EXAMPLES IN THE SYMBOLIC CALCULUS FOR MEASURES

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SUMMARY

The work is a contribution to attempts to frame converses to the generalized Wiener-Levy theorem, that (essentially) only real-analytic functions *operate on* the Gelfand transforms of measures. Methods have been developed by William Moran to exploit analytic structure in the maximal ideal space of the measure algebra of a locally compact abelian group to establish results of the kind wanted. These methods are employed to find measures on any locally compact abelian group on which only analytic functions operate. These measures arise from Bernoulli convolutions.

The extensive machinery necessary is first developed in part 2, after a detailed description of the problem and its context in part 1; in part 3 the three special cases of the circle group, the groups of p-adic integers, and infinite products of finite abelian groups are treated in detail. In the third case it is necessary to distinguish the case where the orders of the groups in the product are bounded. Finally a general statement for all locally compact abelian groups is deduced.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma and to the best of my knowledge and belief contains no material previously published or written by another person except where due reference is made in the text.

Edwin Ronald Coleman.

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PART ONE : THE PROBLEMS CONSIDERED, RESULTS DISCUSSED AND METHODS USED

1.1 THE PROBLEM : FUNCTIONS THAT OPERATE ON FOURIER TRANSFORMS

1.2 SUFFICIENT CONDITIONS FOR OPERATION

1.3 SOME NECESSARY CONDITIONS

1.4 RESULTS OF THE PRESENT WORK

1.5 METHODS USED HERE

1.6 ORGANISATION OF THE WORK

1.1 THE PROBLEM : THE FUNCTIONS THAT OPERATE ON FOURIER TRANSFORMS

The questions we consider are of this form: what conditions ensure that a function F applied to any transform in a set $A^\wedge$ always yields a transform in set $B^\wedge$? We restrict our attention to measures, so we are asking when

$$A^\wedge, B^\wedge \subseteq M(G)^\wedge \Rightarrow (\forall \hat{\mu} \in A^\wedge)(\exists \hat{\nu} \in B^\wedge) : F(\hat{\mu}(\gamma)) = \hat{\nu}(\gamma) \quad \forall \gamma$$

We say that F *operates from* $A^\wedge$ *to* $B^\wedge$ if this is so; when $A^\wedge = B^\wedge$ we say F operates in $A^\wedge$.

The questions of the form indicated may be thought of as seeking converses to the Wiener-Lévy theorem and its generalisations:

Generalised Wiener-Lévy Theorem

Every real-entire function operates in $M(G)^\wedge$;

every real-analytic function operates in $L^1(G)^\wedge$.

We seek conditions on F and/or sets $A^\wedge, B^\wedge$ which entail that F has an analytic property such as being holomorphic in some disc, or real-analytic in some region, or entire or ... .

1.2 SUFFICIENT CONDITIONS

The first result of the kind considered was the

Theorem of Wiener (1932) [WIENER, TT, lemma IIe]

If $f \in L^1(T)$ has absolutely convergent Fourier series, and $\hat{f}$ is never zero, $1/f$ also has ACFS.

This was almost immediately extended by the

Theorem of Lévy (1934) [LÉVY, SCA, théoreme V]

Si $y = f(x)$ est representable par une série F [i.e. has ACFS], et si $z = F(y)$ est une fonction holomorphe pour toute les valeurs de y prises par $f(x)$ pour les valeurs réelles de x, la fonction $F[f(x)]$ est répresentable par une série F.

The following further extensions are proved, e.g. in [FAG, p. 133] by essentially the techniques of Wiener and Levy. The theory of Banach algebras could also be used [AC].

Generalised Theorem of Wiener and Lévy (GWL) [RUDIN, FAG, p. 133]

1. If F is real-entire then F operates in $M(G)^\wedge$.
2. If F is real-analytic in some open set about 0 and $F(0) = 0$, and G is compact, then F operates in $L^1(G)^\wedge$.
3. If F is real-analytic in some open set $E \subseteq C$, $f \in L^1(G)$ and $(\hat{f}(\Gamma))^- \subseteq E$, then $F \circ f \in L^1(G)^\wedge$. (For G not discrete, we need $F(0) = 0$.)

