tau_{max} = 10$ ksi

calculated by $\tau_{max} = 1.38\ \tau_o$. (from Eq.4.27)

<u>Longer slits</u>: torque limited (arbitrarily) by $\sigma_{max} = 20$ ksi

calculated by $\sigma_{max} = 2.44 \frac{L_s}{D} \tau_o$ (from Eq.4.23)

LIMITING TEST TORQUES

FIG. 7.4.

Angle of twist was measured on each test run at suitable increments of torque. Up to 9 3-gauge rosettes and 11 single strain gauges were read on each test run.

4. An end plate was then taken off, the full-length slit tube cut back to 22" span, the end plate re-welded back on, and two test runs carried out on this reduced span full-length slit tube:

(a) torque applied about tube axis

(b) torque applied about shear centre axis

Another similar pair of tests were in progress at further reduced span on the same tube (cut back) when the experiments had to be curtailed due to circumstances beyond the control of the author. A test on a short length of aluminium alloy tube is, however, included in the thesis. This "slit" tube was actually formed from a rectangular sheet of 26 gauge alclad sheet, rolled into cylindrical form and gripped at both ends by massive steel end plates (Fig. 7.2). The specimen was then annealed at 550°C, quenched in water, and age hardened before carrying out the torque and twist tests on a commercial "Avery" machine.

All the commercial machines measured the torque very accurately; the angle of twist in all cases was measured by the clinometer.

Finally from the tests described in paragraph 3 above, certain hypotheses were made by the author relating to the general applicability of semi-empirical solutions based on the tests of the part-length slit tubes with ends constrained to remain plane.

These hypotheses were tested with further experiments on mild steel tubes whose ends (a) were welded to a light steel end plate, and (b) had no end plate. Commercial "Tecquipment" and "Avery" torsion machines were used to apply and to measure the torque on these specimens.

Presentation of results

The next chapter gives the principal derived results in graph form. Detailed experimental data and observed results are in the appendix chapter A.5.

Most derived results are presented non-dimensionally, the measured stresses and angles of twist generally being quoted relative to the appropriate condition in similar unslit tubes subject to the same torque.

Apart from the general appropriateness of such a presentation, the relative results are more accurate than the absolute results because of the inherent calibration provided by the unslit tube results.

Thus the stresses actually derived from the measured results were, from Eqs.7.1, 7.2 and 7.3, to an estimated accuracy of approximately ± 2% for τ/τ_o, and ± 6% for δ/δ_o.

$$\frac{\tau}{\tau_o} = \frac{\sin 2\Theta}{(\varepsilon_{45})_o}\left[\tfrac{1}{2}(\varepsilon_{45} - \varepsilon_z)^2 + \tfrac{1}{2}(\varepsilon_{45} - \varepsilon_s)^2\right]^{\frac{1}{2}} \qquad \text{Eq.7.4}$$

and

$$\frac{\delta}{\delta_o} = \frac{\varepsilon_z + \nu\,\varepsilon_s}{(\varepsilon_{45})_o\,(1 - \nu)} \qquad \text{Eq.7.5}$$

where $(\varepsilon_{45})_o$ was the diagonal strain measured in the unslit tube by the same strain gauges and at the same torque.

Further discussion on the calculation and presentation of theoretical and experimental data, including specimen estimations of accuracy and errors, is given in the appendix chapter A.3.

8. GRAPHICAL PRESENTATION OF THE MAJOR EXPERIMENTAL RESULTS

Figs. 8.1 Stresses in tubes slit full-length

(a) L = 9.5 D

(b) L = 6.5 D

Fig. 8.2 Angle of twist of tubes slit full length

Fig. 8.3 Strains in a tube slit part-length

Fig. 8.4 Stresses in the unslit length of a tube slit part-length

Fig. 8.5 Angle of twist of a tube slit part-length and with massive end plates

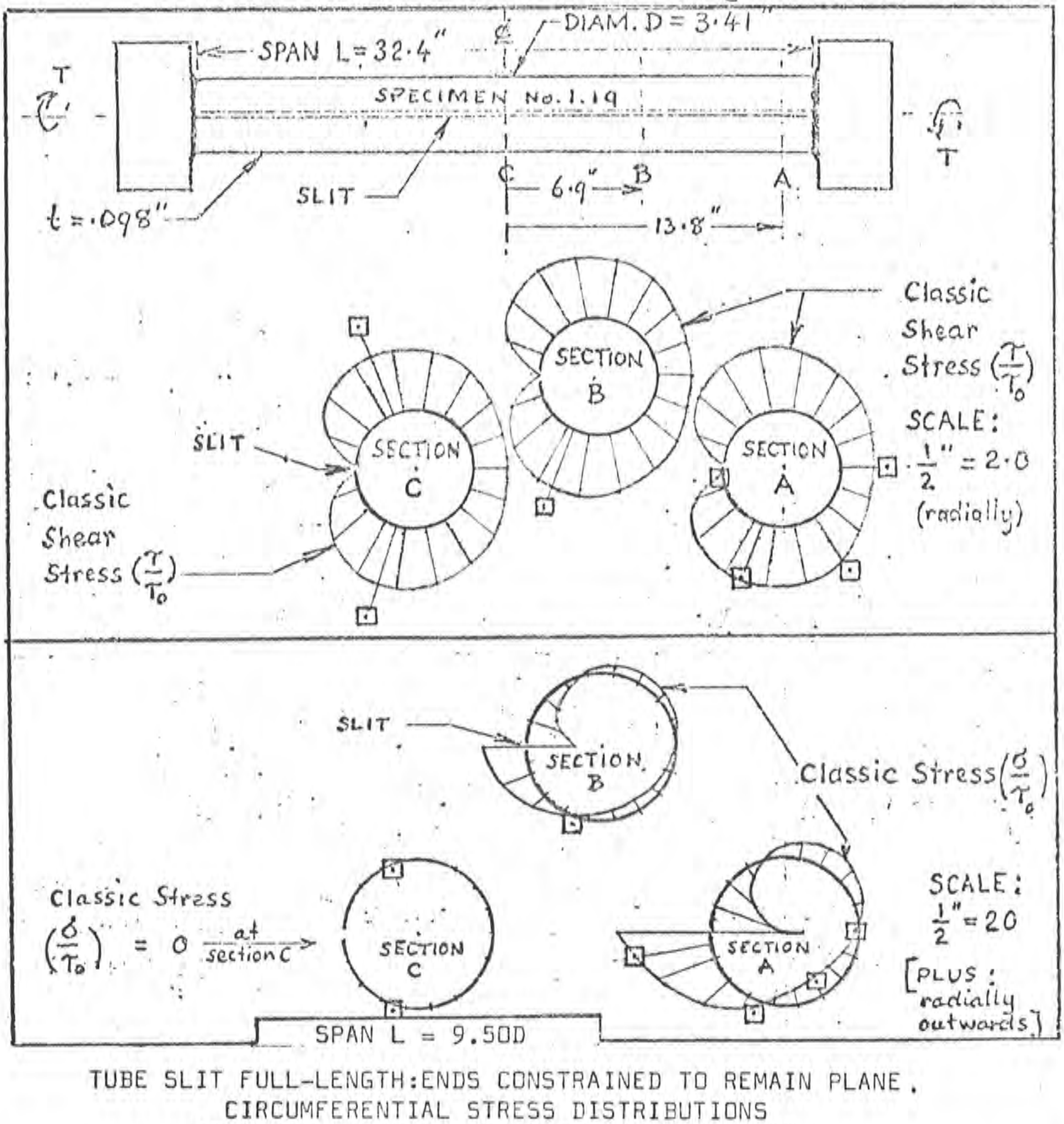
Fig. 8.6 Angle of twist of metal tubes with any end fixing and with a part-length slit

Explanation and discussion of the graphs are in the following chapter 9.

Observed results and other derived results are in the appendix chapter A5.

AXIAL STRESSES – Tensile or Compressive σ , and shear τ
IN A ROUND THIN-WALLED TUBE WITH MASSIVE END PLATES, AND WITH
A FULL-LENGTH NARROW AXIAL SLIT THROUGH THE TUBE WALL ,

– Due to a Uniform Torque;
[Relative to
the shear stress (τ_o) in the original unslit
tube subjected to the same torque]



TUBE SLIT FULL-LENGTH: ENDS CONSTRAINED TO REMAIN PLANE.
CIRCUMFERENTIAL STRESS DISTRIBUTIONS
AT VARIOUS CROSS-SECTIONS ALONG THE SPAN

(i) Thesis ($\equiv$ Classic) Solutions: $\tau/\tau_o = 3.88\ (5.87 - \varphi^2 - 4\cos\varphi)$ Eq.4.27

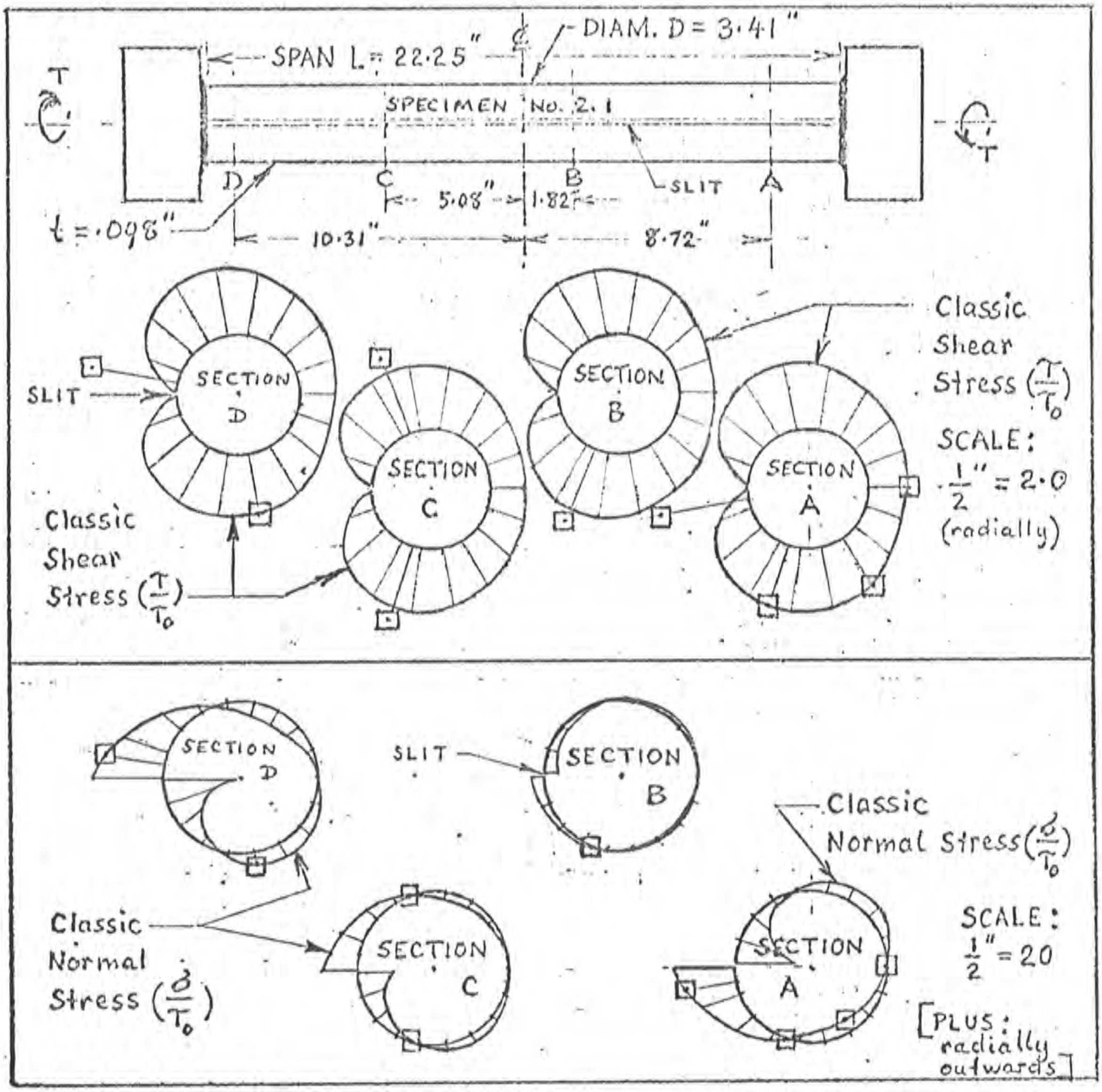
$\sigma/\tau_o = 1.55\ (\varphi - 2\sin\varphi)\ z/D$ Eq.4.23

(ii) Experimental Results: ⊡ (Specimen . 1.19)

FIG. 8.1 (a)

AXIAL STRESSES - Tensile or Compressive σ, and shear τ IN A ROUND THIN-WALLED TUBE WITH MASSIVE END PLATES, AND WITH A FULL-LENGTH NARROW AXIAL SLIT THROUGH THE TUBE WALL,

— Due to a Uniform Torque;
[Relative to
the shear stress (τ_0) in the original unslit
tube subjected to the same torque]



SPAN L = 6.52D

TUBE SLIT FULL-LENGTH:ENDS CONSTRAINED TO REMAIN PLANE.
CIRCUMFERENTIAL STRESS DISTRIBUTIONS
AT VARIOUS CROSS-SECTIONS ALONG THE SPAN

(i) Thesis ($\equiv$ Classic) Solutions: $\tau/\tau_0 = .388\ (5.87 - \varphi^2 - 4 \cos \varphi)$ Eq.4.27

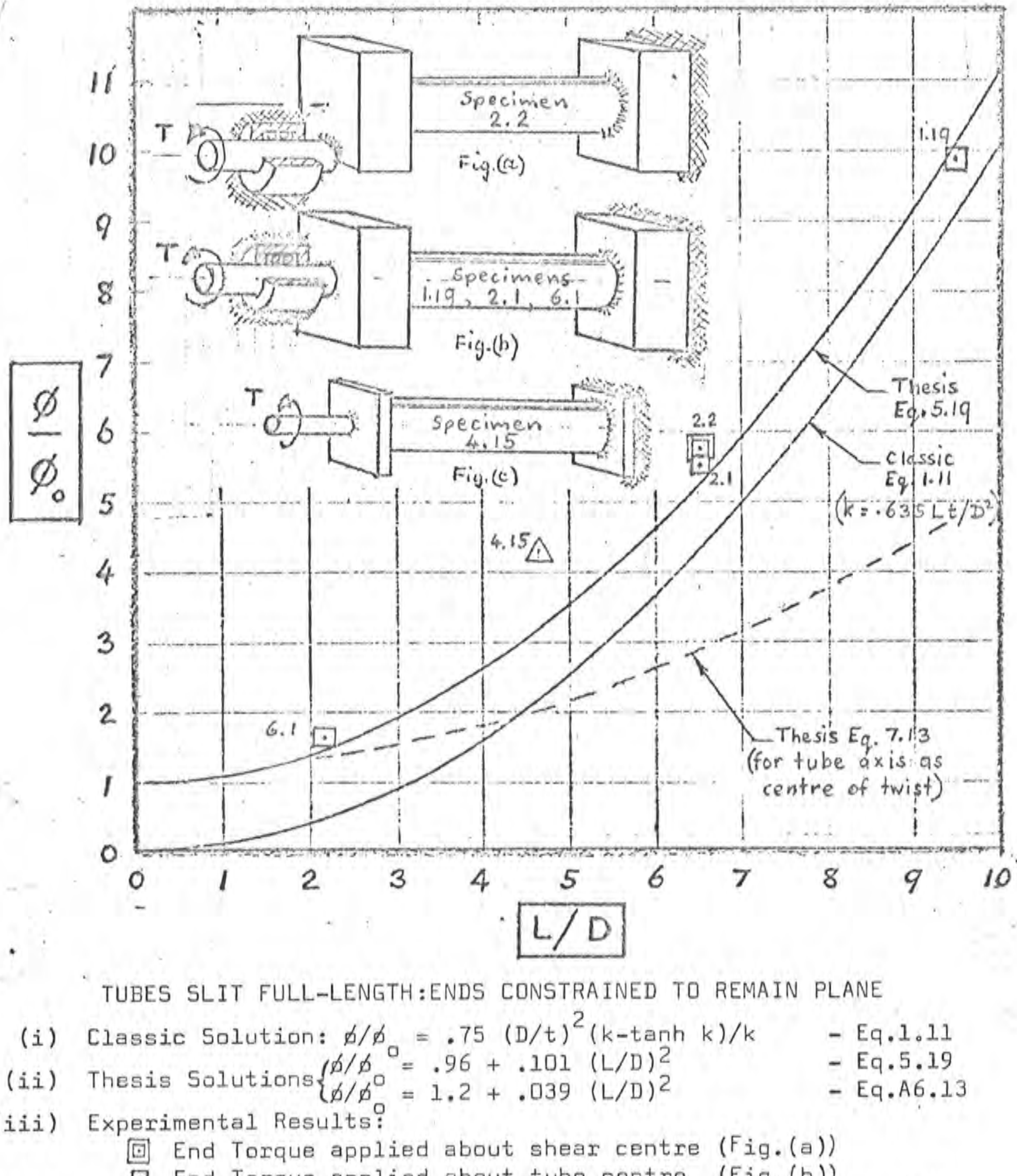
$\sigma/\tau_0 = 1.55\ (\varphi - 2 \sin \varphi)\ z/D$ Eq.4.23

(ii) Experimental Results: ⊡ (Specimen 2.1)

FIG. 8.1 (b)

ANGLE OF TWIST (ϕ)
OF ROUND THIN-WALLED TUBES WITH MASSIVE END PLATES, AND WITH A FULL-LENGTH NARROW AXIAL SLIT THROUGH THE TUBE WALL,

– Due to a Uniform Torque;
[Relative to the angle of twist (ϕ_o) of the original unslit tube subjected to the same torque]



TUBES SLIT FULL-LENGTH: ENDS CONSTRAINED TO REMAIN PLANE

(i) Classic Solution: $\phi/\phi_o = .75\ (D/t)^2 (k - \tanh k)/k$ – Eq.1.11

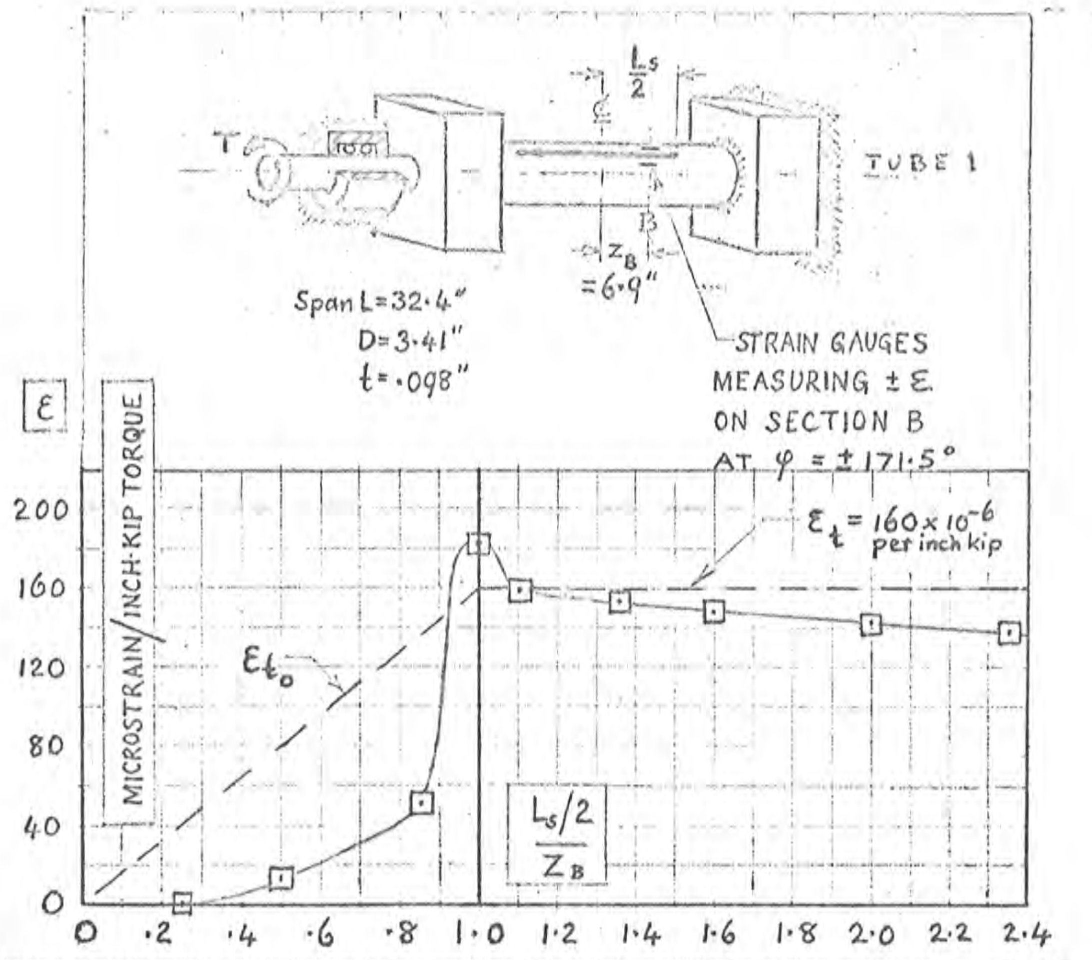
(ii) Thesis Solutions $\begin{cases} \phi/\phi_o = .96 + .101\ (L/D)^2 & \text{– Eq.5.19} \\ \phi/\phi_o = 1.2 + .039\ (L/D)^2 & \text{– Eq.A6.13} \end{cases}$

(iii) Experimental Results:
- ▣ End Torque applied about shear centre (Fig.(a))
- ⊡ End Torque applied about tube centre (Fig.(b))
- △ Additional test : slit-tube with medium size end plates (Fig.(c))

[Specimen no. shown by each point]

FIG. 8.2

MEASURED AXIAL STRAINS (ε)
IN A ROUND THIN-WALLED TUBE, WITH MASSIVE END PLATES
AND CONTAINING A NARROW AXIAL SLIT, OF VARIOUS LENGTHS (L_s)
through the wall of the tube.
[Compared to
theoretical strains (ε_t) which assume
that the cross-sections containing the ends of the
slit stay plane]



TUBE SLIT PART-LENGTH: ENDS CONSTRAINED TO REMAIN PLANE
SPAN L = 9.5D

⊡ = measured strain at section B, 8.5° from the line of the slit

$\varepsilon_t = \sigma_B/E$ (Eq.4.2), where $\sigma_B = 1.55\, z_B(\varphi - 2\sin\varphi)\,\tau_o/D$ (Eq.4.23)

$\varphi = 171.5^\circ$ and $E = 29.5 \times 10^3$ kips per in^2 (assumed)

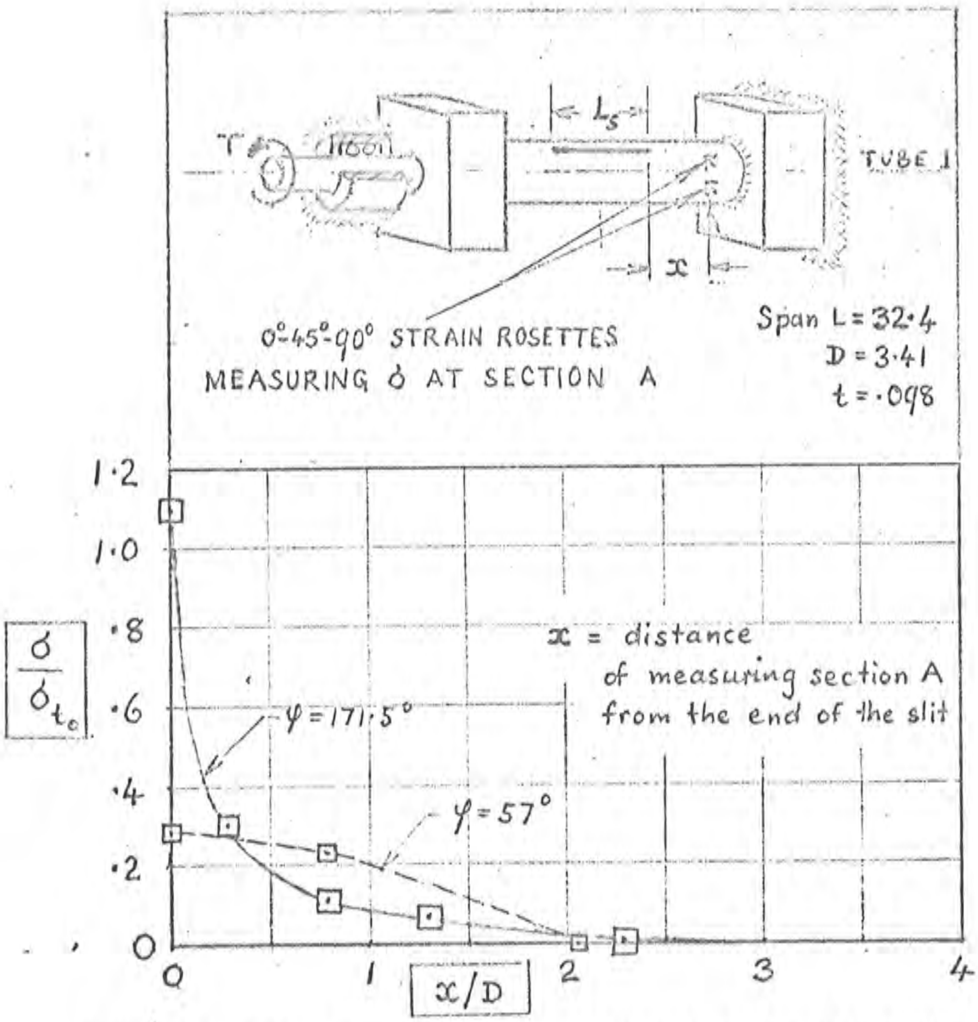
For the case where the slit does not extend to section B, (i.e. $\frac{L_s/2}{z_B} < 1$):

$\varepsilon_{t_o} = 1.55\ \frac{L_s}{2}\ (\varphi - 2\sin\varphi)\ \tau_o/DE$, i.e. a nominal value at the sections containing the end of the slit assuming these sections stay plane.

with $\varphi = 171.5^\circ$

FIG. 8.3

MEASURED AXIAL STRESSES (σ) IN THE UNSLIT LENGTH OF A ROUND THIN-WALLED TUBE WITH MASSIVE END PLATES AND WITH A NARROW AXIAL SLIT OF VARIOUS LENGTHS (L_s) through the wall of the tube.
[Relative to theoretical stresses (σ_{t_o}) on the cross-sections containing the ends of the slit (σ_{t_o} assumes that the slit-end cross-sections stay plane)]



TUBE SLIT PART-LENGTH: ENDS CONSTRAINED TO REMAIN PLANE
SPAN L = 9.5D

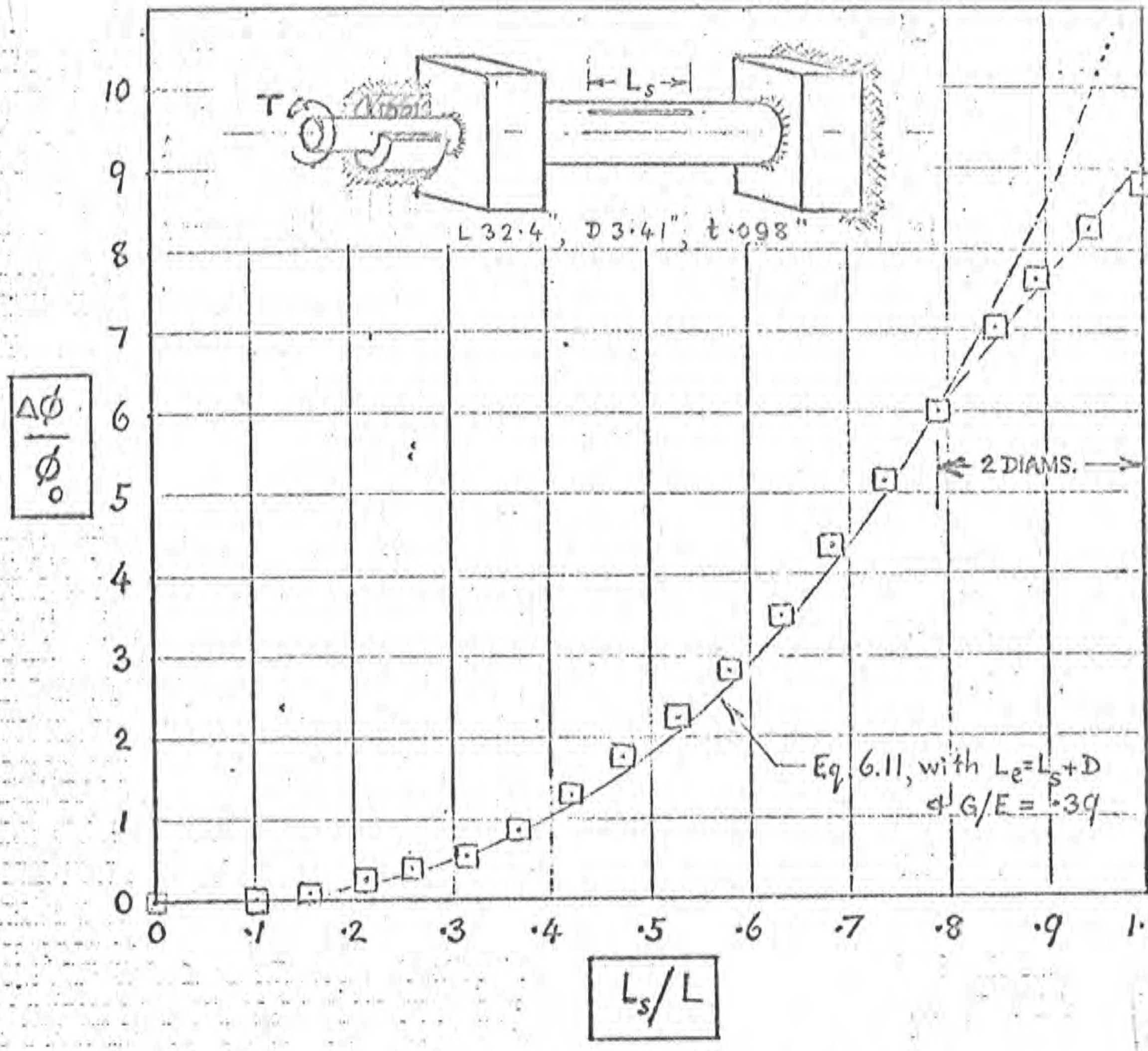
⊡ = measured stresses at section A, plotted relative to the theoretical stress σ_{t_o} on the same generator, where

$$\sigma t_o = 1.55 \frac{L_s}{2} (\varphi - 2\sin\varphi)\, \tau_o / D \text{ (from Eq.4.23)}$$

FIG. 8.4

INCREASE IN ANGLE OF TWIST ($\Delta \phi$)
of a round thin-walled tube with massive end plates,
subjected to a uniform torque
DUE TO A NARROW AXIAL SLIT OF ANY LENGTH (L_s)
through the wall of the tube

[Relative to
the angle of twist (ϕ_o) of the original unslit
tube subjected to the same torque]



TUBE SLIT PART-LENGTH:ENDS CONSTRAINED TO REMAIN PLANE
SPAN L = 9.5D

(i) Experimental results ⊡ (specimens 1.1 to 1.19)
(ii) Thesis semi-empirical solution: Eq.6.11 , ($L_e = L_s + D$) :

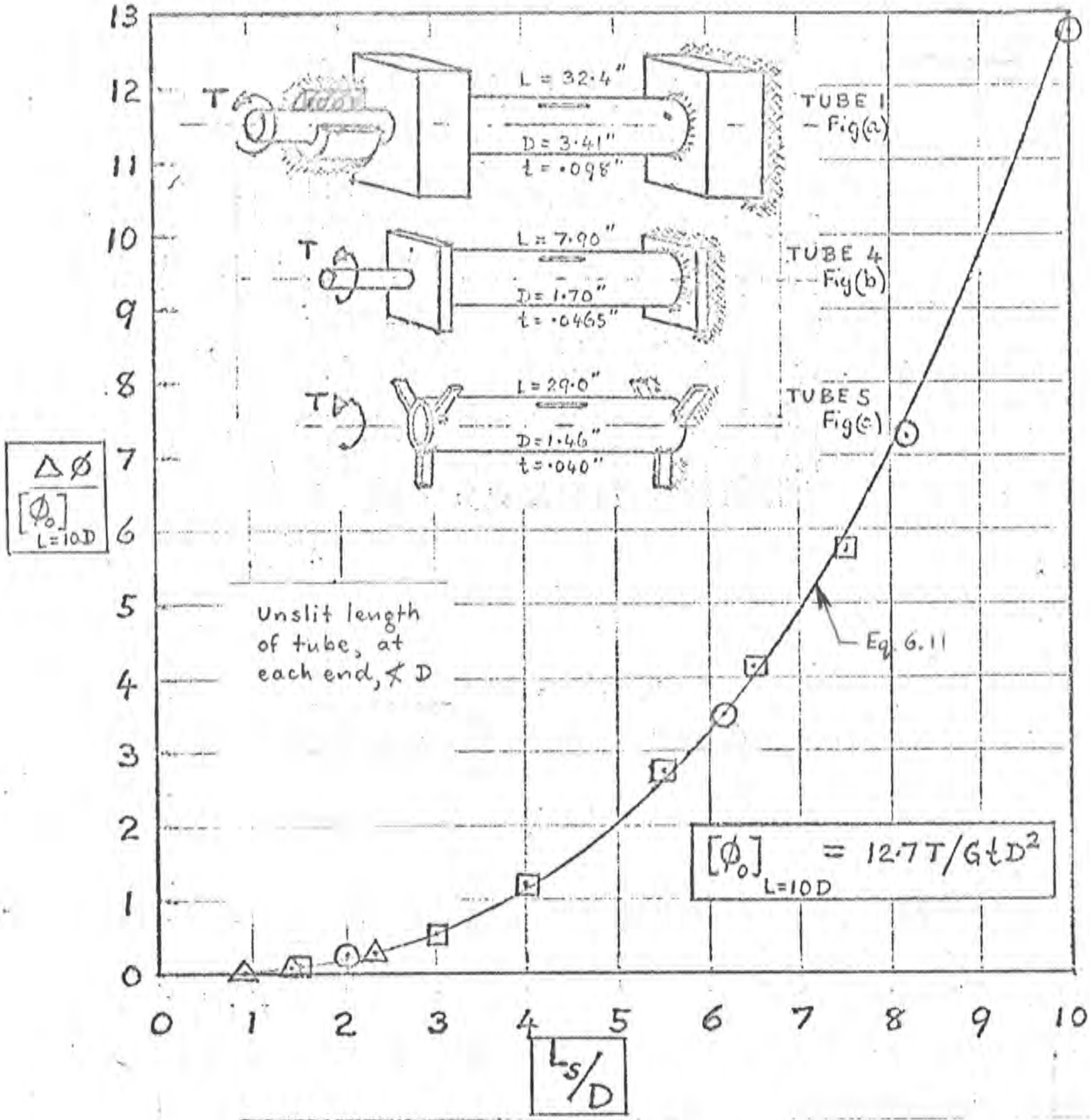
$$\Delta \phi = .101 \left(\frac{L_s}{D}\right)^2 \left(\frac{L_s}{L}\right) \left[3\frac{L_e}{L_s} - 2 \right] \phi_o$$

merged (by straight line) with thesis solution for $L_s = L$: $\phi/\phi_o = 9.06$
From Eq.5.19

FIG. 8.5

INCREASE IN ANGLE OF TWIST ($\Delta\phi$)
OF A ROUND THIN-WALLED TUBE
SUBJECTED TO A UNIFORM TORQUE,
DUE TO A NARROW AXIAL SLIT OF ANY LENGTH (L_s)
through the wall of the tube;

[Relative to
the angle of twist of a 10 diameter length
of the original unslit tube subject to the same
torque]



TUBES SLIT PART-LENGTH: ANY END FITTING

(i) Experimental results: ⊡ very heavy end plates (Fig.(a))
△ medium end plates (Fig.(b))
⊙ open ends (Fig.(c))

(ii) thesis semi-empirical solution: Eq.6.11, with $L_e = L_s + D$:

$$\Delta\phi / [\phi_o]_{L = 10D} = .03\ (L_s/D)^2 + .01\ (L_s/D)^3$$

FIG. 8.6

9. DISCUSSION OF EXPERIMENTAL RESULTS

CONTENTS

9. DISCUSSION OF EXPERIMENTAL RESULTS

Figs. 8.1 - Stresses in tubes slit full-length

Each experimental point plotted on Figs. 8.1 (a) and (b) represent the slope of a torque-stress curve (for a complete test run on the slit tube) divided by the slope of the torque-shear stress curve for the original unslit tube, and is estimated to have an accuracy of better than ± 10%. These curves are drawn in the appendix chapter A5 together with tabulations of all observed strain readings.

Tubes 1 and 2 were used for these tests (see appendix chapter A5 for details) in the apparatus illustrated in Figs. 7.1 (a) and (b).

Figs. 8.1 (a) and (b) show that the stresses in short open-section beams of high warping rigidity subjected to non-uniform torsion conform in general to the classic stress distribution for a rotation about the line of shear centres for a long open-section beam.

The longitudinal tensile and compressive stresses agreed particularly well with the classic solution for the slit tube approximated by Eq.4.23. This indicates first that the short slit tube was twisting essentially about the line of shear centres, secondly that the massive end plates were effective in maintaining both end cross-sections plane, and thirdly that the general assumptions made in chapter 4 are valid for beams with cross-sections of high warping rigidity.

Concerning the second conclusion just mentioned it will be seen from the following figures that in fact the stress distribution in a slit tube is not changed so markedly as is the angle of twist by non-rigid end plates.

The experimental shear stresses were also as expected except for the longer span (9.5D) tested where in the middle length of the span the measured longitudinal shear stresses were 40% to 60% more than that given by Eq.4.27.

This could have been associated with a slight "opening up" of the slit in the middle portion of the span, perhaps due to the release of locked up stresses caused by the welding of the steel tube to the end plates, or perhaps due to the bending of the tube which would occur because of the shift of the centre of twist from the centre of the end cross-section to the shear centres in the middle length of the tube.

In the shorter span tube (6.5D) which was already slit prior to completion of the welding - the shear stresses in the middle portion of the span were approximately 20% greater than theoretical, which ties in with the latter hypothesis regarding the bending of the tube. Alternatively there may have been variations in shear stress through the wall thickness, perhaps partly due to a small warping of the end sections.

Further experiments are required with more extensive strain measurements including a rosette on the inside surface of the tube corresponding to each rosette on the outside surface.

The stress distributions in a slit tube with the torque applied about the geometric centre and the shear centre need to be compared also.

Apart from the variation from theory in the middle length of the longer spans, it will seem in Fig. 8.1 (b) that there is a singularly high value of shear stress close to the slit near the end sections of the shorter tube. This can be likened to a stress concentration at the ends of the slit and is actually predicted by the shell theory of Mizoguchi and Shiota [4] (See their Fig. 7).

Fig. 8.2 - Angle of twist of tubes slit full-length

Each experimental point plotted on Fig. 8.2 has an estimated accuracy of better than ± 5%, and is the slope of a torque-twist curve obtained from a complete test run. All points are within 10% of the angle of twist predicted by the thesis Eq.5.19 except the triangle point which is for a slit tube with light end-plates. The square points plotted are for tubes with massive steel end plates.

Provided that it is accepted that the massive steel end plates prevent the two end sections of the tube from warping it can be said that Eq.5.19 is a valid approximate solution.

Hence the validity of Eqs. 5.31, 5.36 and 5.37 for other arrangements of torque tubes slit full length also seems to be likely; and the general method of determining a shear correction from the strain energy of the classic shear stresses seems to be justified, at least for beams with high warping rigidity.

Other conclusions to be drawn from this graph are:

(i) the overall angle of twist is virtually the same whether the end torque is applied about the axis of shear centres or the tube geometric axis.

This was demonstrated with the $3\frac{1}{2}$" diameter tube of span 6.50 which had the end torque applied about the shear centre axis (double square point on graph) as well as the tube geometric centre.

This is presumably because one end fixing and perhaps the tube itself were flexible enough to be bent sufficiently for the tube to rotate about its line of shear centres for most of its length. (See also Marin [2]).

(ii) the single point which is designated by a triangle on Fig. 8.2 represents a test run on a full length slit tube with more normal end plates, about $\frac{5}{8}$" thick. It appears that these were not sufficiently effective in restricting the warping of the tube end sections because the angle of twist measured here was approximately 30% more than that predicted by Eq.5.19.

This is in line with the experiments of other authors, for example Glasunova, from Mescheryakov [3], and Mizoguchi and Shiota [2] .

Fig. 8.3 - Axial strains in a tube slit part-length

The experimental points plotted in Fig. 8.3 are estimated to be accurate to within $\pm$ 15% and are the average of the numerical readings from two longitudinal strain gauges located at

$\varphi = \pm 171.5^{o}$ and $z = 6.9"$ on the outside surface of the 32.4" long 3" diameter steel tube with massive steel end plates (Tube 1 in chapter A5. See also Figs. 7.1 (a) and (b)).

Each experimental point is the slope of a torque-strain curve obtained from a test run with a particular length of slit. These torque-strain curves are not recorded in the thesis but the observed results are tabulated in the appendix chapter A5.

(a) $L_s/2 > Z_B$

The right hand side of the graph is for those tests where the slit extended beyond the cross-section (B) on which the strains were measured i.e. $\frac{L_s/2}{Z_B} > 1$.

If it were assumed that the slit were full length, or if it were assumed that the cross-sections containing the ends of the slit remained plane, then the classical value for the theoretical strain at $\varphi = 171.5^{o}$ (8.5^{o} from the slit) and $z = 6.9"$ would be given by

$$\varepsilon_z \simeq \frac{\sigma}{E} = 1.55 \ \frac{z}{D} \ (\varphi - 2 \sin \varphi) \ \frac{\tau_o}{E} \quad \text{(from Eq.4.23)}$$

and this would be a constant for any length of slit (symmetrical within the span) greater than 2 x 6.9". This theoretical strain is drawn on the Fig. 8.3 and for $\tau_o = .543$ ksi per inch kip and an assumed value of $E = 29.5 \times 10^{6}$ lbf/in^{2} is

$\varepsilon_z = 160 \times 10^{-6}$ per inch kip of torque

It is seen from the figure that the measured strains are within 20% of this calculated strain, indicating that the longitudinal tensile and compressive stresses close to the slit, in the slit length of a part slit tube, may be predicted by the (classic) formula for a full length slit tube.

This result is in accord with previous reasoning concerning the immobility of the end of the slit itself.

(b) $L_s/2 < \ell_B$

The left hand side of Fig. 8.3 is for those tests where the slit did not extend as far as the measuring section (B) so that the strain readings were being taken in the unslit portion of the tube.

The theoretical strain plotted here is from the same equation as above viz.

$$\varepsilon_z \simeq 1.55 \ \frac{z}{D} \ (\varphi - 2 \sin \varphi) \ \frac{T_D}{E} \qquad \text{(from Eq.4.23)}$$

but with $z = L_s/2$. That is, it is an estimate of the strain on the cross-section containing the end of the slit at $\varphi = 171.5^\circ$ - in line with the strain gauges at section B - and it has just been shown that this is a good estimate of the actual strain.

It is seen from the figure that the strains measured at section B for this case are much less than the strain at the cross-section containing the end of the slit, even when the measurements are taken fairly close to this cross-section.

This rapid dying away of the strains into the unslit portion of the tube is again consistent with the opposite nature of the strains either side of the slit; this phenomenon is further exemplified in Fig. 8.4.

Fig. 8.4 - Axial stresses in the unslit lengths of a tube slit part-length

The longitudinal tensile and compressive stresses plotted on this graph are derived from two strain rosettes situated on the cross section (A) $z = 13.8"$, at $\varphi = 57^{\circ}$ and 171.5° respectively, on the outside surface of the 32.4" long $3\frac{1}{2}"$ diameter tube with massive end plates (Tube 1 in chapter A5. See also Figs. 7.1 (a) and (b)). Each experimental point is the slope of a torque stress curve obtained from a test run for a particular length of slit (L_s), with $L_s \leqslant 2 \times 13.8"$. The accuracy of the points is reckoned to be better than $\pm$ 10%.

The abcissa of Fig. 8.4 is x, the distance of the measuring section (A) from the end of the slit. When $x = 0$ the measuring section coincides with the cross-section containing the end of the slit and when $x > 0$ the measuring sectioning is in the unslit length of tube.

The ordinate in the figure is the measured stress, divided by the theoretical stress for the same value of φ on the cross-section containing the end of the slit. This theoretical stress is similar to the corresponding theoretical strain in Fig. 8.3, and was calculated from

$$\delta = 1.55 \frac{z}{D} (\varphi - 2 \sin \varphi) \tau_o \qquad \text{(Eq.4.23)}$$

with $z = L_s/2$.

Fig. 8.4 shows again that the stress at $\varphi = 171.5^\circ$ is close to the above theoretical value at the section containing the end of the slit (or slightly greater due to stress - concentration effects) but dissipates extremely rapidly into the unslit length of tube.

When $\varphi = 57^\circ$, the actual stress is only one third of the above theoretical value for x = 0, because away from the slit the non-rigidity of the cross-section allows it to warp, thus relaxing the stresses compared to Eq.4.23. The stress along this generator ($\varphi = 57^\circ$) also is seen from Fig. 8.4 to dissipate into the unslit length, vanishing three diameters from the end of the slit.

Fig. 8.5 - Angle of twist of part slit tube with massive end plates

Each experimental point on this graph is probably accurate to better than ± 5% and represents the slope of a torque-twist curve obtained from a complete test run on the 32.4" long 3½" diameter tube in the apparatus illustrated in Figs. 7.1 (a) and (b). The observed results and the separate torque-twist curves are in the appendix chapter A5.

To illustrate better the effect of the slit on the torsional rigidity of the tube compared to the rigidity of the original unslit tube the ordinate of the graph is the increase in angle of twist ($\Delta\phi = \phi - \phi_o$) caused by the slit divided by the original angle of twist of the unslit tube due to the same torque.

The interesting feature of Fig. 8.5 is that the thesis formula

$$\frac{\phi}{\phi_o} = 1 + .1 \left(\frac{L_s}{D}\right)^2 \left[3 \frac{L_e}{L_s} - 2 \right] \frac{L_s}{L} \qquad \text{(Eq.6.11)}$$

can be made to fit the experimental results, up to a point where the slit is just one diameter from each end of the tube, by choosing a value for L_e of approximately $L_s + D$.

This ties in with the stress picture described above and depicted in Figs. 8.3 and 8.4 where the highest stresses on the cross-section containing the end of the slit are seen to be quartered within one diameter into the unslit length.

The equivalent length

$$L_e = L_s + D$$

means that the actual elastic effect of the unslit end lengths on the slit length of tube (compared to rigid end plates at each end of the slit length) is approximately the same as that of a half-diameter length of unslit tube at each end behaving according to the Vlasov principle [12] discussed in chapter 6. Briefly, the Vlasov approach assumes a constant strain along each generator of the unslit length (between the cross-section containing the end of the slit and a rigid end plate), but varying circumferentially according to the strain in the slit part around the cross-section containing the end of the slit.

Because of the good fit obtained with these experimental results, and because the measured stress dissipation was so rapid in the unslit length, the same formula, Eq.6.11 above, with $L_e = L_s + D$, was used to predict the effect of slits of various lengths on the angle of twist of two other tubes of different dimensions and different end fixings.

The results of this prediction are shown in Fig. 8.6.

Fig. 8.6 - Angle of twist of a metal tube with any end fixing and with any length of slit

Each experimental point on this graph is estimated to be accurate to better than $\pm$ 10% and represents the slope of a torque-twist curve obtained from a complete test run for a particular length of slit.

Some results from Fig. 8.5 are reproduced here (tube 1, with massive end plates), together with the results from tubes 4 and 5. (see Figs. 7.1 and 7.2, and see chapter A5 for the observed results and the separate torque-twist curves).

Tube 4 had "normal" or "medium" end plates (as opposed to "massive" end plates) and tube 5 had no end plates - i.e. it was open-ended, the torque being applied through three webs attached to the tube at $\varphi = 0^{\circ}$ and $\pm 110^{\circ}$.

In no case was the unslit length at each end less than one diameter; also the maximum length of slit tested was 10 diameters.

As a prediction of the total angle of twist ϕ of such slit tubes the thesis formula can be said to be adequate, and the method used thereby holds promise for any similar kind of structure.

However the increase in angle of twist ($\Delta\phi = \phi - \phi_o$) due to a short length of slit is very small and the accuracy of this increase is not known. Further investigations are required in this region since such information could be important for dynamic situations.

Note on interpretation of Fig. 8.6

The particular method of plotting the results used in Fig. 8.6 was adopted in order to bring results from any tube on to one curve.

The abcissa is the length of the slit in diameters.

The ordinate is the increase in angle of twist due to the slit ($\Delta\phi = \phi - \phi_o$) divided by the theoretical or actual angle of twist (for the same torque) of a 10 diameter length of the original unslit tube $\left[\phi_o\right](L = 10D) = \frac{10D}{L} \quad \phi_o$

Example of the use of Fig. 8.6.

An aluminium alloy thin-wall round cylinder is 20 feet long and 1 ft. diameter and is subjected to a torque which is constant along its length.

Estimate the relative increase in angle of twist that would be caused by a 5 ft. longitudinal narrow slit through the wall in the middle portion of the span.

Assume no buckling.

1. $\frac{L_s}{D} = \frac{5'}{1'} = 5$

2. From Fig. 8.6

$$\frac{\Delta \phi}{\left[\phi_o\right]_{L = 10D}} = 2.0 = 200\% \text{ increase for a}$$

10 diameter (or 10 ft.) length of tube.

3. Actual percentage increase

$$= 200\% \times \frac{10 \text{ diameters}}{\text{actual length}}$$

$$= 200\% \times \frac{10'}{20'}$$

$$= 100\%$$

ALTERNATIVELY

$$\phi = 2.0 \left[\phi_o\right]_{L = 10D}$$

$$= 2.0 \times \frac{\left[\phi_o\right]_{L = 20D}}{2} = \left[\phi_o\right]_{L = 20D}$$

<u>Note concerning tubes with a slit extending to or near to the ends</u>

Fig. 8.5 illustrates the effect to be expected when the unslit length at both ends of the tube is less than one diameter.

The value of the angle of twist for a slit length between $L_s = L - 2\,D$ and $L_s = L$ will depend on the type of end fixing.

Further work remains to be done to determine the precise nature of this effect but it is expected that, as the length of slit increases from $L_s - 2\,D$ up to L, the equivalent length will vary from $L_s + D$ to $L + y$ respectively, where y will be 0 for rigid end plates and generally between 0 and D for other end fixings.

CONCLUSIONS

CONCLUSIONS

1. Applicability of classical equations to short beams

The classical stress system (for long open-section beams which are subjected to non-uniform torsion) is valid for short beams of high warping rigidity.

This is demonstrated theoretically in Chapters 2 and 4, while Figs. 8.1(a) and 8.1(b) illustrate the experimental confirmation.

The classical solution appreciably underestimates the angle of twist of short beams of high warping rigidity.

This is proved theoretically in Chapter 1. Test results are shown in Fig. 8.2.

CONCLUSIONS (Continued)

2. Thesis correction for angle of twist

The additional angle of twist due to the axial-circumferential shear strain (compared to the classical solution) may be claculated from the strain energy of the classical shear stresses.

The method is given in detail in chapter 5, together with worked solutions for several typical end conditions and load configurations.

The correction is important for short spans of high warping rigidity, especially where the ends of the beam are held from turning or restricted from warping.

The thesis solution was verified by a series of tests on torque tubes which were slit full-length and had their ends constrained to remain plane. See Fig. 8.2.

CONCLUSIONS (Continued)

3. <u>Thesis solution for angle of twist of torque-tube containing part-length slit.</u>

The increase in angle of twist ($\Delta\phi$) caused by an axial slit through the wall of a thin-walled metal torque tube may be estimated from

$$\Delta\phi = \frac{12.7\,T}{GD^2\,t}\left[.03\left(\frac{L_s}{D}\right)^2 + .01\left(\frac{L_s}{D}\right)^3 \right]$$

which is from Eq. 6.11 in the thesis; and where

T = torque, constant along length of tube

G = shear modulus

D = "mean" diameter of tube (= outside diameter minus thickness)

t = thickness of tube (constant throughout)

L_s = length of slit.

Fig. 8.6 shows how the angles of twist of three different tubes with various length of slit were closely predicted by this thesis formula.

CONCLUSIONS (Continued)

4. <u>Miscellaneous</u>

Other conclusions reached in the thesis include the following:

(a) The angle of twist of a tube slit full-length or nearly full length is highly sensitive to the end conditions.

In particular, massive end-plates are needed to prevent the end cross-sections from warping. Marked increases in angle of twist will result from the use of lighter end plates.

(b) The classical equations for the stresses may be used for the slit portion of a tube slit any length, even for light end plates.

There will be stress concentrations in the vicinity of the ends of the slit.

(c) The angle of twist of a slit tube is virtually unaffected by the position of the axis of an applied end torque.

CONCLUSIONS (Continued)

5. Further work indicated.

(a) On the basic problem of non-uniform torsion of open-section beams

(i) experimental and further theoretical investigation into the applicability of the classic long beam stress formulas to very open cross sections of low flexural torsion warping rigidity, thence application of the thesis shear correction to these beams;

(ii) the effect of end conditions on very short or shell-like beams;

(iii) a "large-deflection" theoretical approach including instability effects;

(iv) an investigation into failure modes;

(v) fatigue;

(vi) combined loadings;

(vii) vibrations;

(See Chilver [1] for some recent works on the buckling problems of thin-walled sections)

CONCLUSIONS (continued)

5. Further work indicated

(b) On the effect of a slit in any structure

The application of the same method which produced Eq. 6.11 for slit tubes may provide a solution for any closed-section thin-walled structure that has a longitudinal slit-like aperture.

The increase in angle of twist due to the slit could be given by an equation of the form

$$\Delta \phi = \frac{TL_s^2}{12\ EJ_w} \quad (L_s + 3L_H)$$

where
- T = torque (constant along length of slit),
- L_s = length of slit,
- J_w = warping rigidity of cross-section through slit-length,
- L_H = an empirical equivalent length from model tests. It could be approximately equal to the depth of the cross-section.

APPENDIX

A.1. A REVIEW OF OTHER AUTHORS' SOLUTIONS WHICH TAKE ACCOUNT OF AXIAL SHEAR STRAIN IN NON-UNIFORM TORSION OF THIN-WALL OPEN-SECTIONS: & SOME EXPERIMENTAL RESULTS OF OTHER AUTHORS.

L. Beskin [2]

In the Journal of Aeronautical Sciences in 1944 Beskin obtains a differential equation for angle of twist with account for shear by first adding a shear term to a membrane solution like Eq.4.19 and then including the effect of pure torsion by equating the angles of twist obtained via the flexural torsional system and the pure torsional system. The result is Eq.A.1.5 which is derived here in not quite the same way in order to emphasize the assumptions involved.

The equation to be developed will be built up from the separate components of torque (flexural-torsional T_s and pure torsional T_T) and angle of twist ("classic" θ_{BT} and direct shear θ_s); and assumptions made in the analysis are as follows:

(a) the applied torque is resisted at any point by St. Venant torsion plus the flexural torsion;

(b) the flexural torsion consists of two separate components called "bending" and "shear";

(c) the part of the torque resisted by the St. Venant shear stress is proportional to the rate of twist, at any particular section;

(d) the part of the torque resisted by the flexural torsion is proportional to the rate of twist caused by shear only, and is also proportional to the angle of twist other than that caused by shear differentiated twice with respect to z;

(e) the rate of twist at the fixed end is due to the "shear" component only.

The relation between the actual rate of twist θ' at any section and that part of the torque which is resisted by the moments within the walls due to the variation of shear stress through the thickness is

$$T_T = GJ\theta' \qquad \text{(Eq.16)}$$

The remainder of the torque (T_S) is that resisted by flexural torsion such that (from Eq.18)

$$T_S = -EJ_w\,\theta'''_{BT} \qquad \text{Eq.A.1.1}$$

where θ'_{BT} is the rate of twist excluding the direct shear deflection θ_s; that is

$$\theta_{BT} = \theta - \theta_s \qquad \text{Eq.A.1.2}$$

and hence

$$\theta'''_{BT} = \theta''' - \theta'''_s \qquad \text{- (i)}$$

Substitute this in Eq.A.1.1,

$$T_S = -EJ_w\,(\theta''' - \theta'''_s) \qquad \text{- (ii)}$$

Assume now that the shear deflection per unit length (θ'_s) depends on the torque producing it T_S and define as a constant of proportionality a shear rigidity GV_s such that

$$\theta_s' = \frac{T_S}{GV_s} \qquad \text{Eq.A.1.3}$$

From which

$$\theta_s''' = \frac{T_S''}{GV_s} \qquad \text{– (iii)}$$

Substitute this in eq. (ii) above to give

$$T_S = - EJ_w \left(\theta''' - \frac{T_S''}{GV_s} \right) \qquad \text{– (iv)}$$

The total torque is

$$T = T_S + T_T \qquad \text{Eq.A.1.4}$$

and substituting for T_S and T_T from eq. (iv) and Eq.16 above gives

$$T = - EJ_w \left(\theta''' - \frac{T_S''}{GV_s} \right) + GJ\,\theta' \qquad \text{– (v)}$$

Also from Eq.A.1.4

$$T'' = T_S'' + T_T''$$

or, again using Eq.16 above,

$$T_S'' = T'' - T_T'' = T'' - GJ\,\theta'''$$

Substituting this in eq. (v) gives

$$T - \frac{EJ_w}{GV_s} T'' = - EJ_w \left(\theta''' + \frac{GJ}{GV_s} \theta''' \right) + GJ\,\theta' \qquad \text{Eq.A.1.5}$$

where $T = T\ (z)$ is the torque at any section of the beam.

For a uniform torque,

$$T = GJ\,\theta' - EJ_w\,\theta''' \left(1 + \frac{J}{V_s}\right) \qquad \text{Eq.A.1.6}$$

Let

$$x = \frac{1}{1 + {}^{J}/_{V_s}} \qquad \text{Eq.A.1.7}$$

and then the differential equation may be written

$$\theta''' - \frac{k_1^2}{(L/_2)^2}\,\theta' + x\,\frac{T}{EJ_w} = 0 \qquad \text{Eq.A.1.8}$$

where

$$k_1^2 = x\,k^2 \qquad \text{Eq.A.1.9}$$

and

$$k = \frac{L}{2}\sqrt{\frac{GJ}{EJ_w}} \qquad \text{(Eq.1.2)}$$

as before.

It will be shown later that V_s is not constant along the length of the beam (see also eq.70 of Beskin [1] where $J_b = V_s$) which means Eqs.A.1.5 and A.1.6 are very complex and therefore not of the same type as Eqs.19 and 21 which they replace. However, in order to solve them in a similar manner, assume V_s and hence x are constants.

The general solution of Eq.A.1.8 is then, (from von Karman and Christensen [1], for example),

$$\theta' = C_1 \sinh \frac{2\,k_1\,z}{L} + C_2 \cosh \frac{2\,k_1\,z}{L} + \frac{T}{GJ} \qquad \text{- (vi)}$$

The constants may be evaluated from known end conditions.

For example, for the case of ends prevented from warping, at $z = 0$ at the centre of the span, from the symmetry of the system, $\theta'' = 0$; but from (vi) above,

$$0 = \theta'' = C_1 \frac{2\,k_1}{L} \cosh \frac{2\,k_1\,z}{L} + C_2 \frac{2\,k_1}{L} \sinh \frac{2\,k_1\,z}{L}$$

$$= C_1 \frac{2\,k_1}{L} \text{ (with } z = 0\text{)}$$

$$\therefore \quad C_1 = 0$$

Then, at the fixed ends $z = \pm \frac{L}{2}$, from Eq.A.1.3,

$$\theta_o' = (\theta_s')_o = \frac{T_s}{GV_s} \qquad \text{Eq.A.1.10}$$

$$= \frac{T - T_T}{GV_s} \qquad \text{(from Eq.A.1.4)}$$

$$= \frac{T - GJ\,\theta_o'}{GV_s} \qquad \text{(from Eq.16)}$$

$$\theta_o' \left(1 + \frac{GJ}{GV_s}\right) = \frac{T}{GV_s}$$

or

$$\theta_o' = \frac{x\,T}{GV_s} \qquad \text{Eq.A.1.11}$$

using Eq.A.1.7.

Substituting this in eq. (vi) above, when $z = \frac{L}{2}$

$$\theta_o' = \frac{x\,T}{GV_s} = C_2 \cosh k_1 + \frac{T}{GJ}$$

$$\therefore \quad C_2 = -\frac{T}{GJ} \left(1 - x \frac{J}{V_s}\right) / \cosh k_1$$

$$= -\frac{x\,T}{GJ \cosh k_1}$$

Then eq. (vi) becomes, using this value of C_2 and $C_1 = 0$,

$$\theta' = \frac{T}{GJ} \left(1 - \frac{x \cosh 2\,k_1\,z/L}{\cosh k_1}\right) \qquad \text{Eq.A.1.12}$$

The angle of twist at distance z from the centre section is then

$$\theta = \frac{TL}{GJ} \left(\frac{z}{L} - \frac{x \sinh 2\,k_1\,z/L}{2\,k_1 \cosh k_1}\right) \qquad \text{Eq.A.1.13}$$

and the total angle of twist is twice that at $z = \frac{L}{2}$, that is

$$\phi = \frac{TL}{GJ} \left(1 - \frac{x}{k_1} \tanh k_1\right) \qquad \text{Eq.A.1.14}$$

The last three equations may be compared with the classic equations which have no account for shear strain; for example Eq.1.1. It is seen that k has become k_1 (or $k\sqrt{x}$), and the section constant x appears also as a multiplier of the hyperbolic term.

Relative to the angle of twist of a closed tube of the same material and length and subject to the same torque, Eq.A.1.14 is

$$\frac{\phi}{\phi_o} = \frac{J_o}{J} \left(1 - \frac{x}{k_1} \tanh k_1\right) \qquad \text{Eq.A.1.15}$$

Subtract from this Eq.1.10, which is $\frac{\phi_{BT}}{\phi_o}$, to give the increase in angle of twist due to shear compared with the Vlasov-Timoshenko solution,

$$\frac{\phi_s}{\phi_o} = \frac{J_o}{J}\left(\frac{\tanh k}{k} - x\,\frac{\tanh k_1}{k_1}\right) \qquad \text{Eq.A.1.16}$$

When k is small, say $< .3$, this may be written

$$\frac{\phi_s}{\phi_o} = \frac{J_o}{J}\left[(1 - \frac{k^2}{3}) - x\,(1 - \frac{k^2 x}{3})\right]$$

or, since $\frac{J_o}{J}(1 - x) = x\,\frac{J_o}{V_s}$,

$$\frac{\phi_s}{\phi_o} = x\,\frac{J_o}{V_s}\left[1 - \frac{k^2}{3}\,(1 + x)\right] \qquad \text{Eq.A.1.17}$$

This equation may be compared with the first term of the energy solution Eq.5.17 which gives, (for small values of k again),

$$\frac{\phi_s}{\phi_o} = \frac{J_o}{J_s}\left(1 - \frac{2}{3}\,k^2\right)$$

It will be shown below that as $k \to 0$, $V_s \to J_s$, hence for the slit tube, from Eq.A.1.6 and Fig. 5.1, $x \to 1$. Thus it is seen that the last two equations above are equivalent. For more open sections x is rather less than unity so that Eq.A.1.15 gives a proportionately lower value of the shear deflection than the energy solution of chapter 5.

Fig. 5.4 gives some numerical results for Eq.A.1.15, with $V_s = J_s$.

Evaluation of the cross-section shear modulus V_s

The energy methods of chapter 5 indicate the way in which the shear rigidity GV_s, which is defined by

$$V_s = \frac{T_s}{G\,\theta_s'} \qquad \text{(Eq.A.1.3)}$$

may be evaluated. See also Heskins eq.70 [4].

The "shear" work done by a torque T acting over an elemental length dz is

$$\tfrac{1}{2}\, T \quad d\,\theta_s \qquad \text{- (i)}$$

where $d\,\theta_s$ is the angle of twist resulting solely from the shear strain γ, and this work may be equated to the internal strain energy of direct shear which is

$$\int_A \frac{\tau^2}{2G} \, dA \quad dz \qquad \text{- (ii)}$$

where τ is the shear stress associated with γ.