A number of variations are available, for example:

Theorem of Katznelson

If B is a regular semisimple self-adjoint Banach algebra with unit and f is a continuous function on $\Delta(B)$ such that in an open set around each $m_0 \in \Delta(B)$, f can be written as $F(\hat{x})$, where $F(\zeta) = F(\xi+i\eta)$ is real-analytic in ξ and η in a neighbourhood of $\hat{x}(m_0)$, then $f \in B^\wedge$.

[Katznelson, IHA, p. 236]

The problems of the symbolic calculus for measures concern identifying *necessary* conditions on functions operating from one set of transforms to another.

1.3 SOME NECESSARY CONDITIONS

There are global results, such as the converse of the GWL:

Theorem of Helson, Kahane, Katznelson and Rudin [RUDIN, FAG, §6.9]

1. If F operates in $M(G)^\wedge$ and G is not discrete, then F extends to a real-entire function.
2. If F operates in $L^1(G)^\wedge$ and G is compact, then F is real-analytic in some open set about 0.
3. If F operates in $L^1(G)^\wedge$, G is not compact and E is a closed convex subset of C, then F is real-analytic on E (not just on E^0).

Theorem of Varopoulos [VAROPOULOS, 1965]

If F defined on $[-1,1]$ operates in $M_0(G)^\wedge$ (where G is compact and infinite), then F agrees with an entire function in some open set about 0.

Theorem of Herz and Rider [HERZ 1963; RIDER 1971]

If F operates in $PD(\Gamma)$ (for infinite Γ not a finite group x a group of exponent 2) then

$$F(z) = \sum a_{nm} z^m \bar{z}^n \qquad z \in D(\Gamma)^0$$

$$\text{with } a_{nm} \geq 0 \text{ and } \sum a_{nm} < \infty.$$

Theorem of Moran (1) [MORAN, ICSM, p. 407]

If F operates from $PD(\Gamma)$ to $B(\Gamma) = M(G)^\wedge$ for Γ not "exceptional", then $F(z) = \sum a_{mn} z^m \bar{z}^n$, $z \in D(\Gamma)^0$

$$\text{with } \sum |a_{mn}| < \infty$$

Moran [ICSM1, 414] gives the counterexample of

$$\sum_{n=0}^{\infty} n^k (\tfrac{1}{2}(z_p - \bar{z}_p))^n$$

where G is non-discrete with no perfect Kronecker or K_p set $p > 2$, with finite exponent p. Graham [ECHA, 279] also shows

$F(z) = (2 - \beta z - \bar{z})^{-1}$ for $\beta \in T$ of infinite order to operate from $PD(\Gamma)$ to $B(\Gamma)$ for exceptional G. (Γ is not "exceptional" if infinite and either has no compact open subgroups of exponent 2 or else has a compact open subgroup H s.t. G/H has elements of infinite order.)

In contrast to these global results, in the individual symbolic calculus we investigate which functions can operate on *specific* transforms. The idea is to identify "difficult" measures which force analytic properties onto any function operating on them. Typical results include

Theorem of Katznelson [KATZNELSON, IHA, p. 248]

(A) There is a measure κ on R with real $\hat{\kappa}$, of norm ≤ 2, containing every polynomial with rational coefficients of norm ≤ 1; for this κ

(B) If $F \circ \hat{\kappa}$ is a transform of a measure then F is analytic at 0; if $F \circ (c\hat{\kappa}) \in M(R)\hat{}$ for all c, F is entire.

the

Theorem of Moran (2) [MORAN, ICSM1, p. 401]

Let μ be a continuous probability measure on a Kronecker set in T; if F is continuous on the closed unit disc and operates on μ then $F = \sum a_n z^m \bar{z}^n$, $\sum |a_n| < \infty$.

and the

Theorem of Kaufman [SCBC]

The Bernoulli convolution μ_0 constructed in his paper has the property: any function operating on μ_0 is analytic in the unit disc.

1.4 RESULT OF THIS WORK

We establish the following main result.