Combining eqs. (i) and (ii),

$$\theta_s' = \frac{d\,\theta_s}{dz} = \frac{1}{TG}\int_A \tau^2 \, dA \qquad \text{- Eq.A.1.18}$$

Substitute for θ_s' from Eq.A.1.3

$$\frac{T_s}{GV_s} = \frac{1}{TG}\int_A \tau^2 \, dA$$

from which

$$V_s = \frac{TT_s}{\int_A \tau^2 \, dA} \qquad \text{- Eq.A.1.19}$$

* p. A8 deleted

The real values of T_s and τ are interdependent with V_s, but an approximate value for V_s can be obtained by using the classic relations between T_s and τ. Thus from Eqs. 13 and 18 in the introduction,

$$\tau = -\frac{T_s S_w}{J_w} \qquad \text{- Eq.A.1.20}$$

Substituting this in Eq.A.1.19 gives

$$V_s = \frac{T}{T_s} J_s \qquad \text{- Eq.A.1.21}$$

where

$$J_s = \frac{J_w^2}{S_{ww}} \qquad \text{(Eq.5.3)}$$

J_s is the (constant) shear modulus when the pure torsion is ignored, (see chapter 5).

Eq.A.1.21 is consistent with the result of chapter 5 in so far as $T_s \rightarrow T$ when $k \rightarrow 0$.

However it is seen that in general V_s is not constant along the length of the beam and hence Eqs.A.1.12 to A.1.14 are valid solutions only for the limiting case of $k \rightarrow 0$ when $V_s \rightarrow J_s$. Nevertheless it is worthwhile comparing these solutions with the others because the overall influence of the shear decreases with increasing span.

Substituting from Eq.A.1.21 in Eq.A.1.3 gives, for the rate of twist due to direct shear,

$$\theta_s' = \frac{T_s^2 / T}{GJ_s} \qquad \text{- Eq.A.1.22}$$

This means that θ_s' varies along the length of the beam according to $(T_s)^2$, approaching the constant value of $\frac{T}{GJ_s}$ for short beams when $k \to 0$, and approaching zero for long beams.

The shear stress τ may be substituted back for T_s from Eq.16, so that Eq.A.1.22 becomes

$$\theta_s' = \frac{\tau^2}{TG} \frac{S_{ww}}{S_w^{\,2}} \qquad \text{— Eq.A.1.23}$$

Integrating this expression over a certain length will give the total angle of twist due to direct shear over this length, thus,

$$\theta_s = \frac{S_{ww}}{G\,S_w^{\,2}} \int_0^z \frac{\tau^2}{T}\,dz \qquad \text{— Eq.A.1.24}$$

For example, for the case of a uniform torque T acting on a beam with ends constrained from warping, the shear stress is

$$\tau = \frac{T}{J_w} S_w \frac{\cosh(2\,kz/L)}{\cosh k} \qquad \text{(Eq.2.1)}$$

Substituting this in Eq.A.1.24 and integrating gives (using Eq.5.3)

$$\theta_s = \frac{T}{GJ_s} \left[\frac{z + \frac{L}{4k} \sinh \frac{4\,kz}{L}}{2\cosh^2 k} \right]$$

which is Eq.5.4 obtained directly from the energy of the shear stresses.

Remark

Similar results to all the above are obtainable by relating the angle of twist associated with the axial shear strain to the total torque (like the St. Venant relation) viz.

$$\theta_s' = \frac{T}{GV_s}$$

rather than to T_s (Eq. A.1.3)

This gives

$$\theta_s' = \frac{T_s^{\,2}/T}{GV_s}$$

so again $V_s \rightarrow J_s$ when $k \rightarrow 0$, and with this new definition for the direct shear rigidity the differential equation for angle of twist (Eq.A.1.5) becomes $T - \frac{EJ_w}{GV_s} T'' = GJ\theta' - EJ_w\theta'''$

This means the case of the uniform torque is unchanged from the classic Eq.21.

Eq.A.1.14 for the angle of twist of a beam with constrained ends is replaced by

$$\phi = \frac{TL}{GJ}\left(1 - \frac{x_1}{k}\tanh k\right)$$

where

$$x_1 = 1 - \frac{J}{V_s}$$

<u>Average</u> V_s

It may be of interest to obtain an "average" value of V_s for the case of both ends remaining plane and a uniform torque defined as follows,

$$\phi_s = \frac{1}{G(V_s)_{av}} \int_{-\frac{L}{2}}^{\frac{L}{2}} T_s \, dz \qquad \text{(from Eq.A.1.3)}$$

which, substituting for T from Eq.A.1.20, then substituting for τ from Eq.2.1 and integrating gives,

$$\phi_s = \frac{TL}{G(V_s)_{av}} \quad \frac{\tanh k}{k} \qquad \text{Eq.A.1.25}$$

But

$$\phi_s = k_s \frac{TL}{GJ_s} \qquad \text{(first term of Eq.5.7)}$$

Hence

$$(V_s)_{av} = J_s \frac{\tanh k}{k} \quad \frac{1}{k'_s}$$

$$= J_s \frac{1}{k} \left(\frac{\sinh 2k}{\cosh 2k + 1}\right) \left(\frac{\cosh 2k + 1}{\frac{1}{2k} \sinh 2k + 1}\right)$$

$$(V_s)_{av} = \frac{J_s}{\frac{1}{2} + \frac{k}{\sinh 2k}} \qquad \text{Eq.A.1.26}$$

when $k \to 0$, $(V_s)_{av} \to J_s$,

and when $k \to \infty$, $(V_s)_{av} \to 2 J_s$

Values of J_s for the slit-tube and semi-circle are given in Fig. 5.1.

L. N. VOROBEV [1] (in Russian)

In 1955 in a Russian publication this author uses Castigliano's theorem of least energy and includes the axial shear energy in his expression for total energy which otherwise is taken from Vlasov's general expressions that are for the general loading case of a transverse load which causes ordinary bending as well as non-uniform torsion.

Vorobev's solution [2] is Eq. A.1.5 with $V_s = J_s$; and his end condition for rate of twist for a built-in beam is Eq. A.1.11.

The assumptions involved in these solutions have been discussed above. It has been shown too that it gives results close to that of the direct shear energy method of chapter 5 in spite of its limitations.

It is interesting to note that Vorobev points out that if vertical shear in bending and axial shear in non-uniform torsion are simultaneously important, then they become interdependent such that a simple superposition of the separate parts of the combined loading is not tenable.

E. REISSNER [1] [7]

In the early 1950's E. Reissner carried out extensive studies on a generalised variational stress analysis procedure which he applied to the general case of non-uniform torsion of bars.

In the 1952 Journal of Mathematics and Physics Reissner [1] confirms in general terms, for any doubly symmetric cross-section bar, respectively, the three different solutions following: (depending on the assumptions made regarding the variables affecting longitudinal displacement u or stress σ),

(a) the classic solution

(b) Eq.4.36 for the rate of twist at the fixed end, if the centre of twist is the centre of gravity of the section, (Eq.A.1.27 following), and

(c) the "Beskin-Vorobev" equation (Eq.A.1.5) and the same end condition for rate of twist as Eq.A.1.10.

These three solutions have already been discussed, but it is still worthwhile considering Reissner's methods and comments.

(a) The first result of Reissner [3] comes from the assumption

$$u = \theta'(z) \quad \Phi \quad (x, y)$$

where θ' is variable with z, and Φ is St. Venant's warping function or an approximation to it, x and y being the coordinates of the cross-section.

The condition for u = 0 at a fixed end automatically gives (as explained by Reissner)

$$\theta_0' = 0$$

from which follow the classic solutions.

It is also noted that for thin-walled open sections

$$\Phi \equiv w$$

(b) In the second method of Reissner [4], the longitudinal displacement u is no longer assumed to be proportional to the rate of twist but to some other function of the length, say $\alpha(z)$ so that

$$u = \alpha(z)\ \Phi\ (x,y)$$

This gives, via the variational equation,

$$\theta_o' = \frac{T}{GJ_p} \qquad \text{Eq.A.1.27}$$

where J_p is the polar moment of inertia; and for the differential equation for the angle of twist,

$$GJ\theta' - x_1 E\Gamma\theta''' = T \qquad \text{Eq.A.1.28}$$

where

$$x_1 = \frac{1}{1 - \frac{J}{J_p}} \qquad \text{Eq.A.1.29}$$

(obtained directly from a later publication of Reissner [6].

Γ is the warping constant which for thin-walled open-sections is equivalent to J_w.

Eq.4.36 is similar with the expression above for rate of twist (θ_o') in spite of the fact that in the analysis of chapter 4 u is taken as proportional to θ' . The reason for this is that the determinations of the end condition for rate of twist in chapter 4 are carried out independently of the general analyses of chapter 4, whereas in the variational approach all quantitites are contained in a single expression for energy so that all the quantities must naturally be compatible with each other (within the li its of the assumptions made).

In fact, making the general assumption

$$u = \alpha(z)\ \beta(s)$$

and following the same procedure as in part (a) of chapter 4 gives the same result again, as follows:

Substituting $u = \alpha(z)\ \beta(s)$ into Eq.4.8 which is

$$\tau = G\left(m\,\theta' + \frac{\partial u}{\partial s}\right)$$

gives

$$\tau = G\left(m\,\theta' + \alpha\frac{\partial \beta}{\partial s}\right)$$

And substituting this into Eq.4.10 which is (for a circular arc section),

$$T = \int_{-s_0}^{s_0} \tau\, t\, m\, ds$$

gives

$$T = G\,t\,a\left[\theta'\int_{-\varphi_0}^{\varphi_0} m^2\, d\varphi + \alpha\int_{-\varphi_0}^{\varphi_0} m\,\frac{\partial\beta}{\partial\varphi}\, d\varphi\right]$$

At the fixed end, for $u = 0$, $\alpha = 0$, and assuming here a rotation about the centre of gravity so that the moment arm m becomes p, say

$$T = G\,t\,a\,\theta_0'\int_{-\varphi_0}^{\varphi_0} p^2\, d\varphi$$

or

$$\theta_0' = \frac{T}{GJ_p} \qquad \text{(Eq.A.1.27)}$$

Finally it is noted that the correction factor x, (Eq.A1.29 above) is virtually unity for thin-walled open-sections so that Reissner's second method gives effectively the classic differential equation for thin-walled open sections (Eq.21) plus the end condition of Eq.A.1.1.

(c) Reissner's third method [5] starts with a substitution for the longitudinal stress

$$\sigma = \sigma(z)\,\Phi(x,y)$$

in the variational equation, and for the end condition puts

$$\sigma(0) = 0$$

which would occur at the centre of the span for the beam constrained at both ends and subject to an end torque.

The solution puts in general form for any cross-section subject to an end torque what Vorobev [1] obtains for a thin-walled section subject to arbitrary loading.

The rate of twist at the fixed end is equivalent to

$$\theta_o' = \times \frac{T}{GJ_s} \qquad \text{(Eq.A.1.10)}$$

where the definition for J_s as given by Eq.5.3 is seen to be the particularisation for thin walled open-sections of Reissner's expression (quote)

$$\frac{D(\varphi)}{\Gamma^2} = \frac{\int_A \Phi\,\varphi\,dA}{\left[\int_A \Phi^2\,dA\right]^2} \qquad \text{(i)}$$

where

$$\nabla^2\varphi = \Phi$$

and the differential equation is given as

$$GJ\theta' - E\Gamma\theta'''(1 + \frac{J}{J_s}) = T \qquad \text{(Eq.A.1.6)}$$

wherein

$$J_s \equiv \frac{D(\varphi)}{\Gamma^2} \qquad \text{(see eq. (i) above)}$$

Reissner feels that this solution may be a better one than the former ones, although in an example given some years later [6] he uses Eqs.A.1.27 and A.1.28 and not Eq.A.1.6 above. The greater complication of the last method is presumably not worthwhile for symmetrical closed sections. As already explained these two results give quite different answers for the shear deflection in thin-walled open sections so that experimental work should readily determine the more accurate solution provided that it can be ensured that non-rigid end effects and the position of the centre of twist do not obscure the results.

K. MIZOGUCHI AND H. SHIOTA [1]

In a Japanese journal of 1965 these authors apply a cylindrical shell theory of the former to the case of the torsion of an open cylindrical arc section restrained from warping.

The basic differential equation is a typical shell equation being of 8th order in the radial displacement and its solution is rather involved.

(a) Total Angle of Twist

In a test the authors [2] subject a fixed-ended slit tube $\frac{L}{D} = 3$, $\frac{D}{t} = 50$, $\frac{G}{E} = .384$ to a uniform torque and find that the angle of twist is 1.88 to 1.89 times that of the equivalent closed tube, which is over 25% more than the factor of 1.50 which their theory predicts; (these results have been taken from the above authors' Fig. 5).

For comparison, the solution from this thesis is

$$\frac{\phi}{\phi_0} = .96 \quad (1 - \frac{2}{3} k^2) \; + \; .26 \frac{G}{E} \; (\frac{L}{D})^2 \quad (1 - \frac{2}{5} k^2) \quad \text{(Eq.5.19)}$$

$= 1.85$

It seems surprising that the theory which takes into account the radial deflections should give a lower answer for angle of twist than one that does not. The reason may be that in the Mizoguchi-Shiota analysis [3] the tube is assumed to rotate about its physical axis rather than the shear centre, as a short or medium length shell might be constrained to do.

The general consequences of a rotation about any axis other than the shear centre are discussed in chapter A.6. The effect on the theoretical angle of twist in the example above is given by Eq. A.6.13 (approximately),

$$\frac{\phi}{\phi_0} = 1.2 + .1 \frac{G}{E} \quad (\frac{L}{D})^2$$

which substituting $\frac{G}{E} = .384$, and $\frac{L}{D} = 3$, is

$$\frac{\phi}{\phi_0} = 1.55$$

This is close to the theoretical solution of Mizoguchi and Shiota.

Nevertheless no definite conclusions can be reached here because of the uncertainty of the end effects in the experiment and the lack of a measured stress distribution.

(b) Rate of Twist at Fixed End

The shell theory of Mizoguchi and Shiota [4] allows a differing rotation for each point on a cross-section; however their solution for the mean value of the rate of rotation at the fixed end of the slit tube in the example above is precisely,

$$\theta_o' = \frac{T}{GJ_o}$$

confirming Eq.5.5 for slit tubes.

It is interesting to note here that Von Karman and Wei Zang Chien [1] obtain a similar result for closed sections.

(c) Stresses

Apparently no stresses are measured by the above authors but they do give their theoretical stress distribution for the slit tube (Mizoguchi and Shiota [4]). It again seems that they have considered the tube to rotate about its physical centre because their Fig. 8 shows the longitudinal normal stress σ to be of the same sign from the slit to a point opposite the slit, instead of changing signs at 70° from the slit too - see Figs. 2.1. and A.6.1.

V.S. MESCHERYAKOV [1]

In a Russian publication of 1965 Mescheryakov uses a method of Goldenveyzer which follows the Vlasov procedure for the general loading case of bending and twist of thin-walled open-section beams but, in order that the shear may be included, he does not take into account the pure torsion.

For the simple loading case of a uniform torque on a beam with both ends restrained from warping Mescheryakov [2] obtains a solution which is Eq.5.13 in a different form. However it is not explained in the article that this result is applicable only to shorter length shells or beams.

Solutions from other Russian sources are quoted by Mescheryakov [3] and one of interest is a torsion test carried out by Glasunova on a built-in slit tube 30 cm long, mean diameter 4.797 cms, and wall thickness .295 cms.; ($\frac{L}{D}$ = 6.25, $\frac{D}{t}$ = 16.28). The experimental result is 8% greater than the Mescheryakov solution (i.e. Eq.5.13).

H.L. ENGEL and J.N. GOODIER [1]

In the December 1953 Journal of Applied Mechanics combined bending and torsion tests carried out on thin-walled semi-circular section beams with constrained ends are described. The material used is not specified but is probably aluminium alloy.

The section dimensions were D = 1.982" and t = .050" giving

$\frac{D}{t} = 39.6$

The five lengths tested were 16.12", 28.50", 41.06", 53.25", 65.75", and the last four gave good agreement with a theory based on Eq.1.1 which makes no allowance for shear strain. The first test however with

$$\frac{L}{D} = 8.14$$

showed an angle of twist about 6% greater than the theory. This could have been due partly to lack of complete end restraint as suggested by the above authors but could be partly due to the shear which according to the methods of chapter 5 herein would be

$$\frac{\phi_s}{\phi_{BT}} = \frac{K_s}{K_{BT}} \cdot \frac{J}{J_s} \qquad \text{(Eq.5.14)}$$

$$= \frac{.438}{.355} \cdot \frac{.525}{.0128} \left(\frac{1}{39.6}\right)^2 = 3.2\%$$

A.2 A FIRST APPROXIMATION TO THE SHEAR DEFLECTION IN A SLIT TUBE FROM THE AVERAGE CIRCUMFERENTIAL SHEAR STRESS.

In this chapter it will be shown that the average shear stress around the circumference in a slit-tube as determined from the Vlasov result, Eq.2.1, is close to the shear stress in a closed tube for spans up to L = 10D, so that the direct shear deflection in both tubes is likely to be similar.

The mean value of Eq.2.1 for the whole tube is given by the double definite integral,

$$\tau_{av} = \frac{1}{L} \cdot \frac{1}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\pi}^{\pi} T \; \frac{S_w}{J_w} \; \frac{\cosh 2\,kz/L}{\cosh k} \; d\varphi \, dz$$

$$= \frac{T}{2\pi\, LJ_w \cosh k} \int_{-\pi}^{\pi} S_w \; d\varphi \int_{-\frac{L}{2}}^{\frac{L}{2}} \cosh 2\,kz/L \; dz$$

where

$$S_w = \frac{D^3}{16} \left(5.87 - \varphi^2 - 4 \cos \varphi\right) \qquad \text{(Eq.2.2)}$$

Integrating and substituting for J_w from Eq.1.4,

$$\tau_{av} = \frac{T}{2\pi L \times .253\, tD^5 \times \cosh k} \left(11.74\,\pi - \frac{2\pi^3}{3}\right) 2 \sinh k$$

$$\tau_{av} = .64 \frac{T}{tD^2} \frac{\tanh k}{k} \qquad \text{- Eq.A.2.1}$$

The shear stress in a thin-walled closed tube is

$$\tau_o = \frac{TD}{2J_o} \qquad \text{- Eq.A.2.2}$$

Substituting for J_o for a thin tube (Eq.1.9) gives

$$\tau_o = \frac{2T}{\pi D^2 t} = .636 \frac{T}{t D^2} \qquad \text{- Eq.A.2.3}$$

The average shear stress in the slit tube relative to the shear stress in the closed tube is therefore Eq.A.2.1 divided by Eq.A.2.3, approximately

$$\frac{\tanh k}{k}$$

Some numerical values of $\frac{\tanh k}{k}$ are, for the limiting case of D = 10 t, with L = D, 3D, 6D, and 10D respectively:- 1.0, .99, .96 and .90. For thinner tubes the values are still closer to unity.

Therefore the average circumferential shear deflection and hence the angle of twist due to this deflection may be expected to be similar, for the closed tube and the slit tube.

Hence as a first approximation it would seem that the angle of twist of a closed tube - Eq.1.8 - could be added to the classic equation for long beams - Eq.1.1 (a) - to give a solution for short and medium length slit tubes with constrained ends.

This means adding unity to Eq.1.11 (from which Fig. 1.1 was obtained) to give

$$\frac{\phi}{\phi_o} = \frac{3}{4}\left(\frac{D}{t}\right)^2\left(1 - \frac{\tanh k}{k}\right) + 1 \qquad \text{Eq.A.2.5}$$

To a similar order of accuracy for short slit tubes where $\frac{D}{t} \gg \frac{L}{D}$,

$$\frac{\tanh k}{k} \simeq 1 - \frac{k^2}{3} \qquad \text{(from Eq.1.12)}$$

Substituting for k from Eq.1.5.

$$\frac{\tanh k}{k} \simeq 1 - \frac{(1.017)^2}{3}\left(\frac{L}{D}\right)^2\left(\frac{t}{D}\right)^2\frac{G}{E}$$

Eq.A.2.5 then becomes

$$\frac{\phi}{\phi_o} = \frac{3}{4}\left(\frac{D}{t}\right)^2\frac{1.034}{3}\left(\frac{L}{D}\right)^2\left(\frac{t}{D}\right)^2\frac{G}{E} + 1$$

$$\frac{\phi}{\phi_o} = 1 + .26\,\frac{G}{E}\left(\frac{L}{D}\right)^2 \qquad \text{Eq.A.2.6}$$

which is

$$\frac{\phi}{\phi_o} = 1 + .1\left(\frac{L}{D}\right)^2 \qquad \text{Eq.A.2.7}$$

if $\frac{G}{E}$ is again given the typical value of .39.

Numerical solutions of this equation for $\frac{L}{D}$ ratios of 0, 1, 3, 6, 10 are, respectively, 1, 1.1, 1.9, 4.6, and 11.0. Comparison of these results with Fig. 1.1 show an increase in $\frac{\phi}{\phi_o}$ of the order unity for the short or very thin-walled tubes.

These semi-intuitive results have been included to indicate the reasoning which prompted the author to proceed with the analysis of short open-section beams (and short slit tubes in particular) especially as Eq.A.2.6 gave the correct result of an experiment carried out by Mizoguchi and Shiota [2] with a slit metal tube $\frac{L}{D} = 3$, $\frac{D}{t} = 50$.

(This reference was the only one at the time found to contain results of tests of the twisting of a fixed ended slit tube).

A.3 LIMITS OF ACCURACY OF THEORETICAL FORMULAS AND OF EXPERIMENTAL RESULTS

The formulas chosen to check were, for the angle of twist for any length,

$$\phi = K_s \frac{TL}{GJ_s} + K_{BT} \frac{TL}{GJ} \qquad \text{(Eq.5.7)}$$

and for the stress distribution for any length,

$$\tau = T \frac{S_w}{J_w} \frac{\cosh 2\,kz/L}{\cosh k} \qquad \text{(Eq.2.1)}$$

$$\sigma = TL \frac{w}{J_w} \frac{\sinh 2\,kz/L}{2k \cosh k} \qquad \text{(Eq.2.3)}$$

These formulas are for a uniform torque loading, and for both ends constrained to remain plane. The first differs from the (long beam) classic solution by the addition of the first term on the right hand side which is intended to take account of the axial shear strain so correcting the classic solution and validating it for any span provided the line of shear centres remains the centre of twist. The second two formulas are the (Long beam) classic formulas which according to thesis proposals are valid also for short spans, at least for the slit tube.

A thin-walled round tube fitted to rigid end plates and slit full length through the wall was the particular open-section chosen to test and the reasons for this were:

1. a direct comparison with the same tube prior to slitting it could be observed;

2. tests on part-slit tubes could also be held on the same basic specimen.

Such comparative tests enable a satisfactory accuracy of theoretical calculations of angle of twist and stress to be obtained from easily measured specimen dimensions; in particular decreasing the dependence of angle of twist calculations on diameter from D^5 to D^2. Thus, if Eq.5.7 above is for a circular arc section with median line radius $a = D/2$, then relative to the angle of twist of a round closed tube of median line diameter D, same span L, same (uniform) thickness t and subject to the same torque T, Eq.5.7 becomes,

$$\frac{\phi}{\phi_o} = K_s \frac{J_o}{J_s} + K_{BT} \frac{J_o}{J} \qquad \text{(Eq.5.17)}$$

or, for the slit tube,

$$\frac{\phi}{\phi_o} = .96\ K_s + .75\ \left(\frac{D}{t}\right)^2\ K_{BT} \qquad \text{(Eq.5.18)}$$

When $\frac{L}{D} < .3\ \frac{D}{t} \sqrt{\frac{E}{G}}$ for the slit tube the pure torsion is negligible and the dependence on t is virtually eliminated so that Eq.5.18 may be written

$$\frac{\phi}{\phi_o} = .96\ (1 - \frac{2}{3} k^2) + .26\ \frac{G}{E} \left(\frac{L}{D}\right)^2 \ (1 - \frac{2}{5}\ k^2) \qquad \text{(Eq.5.19)}$$

See Fig. 5.5

$\frac{G}{E}$ can be reduced to the single elastic constant ν (Poisson's Ratio) from the relation

$$\frac{G}{E} = \frac{1}{2\ (1 + \nu\ )} \qquad \text{Eq.A3.1}$$

Substituting this in Eq.5.19 above gives

$$\frac{\phi}{\phi_o} = .96\left(1 - \frac{2}{3}k^2\right) + \frac{.13}{1+\nu}\left(\frac{L}{D}\right)^2\left(1 - \frac{2}{5}k^2\right) \qquad \text{Eq.A3.2}$$

or, with $\nu = .29$

$$\frac{\phi}{\phi_o} = .96\left(1 - \frac{2}{3}k^2\right) + .101\left(\frac{L}{D}\right)^2\left(1 - \frac{2}{5}k^2\right) \qquad \text{Eq.A.3.2(a)}$$

Similarly by comparing the normal and shear stresses with the stress $\tau_o = 2T/\pi D^2 t$ in a closed tube, Eqs. 2.1 and 2.3 become, for a slit tube where $\frac{L}{D} < .3\frac{D}{t}\sqrt{\frac{G}{E}}$, respectively

$$\frac{\tau}{\tau_o} = .39\left(5.87 - \varphi^2 - 4\cos\varphi\right) \qquad \text{(Eq.4.27)}$$

$$\frac{\sigma}{\tau_o} = 1.55\,\frac{z}{D}\left(\varphi - 2\sin\varphi\right) \qquad \text{(Eq.4.23)}$$

<u>Accuracy of theoretical equations using measured properties</u>:

i.e. the calculation of the right hand side of Eqs.A3.2, 4.27 and 4.23.

(a) <u>Angle of twist equation, Eq.A3.2</u>

The effect on ϕ/ϕ_o of a small error Δk in k is, from Brinkworth[1] ,

$$\Delta\left(\frac{\phi}{\phi_o}\right)_k = \frac{\partial}{\partial k}\left(\phi/\phi_o\right)\Delta k$$

$$= -\left(1.28\,k + \frac{.104\,k}{1+\nu}\left(\frac{L}{D}\right)^2\right)\Delta k$$

$$\simeq -\frac{\phi}{\phi_o}\left(\frac{1.28 + \frac{.104}{1+\nu}\left(L/D\right)^2}{.96 + \frac{.13}{1+\nu}\left(L/D\right)^2}\right) k\ \Delta k$$

for small k. Or, in the order

$$\frac{\Delta\left(\phi/\phi_o\right)_k}{\phi/\phi_o} = k^2 \left(\frac{\Delta k}{k} \right) \qquad \text{Eq.A3.3}$$

Since for the slit tube, $k = 1.017 \frac{Lt}{D^2} \sqrt{\frac{G}{E}}$ (Eq.1.5),

maximum errors in k are given by

$$\frac{\Delta k}{k} = \pm \left| \frac{\Delta L}{L} + \frac{\Delta t}{t} + 2 \frac{\Delta D}{D} + \frac{\Delta G}{2G} + \frac{\Delta E}{2E} \right| \qquad \text{Eq.A3.4}$$

so that when $k < .3$, say, satisfactory measurements of the quantities in Eq.A4.4 are easily obtained to make Eq.A3.3 negligible.

The maximum total effect of small errors in the other variables ν, L and D are similarly calculated to be

$$\frac{\Delta\left(\phi/\phi_o\right)_{\nu, L, D}}{\phi/\phi_o} \simeq \pm \frac{\left| \frac{\Delta\nu}{4\nu} + \frac{2\Delta L}{L} + \frac{2\Delta D}{D} \right|}{1 + 1/ \frac{.13}{1+\nu}\left(\frac{L}{D}\right)^2}$$

Due to the presence of the denominator on the right hand side Eq.A3.5 becomes negligible for very short beams. Otherwise all terms may be important and are discussed below.

L is measured to better than .02". The biggest maximum error for $\frac{\Delta L}{L}$ would be, for L = 10" say,

$$\frac{\Delta L}{L} = \frac{.02}{10} = .002 \equiv .2\% \qquad \text{Eq.A3.6}$$

The mean diameter D was obtained from:

outside diameter - t. The outside diameter (nominal 3.5") was measured by calibrated micrometer to an accuracy of $\pm$ .0005" for each reading, but varied .040" along and round the tube. The average (effective) outside diameter was estimated to have an accuracy

$$\frac{\Delta O.D}{O.D} = \frac{.020}{3.5} \simeq .006$$

Hence the possible error in D is about the same since $\frac{\Delta t}{D}$ is negligible. That is

$$\frac{\Delta D}{D} = .006 \qquad \text{Eq.A3.7}$$

Due to the fractional dependence of $\frac{\phi}{\phi_o}$ on ν a handbook value for poisson's ratio is acceptable (e.g. Marks [1]), Say ν = .29 $\pm$ 8%. Then

$$\frac{\Delta \nu}{\nu} = .08 \qquad \text{Eq.A3.8}$$

<u>Calculation</u> of ϕ/ϕ_o

Substitute Eqs. A3.6, A3.7 and A3.8 in Eq.A3.5 to give the estimated maximum error in the calculated value of ϕ/ϕ_o in Eq.A3.2

$$\frac{\Delta(\phi/\phi_o)}{\phi/\phi_o} = \frac{\mp(.02 + .004 + .012)}{1 + 1/\frac{.13}{1+\nu}\left(\frac{L}{D}\right)^2}$$

$$= \frac{\mp .04}{1 + 1/\frac{.13}{1+\nu}\left(\frac{L}{D}\right)^2} \qquad \text{Eq.A3.10}$$

This varies from approximately 2% for $\frac{L}{D}$ = 3 to 3.5% for $\frac{L}{D}$ = 10.

Probable maximum errors for calculated theoretical ϕ/ϕ_o are approximately 1.5% for $\frac{L}{D} = 3$ and 2.5% for $\frac{L}{D} = 10$, using rms values according to Brinkworth [2].

(b) <u>Stress equations, Eqs.4.27 and 4.23</u>

The effect of a small error in φ , on τ and δ could be large because of the rapidly varying value of the stresses with φ (see Fig. 2.1).

The calculated relative shear stress is

$$\frac{\tau}{\tau_o} = .39\ (5.87 - \varphi^2 - 4\cos\varphi) \qquad \text{(Eq.4.27)}$$

Hence

$$\frac{\Delta\left(\tau/\tau_o\right)_\varphi}{\tau/\tau_o} = \left(\frac{-2\varphi^2 + 4\varphi\sin\varphi}{5.87 - \varphi^2 - 4\cos\varphi}\right)\frac{\Delta\varphi}{\varphi} \qquad \text{Eq.A3.11}$$

which demonstrates the large possible errors near to the slit ($\varphi = \pm\ \pi$).

Similarly

$$\frac{\Delta\left(\delta/\tau_o\right)_\varphi}{\delta/\tau_o} = \left(\frac{\varphi - 2\varphi\cos\varphi}{\varphi - 2\sin\varphi}\right)\frac{\Delta\varphi}{\varphi} \qquad \text{Eq.A3.12}$$

and the calculated relative normal stress is also seen to be in large percentage error if great care is not taken in locating the strain gauges accurately on the test piece.

<u>Estimation of experimental errors involved in measuring T, ϕ, δ, τ in the author's built-up torsion testing apparatus.</u>

<u>Torque</u> (T)

The stationary end of the specimen was fixed to one end of an existing torsion machine while the other end of the specimen was 'free' to rotate in a specially made bearing. See Figs. 7.1.

The torque was applied at the bearing end, as shown, by equal dead weights applied to each ends of an arm extending from both sides of the specimen.

The friction in the bearing was measured and found to be negligible, as was the hysteresis.

The friction in the pulley was measured too and also found to be negligible.

All weights were checked and found accurate.

The error in measuring moment arms was negligible.

The maximum possible error in any particular torque reading was therefore very small; say $\pm$ 1%

In any case T does not appear in the comparative Eqs.A3.2 4.27 and 4.23 and it is possible that the degree of repeatability of measurement of T is of a higher order of accuracy than its absolute value.

Angle of Twist ϕ

The theoretical calculations showed that total angles of twist would be as low as 25 minutes of arc and generally in the order of 30 minutes.

The method chosen to measure ϕ_o and ϕ was simple. A clinometer (bubble level) calibrated in minutes of arc- and readable to $\frac{1}{4}$ minute - was placed, in turn, on knife edges welded to each end block of the specimen. The change in angle of twist for a particular change in torque was then the change in the difference of the clinometer readings.

Therefore a maximum error of approximately $\pm$ 1 minute arc could be expected for each change in angle (and a probable error of $\frac{1}{2}$ minute).

Depending on the actual consistency and linearity of the actual results, a low degree of error in the order of 1% could be expected for the measured slope of experimental T - ϕ curves.

Strains and Stresses

Paper strain gauges were used throughout. They were Shinkoh 120 Ω $0^{o} - 45^{o} - 90^{o}$ rosettes, gauge length 8 mm. The repeatability of anyone gauge would be good such that small error (say $\pm$ 1%) would result in the measurement of the relative shear stress

$$\frac{\tau}{\tau_o} = \frac{\sin 2\theta}{(\varepsilon_{45})_o} \left[\tfrac{1}{2} (\varepsilon_{45} - \varepsilon_z)^2 + \tfrac{1}{2} (\varepsilon_{45} - \varepsilon_s)^2 \right]^{\frac{1}{2}} \quad \text{(Eq.7.4)}$$

However the maximum error in the calculation of the relative normal stress from the measured strains could still be appreciable due to the involvement of the elastic constant ν or G/E .

Thus

$$\frac{\delta}{\tau_o} = \frac{\varepsilon_z + \nu \varepsilon_s}{(\varepsilon_{45})_o (1 - \nu)} \quad \text{(Eq.7.5)}$$

depends mainly on ν , as follows,

$$\Delta (\delta/\tau_o)_\nu = \frac{\partial}{\partial \nu}\left(\frac{\delta}{\tau_o}\right)\Delta\nu$$

$$= \frac{\varepsilon_s (\varepsilon_{45})_o (1-\nu) + (\varepsilon_{45})_o (\varepsilon_z + \nu\varepsilon_s)}{(\varepsilon_{45})_o^2 (1-\nu)^2} \Delta\nu$$

and therefore

$$\frac{\Delta(\delta/\tau_o)_\nu}{\delta/\tau_o} = \left(\frac{\nu}{\frac{\varepsilon_z}{\varepsilon_s} + \nu} + \frac{\nu}{1-\nu} \right) \frac{\Delta\nu}{\nu} \quad \text{Eq.A3.13}$$

The first term will generally be negligible because ξ_s will be small (theoretically zero); hence, if ν is assumed to be .29,

$$\frac{\Delta(\delta/\tau_o)_\nu}{\delta/\tau_o} = .41\ \frac{\Delta\nu}{\nu} \qquad \text{Eq.A3.14}$$

which would mean an error of 3% in the relative normal stress for an error of 8% in Poissons ratio ν .

(It is felt that measured values of ν , or $\frac{G}{E}$, for the particular material would not have an accuracy appreciably better than the 8% error quoted above).

Finally, when comparing the measured quantities with the theoretical ones, three further points must be remembered:

(a) the measured stresses were from strain gauges on the outside surface of the tube, whereas the theoretical equations are effectively for the average stresses through the wall thickness. On this account, the measured shear stress can be expected to be approximately

$$\frac{t}{D} \times 100\%$$

greater than theoretical.

(b) following the classic procedure, the theoretical analyses in the thesis have not taken into account the reduced elastic modulus $\frac{E}{1-\nu^2}$. Because of this, the measured stresses might be 9% more than the theoretical, and the measured angles of twist might be less than the theoretical by 1% for spans L = D, or 8% for spans L = 10D.

(c) All the thesis analyses on slit tubes have assumed that the slit-while changing the "closed" tube section to an "open" section - is of negligible width; in other words, the remaining section is said to be a 360° circular arc open section.

The width of actual slits in the test specimens was about one hundredth of the diameter, i.e. always less than $1\frac{1}{2}^{\circ}$ of arc.

The theoretical stress distribution for such a finite width slit is virtually the same as for the zero-width case, provided circumferential measurements are made relative to the actual edge of the slit.

But the measured angles of twist could be expected to be greater than the theoretical by amounts shown in the following table (Fig. A3.1)

ACTUAL WIDTH OF SLIT (Degrees of arc)	% Correction to ϕ (ADD)		
	L = 3D	L = 6D	L = 10D
1°	1	1.5	1.8
2°	2	3	3.6
3°	3	4.5	5.4

% CORRECTION TO THESIS ANGLE OF TWIST FORMULAS

(e.g. $\phi/\phi_o = .96 + .101\ (L/D)^2$ - Eq.5.19)

FOR FINITE SLIT WIDTHS

FIG. A3.1

The above corrections are due entirely to the decrease in the warping rigidity (J_w) with increasing width of slit.

Tests on other machines

Tubes 1 and 2 (the 3½" diameter tubes - see chapter A.5 following) were tested in the author's own set-up, and the other specimens, which were smaller and much lighter, were tested in commercial laboratory torsion machines. No stresses are recorded in the thesis from these other tests. The torque and angle of twist were measured to the same accuracies as those quoted above, but the physical dimensions and properties were generally less carefully measured than the same features for the 3½" diameter tubes.

A4. THE OLIVETTI 101 DESK COMPUTER

Slide rule accuracy was considered sufficient for all calculations in the thesis. However a desk computer was used for most repetitive calculations such as the theoretical stress equations

$$\frac{\tau}{\tau_o} = .39\ (5.87 - \varphi^2 - 4 \cos \varphi) \qquad \text{(Eq.4.27)}$$

and

$$\frac{\delta}{\tau_o} = 1.55\ \frac{z}{D}\ (\varphi - 2 \sin \varphi) \qquad \text{(Eq.4.23)}$$

and the experimental stresses calculated from the strain rosette measurements

$$\frac{\tau}{\tau_o} = \frac{\sin 2\theta}{(\varepsilon_{45})_a} \left[\tfrac{1}{2} (\varepsilon_{45} - \varepsilon_z)^2 + \tfrac{1}{2} (\varepsilon_{45} - \varepsilon_s)^2 \right]^{\frac{1}{2}}$$

(Eq.7.4)

and

$$\frac{\delta}{\tau_o} = \frac{\varepsilon_z + \nu \varepsilon_s}{(\varepsilon_{45})_o (1 - \nu)} \qquad \text{(Eq.7.5)}$$

where

$$\tan 2\theta = \frac{2\varepsilon_{45} - \varepsilon_z - \varepsilon_s}{\varepsilon_z - \varepsilon_s}$$

The two programs used on the Olivetti 101 desk computer to evaluate the above expressions are given in the following tables.

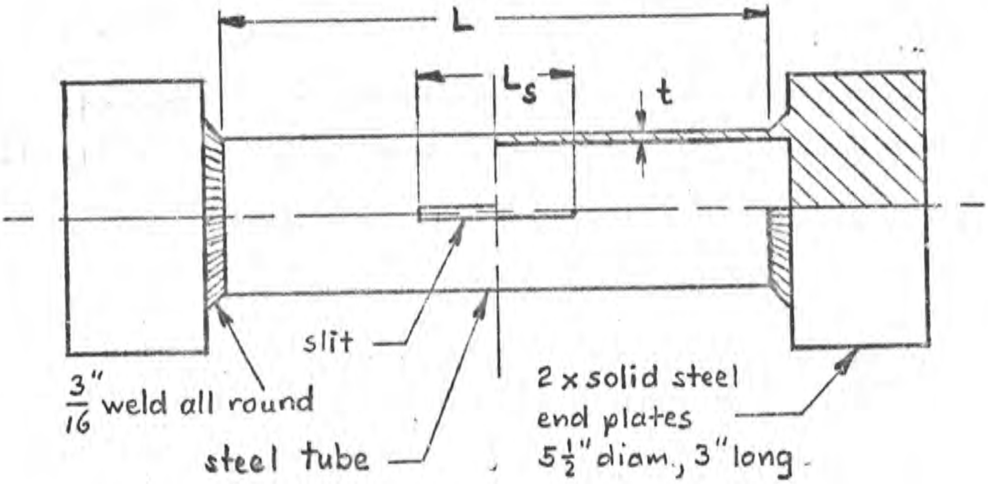
Variable to feed in	Olivetti 101 Program	Solution Printed Out
	AV	
φ	SB ↑	
$\cos\varphi$	S ↓ E X B/ ↕ B ↓ AX	
	B/ + C ↕ D/ ↓ C − DX	
	A ◇	τ/τ_o (Eq.4.27)
$\sin\varphi$	S ↓ F X C/ ↕ B ↓	
	C/− E/X F/ ↕	
	W	
	AWF/ ↓	
z	SX	
	A ◇	δ/τ_o (Eq.4.23)
	W	
Constants	.39(D), 5.87(D/), 4(E), $\frac{1.55}{D}$ (E/), 2(F)	

Variable to feed in	Olivetti 101 Program	Solution Printed Out
	AW	
ε_z	SB ↑	
ε_s	SB/ ↑ B ↓ B/− C/ ↕	
ε_{45}	SD ↑ D ↓ D/X B−B/−C/÷	
	A ◇	tan 2θ (Eq.7.2)
	Y	
	A Y D ↓ B/− A X C ↕ D ↓	
	B−A X C + D/÷ A √	
sin 2θ	S X E ÷	
	A ◇	τ / τ_0 (Eq.7.4)
	V	
	A V E/ ↓ B/ X B + F ÷	
	A ◇	δ / τ_0 (Eq.7.5)
	W	
Constants	2 (D/), $(\varepsilon_{45})_o$ (E), ν (E/), $(1-\nu)(\varepsilon_{45})_o$ (F)	

A5. DETAILS OF THE TEST SPECIMENS & EXPERIMENT RESULTS

SPECIMEN NO.	DESCRIPTION OF SPECIMEN	RESULTS	FIG. NO.
	TUBE 1. Mild steel tube with massive steel end plates L 32.4", OD $3\frac{1}{2}$", t $\frac{1}{10}$"	Dimensions & Properties	A5.1 (See also 7.1)
1.1	Original unslit tube	Strains Stress Twist	A5.2, 6(a), 16 A5.16 A5.11(a), 16
1.2	Length of slit (L_s) = 3.41"	Strains Twist	A5.6(b) A5.11(a), 17(a)
1.3	L_s 5.10"	Twist	A5.11(a), 17(a)
1.4	L_s 6.82"	Strains Twist	A5.6(b) A5.11(a), 17(a)
1.5	L_s 8.50"	Twist	A5.11(a), 17(a)
1.6	L_s 10.23"	Strains Twist	A5.6(c) A5.11(b), 17(b)
1.7	L_s 11.93"	Strains Twist	A5.6(c) A5.11(b), 17(b)
1.8	L_s 13.64"	Strains Twist	A5.6(d) A5.11(b), 17(b)
1.9	L_s 15.34	Strains Twist	A5.6(d) A5.11(b), 17(b)
1.10	L_s 17.05	Twist	A5.11(b), 17(b)
1.11	L_s 18.75	Strains Twist	A5.6(e) A5.11(c), 17(c)
1.12	L_s 20.46"	Twist	A5.11(c), 17(c)
1.13	L_s 22.16"	Strains Twist	A5.6(e) A5.11(c), 17(c)
1.14	L_s 23.87"	Twist	A5.11(c), 17(c)
1.15	L_s 25.57"	Strains Twist	A5.6(f) A5.11(d), 17(d)
1.16	L_s 27.28"	Strains Twist	A5.6(f) A5.11(d), 17(d)
1.17	L_s 28.98"	Twist	A5.11(d), 17(d)
1.18	L_s 30.69	Twist	A5.11(d), 17(d)
1.19	Full length slit $L_s = L = 32.4$"	Strains Stresses Twist	A5.6(g), 8 A5.10 A5.11(d), 17(d)
1.1 to 1.19	Further derived results	Strains Stresses Twist	8.3 8.1(a), 4 8.2, 5, 6

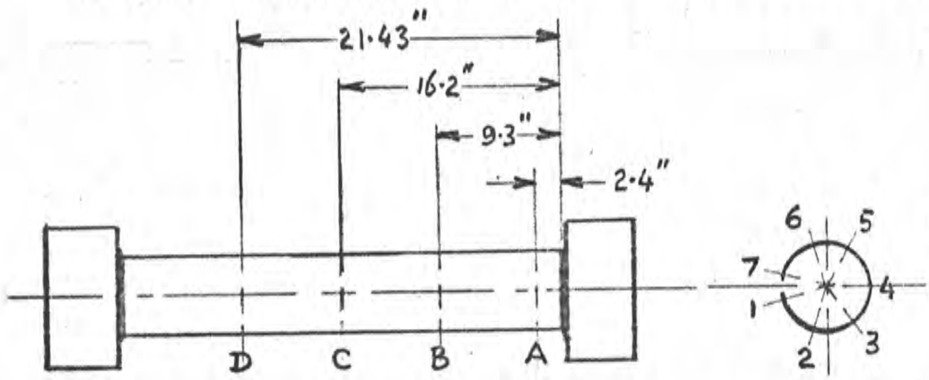
SPECIMEN NO.	DESCRIPTION OF THE SPECIMEN	RESULTS	FIG. NO.
	TUBE 2. Tube I cut back, same end plates, same strain gauges Full length slit $L_s = L = 22{\cdot}25''$		See Tube 1
2.1	End torque applied about tube centre	Strains Stresses Twist	A5.7, 9 A5.10 A5.12
2.2	End torque applied about shear centre	Twist	A5.12
2.1 & 2.2	Further derived results		8.1(b), 2
	TUBE 4. Mild steel tube, with medium size steel end plates L $7{\cdot}90''$ OD $1\frac{3}{4}''$ t $\frac{3}{64}''$	Dimensions & Properties	A5.3 (See also 7.2)
4.1	Original unslit tube	Twist	A5.13(a), 19(a)
4.2	L_s $0{\cdot}84''$	Twist	A5.13(a)
4.3	L_s $1{\cdot}60''$	Twist	A5.13(a), 19(a)
4.4	L_s $2{\cdot}50''$	Twist	A5.13(b), 19(a)
4.5	L_s $3{\cdot}00''$	Twist	A5.13(b), 19(a)
4.6	L_s $3{\cdot}50''$	Twist	A5.13(b), 19(a)
4.7	L_s $4{\cdot}00''$	Twist	A5.13(c), 19(b)
4.8	L_s $4{\cdot}50''$	Twist	A5.13(c), 19(b)
4.9	L_s $5{\cdot}0''$	Twist	A5.13(c), 19(b)
4.10	L_s $5{\cdot}5''$	Twist	A5.13(d), 19(b)
4.11	L_s $6{\cdot}0''$	Twist	A5.13(d), 19(b)
4.12	L_s $6{\cdot}5''$	Twist	A5.13(d), 19(c)
4.13	L_s $7{\cdot}0''$	Twist	A5.13(e), 19(c)
4.14	L_s $7{\cdot}5''$	Twist	A5.13(e), 19(c)
4.15	$L_s = L = 7{\cdot}90''$	Twist	A5.13(e), 19(c)
4.1 to 4.15	Further derived results		A5.20; 8.2, 6
	TUBE 5. Mild steel tube, with open ends, (torque applied through webs) L $29{\cdot}0''$ OD $1\frac{1}{2}''$ t $\frac{1}{25}''$	Dimensions & Properties	A5.4 (See also 7.2)
5.1	Original unslit tube	Twist	A5.14(a), 21(a)
5.2	L_s $3{\cdot}0''$	Twist	A5.14(a), 21(b)
5.3	L_s $9{\cdot}0''$	Twist	A5.14(b), 21(c)
5.4	L_s $12{\cdot}0''$	Twist	A5.14(b), 21(d)
5.5	L_s $14{\cdot}6''$	Twist	A5.14(b), 21(d)
5.1 to 5.5	Further derived results		8.6
	TUBE 6. Aluminium alloy tube, with massive steel end plates L $7{\cdot}33''$ OD $3\frac{1}{2}''$ t $\frac{1}{40}''$	Dimensions & Properties	A5.5 (See also 7.2)
6.1	$L_s = L = 7{\cdot}33''$ Further derived result	Twist	A5.15, 22 8.2



SKETCH FOR TUBES 1 & 2 (not to scale)

LENGTH L inches	TUBE 1 32·4 ± ·05 , TUBE 2 22·25 ± ·05
MATERIAL	Cold rolled mild steel plate, seam welded
OUTSIDE DIAMETER OD, inches	Circumferential variation $3{\cdot}500\,^{+\cdot 027}_{-\cdot 013}$ Longitudinal mean of circumferential average 3·508 ± ·003
WALL THICKNESS t, inches	Circumferential variation $\cdot 100\,^{+\cdot 002}_{-\cdot 0035}$ Longitudinal variation, zero Estimated mean ·098 ± ·0005
OD − t D, inches	3·41 ± ·004
POISSON'S RATIO, ν	Estimated ·29 ± ·02

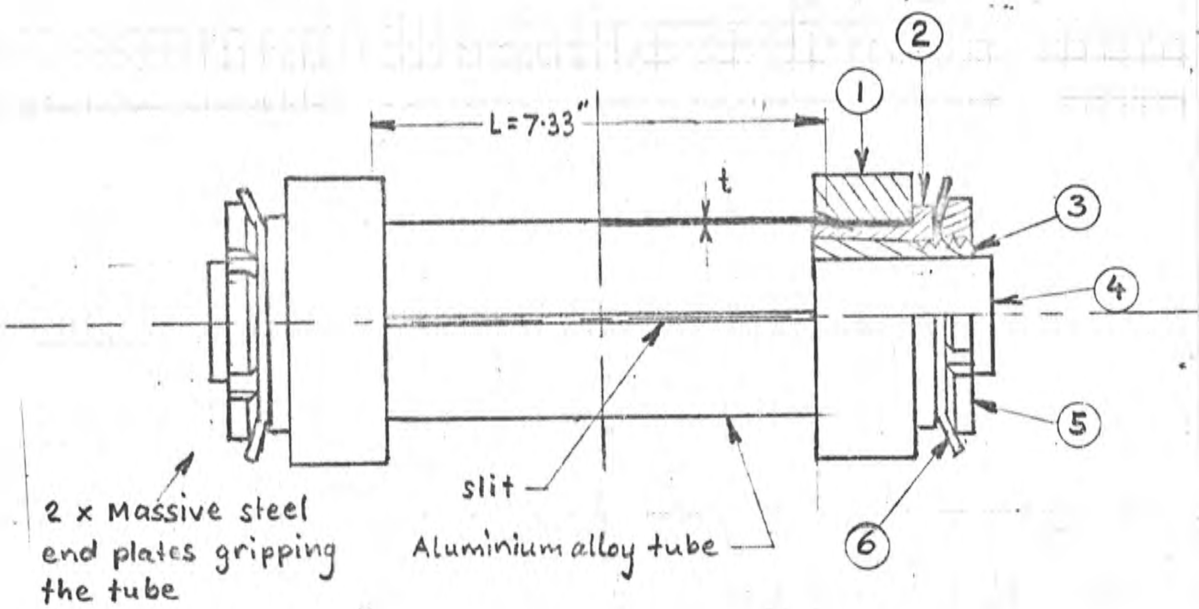
FIG. A5.1 DIMENSIONS & MATERIAL PROPERTIES OF TUBES 1 & 2



Approximate angular locations, 1 to 7, at the measuring section A, B, C & D.

Example: designation $A1_{45°}$ means a strain gauge approximately 8° from the slit, 2·4" from the end of the tube, measuring the strain at 45° to the axis of the tube.

FIG. A5.2 LOCATION OF STRAIN GAUGES ON TUBES 1 & 2



MATERIAL Alclad 23 S, cold rolled to form "slit tube", annealed at 650°C, quenched, age hardened

OD	3·478" ± ·01	
t	·0225" ± ·0005	
D	3·455" ± ·01	
ν	·295 ± ·02	(measured)
E	(10·0 ksi ± ·3) × 10³	(measured)

1. Keeper ring
2. Split ring (tapered)
3. Adapter sleeve (tapered)
4. Plug
5. Adapter nut
6. Lock washer

Note.

The end plates were originally bearing adapter sleeve assemblies. Because of the taper, tightening the adapter nut grips the tube between the keeper ring and the split ring.

FIG. A5.5 DESCRIPTION OF TUBE 6, INCLUDING DIMENSIONS

FIG. A5.6 (a)

OBSERVED STRAINS - SPECIMEN 1

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment x 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

TORQUE T lb.ins.	A1			A2			A3			A4			A5	A6		A7	B1	B3	B7	C1	C2		C3	C7
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	0°	45°	0°	0°	0°	0°	0°	0°	45°	0°	0°
0				959	2185						375										597	939		
2925					2109						307										598	869		
5850				949	2039						240										600	802		
8775					1970						169										601	731		
11700					1904						101										604	663		
14625				936	1837						34										605	593		
13125																						627		
11700					1901						100											659		
10200																						696		
8775					1969						168											726		
7350																						759		
5850					2036						238											796		
4425																						828		
2925					2105						306											865		
1500																						897		
0				954	2176						374										594	934		

OBSERVED STRAINS

SPECIMEN 1.2 (Centre of slit offset ·18" from centre-section of tube)

SPECIMEN 1.4 (Centre of slit length offset 1·86" from centre-section of tube)

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment × 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

TORQUE T lb.ins.	A1			A2			A3			A4			A5	A6		A7	B1	B3	B7	C1	C2		C3	C7
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	0°	45°	0°	0°	0°	0°	0°	0°	45°	0°	0°
0	630	498	923	927	2155	1953						730	812	15	724	70	626	763		198		846	12	57
1500		960			2115										688					205		801		52
2925		928			2082										653					211		763		46
4425		890			2044										617					218		719		39
5850		857			2012										583					226		680		34
7350		818			1977										546					236		635		29
8775		786			1948										514					244		595		23
10275		746			1910										483					251		549		19
11700	630	711	922	917	1879	1960						729	815	29	450	74	611	763		259		509	12	16
0	29	2026	72		176					837	361				743	41	602		747	116	88	828	93	328
1425	29	1990			143						325				710	42	586		765	197	73	783		251
2925	24	1949			106						288				675	43	567		785	282	56	734		166
4350	24	1914			73						254				642	44	552		800	361	43	689		091
5850	18	1876			35						216				609	44	534		820	445	28	634		013
7275	21	1840	72		6					836	183				575	49	523		837	528	14	591	96	-59

FIG. A5.6(b)

A46

FIG.A5. 6(c)

OBSERVED STRAINS

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment × 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

TORQUE T lb.ins.	A1			A2			A3			A4			A5	A6		A7	B1	B3	B7	C1	C2		C3	C7
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	0°	45°	0°	0°	0°	0°	0°	0°	45°	0°	0°
SPECIMEN 1.6																								
0					194																78	776		
570					178																	750		
1170					164																	722		
1740					148																	697		
2340					132																	670		
2910					119																	645		
3510					104																	618		
4080					89																	595		
4680					75																73	566		
SPECIMEN 1.7																								
0	640	2017	53	917	176	36	12	958	83	79	344	25	96	889	758	48	703	60	790	10	17	749	76	45
570	635	2004		913	162			945					94	992	747		676	64	822			720		
1170	635	1985		906	147			931			315		94	998	735		645	68	855			692		
1740	635	1970		902	134			919			300		91	1004	721		618	72	887			666		
2340	632	1954		895	115			905			285		91	1010	710		587	76	921			635		
2910	632	1934		890				892			270		90	1016	700		561	80	954			609		
3510	627	1924		882	88			879			255		90	1021	686		531	84	989			578		
4080	627	1909	53	879	74	46	12	868	83	78	242	25	88	1028	675	48	506	89	1021	10	17	550	66	45

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment x 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

SPECIMEN 1.8

TORQUE T lb.ins.	A2 0°	A2 45°	A2 90°	A3 0°	A4 0°	A4 45°	A4 90°	B1 0°	B7 0°	C2 0°	C2 45°
0	914	162	42	77	22	335	95	1335	436	11	719
570	906					321		1229	541		615
1170	896	131				305		1114	654		664
1740	891	117				289		1008	761		635
2340	881	101				274		898	866		603
2910	873	86				259		798	966		575
3510	863	71	42	77	22	224	95	694	1073	11	543

SPECIMEN 1.9

TORQUE T lb.ins.	A2 0°	A2 45°	A2 90°	A3 0°	A4 0°	A4 45°	A4 90°	B1 0°	B7 0°	C2 0°	C2 45°
0								817	694		705
570								734	762		675
1170								641	880		640
1740								556	967		610
2340								466	1061		578
2910								386	1156		547
3510								303	1254		515

OBSERVED STRAINS

FIG. A5.6 (d)

A48

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment x 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

OBSERVED STRAINS

SPECIMEN 1.11

TORQUE T lb.ins.	A1			A2			A3			A4			A5	A6		A7	B1	B3	B7	C1	C2		C3	C7
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	0°	45°	0°	0°	0°	0°	0°	0°	45°	0°	0°
0	90	1048	37	906	74	70	53	90	96	73	359	91	45	9	720	63	759	51	689	65	75	726	93	14
285	87	1038		896	67	71	57	83	94		350		42	16	716	64	716	52	734		71	710	91	
585	82	1025		886	59	78	61	76	91		341		38	25	714	67	671	55	781		69	692	87	
870	79	1016		877	52	83	66	70	89		333		35	34	710	70	630	57	825		67	674	85	
1170	74	1005		866	44	86	69	64	87		324		31	41	706	71	584	59	871		65	661	82	
1455	71	994		857	40	92	74	58	85		315		26	50	703	75	542	62	916		63	644	80	
1755	66	983		846	31	94	78	53	83		306		23	59	700	77	500	64	963		61	627	76	
2040	64	974		837	26	101	83	47	81		299		19	66	695	81	461	66	1010		59	611	75	
2340	59	963	37	826	18	104	86	41	79	73	290	91	17	79	692	83	420	69	1061	65	56	595	73	14

SPECIMEN 1.13

TORQUE T lb.ins.	A1			A2			A3			A4			A5	A6		A7	B1	B3	B7	C1	C2		C3	C7
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	0°	45°	0°	0°	0°	0°	0°	0°	45°	0°	0°
0	60	1034	68	801	157	5	61	79	86	69	154	83	45	740	703	51	620	400	615	73	78	716	200	29
285	52	1023				14	67	72	78		145		39			56	577		660			697		
585	43	1006		776		24	75	65	71		135		33	754		63	533		708		83	618		
870	36	991		763	144	33	81	58	65		128		26	761		71	489		753			660		
1170		975		748		43	88	51	57		119		19	768	696	77	445	412	798		87	640	186	
1455	20			734		52	96	45	50		110		14	775		85	405		840			621		
1755	11	943		720		64	102	38	43		102		8	783		91	336		884			602		
2040	5	930	64	707	129	74	109	33	37	64	95	89	4	791	692	98	325	421	927	73	94	585	176	32

FIG. A5.6 (e)

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment × 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5. 2]

TORQUE T lb. ins.	A1			A2			A3			A4			A5	A6		A7	B1	B3	B7	C1	C2		C3	C7
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	0°	45°	0°	0°	0°	0°	0°	0°	45°	0°	0°
SPECIMEN 1.15																								
0	120	939	130																					
600	66	860	136																					
1170	18	792	142																					
600	64	862	135																					
0	116	935	128																					
SPECIMEN 1.16																								
0	1343	770	373	741	350	994	55	90	84	17	402	12	55	29		295	474	396	494	132	34	676	36	74
300	1227	725	405				65	81								390	434		538			656		
585	1134	680	436	724	341		73	71	64		385		38			480	394		582		25	636	25	
885	1024	632	470				82	63			378					580	358		626			616		
1170	920	585	500	708	333	1006	91	55	45	25	370	12	21	44		675	315	408	670	132	15	596	15	74

OBSERVED STRAINS

FIG. A5. 6 (f)

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment × 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

TORQUE T lb.ins.	A1 0°	A1 45°	A1 90°	A2 0°	A2 45°	A2 90°	A3 0°	A3 45°	A3 90°	A4 0°	A4 45°	A4 90°	B2 0°	B2 45°	B2 90°	C2 0°	C2 45°	C2 90°	C6 0°	C6 45°	C6 90°
0	909	565	416	84	69	–	56	92	42	89	60	56	84	364	9	78	726		35	108	11
300	821	534	437	68	54	774	69	85	33		50		70	343		70	706		42	94	
585	741	506	460	56	42	789	81	79	27		41		65	326		60	685		49	80	
885	656	478	481	40	29	795	94	70	19		33		53	308		52	666		56	68	
1170	570	450	504	26	17	804	108	65	14	94	26	53	46	292	-2	40	646		62	55	11
885	653	478	481	41	29	794	95	69	19		32		51	308		50	665		55	66	
585	740	506	458	56	40	786	83	75	24		39		64	326		57	681		48	78	
300	820	535	435	69	52	777	68	80	30		46		71	343		63	700		41	90	
0	909	562	411	84	64	768	53	86	36	87	53	56	80	361	7	70	719		34	104	11
1170	570	447	501	24	14	803	107	62			22		43	286		40	644		61	54	
0	907	563	411	81	61	768	52	84			50		79	360	-	69	718		34	103	

TORQUE T lb.ins.	A5 0°	A6 0°	A6 45°	A7 0°	B1 0°	B3 0°	B7 0°	C1 0°	C3 0°	C7 0°
0	761	784	390	255	429	446	496	593	259	96
1170	716	824	360	575	276	455	666	591	229	96

FIG. A5.6 (9)

OBSERVED STRAINS

SPECIMEN 1.19 {ROSETTE READINGS & SINGLE GAUGES}

TABLE OF OBSERVED STRAIN BRIDGE READINGS [differences = strain increment × 10^6]

LOCATION OF STRAIN GAUGES [For meaning of designations see FIG. A5.2]

TORQUE T lb.ins.	A1			A2			A3			A4			B2			C2			C6			D 2-3		
	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°
0	1026	471	184	502	131	48	871	715	70	311	306	895	700	204	77	55	710		335	223	21	858	252	59
570	915	430	211		113		900		55		294			179			685			199		831	225	66
1170	801	387	245		96	60	931	700	45		278			155	66		660			174		806	199	74
1740	685	348	276		80		961		33		264			130	~~82~~		635			149		780	170	81
2040	569	307	306	502	63	71	991	685	21	300	249	895	710	106	56	55	612		335	125	21	756	143	88
1740	686	348	275		80		961		32		260			130			635			148		780	170	81
1170	801	387	243		96	56	929	698	41		273			155	65		658			171		807	197	73
570	905		213		111		901		52		290						680			196			221	66
0	1024	467	179	502	128	44	868	710	64	310	302	895	698	200	74	55	705		335	219	21	859	249	55

	D7		
	0°	45°	90°
0	1020	840	614
570	856	760	653
1170	752	690	689
1740	611	613	724
2040	473	538	761
1740	608	613	726
1170	746	688	688
570	373	760	652
0	1020	836	611