1.4.1 For any nondiscrete LCA group G there is a class of infinite convolution probability measures on which a continuous function on $[-1,1]$ can only operate if it has the form

$$F(x) = \sum_{n=0}^{\infty} b_n x^n, \quad \sum_{n=0}^{\infty} |b_n| < \infty .$$

1.5 METHODS USED

The results use the methods of Moran [ICSM2]. The characteristic of this approach is the exploitation of analytic structure in $\Delta(M(G))$. The key ideas are these:

(1) if $F \circ \hat{\mu}(\gamma) = \hat{\nu}(\gamma) \quad \gamma \in \Gamma$

then we also have, if we assume or show F continuous, that

$$F(\hat{\mu}(\zeta)) = \hat{\nu}(\zeta)$$

for every $\zeta \in \Gamma^-$, the closure of Γ in $\Delta M(G)$.

(2) for the measures of interest, there are generalised characters $\zeta \in \Gamma^-$ such that ζ_μ is c where c is any constant in the interval $[-1,1]$.

(3) using these generalised characters it is shown via Choquet's theorem that a function operating on μ must be a convex combination of functions of the form x^t, $t \geq 0$.

(4) Finally one investigates which values of t are actually possible.

1.6 ORGANISATION OF THE WORK

In part 2, the standard terminology, machinery and needed facts from the classical theories of LCA groups, the Fourier transform, measures, complex functions and Banach algebras are set out. The aims of this exposition are to establish a consistent notation and to make the work conceptually self-contained as far as practical.

Also in part 2 relevant elements of the more recent "convolution algebra" theory of the maximal ideal space of $M(G)$ for LCA groups G are set out. The basis of this is the representation of ΔMG a space of generalised characters in which a number of operations may be defined, namely multiplication $\zeta.\eta$, multiplication by measures $\zeta.\mu$, conjugation $\bar{\zeta}$, absolute value $|\zeta|$, polar decomposition ζ^0, exponentiation ζ^z, and adjoint $\tilde{\zeta}$. Exponentiation is particularly important for us because the combination of continuous operating function and measure μ for which there are generalised characters ζ with constant μ coordinate ζ_μ enable us to extend the operation of γ to the closure Γ^- of Γ in ΔMG in such a way that the analytic properties of F may be proved.

In part 3 we define the class of measures μ to be investigated and establish the necessary facts about $C(\mu)$, the set of constants in Γ^-. Generalisation of results of Moran in [ICSM2] enable us then to establish our results about infinite convolutions.

PART TWO : THE CONVOLUTION ALGEBRA M(G) AND ITS DUAL B(Γ)

2.1 LCA GROUPS

2.1.1 LCA GROUPS AND THEIR DUALS

2.1.2 STRUCTURE THEOREMS

2.2 BANACH ALGEBRAS

2.2.1 DEFINITION, EXAMPLES

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2.3 MEASURE ALGEBRAS

2.3.1 MEASURES

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2.4.1 B(Γ), THE TRANSFORMS OF M(G)

2.4.2 TRANSFORMS ON SUBGROUPS AND QUOTIENTS

2.5 M(G) AS A SPACE OF GENERALIZED CHARACTERS

2.5.1 COMPLEX HOMOMORPHISMS AS GENERALIZED CHARACTERS

2.5.2 CALCULUS OF GENERALIZED CHARACTERS

2.5.3 CONVERGENCE OF GENERALIZED CHARACTERS

A Note on References

I have not given explicit references for every item in this chapter; it is all standard material and any item will be found in RUDIN [RCA],[FAG], KATZNELSON [IHA], BROWN and MORAN [BMA], GRAHAM and McGEHEE [ECHA], or GELFAND, RAIKOV and SHILOV [CNR].

2.1 LCA GROUPS

2.1.1 LCA GROUPS AND THEIR DUALS

2.1.1.1 LCA groups

A *locally compact abelian group* G (LCA group) is an abelian group which is also a locally compact Hausdorff space such that

$$\langle x , y \rangle \mapsto x + y \quad \text{and} \quad x \mapsto -x$$

are continuous.

(DIGRESSION ON NOTATION

I follow the usual convention of referring to a complicated object by a simple name, as G, in keeping with the remarks of RUDIN [RCA, 18] which end

"it is a safe bet that very few mathematicians think of the real field as an ordered quadruple.")