FIG. A5.7

OBSERVED STRAINS

SPECIMEN 2.1

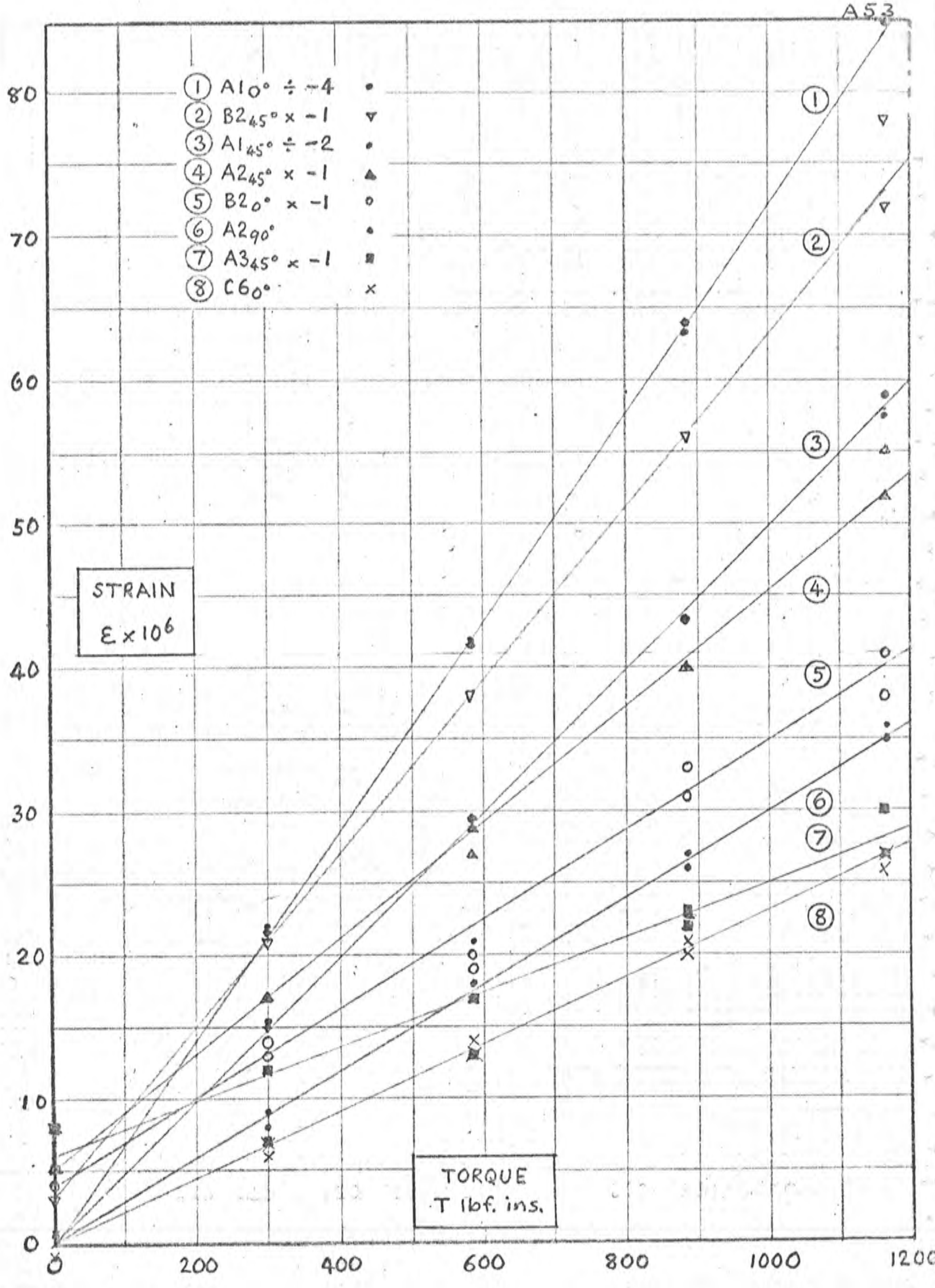


FIG. A5.8(a) TORQUE ~ STRAIN CURVES FOR SPECIMEN 1.19

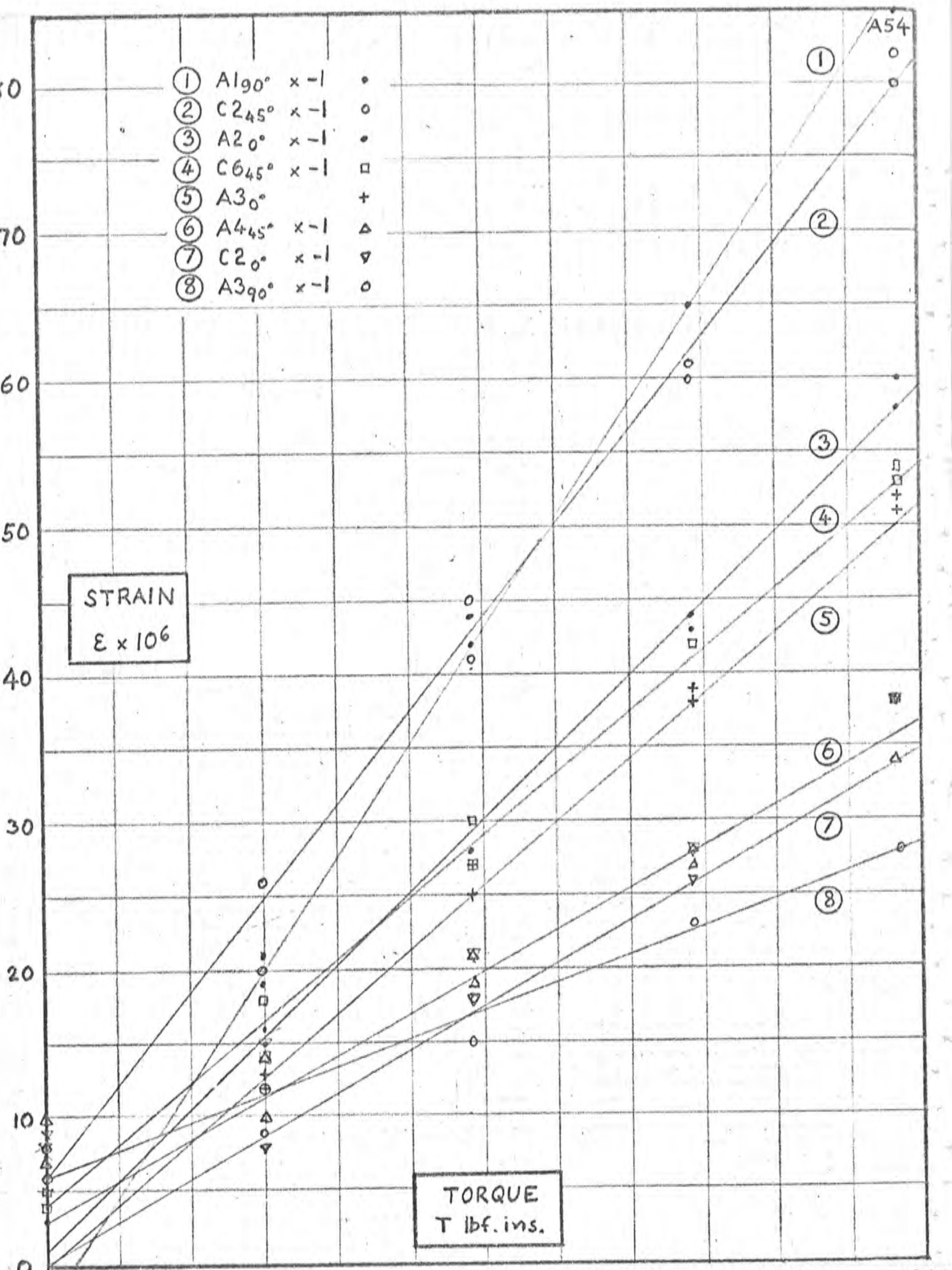


FIG. A5. 8 (b) TORQUE~STRAIN CURVES FOR SPECIMEN 1.19

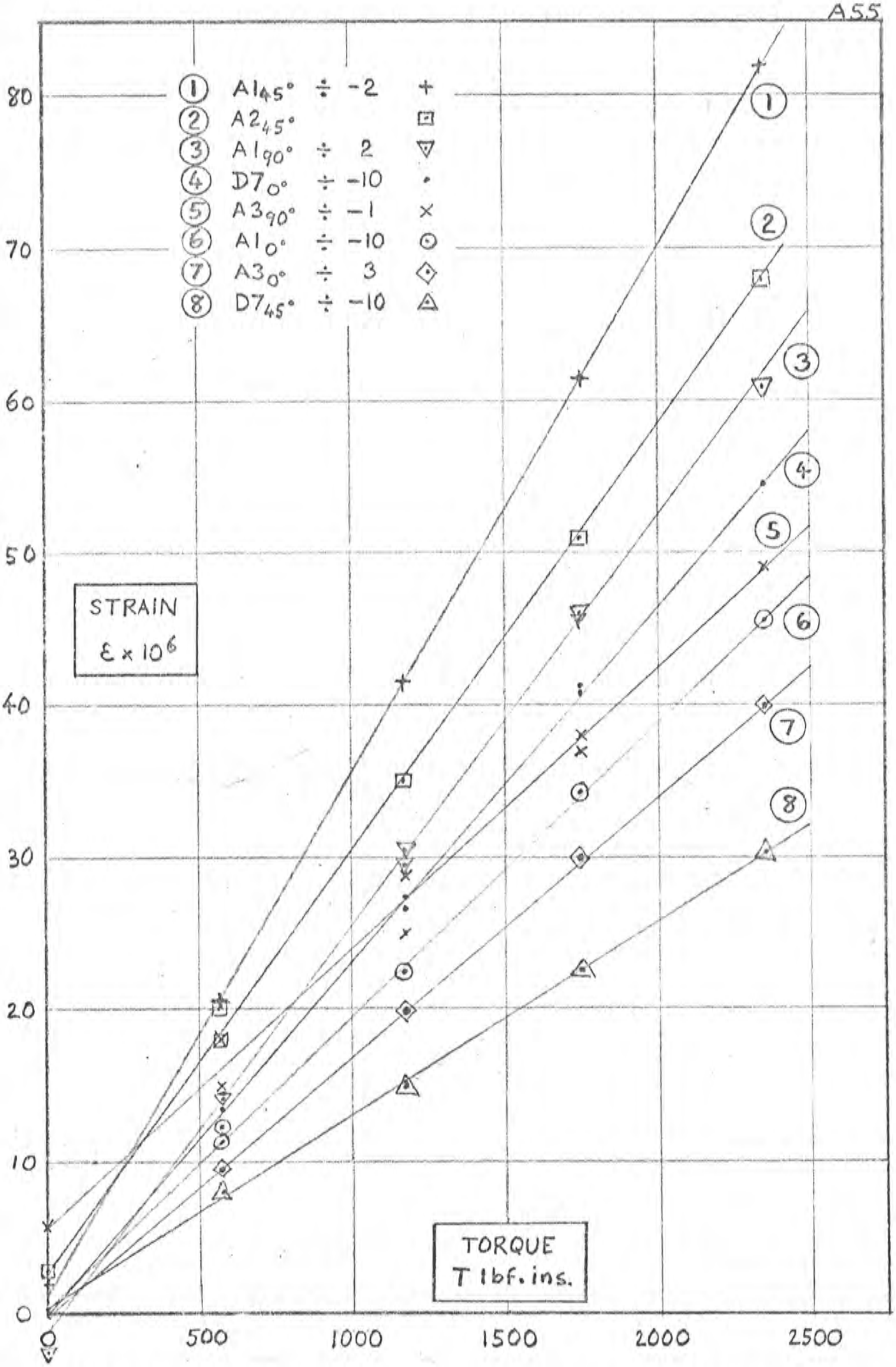


FIG. A5. 9(a) TORQUE ~ STRAIN CURVES FOR SPECIMEN 2.1

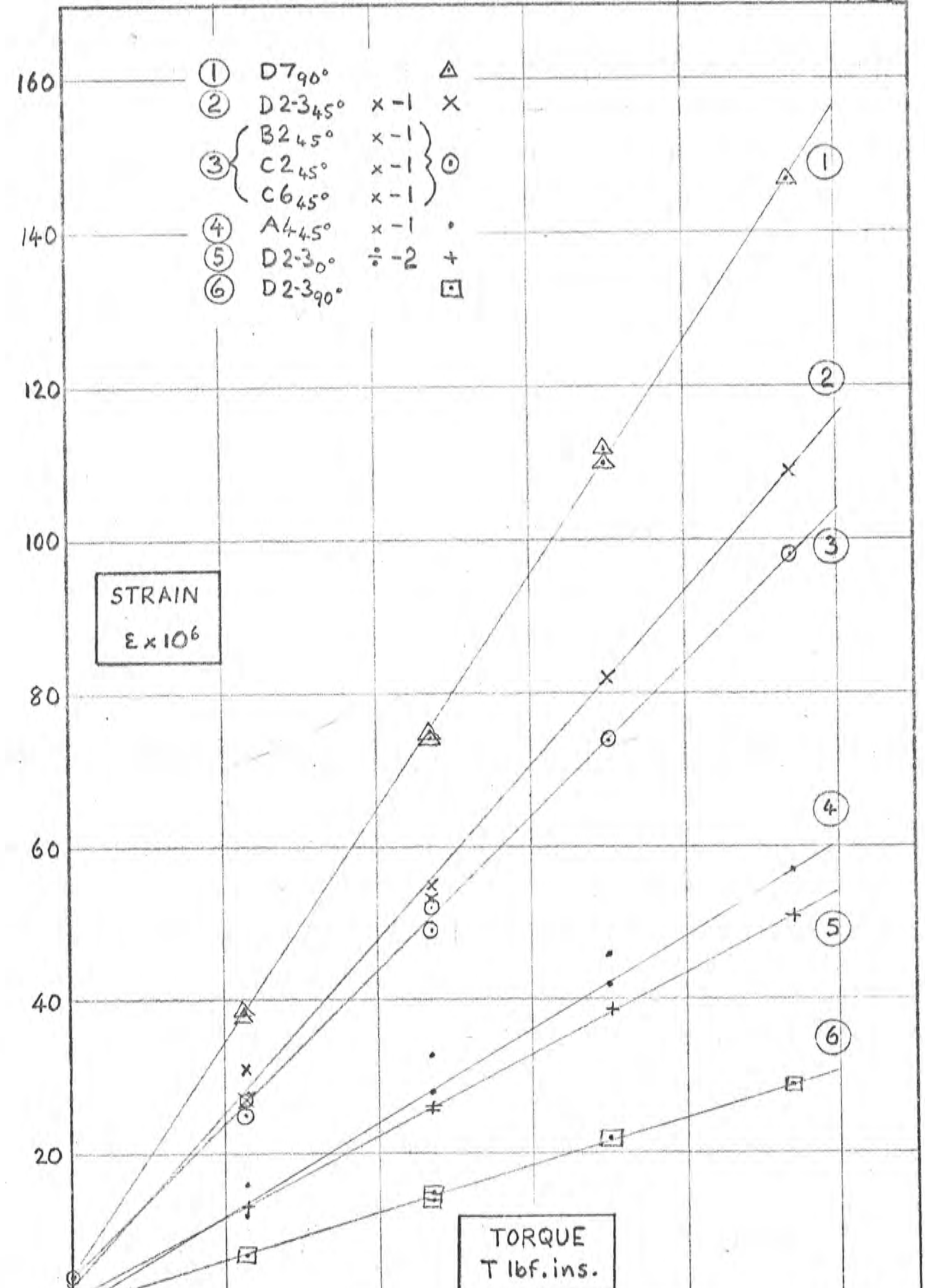


FIG. A5.9(b) TORQUE ~ STRAIN CURVES FOR SPECIMEN 2.1

LOCATION			AXIAL STRESS σ ksi per inch kip torque		SHEAR STRESS τ ksi per inch kip torque	
	φ°	z ins.	Theoretical Eq. 4.23	Measured Eq. 7.3	Theoretical Eq. 4.21	Measured Eq. 7.1
SPECIMEN 1.19						
A 1	171·5	13·8	−9·5	−8·6	−·18	−·07
A 2	112·3	13·8	−·37	−1·30	−·75	−·71
A 3	57·3	13·8	2·32	1·35	−·57	−·75
A 4	0	13·8	0	·11	−·39	−·57
B 2	112	6·9	−·15	−1·08	−·75	−1·01
C 2	109	0	0	−·80	−·75	−1·17
C 6	−112	0	0	−·47	−·75	−1·13
SPECIMEN 2.1						
A 1	171·5	8·72	−6·0	−5·8	−·18	−1·04
A 2	112·3	8·72	−·23	·10	−·75	−·75
A 3	57·3	8·72	1·47	2·50	−·57	−·64
A 4	0	8·72	0	−·14	−·39	−·46
B 2	112	1·82	−·04	−·09	−·75	−·90
C 2	109	−5·08	0	0	−·75	−·91
C 6	−112	−5·08	−·10	0	−·75	−·92
D 2-3	80·9	−10·31	−1·46	−1·31	−·68	−·69
D 7	−171	−10·31	−7·0	−7·0	−·19	−1·00

FIG. A5.10 STRESSES IN TUBES SLIT FULL LENGTH

(See also Figs. 8.1)

TUBE 1			SPECIMEN 1.1			SPECIMEN 1.2			SPECIMEN		
Moment Arm		TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
30"	28.5"	T									
Weight lbf.	Weight lbf.	lbf. ins.	Left ° '	Right ° '	Ø mins.	Left ° '	Right ° '	Ø mins.	Left ° '	Right ° '	Ø mins.
0 say	0 say	0 say	0° 16¼'	1° 57'	0 say	0° 18'	1° 58¾'	0 say			
50	50	2925	17¼	2° 7¼	9¼	18¼	2 8½	9½			
100	100	5850	18	2 17½	18¾	20	2 18¾	18			
150	150	8775	19¼	2 27¾	28¼	20	2 29¼	28½			
200	200	11700	20	2 37½	37¼	20¾	2 39¼	37¾			
200	250	13125	20¾	2 42¼	41½						
250	250	14625	21	2 47½	46¼						
200	200	11700	20¾	2 37¾	36¾	20¾	2 40	38½			
150	200	10200	20¼	2 32½	32	20½	2 34¾	33½			
150	150	8775	20½	2 28¼	27½	20¼	2 30	29			
150	100	7350	19¾	2 22¾	23¼						
100	100	5850	19¾	2 18¼	18¾	19¾	2 20¼	19¾			
100	50	4425	19	2 13½	14¼						
50	50	2925	19	2 8½	9¼	19	2 9½	9¾			
50	0	1500	19	2 3¾	4½						
0	0	0	18½	1 59	¼	18½	1 59	-¼			

			SPECIMEN 1.3			SPECIMEN 1.4			SPECIMEN 1.5		
0 say	0 say	0 say	0° 17½'	1° 51'	0 say	0° 17¾	1° 51'	0 say	0° 17¾'	1° 49¼'	0 say
0	50	1425	17¾	1 56	4¾	18	1 56¾	5½	18	1 56	6½
50	50	2925	19	2 02	9½	18¼	2 3¼	11¾	18	2 3¼	13¾
50	100	4350	19	2 7¼	15	18½	2 9¾	18	18¼	2 9¾	20
100	100	5850	19	2 12¾	20¼	19	2 15¼	23	18¾	2 17¼	27
100	150	7275	19¼	2 18	25¼	19½	2 21	28¼	19¼	2 23½	32¾
150	150	8775	19½	2 23¾	30¾						
100	150	7275	19½	2 18¼	25¼						
100	100	5850	19	2 13	20½	19¼	2 15¾	22¾	19¼	2 17¼	26½
50	100	4350	19	2 7½	14½	19	2 9¼	17	18¾	2 10¼	20
50	50	2925	18¾	2 2½	10¼	19	2 3½	11¼	18¾	2 4	14¼
0	50	1425	18	1 56¼	4¾	18½	1 57¼	5½	18¼	1 56½	6¾
0	0	0	17¾	1 51¼	0	17¾	1 51	0	17¾	1 50¼	¾

FIG. A5.11(a) OBSERVED ANGLES OF TWIST FOR SPECIMENS 1.1 TO 1.5

TUBE 1			SPECIMEN 1.6			SPECIMEN 1.7			SPECIMEN		
Moment Arm		TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
30"	28.5"	T									
Weight lbf.	Weight lbf.	lbf. ins.	Left ° '	Right ° '	Ø mins.	Left ° '	Right ° '	Ø mins.	Left ° '	Right ° '	Ø mins.
0 say	0 say	0 say	0° 17½'	1° 47½'	0 say	0° 16¾'	1° 46¾'	0 say			
0	20	570	17¾	1 50¼	2½	16¾	1 50½	3¾			
20	20	1170	17¾	1 54	6¼	17½	1 54¼	7			
20	40	1740	18	1 56¾	8¾	17½	1 58	10½			
40	40	2340	18	2 00	12	17¾	2 1½	13¾			
40	60	2910	18	2 2¾	15¾	17¾	2 5½	17¾			
60	60	3510	18¼	2 6¼	18	18	2 9	21			
60	80	4080	18½	2 9½	21	18	2 13	25			
80	80	4680	18½	2 12¼	23¾						
60	80	4080	18½	2 9¼	20¾						
60	60	3510	18½	2 6½	18	18	2 9¼	21¼			
40	60	2910	18¼	2 3¼	15	18	2 5½	17½			
40	40	2340	18	2 0½	12¼	18	2 1½	13½			
20	40	1740	18	1 57	9	18	1 58¼	10¼			
20	20	1170	17¾	1 54¼	6½	18	1 54¾	6¾			
0	20	570	17½	1 50¾	3	17¾	1 50½	2¾			
0	0	0	17½	1 47¾	¼	17½	1 47	-½			
			SPECIMEN 1.8			SPECIMEN 1.9			SPECIMEN 1.10		
0 say	0 say	0 say	0° 15½'	1° 44¾'	0 say	0° 16'	1° 44¾	0 say	0° 15½	1° 45¼'	0 say
0	20	570	16	1 49	3¾	16	1 50	5¼	16	1 50¾	5
20	20	1170	16	1 53½	8¼	16	1 55¼	10½	16	1 57½	11¾
20	40	1740	16	1 57¼	12½	16	2° 0½	15¾	16¼	2 3¾	17¾
40	40	2340	16	2 2¾	17½	16¼	2 5¾	20¾	16¼	2 9¾	22¾
40	60	2910	17¼	2 6¾	20¼	16½	2 10½	25¼	16½	2 16	29¾
60	60	3510	17¼	2 11½	25	17	2 16¼	30½			
40	60	2910	16½	2 7	21¼	17	2 11	25¼			
40	40	2340	16½	2 3	17¼	17	2 6½	20¾	16½	2 10¼	24
20	40	1740	16¼	1 58½	13	17	2 0½	15¾	16½	2 4¼	18
20	20	1170	16¼	1 54	8½	16½	1 56	10¾	16¼	1 58	12
0	20	570	16	1 49½	4¼	16½	1 51	5¾	16	1 51¾	6
0	0	0	16	1 45½	0	15¾	1 45½	0	16	1 46	0¼

FIG. A5.11(b) OBSERVED ANGLES OF TWIST FOR SPECIMENS 1.6 TO 1.10

TUBE 1			SPECIMEN 1.11			SPECIMEN 1.12			SPECIMEN 1.13		
Moment Arm 30"	Moment Arm 28.5"	TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
Weight lbf.	Weight lbf.	T lbf.ins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.
0 say	0 say	0 say	0° 16′	1° 46¼	0 say	0° 16′	1° 48	0 say	0° 16′	1° 50¼	0 say
0	10	285	16	1 50	3¾	16	1 52	4	16	1 55	4¾
10	10	585	16	1 54	7¼	16	1 56½	8½	16¼	2 0	9½
10	20	870	16	1 57½	11¼	16¼	2 0	11¾	16¼	2 5	14½
20	20	1170	16	2 1	14¾	16¼	2 4¼	16	16¼	2 10	19½
20	30	1455	16	2 4¼	18	16¼	2 8½	20¼	16¼	2 15	24½
30	30	1755	16¼	2 8	21½	16½	2 13¼	24¾	16½	2 20	29¼
30	40	2040	16¼	2 12	25½	16½	2 17½	29	16½	2 25	34¼
30	30	1755	16¼	2 8¾	22¼	16½	2 13¼	24¾	16½	2 20	29¼
20	30	1455	16¼	2 5	18½	16½	2 9	20½	16½	2 15¼	24½
20	20	1170	16	2 1¼	15	16½	2 5¼	16¾	16½	2 10½	19¾
10	20	870	16	1 57¼	11	16½	2 1	12½	16½	2 5½	14¾
10	10	585	16	1 54	7¾	16½	1 57	8½	16½	2 0½	9¾
0	10	285	16	1 50¼	4	16½	1 52¾	4¼	16½	1 55½	4¾
0	0	0	16	1 46½	¼	16½	1 48¾	−¼	16½	1 50¾	0
			SPECIMEN 1.14								
0 say	0 say	0 say	0° 16¼′	1° 53¾′	0 say						
0	10	285	16¼	1 59	5¼						
10	10	585	16¼	2 4½	10¾						
10	20	870	16½	2 10	16						
20	20	1170	16½	2 16¼	22¼						
20	30	1455	16½	2 22	28						
30	30	1755	16½	2 27¾	33¾						
30	40	2040									
30	30	1755									
20	30	1455	16½	2 22	28						
20	20	1170	16½	2 16¾	22¾						
10	20	870	16½	2 11	17						
10	10	585	16½	2 5	11						
0	10	285	16½	1 59½	5½						
0	0	0	16¼	1 53¾	0						

FIG. A5.11(c) OBSERVED ANGLES OF TWIST FOR SPECIMENS 1.11 TO 1.14

TUBE 1			SPECIMEN 1.15			SPECIMEN 1.16			SPECIMEN 1.17		
Moment Arm 30"	Moment Arm 28.5"	TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
Weight lbf.	Weight lbf.	T lbf.ins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.
0 say	0 say	0 say	0° 16′	1° 57′	0 say	0° 16¼′	1° 58¼′	0 say	0° 16′	2° 1′	0 say
5	0	150	16	2 0¾	3¾	16¼	2 2	3¾	16	5	4
5	5	292	16¼	2 4¾	6½	16¼	2 5¼	7½	16	9	8
10	5	442	16¼	2 6¾	9½	16¼	2 9½	11¼	16	13	12
10	10	585	16¼	2 10	12¾	16¼	2 13	14¾	16	17	16
15	10	735	16¼	2 13½	16¼	16¼	2 17	18¾	16	21	20
15	15	877	16¼	2 16¾	19½	16¼	2 20½	22¼	16	25	24
20	15	1027	16¼	2 19¾	22½	16¼	2 24	25¾	16	29	28
15	15	877	16¼	2 17¼	20	16¼	2 20½	22¼	16¼	25¼	24
15	10	735	16¼	2 14	16¾	16¼	2 17	18¾	16¼	21¼	20
10	10	585	16¼	2 11	13¾	16¼	2 13¼	15	16	17¼	16¼
10	5	442	16¼	2 7¾	10½	16¼	2 9½	11¼	16	13½	12½
5	5	292	16¼	2 4½	7¼	16¼	2 5¾	7½	16	9½	8½
5	0	150	16¼	2 1½	4¼	16¼	2 2	3¾	16	5½	4½
0	0	0	16¼	1 57¾	0½	16¼	1 58¼	0	16	1¼	0¼

			SPECIMEN 1.18			SPECIMEN 1.19		
Weight lbf. (30")	Weight lbf. (28.5")	T lbf.ins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.
0 say	0 say	0 say	0° 16′	2° 4½′	0 say	0° 16′	2° 9½′	0 say
5	0	150	16	2 8¾	4¼	16	2 14	4½
5	5	292	16	2 13	8½	16	2 18½	9
10	5	442	16	2 17¼	12¾	16	2 23	13½
10	10	585	16	2 21½	17	16	2 27½	18
15	10	735	16	2 26	21½	16¼	2 32¼	22½
15	15	877	16	2 30	25½	16¼	2 36¾	27
20	15	1027	16	2 34½	30	16¼	2 41¼	31½
15	15	877	16¼	2 30½	25¾	16¼	2 37½	27¾
15	10	735	16¼	2 26¾	22	16¼	2 33	23¼
10	10	585	16	2 22¼	17¾	16¼	2 28½	18¾
10	5	442	16	2 18½	14	16¼	2 24	14¼
5	5	292	16	2 13¾	9¼	16	2 19¼	9¾
5	0	150	16	2 9¾	5¼	16	2 14¾	5¼
0	0	0	16	2 4¾	0¼	16	2 10	0½

FIG. A5.11 (d) OBSERVED ANGLES OF TWIST FOR SPECIMENS 1.15 TO 1.19

TUBE 2			SPECIMEN 2.1			SPECIMEN 2.2			SPECIMEN		
Moment Arm		TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
30"	28.5"	T	Left	Right	Ø	Left	Right	Ø	Left	Right	Ø
Weight lbf.	Weight lbf.	lbf.ins.	° ′	° ′	mins.	° ′	° ′	mins.	° ′	° ′	mins.
0 say	0 say	0 say	0° $15\frac{3}{4}$	1° $20\frac{3}{4}$	0 say	1° $55\frac{1}{2}$	3° 2′	0 say			
0	10	285	$15\frac{3}{4}$	1 $23\frac{3}{4}$	3	1 56	3 $5\frac{3}{4}$	$3\frac{1}{4}$			
10	10	585	$15\frac{3}{4}$	1 $27\frac{1}{4}$	7	1 $56\frac{1}{4}$	3 $10\frac{1}{4}$	$7\frac{1}{2}$			
10	20	870	16	1 $31\frac{1}{4}$	$10\frac{1}{4}$	1 $57\frac{1}{4}$	3 $14\frac{3}{4}$	11			
20	20	1170	16	1 $34\frac{1}{2}$	$13\frac{1}{2}$	1 58	3 19	$14\frac{1}{2}$			
20	30	1455	16	1 $38\frac{1}{4}$	$17\frac{1}{4}$	2 0	3 24	$17\frac{1}{2}$			
30	30	1755	16	1 $42\frac{1}{2}$	$21\frac{1}{2}$	2 1	3 $28\frac{1}{2}$	21			
30	40	2040	$16\frac{1}{4}$	1 $46\frac{1}{4}$	25	2 1	3 33	$25\frac{1}{2}$			
30	30	1755	$16\frac{1}{4}$	1 $43\frac{1}{2}$	$22\frac{1}{4}$	2 1	3 $24\frac{1}{2}$	22			
20	30	1455	$16\frac{1}{2}$	1 40	$18\frac{1}{2}$	2 1	3 26	$18\frac{1}{2}$			
20	20	1170	$16\frac{1}{2}$	1 $36\frac{3}{4}$	$15\frac{1}{4}$	2 1	3 $22\frac{1}{2}$	15			
10	20	870	$16\frac{1}{4}$	1 33	$11\frac{3}{4}$	2 1	3 19	$11\frac{1}{2}$			
10	10	585	$16\frac{1}{4}$	1 $29\frac{1}{2}$	$8\frac{1}{4}$	2 1	3 $14\frac{1}{2}$	7			
0	10	285	$16\frac{1}{4}$	1 26	$4\frac{3}{4}$	2 1	3 11	$3\frac{1}{2}$			
0	0	0	$16\frac{1}{4}$	1 22	$0\frac{3}{4}$	2 $0\frac{1}{2}$	3 $6\frac{1}{2}$	$-\frac{1}{2}$			

FIG. A5.12 OBSERVED ANGLES OF TWIST FOR SPECIMENS 2.1 & 2.2

TUBE 4			SPECIMEN 4.1					SPECIMEN 4.2					SPECIMEN 4.3				
Moment Arm 30"	Moment Arm 28.5"	TOTAL TORQUE T	Clinometer Readings				ANGLE OF TWIST φ	Clinometer Readings				ANGLE OF TWIST φ	Clinometer Readings				ANGLE OF TWIST φ
Weight lbf.	Weight lbf.	lbf.ins.	Left °	Left ′	Right °	Right ′	mins.	Left °	Left ′	Right °	Right ′	mins.	Left °	Left ′	Right °	Right ′	mins.
		0															
		100	6	34¼	6	29½	0 say	6	7	6	1	0 say	5	45	5	40¼	0 say
		300	6	20¾	6	12½	3	5	58½	5	49¼	3¼	5	36	5	28½	3
		500	6	8¼	5	57¼	5¾	5	50	5	39	5	5	27½	5	17¾	5¼
		700	5	56¾	5	43½	8	5	42	5	28	8	5	17	5	4¾	7¾
		900	5	46	5	30	10¾	5	32¾	5	16½	10¼	5	9	4	53½	11
		1100	5	36½	5	17¾	13½	5	24	5	5	13	4	59¼	4	41¼	14
		1300	5	27½	5	6½	15¾	5	16½	4	55¼	15¼	4	52	4	31	16½
		1500	5	18¾	4	55	18½	5	9¼	4	45	18¼	4	44¼	4	20	19¾
		1700	5	9	4	43¾	20	5	1¼	4	35	20¼	4	35¾	4	8½	22¾
		1900	4	52¾	4	24½	22¼	4	53½	4	25	22½	4	27¾	3	57	26¼
		2000	4	39	4	9½	24¼	4	49¾	4	19½	24¼	4	23¾	3	51½	27½
		1900	4	38¾	4	10½	23	4	51¼	4	22½	22¾	4	24¾	3	54¼	26¼
		1800	4	40½	4	13¾	21½	4	53	4	25¾	21¼	4	26¼	3	57¼	25
		1700	4	43	4	17¼	20½	4	55	4	28¼	20¾	4	28¾	4	1	23¼
		1600	4	45	4	20½	19¼	4	57	4	31¼	19¼	4	30¼	4	4½	21¾
		1500	4	47½	4	24½	17¾	4	59	4	35¼	17¾	4	32¾	4	8	20¾
		1400	4	50	4	28¼	16½	5	1¾	4	39	16¼	4	35¼	4	11¾	19
		1300	4	53	4	32	15¾	5	4	4	43	15	4	38¾	4	16	18¼
		1200	4	55¾	4	36½	14	5	6¾	4	46¼	14½	4	41¾	4	20½	16¾
		1100	4	58¾	4	40¼	13¼	5	9¼	4	51¼	12½	4	44¾	4	25	15¼
		1000	5	1½	4	44½	11¾	5	12¾	4	55	11¾	4	48	4	28¼	14¾
		900	5	5	4	49	10¾	5	16	4	59¼	10¾	4	51	4	35½	13
		800	5	7¾	4	53¼	9¼	5	19½	5	4¼	9¼	4	54¾	4	39	11¼
		700	5	11¾	4	58½	8	5	23	5	9½	7½	4	58½	4	44	10
		600	5	15½	5	3¾	6¼	5	26½	5	14	6½	5	2	4	49	8½
		500	5	19¾	5	8½	6	5	30¼	5	19½	4¾	5	6	4	54¼	7¼
		400	5	24	5	14	4¾	5	34½	5	24½	4	5	10¼	5	0	6¼
		300	5	27¾	5	19¼	3¼	5	38¼	5	30	2¼	5	16	5	6¾	4¾
		200	5	33	5	26	1¼	5	43¼	5	35¾	1½	5	19¾	5	12	3¼
		100	5	38½	5	32½	0¾	5	48½	5	42¼	0¼	5	27	5	20¼	1¾
		0															

FIG. A5.13(a) OBSERVED ANGLES OF TWIST FOR SPECIMENS 4.1 TO 4.3

TUBE 4			SPECIMEN 4.4			SPECIMEN 4.5			SPECIMEN 4.6		
Moment Arm 30" Weight lbf.	Moment Arm 28.5" Weight lbf.	TOTAL TORQUE T lbf. ins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.
		0									
		100	5 47½	5 41½	0 say	5 40¼	5 31¾	0 say	5 25¼	5 17	0 say
		200				5 35¼	5 24¾	2	5 22¾	5 12	2
		300	5 40¾	5 31½	3¼	5 32½	5 20	3½	5 18¼	5 6	3½
		400				5 25	5 13¾	5¼	5 14¼	5 0	5½
		500	5 33	5 20½	6½	5 24	5 8	7	5 10½	4 54¼	7½
		600				5 21	5 3½	8½	5 6½	4 48	9¾
		700	5 26	5 10¼	9¾	5 17	4 57¾	10¼	5 2	4 42	11¼
		800				5 13¾	4 52½	12¼	4 59	4 37	13¼
		900	5 20	5 1½	12½	5 10½	4 47½	14	4 55½	4 32	14¾
		1000				5 7¼	4 42	16¼	4 51½	4 25½	17¼
		1100	5 12¾	4 51	15¾	5 3½	4 37	17½	4 47½	4 19¼	19½
		1200				5 0½	4 32	19½	4 43¼	4 13	21½
		1300	5 6¼	4 41¾	18¾	4 56¾	4 26½	21¼	4 40	4 7½	23¾
		1400				4 54¼	4 22	23¾	4 36	4 1¼	26
		1500	4 59¾	4 30¾	23	4 50½	4 16¾	24¾	4 33¼	3 56½	28
		1400	4 53¼	4 22¼	25	4 52¼	4 20	23¼	4 35½	4 0	26¾
		1300	4 55¾	4 26¼	23½	4 54½	4 24	21½	4 37¾	4 4	25
		1200	4 58¾	4 30¾	22	4 56¼	4 27¼	20	4 41	4 9½	22¾
		1100	5 1	4 34½	20½	4 59	4 31¾	18¼	4 42¼	4 13	20½
		1000	5 4¼	4 39¾	18½	5 0¾	4 34¾	17	4 45	4 17¼	19
		900	5 7¼	4 44¼	17	5 4	4 40	15	4 47¾	4 22¼	16¾
		800	5 10¼	4 48½	15¾	5 7	4 44½	13½	4 51¼	4 27	15½
		700	5 13¼	4 53	14¼	5 10	4 49¼	11¾	4 54½	4 33	12¾
		600	5 17½	4 58½	13	5 13¾	4 54½	10	4 59	4 39	10¼
		500	5 21¼	5 4¼	11	5 17½	5 0	8½	5 3	4 44½	9¾
		400	5 24½	5 9	9½	5 21	5 5¾	6¼	5 7	4 51	7¼
		300	5 29	5 15	8	5 24¾	5 11¼	4½	5 11½	4 57¼	5½
		200	5 35½	5 21¼	6¼	5 29¼	5 17½	2¾	5 16½	5 4	3½
		100	5 39¼	5 28¾	4½	5 35	5 25	1	5 22	5 12	1¼
		0									

FIG. A5.13(b) OBSERVED ANGLES OF TWIST FOR SPECIMENS 4.4. TO 4.6

TUBE 4			SPECIMEN 4.7			SPECIMEN 4.8			SPECIMEN 4.9		
Moment Arm 30"	Moment Arm 28.5"	TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
Weight lbf.	Weight lbf.	T lbf. ins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.
		0	–	–	–	–	–	–	–	–	–
		100	5° 45′	5° 36′	9 say	5 26	5 17	9 say	5° 35¼	5 27′	8¼ say
		200	5 40¼	5 29¼	11	5 22¾	5 11½	11¼	5 31½	5 20	11½
		300	5 36	5 23¼	12¾	5 18¾	5 5	13¾	5 28¼	5 14	14¼
		400	5 32½	5 17½	15	5 14¼	4 58¼	16	5 24	5 7¼	16¾
		500	5 29	5 11½	17½	5 11¼	4 52¼	19	5 20	4 59¾	20¼
		600	5 26	5 6½	19½	5 7	4 45½	21½	5 16¾	4 54	22¾
		700	5 22	5 0½	21½	5 3¼	4 39¼	24	5 14	4 48	26
		800	5 18¼	4 54¼	24	4 59	4 32¾	26¼	5 10	4 41	29
		900	5 15¼	4 49¼	26	4 55½	4 26¼	29¼	5 6	4 34¼	31¾
		1000	5 11½	4 43	28½	4 52¾	4 20½	32¼	5 2¾	4 28	34¾
		1100	5 9½	4 38¼	31¼	4 48	4 13½	34½	4 59	4 20½	38½
		1200	5 5	4 31¾	33¼	4 45¾	4 8	37¾	4 55¾	4 15	40¾
		1300	5 1¾	4 25¾	36	4 42	4 1¾	40¼			
		1400	4 58¼	4 19¾	38½	4 38	3 54	44			
		1500	4 56¾	4 14½	42¼						
		1400	4 58	4 17¼	40¾						
		1300	4 59½	4 21¼	38¼	4 40¾	3 59	41¾			
		1200	5 1¾	4 25¼	36½	4 43	4 4	39			
		1100	5 4	4 30	34	4 44¾	4 7¾	37	4 58	4 20	38
		1000	5 7	4 35	32	4 47¼	4 13	34¼	4 59¼	4 23½	35¾
		900	5 8½	4 39	29½	4 50¼	4 18½	31¾	5 1½	4 28¾	32¾
		800	5 11½	4 43	28	4 54	4 24½	29½	5 4¼	4 34	30¼
		700	5 15	4 49¼	25¾	4 57¼	4 30¼	27	5 7¼	4 40	27¼
		600	5 18	4 55	23	5 0	4 35¾	24¼	5 10¼	4 46	24¼
		500	5 21¾	5 1	20¾	5 4¼	4 42¼	22	5 14¼	4 52	22¼
		400	5 25¼	5 7	18¼	5 8¾	4 49	19¾	5 18¾	4 59½	19¼
		300	5 29¼	5 13¼	16	5 13	4 56¼	16¾	5 22	5 5¼	16¾
		200	5 33¼	5 19¼	14	5 17¾	5 4	13¾	5 26½	5 13¼	13¼
		100	5 38¼	5 27	11¼	5 22¾	5 11½	11¼	5 31	5 21	10¼
		0	–	–	–	–	–	–	–	–	–

FIG. A5.13 (c) OBSERVED ANGLES OF TWIST FOR SPECIMENS 4.7 TO 4.9

TUBE 4			SPECIMEN 4.10			SPECIMEN 4.11			SPECIMEN 4.12		
Moment Arm		TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
30"	25.5"	T									
Weight lbf.	Weight lbf.	lbf. ins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.	Left ° ′	Right ° ′	Ø mins.
		0	–	–	–	–	–	–	–	–	–
		100	5° 33½	5° 25½	8 say	5° 35¾	5 29	6¼ say	5° 36¼	5 30¼	6 say
		200	5 30	5 18½	11½	5 32¼	5 22½	10¼	5 31¼	5 21¼	10
		300	5 25¼	5 11¾	14	5 29	5 14¾	14¼	5 27½	5 13¼	14¼
		400	5 22¼	5 5	17¼	5 25½	5 7¼	18¼	5 23¼	5 5	18¼
		500	5 18	4 57¾	21¼	5 22½	5 0¾	21¾	5 19¾	4 57½	22
		600	5 14½	4 50½	24	5 19	4 53¼	25¾	5 17	4 50½	26½
		700	5 12½	4 45½	27	5 15	4 46½	28½	5 12¾	4 42¾	30
		800	5 7½	4 37	30½	5 12	4 40	32	5 9½	4 35¼	33¾
		900	5 5	4 31½	33½	5 8¾	4 32¾	36	5 6¼	4 28	38¼
		1000	5 1¾	4 24½	37¼	5 5¼	4 25½	39¾	5 3¼	4 20¼	43
		1100	4 57¼	4 16¼	41	5 1	4 17	44	4 59¾	4 12	47¾
		1200	4 54½	4 9½	45	4 58	4 9¼	48¼	4 56¾	4 4¾	52
		1100	4 54¾	4 13	41¾	5 0	4 15¼	44¾	4 57¾	4 9	48¾
		1000	4 55	4 18½	39½	5 2¾	4 21½	41¼	5 0¾	4 15¾	45
		900	5 0¼	4 24	36¼	5 4¼	4 26¼	38	5 3¼	4 22¼	41
		800	5 2½	4 29½	33	5 6½	4 32	34½	5 6	4 28¼	37¼
		700	5 5¼	4 35	30¼	5 10	4 38¾	31¼	5 7¾	4 35½	32¼
		600	5 9	4 42¼	26¾	5 12½	4 45¼	27¼	5 10¼	4 42	28¾
		500	5 13¼	4 49½	23¾	5 16¾	4 52¾	24	5 15	4 49¼	25¼
		400	5 16½	4 56¼	20¼	5 20½	5 0¼	20¼	5 19	4 57¾	21¼
		300	5 20¼	5 3½	16¾	5 24½	5 8¼	16¼	5 22¾	5 5¼	17
		200	5 24¾	5 10¾	13½	5 28	5 15¾	12¼	5 27	5 14¾	12¼
		100	5 29¾	5 20¼	9½	5 33½	5 24¾	8¾	5 31¾	5 23¾	8
		0	–	–	–	–	–	–	–	–	–

FIG. A5.13(d.) OBSERVED ANGLES OF TWIST FOR SPECIMENS 4.10 TO 4.12

TUBE 4			SPECIMEN 4.13			SPECIMEN 4.14			SPECIMEN 4.15		
Moment Arm 30" Weight lbf.	Moment Arm 28.5" Weight lbf.	TOTAL TORQUE T lbf.ins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.
		0	–	–	–	–	–	–	–	–	–
		100	5 43¾	5 37¼	6½ say	5 44	5 39	5 say	5 39¾	5 35¼	4 say
		200	5 40	5 29½	10½	5 40¼	5 30¼	9½	5 35½	5 26¼	9¼
		300	5 36¼	5 21¼	15	5 35½	5 21¼	14¼	5 32¼	5 18	14¼
		400	5 32½	5 13¼	19¼	5 31	5 11	20	5 29	5 9	20
		500	5 28½	5 4¾	23¾	5 26	5 2¼	23¾	5 25¼	5 0½	25¼
		600	5 24¾	4 56	28¾	5 22½	4 53½	29	5 21½	4 50¾	30¾
		700	5 20¾	4 48¼	32½	5 19¾	4 46	33¾	5 18½	4 43	35¼
		800	5 19	4 42	37	5 17	4 37¼	39¾	5 14¾	4 33¾	41
		900	5 15½	4 34	41½	5 12¼	4 29½	42¾	5 11¼	4 25	46¼
		1000	5 11¾	4 25¼	46½	5 10	4 21¼	48¾	5 8¼	4 16	52¼
		900	5 13¼	4 30¾	42½	5 12	4 27¾	44¼	5 9¾	4 22	47¾
		800	5 15¾	4 36¾	39	5 13¼	4 33¼	40	5 11¾	4 28¼	43½
		700	5 17½	4 43¼	34¼	5 16	4 41¾	34¼	5 14¾	4 36¾	38
		600	5 21¾	4 51	30¾	5 18½	4 47¾	30¾	5 17½	4 43¾	33¾
		500	5 23¾	4 58½	25¼	5 21¾	4 56	25¾	5 20	4 51¾	28¼
		400	5 27	5 5¾	21¼	5 24¾	5 4	20¾	5 24¾	4 1½	23¼
		300	5 30¾	5 14¼	16½	5 30	5 14	16	5 27¾	4 10¼	17½
		200	5 34¾	5 23¼	11½	5 33¾	5 22½	11	5 32¼	4 19¾	12½
		100	5 40	5 32½	7	5 38	5 32¼	5¾	5 37¼	4 30¼	7
		0	–	–	–	–	–	–	–	–	–

FIG. A5.13(e) OBSERVED ANGLES OF TWIST FOR SPECIMENS 4.13 TO 4.15

TUBE 5 — SPECIMEN 5.1

Moment Arm 30" Weight lbf.	Moment Arm 28.5" Weight lbf.	TOTAL TORQUE lbf. ins.	Clinometer Readings Left ° ′	Clinometer Readings Right ° ′	ANGLE OF TWIST Ø mins.
		90	10° 5.8	10° 24.5	-18.8 say
		190	10 32.4	10 30.4	+ .7
		290	10 55.9	10 53.2	2.7
		400	11 17.2	11 3.2	14.0
		590	11 51.1	11 19.8	31.3
		800	12 24.4	11 33	51.4
		1000	12 53	11 43.2	69.8
		1200	13 20.8	11 52	88.8
		1400	13 45.4	11 57.5	107.9
		1590	14 9	12 3	126.0
		1800	14 35	12 8.8	146.2
		2100	15 1.4	12 14.7	166.7
		1800	14 34.1	12 12.2	141.9
		1600	14 18.2	12 10.5	127.7
		1400	13 57	12 5	105.0
		1200	13 35.6	12 5.3	90.3
		1000	13 13.5	12 2.4	71.1
		800	12 51.8	12 1	50.8
		600	12 29.8	11 56.8	33.0
		400	12 7.8	11 53.7	14.1
		290	11 54.5	11 50.9	3.9
		190	11 44	11 50	-6.0
		90	11 32.2	11 47.2	-15.0

SPECIMEN 5.2

TORQUE T lbf. ins.	Clinometer Readings Left ° ′	Clinometer Readings Right ° ′	ANGLE OF TWIST Ø mins.
22	10° 18.2	9° 57	-1.2 say
50	10 25.1	10 4.9	1.6
75	10 28.4	10 13.3	4.9
100	10 33.2	10 20.5	7.3
125	10 37.3	10 28	10.2
155	10 43.2	10 30.5	12.7
173	10 45.4	10 40.6	15.2
195	10 51	10 48.2	17.2
225	10 56	10 56.5	20.5
260	10 59.2	11 3.2	24.0
260	10 59	11 3.5	24.5
225	10 53.5	10 57.4	15.6
200	10 57.2	10 54.2	17.0
178	10 55.9	10 57.4	15.8
150	10 54.0	10 46.3	12.8
125	10 52.7	10 43.5	10.8
100	10 52	10 38.2	6.2
75	10 50.2	10 36	5.8
50	10 49	10 31	2.0
25	10 46.1	10 25.5	-.3

FIG. A5.14(a) OBSERVED ANGLES OF TWIST FOR SPECIMENS 5.1 & 5.2

TUBE 5

SPECIMEN 5.3

Moment Arm 30" Weight lbf.	Moment Arm 28.5" Weight lbf.	TOTAL TORQUE T lbf. ins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.
		8	10° 13'	9° 52'	0 say
		15	10 14.5	9 56.8	3.3
		30	10 18.2	10 3.8	6.6
		42	10 22.8	10 10	8.2
		50	10 22.1	10 13.6	12.5
		60	10 25.2	10 13.2	14
		75	10 27	10 23.2	17.2
		82	10 29.1	10 27.8	19.7
		90	10 33	10 32.2	20.2
		90	10 32.9	10 33.2	20.7
		80	10 30.3	10 30.4	20.9
		70	10 31.7	10 26.6	15.9
		62	10 31	10 24.5	14.5
		50	10 30.1	10 20.8	11.7
		41	10 28.8	10 17	10.2
		30	10 28.5	10 13.2	5.7
		20	10 26.6	10 9.2	3.6
		12	10 25.8	10 6	1.2

SPECIMEN 5.4

TORQUE T lbf. ins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.
5	10° 8'	9° 48½'	0 say
10	10 10	9 52	1½
15	10 13½	9 58½	4½
20	10 15½	10 1¼	5¼
25	10 16	10 5	8½
30	10 18	10 10	11½
35	10 18½	10 13	14
40	10 20½	10 16½	15
45	10 21	10 20¼	18¼
50	10 21¾	10 22½	19½
30	10 20	10 12	11½
10	10 18¼	10 0	1¼

SPECIMEN 5.5

TORQUE T lbf. ins.	Clinometer Readings Left ° '	Clinometer Readings Right ° '	ANGLE OF TWIST Ø mins.
5.5	10° 8.6	9° 53.8	0 say
10	10 10.8	9 58.3	3.3
15	10 11.3	10 2.2	5.7
20.5	10 12.2	10 6.3	8.9
25	10 14.5	10 11.1	11.4
30	10 14.5	10 16.0	16.3
35	10 15.1	10 20	19.7
40	10 16	10 25	23.8
45	10 17.1	10 29.8	27.5
50	10 18.9	10 35	30.9
45	10 18.3	10 30.9	27.4
40	10 18.8	10 27	23.0
34.5	10 18.5	10 22.4	18.7
30	10 17.7	10 19.1	16.2
25	10 17.1	10 15.3	13.0
20	10 16.8	10 11.4	9.4
15	10 16.1	10 7.6	6.3
10	10 15.2	10 3	2.6
5	10 13.1	9 57	-1.5

FIG. A5.14 (b) OBSERVED ANGLES OF TWIST FOR SPECIMENS 5.3 TO 5.5

TUBE 6			SPECIMEN 6.1			SPECIMEN			SPECIMEN		
Moment Arm		TOTAL TORQUE	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST	Clinometer Readings		ANGLE OF TWIST
30"	28.5"	T									
Weight lbf.	Weight lbf.	lbf. ins.	Left ° '	Right ° '	Ø mins.	Left ° '	Right ° '	Ø mins.	Left ° '	Right ° '	Ø mins.
		100	6° 1'	5° 25¾	0 say						
		200	5 58¾	5 21	2½						
		300	5 56	5 16¾	4						
		400	5 53	5 12¼	5½						
		500	5 50	5 8	6¾						
		600	5 47½	5 4	8¼						
		700	5 45½	5 0	10¼						
		800	5 43	4 56¾	11						
		900	5 41	4 52¾	13						
		1000	5 38½	4 49	14¼						
		900	5 40¼	4 52	13						
		800	5 42	4 54¾	12.						
		700	5 44	4 59	9¾						
		600	5 45¾	5 2½	8						
		500	5 48	5 6½	6¼						
		400	5 50¼	5 10	5						
		300	5 53¾	5 14¼	4¼						
		200	5 57	5 19	2¾						
		100	6 1	5 25¾	0						