We are interested in these groups:

$\mathbf{Z}$, the integers under addition;

$\mathbf{Z}_k = Z/kZ$, integers under addition modulo k;

$\mathbf{R}$, the real numbers under addition;

$\mathbf{T} = R/Z$, the circle group of complex numbers of norm 1;

(R and Z have the usual topology, Z_n and T the usual induced quotient topology.)

Δ_p, the group of p-adic integers; and

$Z(p^\infty)$, the group of all p-roots of unity.

The last two groups and their topology are described in 2.1.1.3 below.

2.1.1.2 Character group

A *character* χ of group G is a group homomorphism to T

$$\chi : G \mapsto T$$

i.e. $\chi(g_1 g_2) = \chi(g_1)\chi(g_2) \quad g_1, g_2 \in G.$

The set of all continuous characters under the operation defined by

$$\chi_1 \cdot \chi_2 \, (g) = \chi_1(g) \, . \, \chi_2(g)$$

is a topological group called the *dual group of* G, written Γ.

The *Pontryagin duality theorem* shows that the natural topology on Γ is derived from that on G in such a way that the dual of Γ defined in the same way is G. In view of this result it is standard to write $\chi(g)$ more neutrally as (g,χ), and with this notation the topologies of G and Γ are based respectively on the sets of translates of

$$N(K,r) = \{\gamma \in \Gamma \,|\, (g,\gamma) \in U_r \,, \quad g \in K\}$$

and

$$M(C,r) = \{g \in G \,|\, (g,\gamma) \in U_r \,, \quad \gamma \in C\}$$

where

$$U_r = \{z \in C \,|\, |1-z| < r\}$$

and K and C range over compact subsets of

G and Γ and $r \geq 0$.

2.1.1.3 <u>Δ_p and $Z(p^\infty)$</u>

Each of T and Z is the dual of the other, while R is its own dual as is each of the discrete groups Z_k. (Here we make the standard identifications, for example, of $z \in Z$ and the continuous character of T given by

$$t \mapsto \exp 2\pi izt.)$$

For each prime p, the groups Δ_p and $Z(p^\infty)$ are defined as follows:

Δ_p is the set of all sequences $(x_n)_{n=0}^{\infty}$

$$x_n \in \{0,1,2,\ldots,p-1\} = I_p$$

under addition defined inductively thus:

let $(x_n),(y_n) \in \Delta_p$ and suppose

$$x_{m_0} \neq 0 \text{ but } x_n = 0 \quad n < m_0$$

$$y_{n_0} \neq 0 \text{ but } y_n = 0 \quad n < n_0$$

put $p_0 = \min(m_0,n_0)$ and $z_n = 0$ for $n < p_0$.

Write $$x_{p_0} + y_{p_0} = z_{p_0} + t_{p_0}.p$$

where $$z_{p_0} \in I_p, \quad t_{p_0} = 0 \text{ or } 1.$$

Then assuming $z_{p_0}, z_{p_0+1},\ldots,z_k$ and $t_{p_0}, t_{p_0+1},\ldots,t_k$ have been defined we write

$$x_{k+1} + y_{k+1} + t_k = z_{k+1} + t_{k+1}.p$$

with $z_{k+1} \in I_p$ and $t_{k+1} = 0$ or 1.

This defines (z_n) by induction on n as the sum of (x_n) and (y_n). The zero for this operation is the all zero sequence and Δ_p is easily seen to be an abelian group under it, and a compact group under the topology induced by the metric

$$d((x_n),(y_n)) = 2^{-m}$$

where m is the least integer with $x_m \neq y_m$.

The dual group of Δ turns out to be the subgroup of T of p-power roots of unity

$$Z(p^\infty) = \{t \in T | t^{p^n} = 1 \text{ for some } n \in Z\}$$

(under the subgroup topology).

(*Two asides.* (1) Multiplication of p-adic integers, or numbers, is readily defined but we do not use it. (2) These definitions are easily generalised to the a-adic integers Δ_a as in HEWITT and ROSS [AHA 1, 108ff, 402] with dual

$$Z(a^\infty) = \{\exp 2\pi i(\frac{\ell}{a_0,\ldots,a_r}) | \ell \in Z, r \in Z^+\}$$

and most of what we prove about Δ_p could be carried over with little change. But there is no gain in our main result, so we eschew pointless generality.)