FIG. A5.15 OBSERVED ANGLES OF TWIST FOR SPECIMEN 6.1

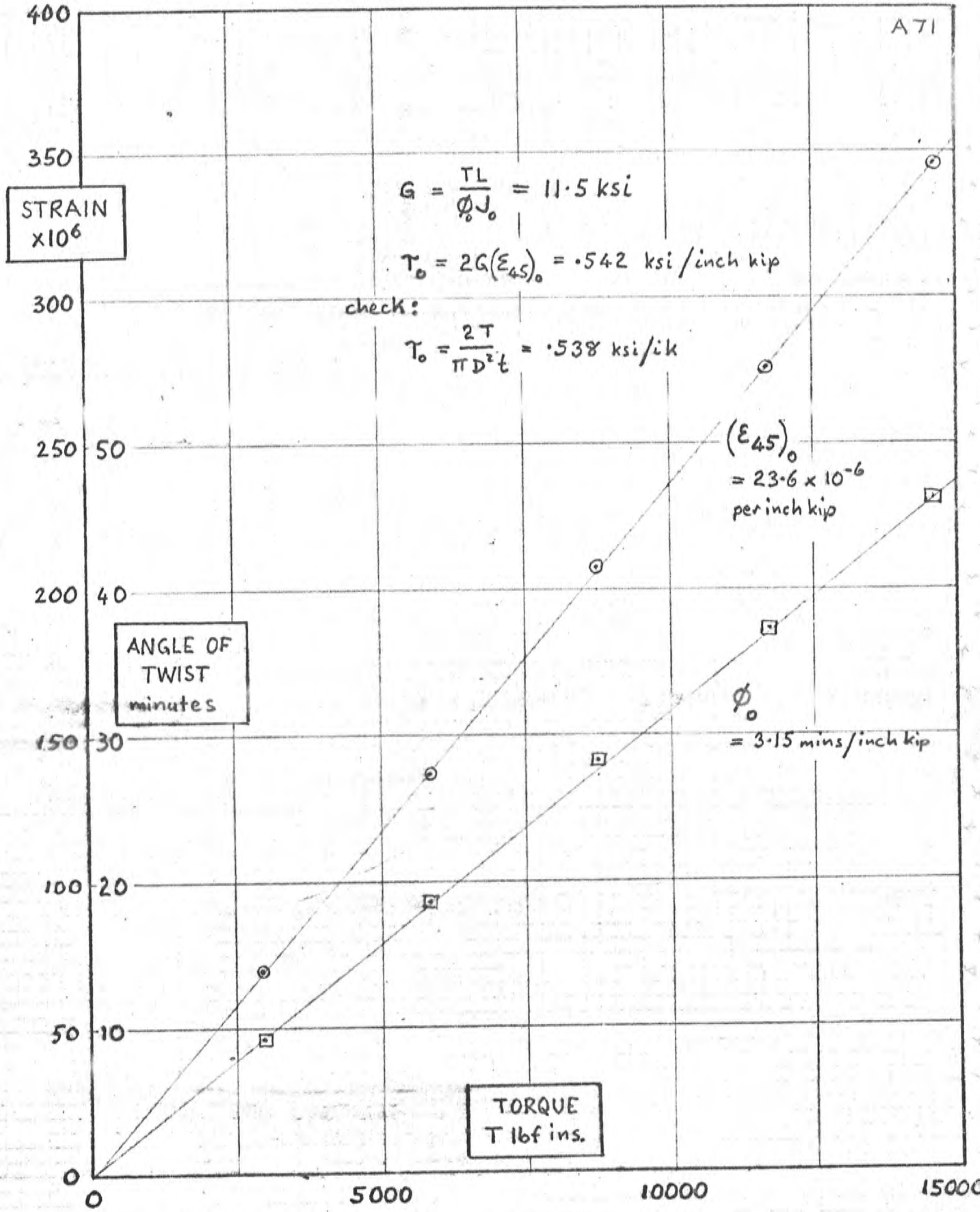


FIG. A5.16 TORQUE, STRAIN & TWIST RELATIONS FOR SPECIMEN 1.1

— TUBE 1 WITH NO SLIT —

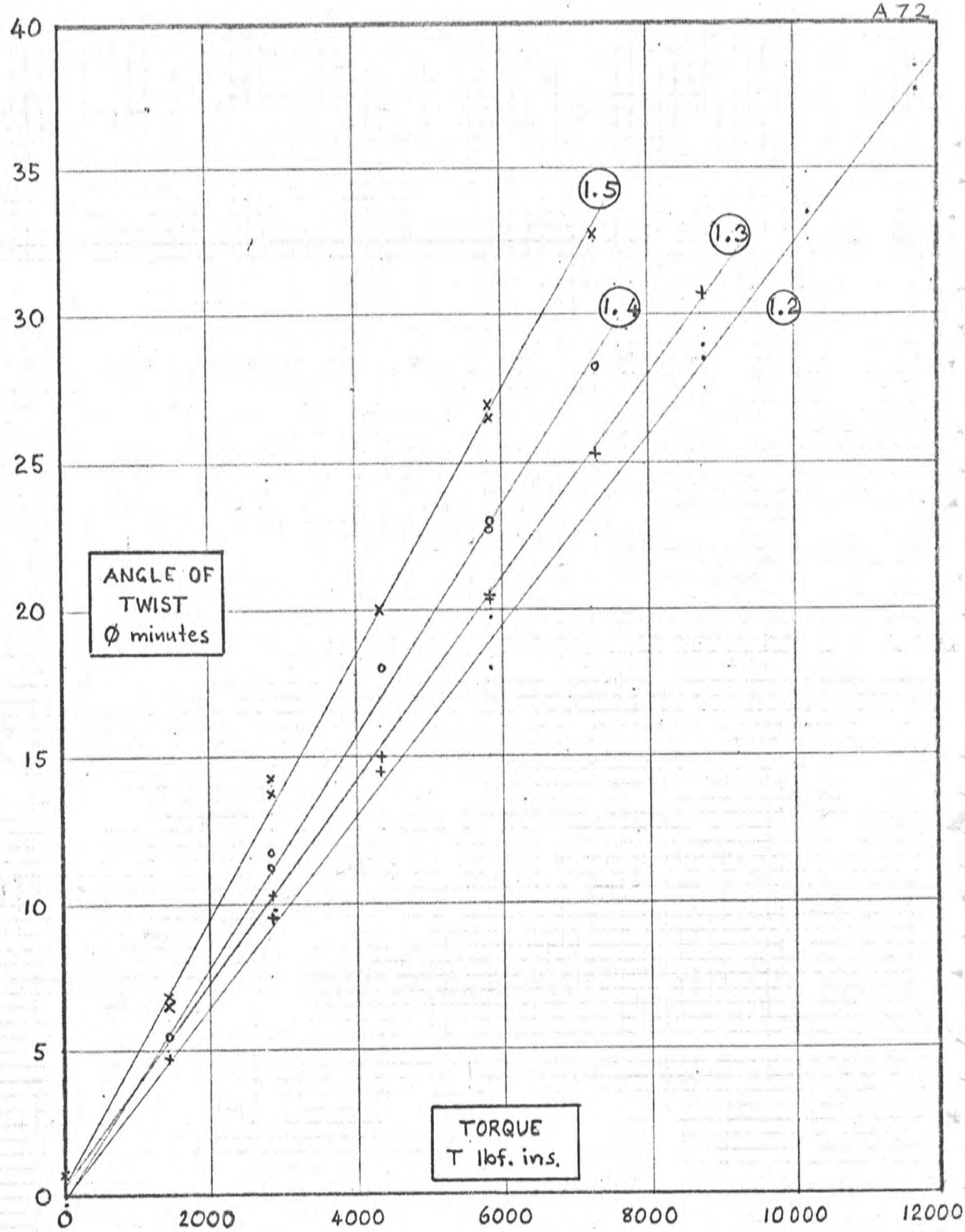


FIG. A5.17(a) TORQUE ~ TWIST CURVES FOR SPECIMENS 1.2 TO 1.5

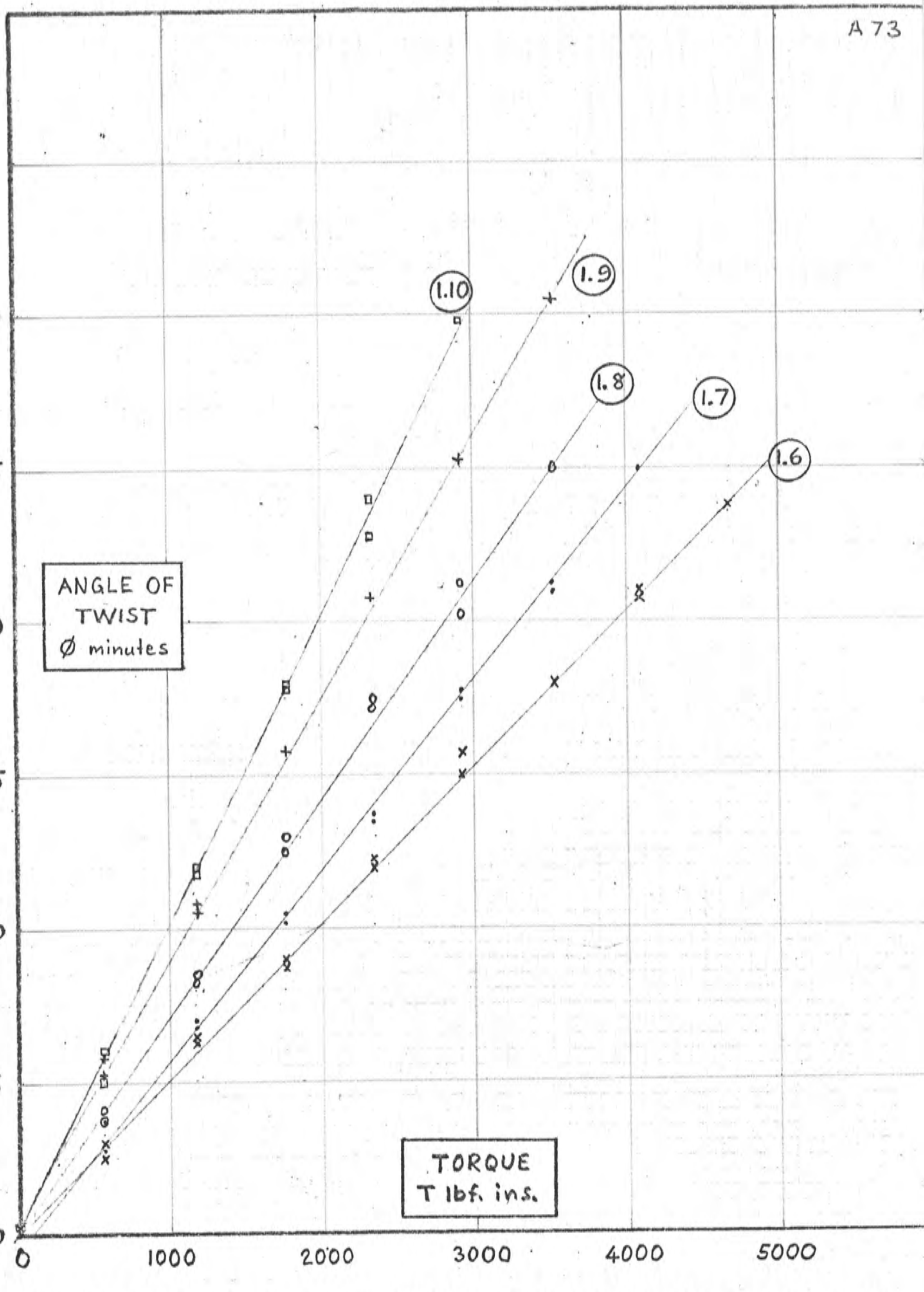


FIG. A5.17(b) TORQUE ~ TWIST CURVES FOR SPECIMENS 1.6 TO 1.10

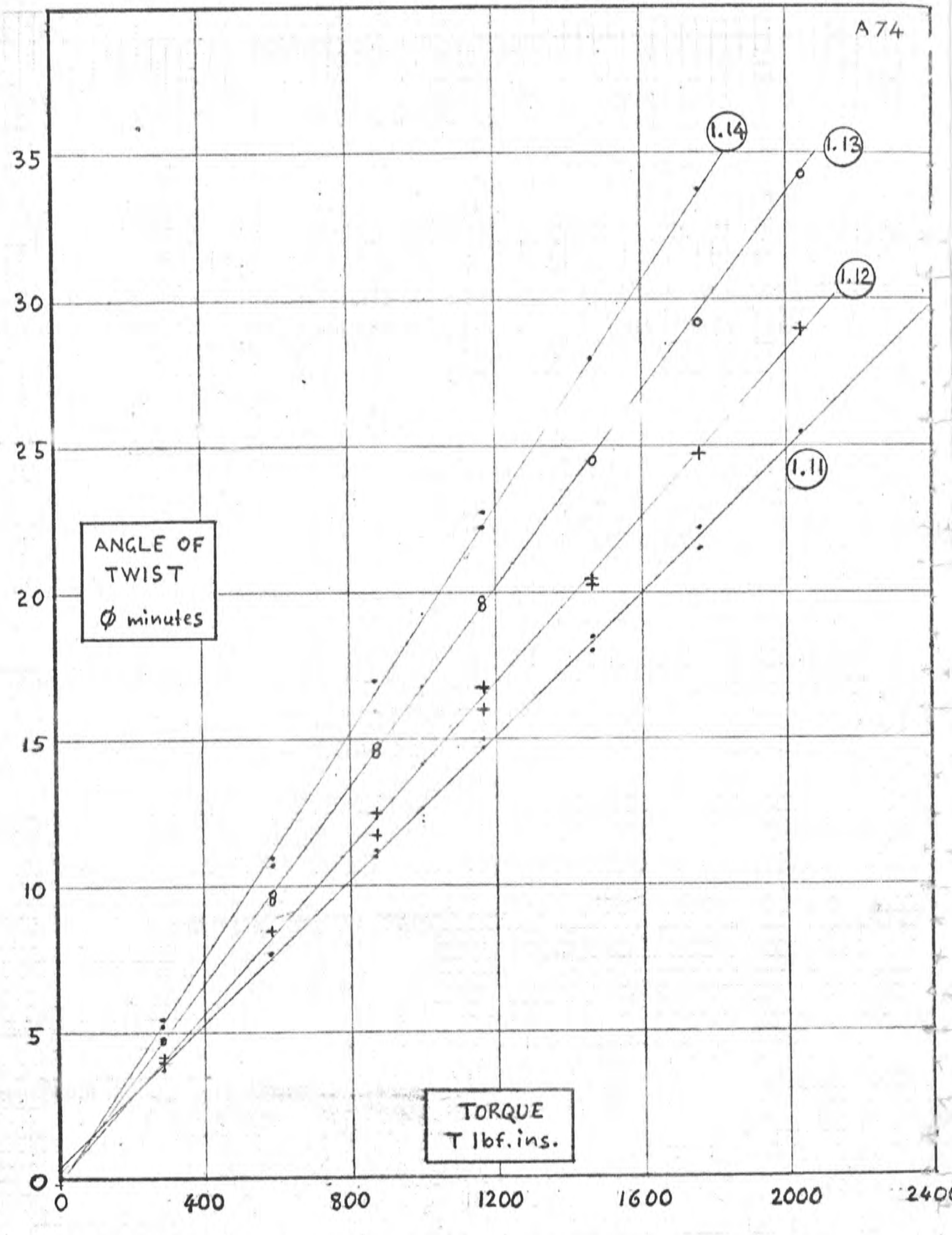


FIG. A5.17(c) TORQUE ~ TWIST CURVES FOR SPECIMENS 1.11 TO 1.14

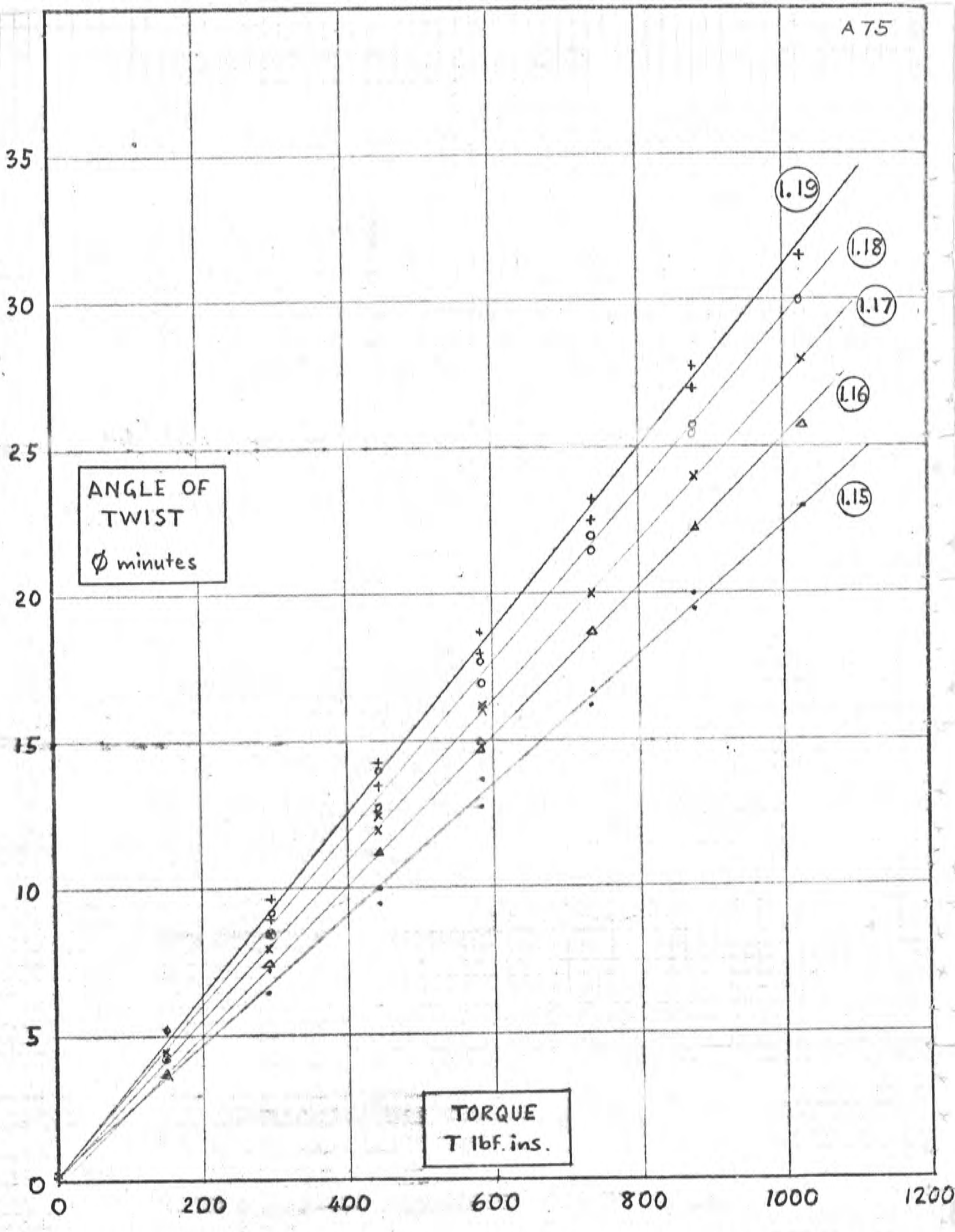


FIG.A5.17(d) TORQUE ~ TWIST CURVES FOR SPECIMENS 1.15 TO 1.19

(SPECIMEN 1.19 :- FULL LENGTH SLIT)

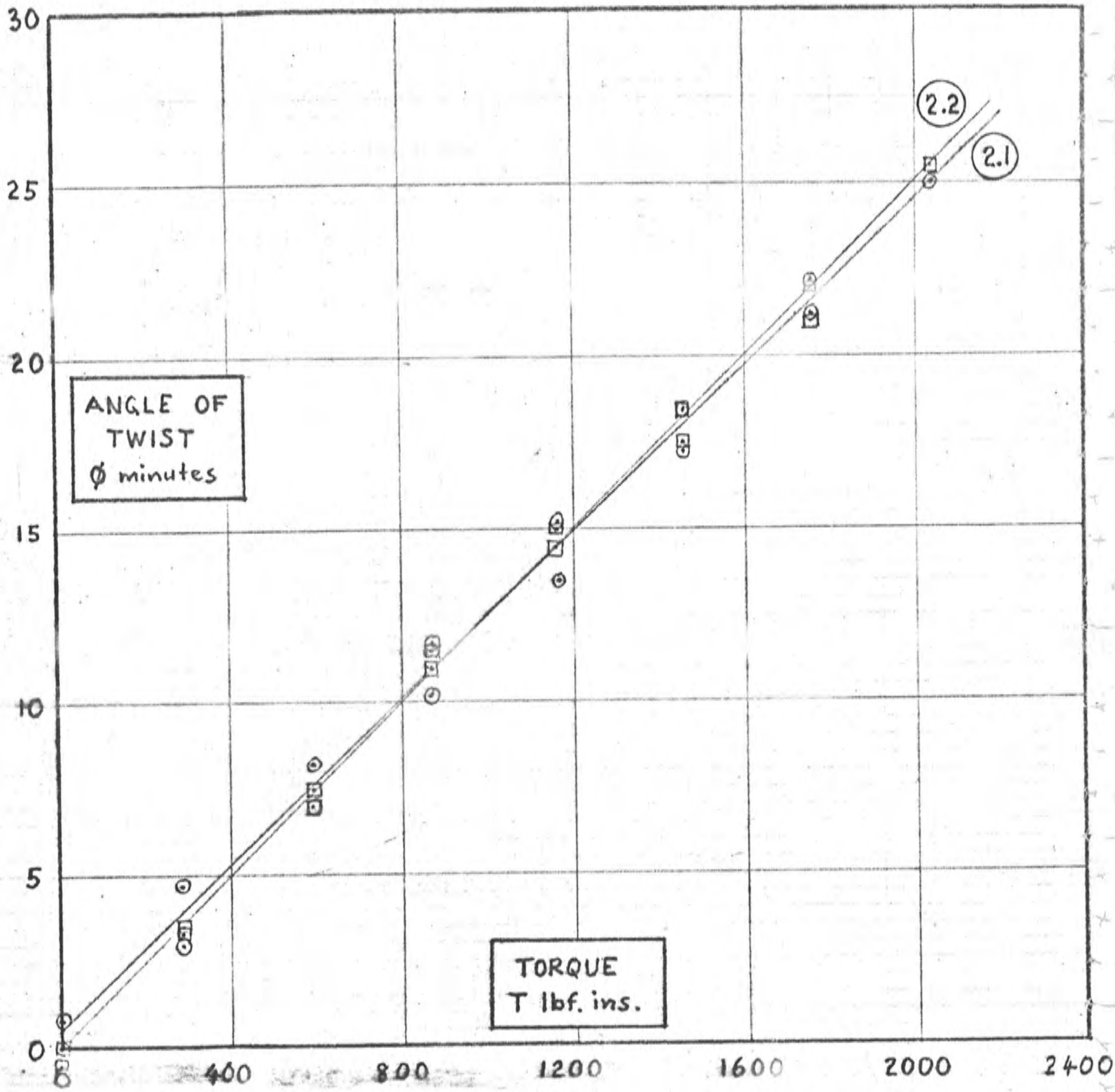


FIG. A5.18 . TORQUE ~ TWIST CURVES :

SPECIMEN 2.1 ⊙

SPECIMEN 2.2 ⊡

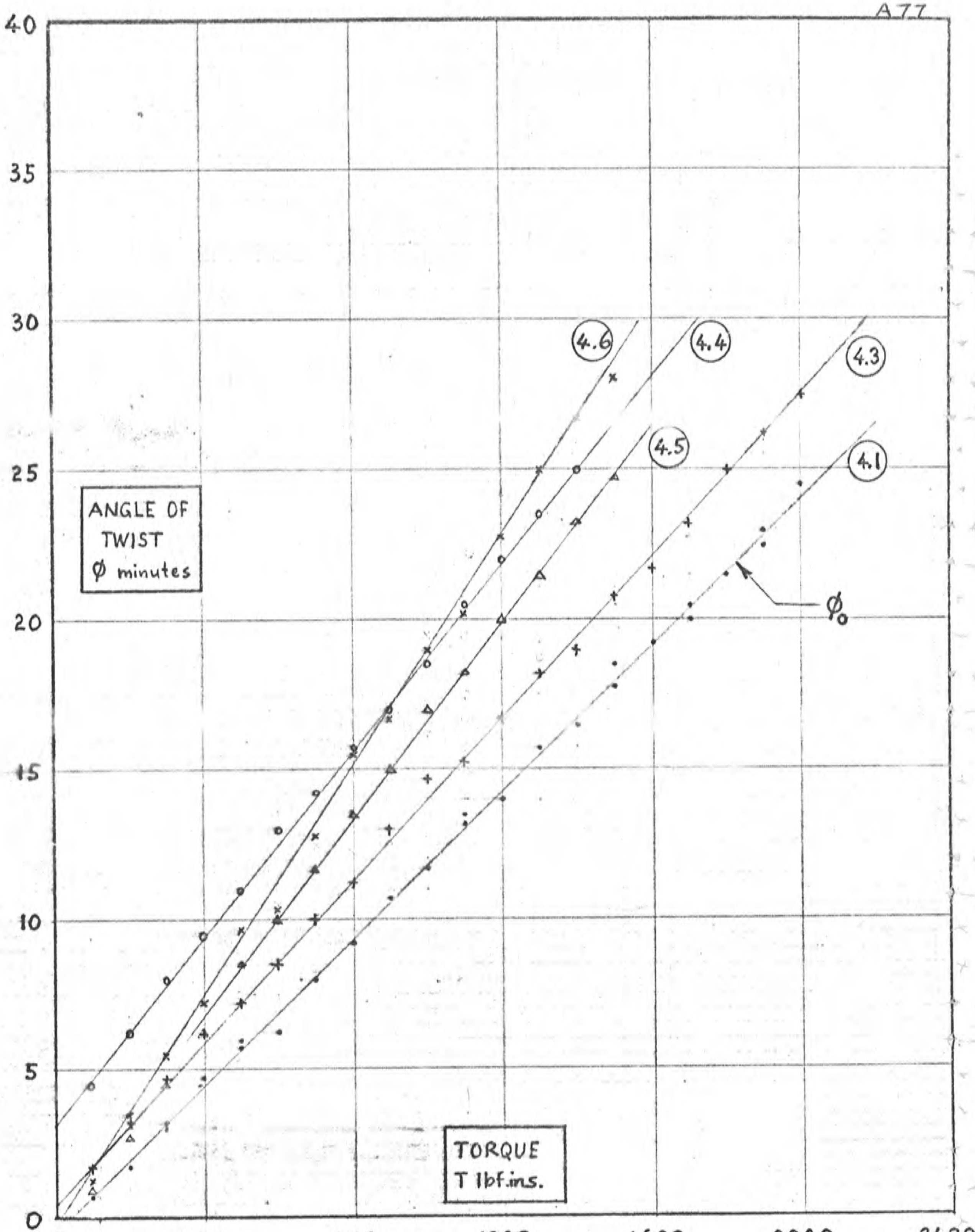


FIG. A5.19.(a) TORQUE ~ TWIST CURVES FOR SPECIMENS 4.1 TO 4.6

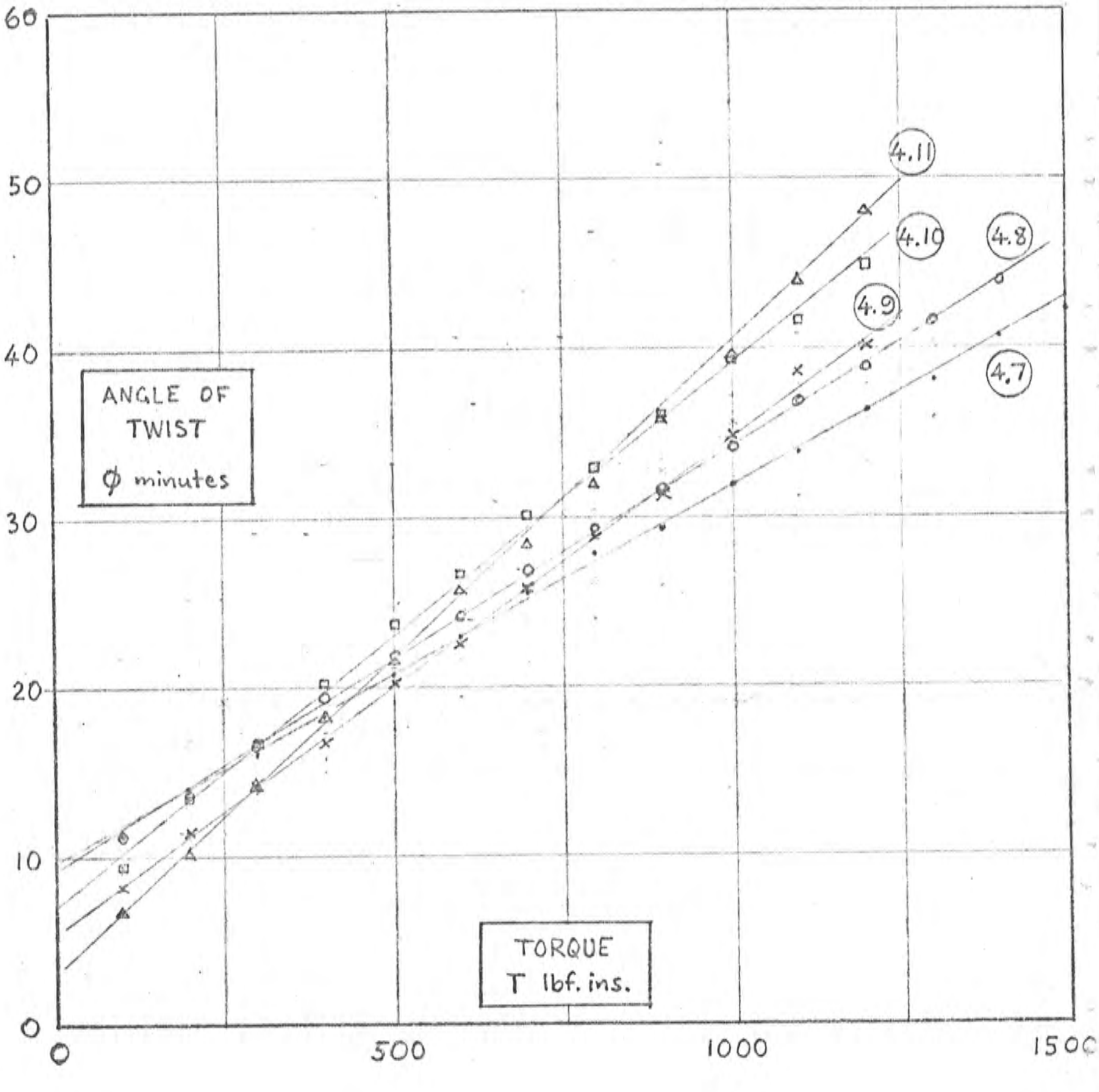


FIG.A5.19(b) TORQUE~TWIST CURVES FOR SPECIMENS 4.7 TO 4.11

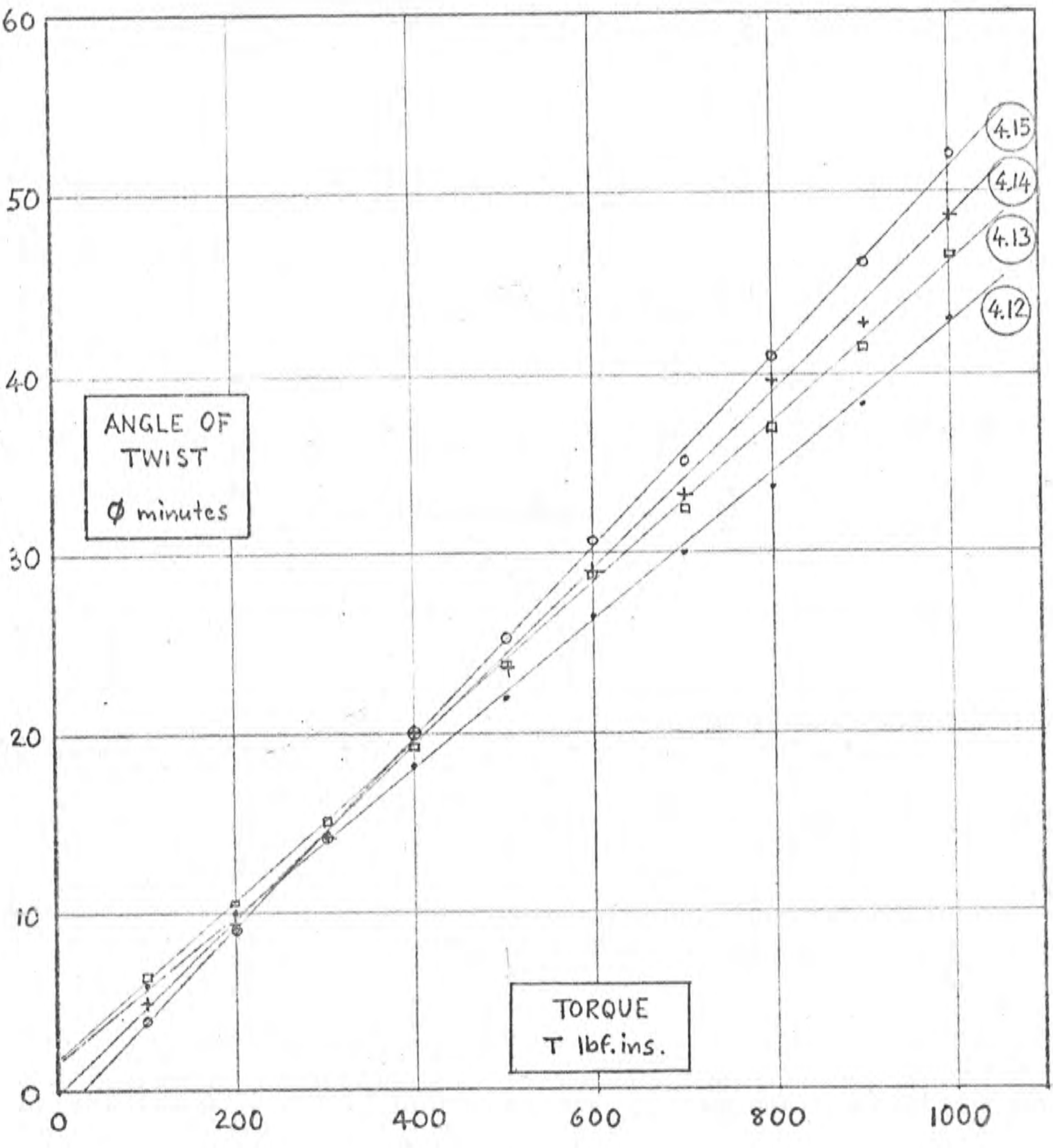


FIG.A5.19(c) TORQUE ~ TWIST CURVES FOR SPECIMENS 4.12 TO 4.15

(SPECIMEN 4.15 - FULL LENGTH SLIT)

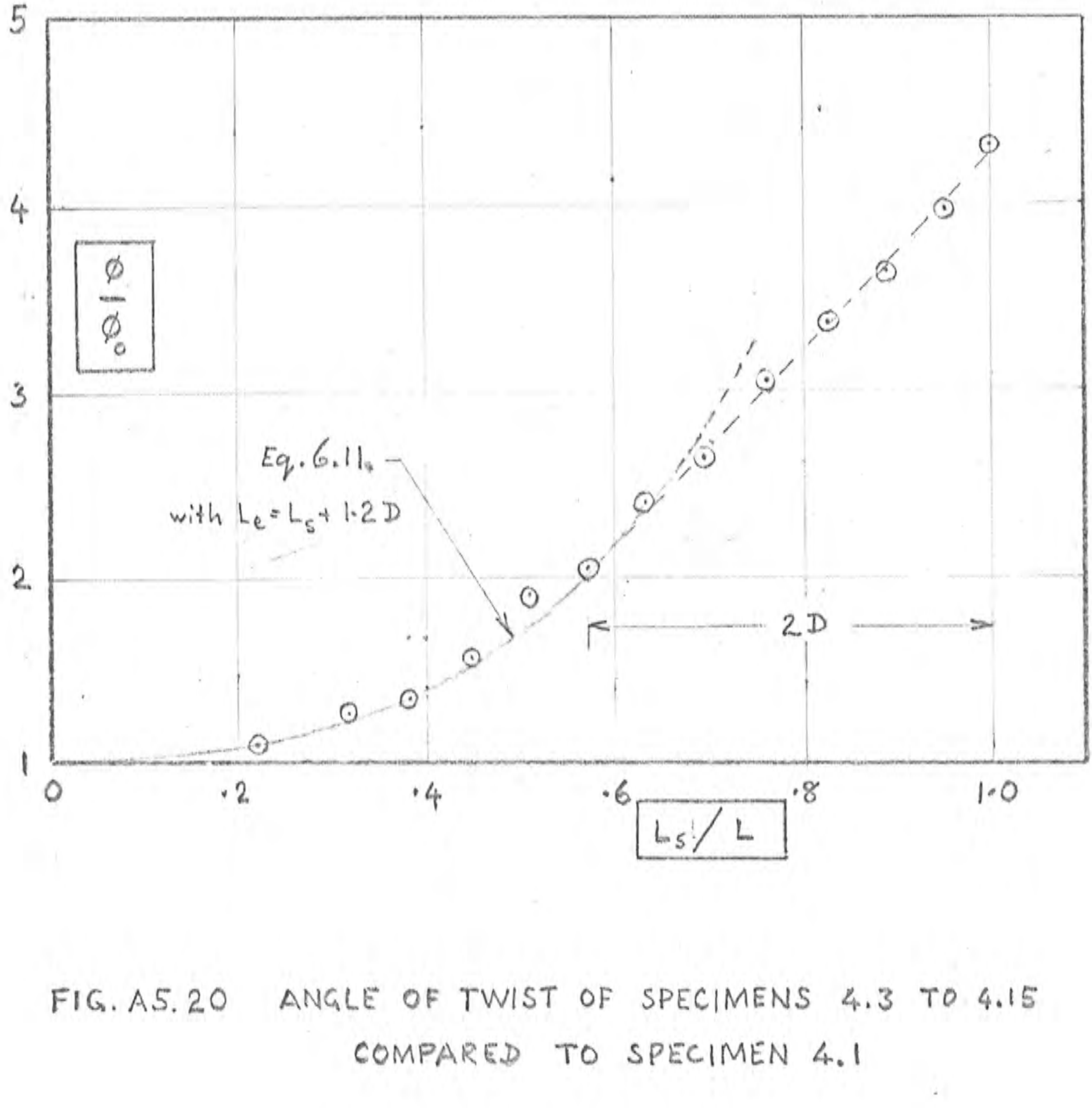


FIG. A5.20 ANGLE OF TWIST OF SPECIMENS 4.3 TO 4.15 COMPARED TO SPECIMEN 4.1

(See also Figs. 8.2 and 8.6)