2.1.1.4 Infinite products of discrete groups

Z_{n_k}, for any $n_k > 0$, $n_k \in Z$, is of course both compact and discrete, being finite. The topological product of countably many such groups, where n_k may vary with k

$$\prod_{k=1}^{\infty} Z_{n_k}$$

however is a compact abelian group (under coordinate-wise addition) whose dual is the *weak product* $\prod_{k=1}^{\infty} {}^*Z_{n_k}$, that is, the subgroup of elements with only finitely many nonzero components, under the *discrete* topology. (cf RUDIN [FAG, 37]).

2.1.1.5 The Bohr compactification

Any LCA group G can be embedded as a dense subgroup of a compact abelian group $\bar{G}$ thus: let Γ be the dual of G, Γd be Γ with the discrete topology, $\bar{G}$ the dual of Γd. The map $\beta : G \mapsto \bar{G}$ defined by

$$(g,\gamma) = (\gamma,\beta(g)) \qquad g \in G, \gamma \in \Gamma$$

is a continuous isomorphism of G onto a dense subgroup $\beta(G)$ of $\bar{G}$ (but βG is *not* a locally compact subset of $\bar{G}$). G being the group of continuous characters on Γ, $\bar{G}$ is the group of *all* characters. (cf. RUDIN [FAG, 31]).

2.1.2 STRUCTURE THEOREMS

2.1.2.1 The principal structure theorem

Any LCA group G has an open subgroup G_1 which is the direct sum of a compact group H and a Euclidean space R^n, $n \geq 0$. (RUDIN [FAG, 40ff]).

This theorem will subsequently be used to reduce our problem to the compact case. In connection with this reduction we shall need these concepts:

the *order* of an element g is the least positive integer n so that $ng = 0$, or infinity; a group is *torsion* if every element has finite order; a group is *divisible* if for every $g \in G$ and $n \neq 0$, $n \in Z$, there is at least one h s.t. $nh = g$.

2.1.2.2 Infinite discrete torsion groups

Any infinite discrete torsion group has a subgroup isomorphic to $Z(p^\infty)$, or one isomorphic to a weak product $\prod_{k=1}^{\infty} {}^*Z_{n_k}$.

2.1.2.3 Theorem

Let P be the set of all primes. For all $p \in P$, let α_p be an arbitrary cardinal, possibly 0, let I_p be an arbitrary index class, possibly empty, and let r_i be an arbitrary positive integer for each $i \in I_p$. Let n be a cardinal that is 0 or 2^m for an infinite cardinal m. For all $p \in P$ let b_p be a cardinal not exceeding n such that b_p is finite or has the form 2^{e_p} for an infinite cardinal $e_p \leq m$. Every compact Abelian group is algebraically isomorphic with a group

$$\prod_{p \in P} [\Delta_p^{a_p} \times \prod_{i \in I_p} Z(p^{r_i})] \times \prod_{p \in P}{}^* Z(p^\infty)^{b_p^*} \times Q^{n^*}$$ [HEWITT & ROSS, AHA 1, 414]

2.2 BANACH ALGEBRAS

2.2.1 DEFINITION, EXAMPLES

2.2.1.1 Banach algebras

A *Banach space* X is complex normed vector space, complete in the norm metric. (A norm is a map $\|\cdot\|: X \to [0,\infty)$ s.t. $\|x+y\| \le \|x\| + \|y\|$, $\|\alpha x\| = |\alpha| \|x\|$, $\|x\| = 0 \Leftrightarrow x = \underline{0}$ for any $x,y \in X$ and $\alpha \in C$.)

A *Banach algebra* A is a Banach space in which a multiplication is defined making it also an algebra and so that

$$\|xy\| \le \|x\| . \|y\| \qquad x,y \in X .$$

(Note: we shall assume A is commutative $(xy = yx)$ and unital $(\exists e: ex = xe = x)$ since this is so for all examples of interest to us.)

2.2.1.2 Examples

The simplest example is C under 1.1. More interesting is $C(X)$, the algebra of all continuous complex-valued functions on a compact Hausdorff space X under pointwise addition and multiplication and the *sup norm*

$$\|f\|_\infty = \sup_{x \in X} |f(x)| .$$

The examples we are concerned with require convolution as multiplication and will be given below.

2.2.1.3 Quotient algebras

For any ideal I in a Banach algebra A, a quotient algebra A/I is defined by the natural multiplication

$$(x+I)(y+I) = xy + I$$

and the quotient norm

$$\|x+I\|_{A/I} = \inf_{y \in I} \|x+y\|_A .$$

2.2.2 GELFAND REPRESENTATION

2.2.2.1 Maximal ideal space

We denote by $\Delta(A)$ the set of complex homomorphisms of a Banach algebra A, i.e. the multiplicative linear functionals from A to C. In view of the following Gelfand theory, it is called the *maximal ideal space of* A.

(1) for any maximal ideal I of A, the canonical map $h: A \to A/I$ is in $\Delta(A)$, for A/I is C [Gelfand-Mazur theorem].

(2) for any $h \in \Delta(A)$, the kernel of h is a maximal ideal.

(3) $x \in A$ has a multiplicative inverse $\Leftrightarrow h(x) = 0$ for no $h \in \Delta(A)$;

$xy = x+y$ has a solution $y \Leftrightarrow h(x) = 1$ for no $h \in \Delta(A)$.

(4) any $h \in \Delta(A)$ is bounded with norm 1, hence continuous.

(5) The *Gelfand transform* is the map $x \to \hat{x}$ from A to $\Delta(A)$ given by

$$\hat{x}(h) = h(x) \qquad h \in \Delta(A).$$

Under the weak topology determined by the set of all $\hat{x}$, $\Delta(A)$ is a locally compact Hausdorff space, in fact a subspace of $C_0(\Delta(A))$, the bounded continuous functions from $\Delta(A)$ to C vanishing at ∞.

(6) In fact the Gelfand transform is a homomorphism mapping A to a subalgebra of $C_0(\Delta(A))$, for

$$(\hat{xy})(h) = h(xy) = h(x)h(y) = \hat{x}(h).\hat{y}(h)$$
$$\text{for all } x,y \in A, \quad h \in \Delta(A)$$

and so on. Notice that $\|\hat{x}\|_\infty \leq \|x\|$ since $\|h\| \leq 1$.

2.2.3 THE SPECTRUM AND THE CAUCHY FORMULA

2.2.3.1 Spectrum

For an element $x \in A$, the *spectrum of* x, $\sigma(x)$, is the set of $\lambda \in C$ for which $x - \lambda$ (i.e. $x - \lambda e$) is not invertible.

The spectral radius formula is contained in the theorem of Gelfand that

$$\rho(x) = \sup_{\lambda \in \sigma(x)} |\lambda| = \lim_{n \to \infty} \|x^n\|^{1/n}$$

for any $x \in A$.

In fact $\sigma(x) = \{\hat{x}(h) \mid h \in \Delta(A)\}$ and so we also have $\rho(x) = \|\hat{x}\|_\infty$.

2.2.3.2 Cauchy formula

If A is a *semisimple* Banach algebra, (i.e. the intersection of all maximal ideals is zero), and F is a function analytic in a region U of C containing $\sigma(x)$ for some $x \in A$, and if γ is any closed rectifiable curve in U enclosing $\sigma(x)$ with index 1 w.r.t. $\hat{x}(h)$ for all $h \in \Delta(A)$ and 0 for any point outside U, then

$$F(x) = \frac{1}{2\pi i} \int_\gamma \frac{F(z)}{z-x}\, dz$$

is a well-defined element in A not depending on γ such that

$$F(\hat{x})(h) = F(\hat{x}(h)) \quad \text{for each} \quad h \in \Delta(A).$$

2.2.3.3 Wiener-Levy theorem

For the algebra $A(T)$ of functions on T with absolutely convergent Fourier series

(i.e. $f(t)$ s.t. $f(t) = \sum \hat{f}(j) e^{ijt}$

with $\sum |\hat{f}(j)| < \infty$ and $\hat{f}(j) = \int_T e^{-ijt} f(t)dt$

as discussed in 2.4 below)

the preceding result specializes to the

WIENER-LEVY THEOREM

If F is a function analytic on an open set containing the range of f for $f \in A(T)$, then

$$g(t) = F(f(t))$$

is also in $A(T)$.