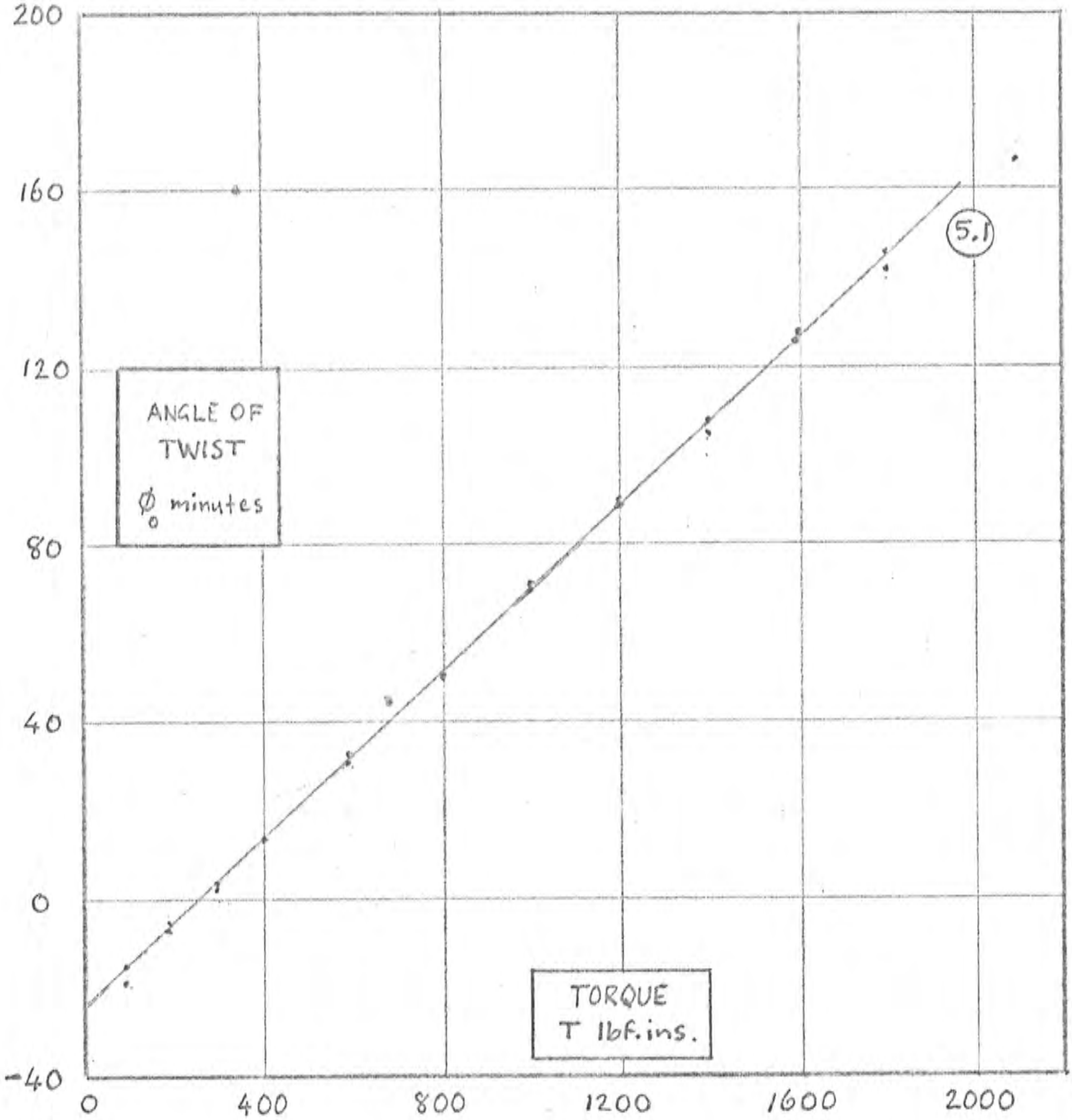


FIG. A5.21 (a) TORQUE ~ TWIST CURVE FOR SPECIMEN 5.1

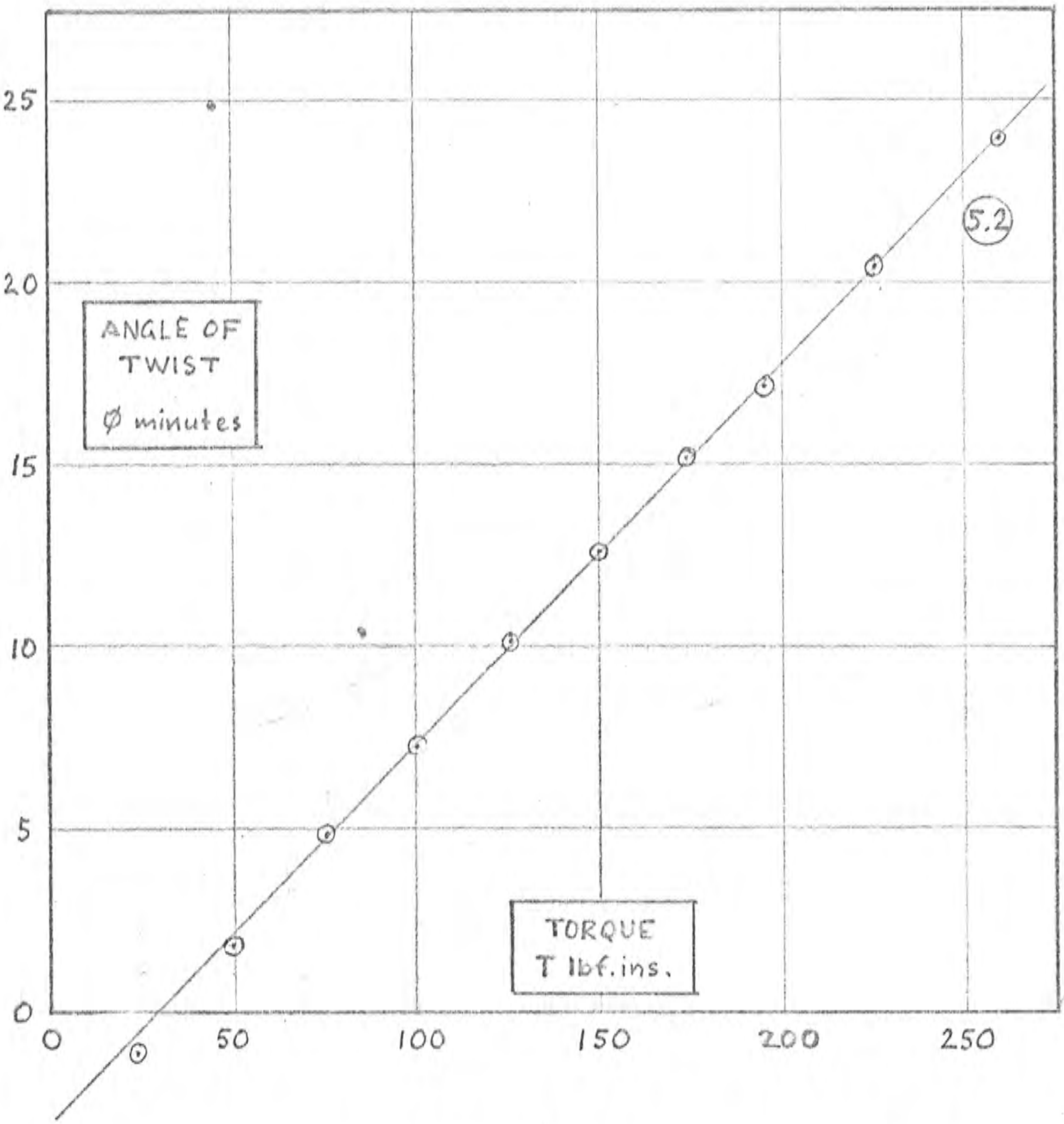


FIG. A5.21 (b) TORQUE ~ TWIST CURVE FOR SPECIMEN 5.2

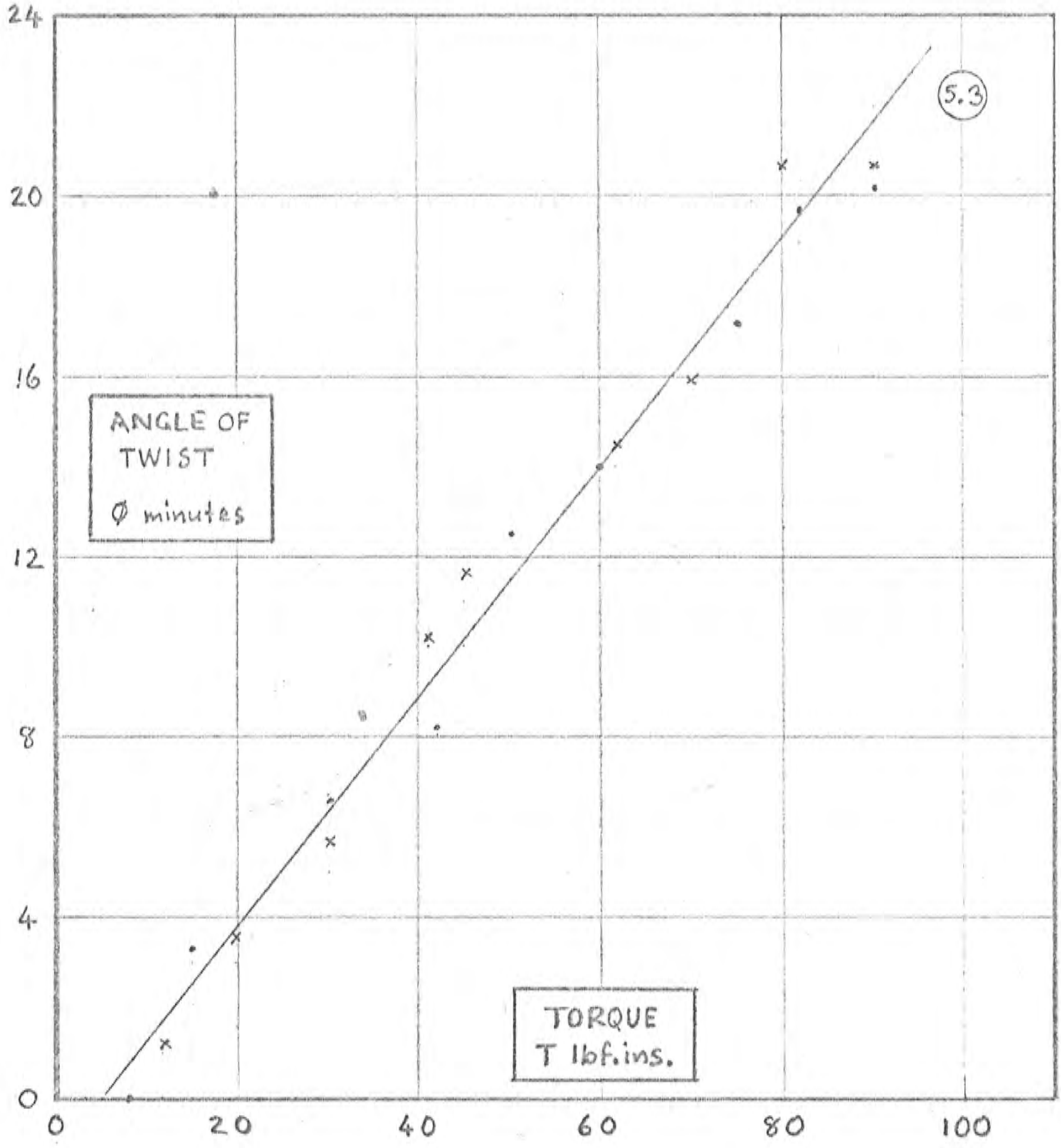


FIG. A5.21(c) TORQUE~TWIST CURVE FOR SPECIMEN 5.3

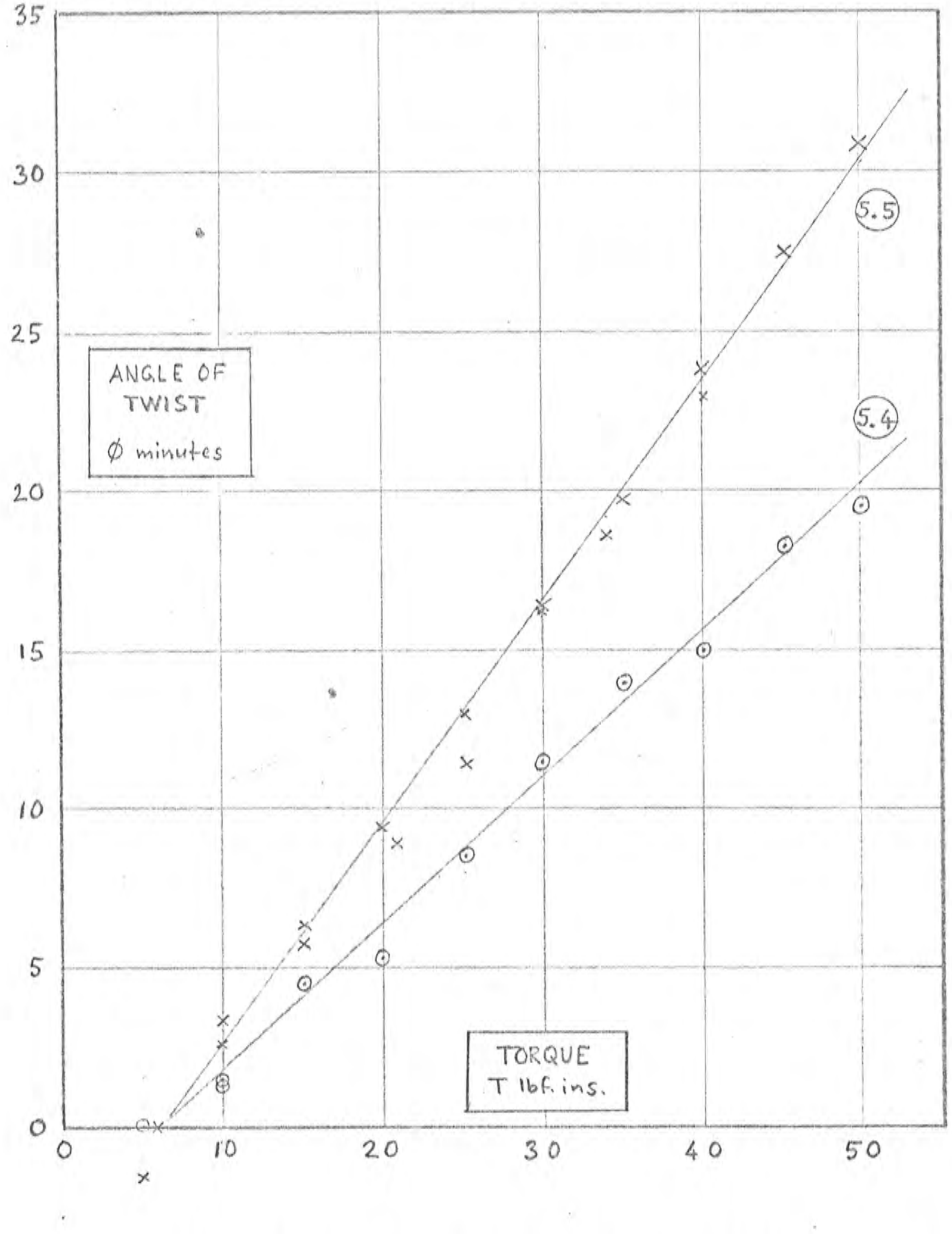


FIG. A5.21 (d) TORQUE ~ TWIST CURVES FOR SPECIMENS 5.4 & 5.5

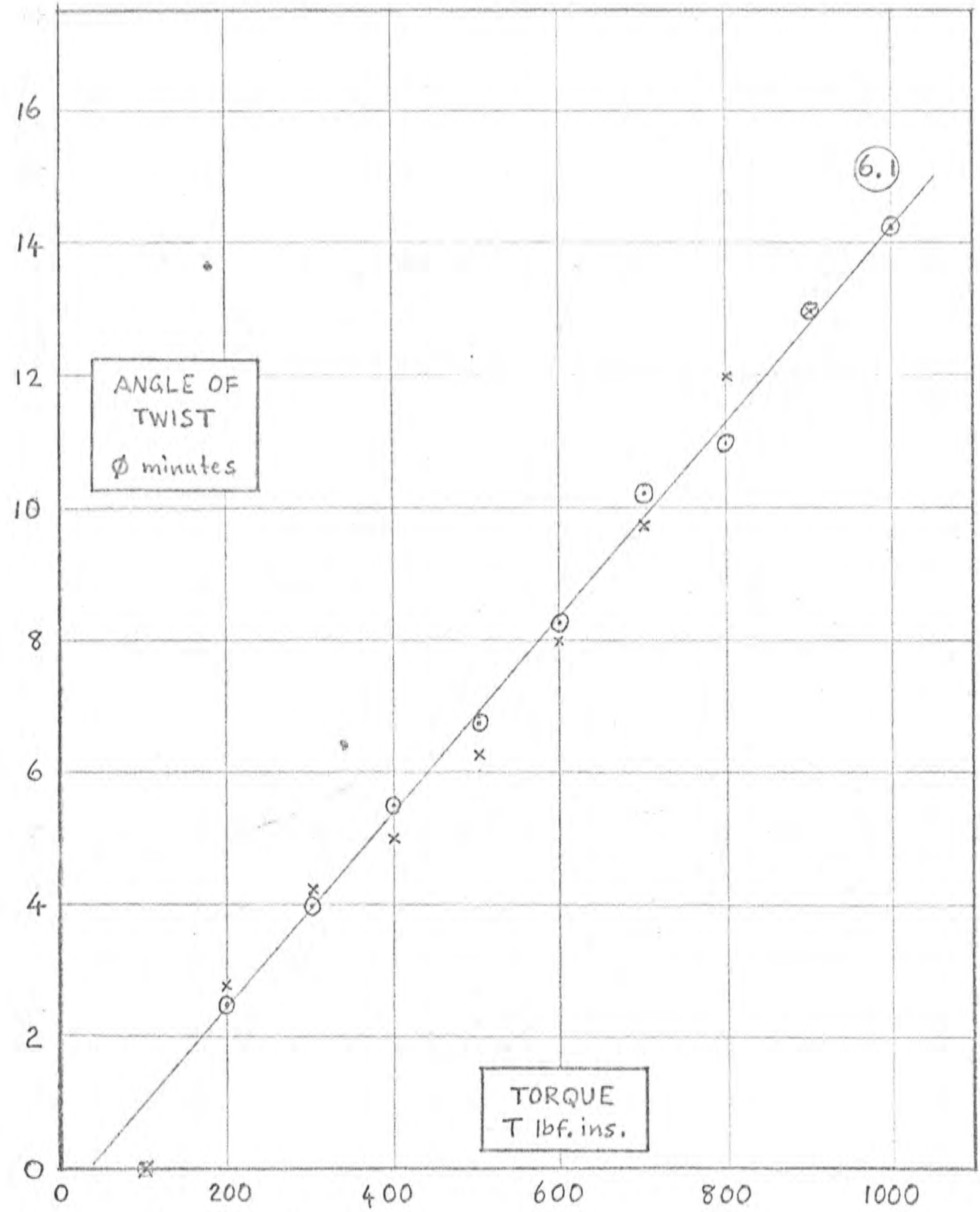


FIG. A5.22 TORQUE ~ TWIST CURVE FOR SPECIMEN 6.1

A6. THE CENTRE OF TWIST

It is shown in this chapter that if the thin-walled open-section beam twists about an axis other than the shear centre this will be shown up by changes in the angle of twist and the stress distribution.

The point in the plane of each cross-section through which a transverse load system must be applied to a beam such that only simple bending and no torsion are present is the shear centre of each cross section. The shear centre of a cross section is thus determined by the distribution of shear stresses across the section due to the above loading case (See for example Williams [2].

Conversely, under elastic conditions, any load which has a moment about the shear centre of any section of a beam (including reactions, and pure rotational torques) will tend to produce that same shear stress distribution and the beam will tend to rotate about the shear centre. (Reciprocal Theorem).

However the shear centre is not always the centre of twist even under elastic conditions; it will depend on other loadings present and on end conditions and other restraints on the structure. L. Beskin [3] gives a summary of the effects of some systems on the location of the centre of twist.

In particular, for very short span structures - i.e. shells - it will be the applied end shear flows that will be the important factor in determining the centre of twist. Where this "applied centre of twist" differs from the natural shear centre it can be expected that a certain transition length is needed wherein the centre of twist moves from its forced position at the end section to the shear centre. See Mar'in [2]

Such a transition case will not be investigated analytically here, merely the case of complete rotation of a tube about its geometrical axis to indicate a limiting situation.

The way in which the location of the centre of twist is related to the stress distribution and angle of twist is shown by taking the moment arm m, in all previous equations, to be measured from the new centre of twist instead of the shear centre.

Consider the tube mentioned above, constrained to rotate about the structural axis when subjected to an end torque. The variable moment arm m becomes the constant mean radius of the tube, a. Then

$$w = \int_0^s m \, ds \qquad \text{(Eq.4.11)}$$

$$w_a \rightarrow \int_0^s a \, ds$$

$$= as$$

or

$$w_a = a^2 \varphi \qquad \text{Eq.A.6.1}$$

Also

$$S_w = \int_s^{s_o} w \, ds \qquad \text{(Eq.4.15)}$$

and substituting from Eq.A.6.1

$$S_{w_a} \rightarrow \int_{\varphi}^{\varphi_o} a^3 \varphi \quad d\varphi$$

giving

$$S_{wa} = \frac{a^3}{2} \left(\varphi_0^2 - \varphi\right) \qquad \text{Eq.A.6.2}$$

which for the slit tube is

$$S_{wa} = \frac{a^3}{2} \left(\pi^2 - \varphi^2\right) \qquad \text{Eq.A.6.3}$$

The flexural warping rigidity is changed as follows,

$$J_w = t \int_{-s_0}^{s_0} m\, S_w \, ds \qquad \text{(Eq.4.18)}$$

substituting from Eq.A.6.2

$$J_{w_a} \rightarrow t \int_{-\varphi_0}^{\varphi_0} \frac{a^5}{2} \left(\varphi_0^2 - \varphi^2\right) \, d\varphi$$

$$= \frac{a^5 t}{2} \left[\varphi_0^2 \varphi - \frac{\varphi^3}{3}\right]_{-\varphi_0}^{\varphi_0}$$

$$J_{w_a} = \frac{2}{3} \; a^5 t \; \varphi_0{}^3 \qquad \text{Eq.A.6.4}$$

which for the slit tube is

$$J_{w_a} = \frac{2\pi^3}{3} \; a^5 t \qquad \text{Eq.A.6.5}$$

Compared with Eq.1.4, this means the tube is now 2.55 times stiffer in terms of its flexural torsion warping rigidity EJ_w. Also the number k will be divided by $\sqrt{2.55} = 1.6$ compared with its value for the same beam rotating about the shear centre.

The last section constant affected is J_s, which occurs in the flexural torsional shear rigidity GJ_s.

Let $S_{ww} = t \int_{-s_0}^{s_0} S_w^{\;2} \; ds$

substituting from Eq.A.6.2

$$S_{wwa} = t \int_{-\varphi_0}^{\varphi_0} \frac{a^7}{4} \left(\varphi_0^{\,4} + \varphi^4 - 2 \varphi_0^{\,2} \varphi^2 \right) d\varphi$$

$$= \frac{ta^7}{4} \left[\varphi_0^{\,4} \varphi + \frac{\varphi^5}{5} - \frac{2}{3} \varphi_0^{\,2} \varphi^3 \right]_{-\varphi_0}^{\varphi_0}$$

$$S_{wwa} = \frac{4}{15} \; a^7 \; \varphi_0^{\,5} t \qquad \text{Eq.A.6.6}$$

which for the slit tube is

$$S_{wwa} = \frac{4 \; \pi^5}{15} \; a^7 \; t \qquad \text{Eq.A.6.7}$$

This changes J_s, which is

$$J_s = \frac{(J_w)^2}{S_{ww}} \qquad \text{(Eq.5.3)}$$

to

$$J_{sa} = \frac{5}{3} \; a^3 t \; \varphi_0 \qquad \text{Eq.A.6.8}$$

which for the slit tube is

$$J_{sa} = \frac{5 \; \pi}{3} \; a^3 t \qquad \text{Eq.A.6.9}$$

which makes the shear rigidity $\frac{4}{5}$ that for a rotation about the shear centre.

Normal stress distribution, rotation about the centre

The membrane stress is

$$\sigma = \frac{T}{J_w} \quad w\ z \qquad \text{(Eq.4.21)}$$

With the new values of w and J_w from Eq.A.6.1 and Eq.A.6.5 this gives, for the slit tube, for rotation about the tube centre,

$$\sigma_a = \frac{3}{2} \quad \frac{T}{\pi^3 \ ta^3} \quad \varphi\ z$$

or

$$\sigma_a = .387 \ \frac{T}{td^3} \ \varphi\ z \qquad \text{Eq.A.6.10}$$

Compared with

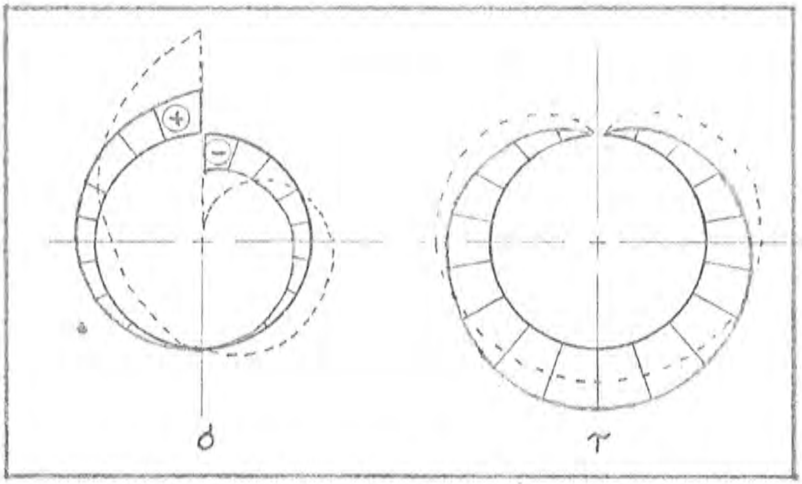
$$\sigma = .988 \ \frac{T}{td^3} \ (\ \varphi - 2 \sin \varphi\ )\ z \qquad \text{(Eq.4.22)}$$

which is the stress distribution for rotation of the slit tube about the shear centre, the stress now changes linearly around the circumference changing sign only opposite the slit. This is illustrated in Fig. A.6.1. The maximum values, at the slit, are only 39% of the "shear centre" values.

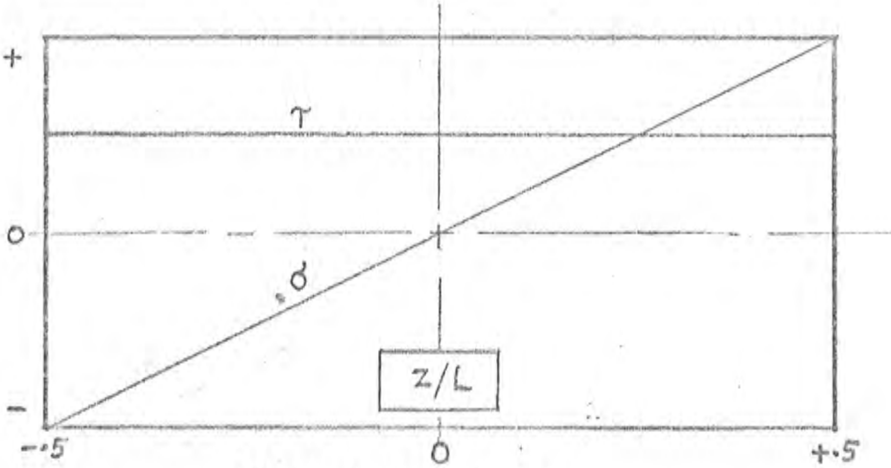
Shear stress distribution, rotation about tube centre

Substituting Eqs.A.6.3 and A.6.5 in the membrane shear stress equation,

$$\tau = \frac{T}{J_w} \quad S_w \qquad \text{(Eq.4.25)}$$



(a) Circumferential Distribution



(b) Axial Distribution

(Dotted lines in (a) show "shear centre" distribution) from FIG. 2.1

FIG. A6.1 THEORETICAL STRESSES FOR SLIT TUBE WITH BOTH ENDS CONSTRAINED TO REMAIN PLANE AND SUBJECTED TO A UNIFORM TORQUE, IF ENTIRE TUBE ROTATES ABOUT ITS GEOMETRIC AXIS (RATHER THAN THE LINE OF SHEAR CENTRES) gives

$$\tau_a = \frac{3T\,(\pi^2 - \varphi^2)}{2\,\pi^3\,a^5 t}\;\frac{a^3}{2}$$

or

$$\tau_a = .0968\;\frac{T}{td^2}\;(9.87 - \varphi^2) \qquad \text{Eq.A.6.11}$$

Compared with

$$\tau = .247\;\frac{T}{td^2}\;(5.87 - \varphi^2 - 4\cos\varphi) \qquad \text{(Eq.4.26)}$$

which is the shear stress for rotation about the shear centre, the shear stress is now a simple parabolic distribution from zero at the slit to a maximum opposite the slit that is 10% higher than the maximum "shear centre" values which occur 70° either side of the slit.

See Fig. A.6.1 .

<u>Angle of twist, rotation about tube centre</u>

Consider, for example, the energy solution Eq.5.43 in the form

$$\frac{\phi}{\phi_o} = K_s\;\frac{J_o}{J_s} + K_B\;\frac{L^2}{12}\;\frac{G}{E}\;\frac{J_o}{J_w} + K_T\;\frac{J_o}{J} \qquad \text{(Eq.5.46)}$$

For a rotation about the tube centre, the values of J_s and J_w change according to Eqs.A.6.8 and A.6.5 respectively, and the value of k in the hyperbolic factors K changes.

For a short slit tube, k $< .3$ say, K_T is negligible and Eq.5.46 may be written, for a rotation about the tube centre, (from Eq.5.44),

$$\frac{\phi}{\phi_o} = \frac{1 + \frac{k_a^2}{3}}{1 + k_a^2} \frac{J_o}{J_s} + \frac{1}{1 + k_a^2} \frac{L^2}{12} \frac{G}{E} \frac{J_o}{J_w} \quad - \text{Eq.A.6.12}$$

Substituting for J_{sa} and J_{wa} from Eq.A.6.9 and A.6.5

$$\frac{\phi}{\phi_o} \simeq 1.2 \left(1 - \frac{2}{3} k_a^2\right) + .1 \left(1 - k_a^2\right) \frac{G}{E} \left(\frac{L}{D}\right)^2$$

where

$$k_a^2 = \frac{k^2}{2.55}$$

For small values of k_a, Eq.A.6.12 tends to

$$\frac{\phi}{\phi_o} = 1.2 + .1 \frac{G}{E} \left(\frac{L}{D}\right)^2 \qquad \text{Eq.A.6.13}$$

This equation may be compared with Eq.5.19 to give an indication of the increase in stiffness caused by a rotation about the tube centre, apart from the case of the very short span shell, $\frac{L}{D} < 1.95$, where a rotation about the tube centre gives a greater angle of twist.

If $\frac{G}{E}$ is again given the typical value .39, Eq.A.6.13 gives, for $\frac{L}{D}$ values 1, 3, 6, 10, respectively $\frac{\phi}{\phi_o}$ = 1.24, 1.55, 2.6 and 5.1; whereas Eq.5.19 gives respectively, 1.1, 1.9, 4.6 and 11.

Consider a <u>real tube</u> slit full-length and with ends constrained to remain plane. If the end torques are applied about the tube centre-line, it might be expected that a certain transition length would be needed for the centre of twist to shift from the tube centre line at the end plates before coinciding with the natural shear centre in the central portion of the tube span. This transition distance might depend on the bending stiffness of the tube.

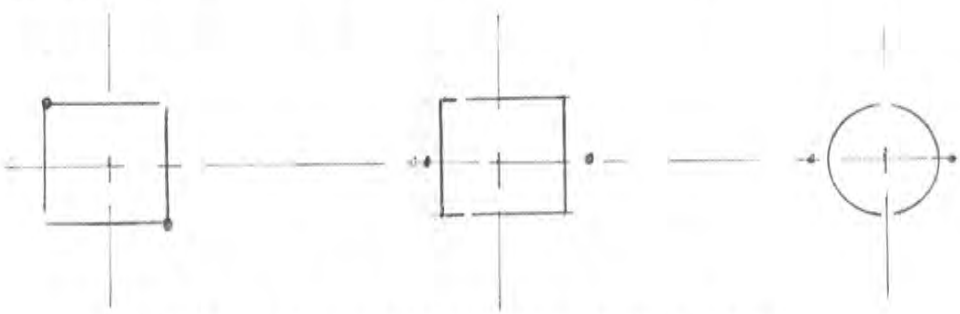
The angle of twist of such a real tube can be expected to lie between the "shear-centre" values (Eq.5.19) and the "tube-centre" values (Eq.7.13)

Part-length slit tube

In the case of the part-length slit tubes considered in the previous chapter, it is possible that the slit portion of the span will twist effectively about its shear centre provided the slit does not intrude far into the transition length mentioned above.

Double-slit tube - angle of twist

An interesting case of the effect of change of centre of twist occurs when a beam with a doubly symmetric section is formed from two separate equal beams which may have symmetry about one axis or be asymmetric; the separate beams are not welded together along the length but are fixed to the same end plates so that the twin units act as a single beam



SYMMETRICAL COMPOSITE SECTIONS FORMED FROM

UNSYMMETRICAL COMPONENT SECTIONS

FIG. A.6.2

The heavy dots in Fig.A.6.2 represent the approximate location of the shear centre of the separate angles, channels or semi-circles. (see Roark [1]). The shear centre for the composite unit may be expected to be at or near the centre of gravity of the section, when the torsional stiffness may be increased in the order of a hundred fold compared to the stiffness of the separate sections.

Consider the double semi-circular section represented by Fig.A.6.2 (c). This may be thought of as a thin-walled round tube welded to an end plate at both ends, then two full length slits sawn through the tube wall diametrically opposite each other.

For a rotation about the centre of the radius of the arc of the single semi-circle,

$m = a, \quad \varphi_0 = \pi/2$

from Eq.A.6.4,

$$J_{wa} = \frac{\pi^3}{12} a^5 t \qquad \text{Eq.A.6.14}$$

from Eq.A.6.8

$$J_{sa} = \frac{5\pi}{6} a^3 t \qquad \text{Eq.A.6.15}$$

The new value of k is

$$k_a = \frac{L}{2} \sqrt{\frac{GJ}{EJ_{wa}}} \qquad \text{(from Eq.1.2)}$$

$$= \frac{L}{2} \sqrt{\frac{G}{E}} \sqrt{\frac{2}{3} \pi a t^3 / \frac{\pi^3}{12} a^5 t}$$

$$k_a = 1.8 \; \frac{L}{D} \cdot \frac{t}{D} \cdot \sqrt{\frac{G}{E}}$$ Eq.A.6.16

For example take the case when $k_a < .3$ say, so that k_a^2 is small, and Eq.5.46 may be used again in the form

$$\frac{\phi}{\phi_o} = \frac{1 + \frac{k_a^2}{3}}{1 + k_a^2} \; \frac{J_o}{J_{sa}} + \frac{1}{1 + k_a^2} \; \frac{L^2}{12} \; \frac{G}{E} \; \frac{J_o}{J_{wa}}$$

(from Eq.5.44)

Substituting from Eqs.A.6.14 and A.6.13 for J_{sa} and J_{wa} respectively,

$$\frac{\phi}{\phi_o} \simeq 2.4 \left(1 - \frac{2}{3} \; k_a^2\right) + .75 \left(1 - k_a^2\right) \frac{G}{E} \left(\frac{L}{D}\right)^2$$

For the double slit tube the angle of twist is half this, viz.

$$\frac{\phi}{\phi_o} = 1.2 \left(1 - \frac{2}{3} \; k_a^2\right) + .375 \left(1 - k_a^2\right) \frac{G}{E} \left(\frac{L}{D}\right)^2$$

Eq.A.6.17

This result may be compared with the equivalent equation for a tube containing a single full length slit, (which normally will rotate about the shear centre). Eq.5.44 gives this as

$$\frac{\phi}{\phi_o} \simeq .96 \left(1 - \frac{2}{3} \; k^2\right) + .26 \left(1 - k^2\right) \frac{G}{E} \; \left(\frac{L}{D}\right)^2$$

Eq.A.6.18

where

$$k_a^2((\text{double slit tube})) = \frac{k^2}{.31} \; ((\text{slit tube}))$$

Now the angle of twist of a single semi-circular section (which will normally rotate about the shear centre) is not twice Eq.A.6.17, but, from Eq.5.44, and Fig. 5.1,

$$\frac{\phi}{\phi_o} = 61.3\ K_s + 55.3\ \frac{G}{E}\left(\frac{L}{D}\right)^2 K_B + 1.5\left(\frac{D}{t}\right)^2 K_T$$

– Eq.A.6.19

The difference in order between this and Eq.A.6.17 is easily seen.

Such effects could be important in determining the flexural axis and the torsional stiffness of aircraft wing structures. See Shanley [1].

<u>Stresses in double-slit tube</u>

The membrane stress equations are

$$\sigma = \frac{T}{J_w}\ w\ z \qquad \text{(Eq.4.21)}$$

and

$$\tau = \frac{T}{J_w}\ S_w \qquad \text{(Eq.4.25)}$$

Values of w, S_w and J_w are obtained for a semi-circle rotating about the arc radius centre as follows, (substituting $\varphi_o = \pi/2$),

$$w_a = a^2\varphi \quad \ldots\ldots(i) \qquad \text{(Eq.A.6.1)}$$

$$S_{wa} = \frac{a^3}{2}\left(\varphi_o^{\,2} - \varphi^2\right) \qquad \text{(Eq.A.6.2)}$$

$$= \frac{a^3}{2} \left(\frac{\pi^2}{4} - \varphi^2 \right) \quad ....(ii)$$

$$J_{wa} = \frac{2}{3} \; a^5 t \quad \varphi_o^{\,3} \qquad \text{(Eq.A.6.4)}$$

$$= \frac{\pi^3}{12} \; a^5 t \qquad ....(iii)$$

Substituting (i), (ii) and (iii) in Eqs.4.21 and 4.25 gives the approximate stress distribution for the double slit tube with rigid end plates,

$$\sigma_a = \frac{T\, a^2 \, \varphi \, z}{\frac{\pi^3}{12} \; a^5 \; t}$$

$$= \frac{12\, T \, \varphi \, z}{\pi^3 \, a^3 \, t}$$

or

$$\sigma_a = 3.1 \; \frac{T}{D^3 t} \quad \varphi \;\; z \qquad \text{Eq.A.6.20}$$

which has a maximum value with $\varphi = \pm \dfrac{\pi}{2}$ and $z = \pm \dfrac{L}{2}$,

$$\sigma_{max} = 2.43 \; \frac{TL}{D^3 t}$$

And

$$\tau_a = \frac{T \; \frac{a^3}{2} \left(\frac{\pi^2}{4} - \varphi^2 \right)}{\frac{\pi^3}{12} \; a^5 \; t}$$

$$= \frac{6T}{\pi^3 a^2 t} \left(\frac{\pi^2}{4} - \varphi^2 \right)$$

or

$$\tau_a = .775 \frac{T}{D^2 t} \quad (2.47 - \varphi^2) \qquad \text{Eq.A.6.21}$$

which has a maximum value with $\varphi = 0$

$$\tau_{max} = 1.915 \frac{T}{D^2 t}$$

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