That is, analytic functions operate in $A(T)$.

2.3 MEASURE ALGEBRAS

2.3.1 MEASURES

2.3.1.1 Definitions

(We only discuss measures on LCA groups, but the concepts in this section apply to any locally compact Hausdorff space.)

The *Borel sets* $\mathcal{B}$ of G are those in the smallest family of subsets of G containing the closed subsets, and closed under complementation and countable union.

A *measure* on G is a (set) function

$$\mu : \mathcal{B} \to C$$

from the Borel sets to C which is

(a) *countably additive*, i.e.

$\mu(E) = \sum_n \mu(E_n)$ for any countable partition $\{E_n\}$ of E for $E \in \mathcal{B}$;

(b) *regular*, that is

$$|\mu|(E) = \sup_K |\mu|(K) = \inf |\mu|(V)$$

where K ranges over compact subsets of E,

and V ranges over open supersets of E

and $|\mu|(E) = \sup \sum_n |\mu(E_n)|$

(the sup being taken over all Borel partitions of E) is the *total variation of* μ which is also a countably additive set function on $\mathcal{B}$; and

(c) *finite*, that is

$$\|\mu\| = |\mu|(X) < \infty.$$

$M(G)$ is the set of all measures on G. We assume the standard theory of Lebesgue integration with respect to measures. $L^1(\mu)$

is the space of μ-integrable functions, etc. Occasionally we refer to "positive measures": these are not necessarily in $M(G)$ in having range $[0,\infty]$.

2.3.1.2 Support, etc.

For any $E \in \mathcal{B}$, the *restriction of* μ *to* E, μ_E, is defined by

$$\mu_E(B) = \mu(B \cap E) \text{ for each } B \in \mathcal{B}.$$

Iff $\mu = \mu_E$, μ is *concentrated on* E. The *support of* μ, $\text{supp}(\mu)$, is the intersection of all closed $B \in \mathcal{B}$ on which μ is concentrated.

Two measures are *mutually singular*, $\mu_1 \perp \mu_2$, iff they are concentrated on disjoint sets. μ_1 is *absolutely continuous* with respect to a positive measure μ_2, $\mu_1 \ll \mu_2$, iff $\mu_2(E) = 0 \Rightarrow \mu_1(E) = 0$ for $E \in \mathcal{B}$. A measure μ is *discrete* iff $\text{supp}\,\mu$ is countable, *continuous* iff every countable $E \in \mathcal{B}$ has $\mu(E) = 0$.

2.3.1.3 Decompositions

Every $\mu \in M(G)$ has a unique *Jordan decomposition*

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

with $\mu_i \geq 0$ (i.e. $\mu_i(E) \geq 0$, $E \in \mathcal{B}$), $\mu_i \in M(G)$, $i=1,2,3,4$ and $\mu_1 \perp \mu_2$, $\mu_3 \perp \mu_4$.

Every $\mu \in M(G)$ has a unique decomposition

$$\mu = \mu_d + \mu_c$$

with μ_d discrete and μ_c continuous.

Every $\mu \in M(G)$ has a unique *Lebesgue decomposition* with respect to any positive measure m

$$\mu = \mu_s + \mu_a$$

with $\mu_a << m$ and $\mu_s \perp m$.

2.3.1.4 $\tilde{\mu}$

There is an involution on $M(G)$ $\mu \to \tilde{\mu}$ given by $\tilde{\mu}(E) = \overline{\mu(-E)}$ for each Borel E. When $\mu = \tilde{\mu}$, μ is called Hermitian, and the Fourier-Stieltjes (not Gelfand) transform is real-valued.

2.3.1.5 M(G) is a Banach space

That $M(G)$ is a Banach space follows from the *Riesz Representation Theorem* (RRT):

for any bounded linear functional (BLF) Λ on $C_0(G)$ there is a unique $\mu \in M(G)$ s.t.

$$\Lambda f = \int_G f \, d\mu \qquad f \in C_0(G)$$

with
$$\sup_{\|f\| \leq 1} |\Lambda f| = \|\mu\| .$$

Thus $M(G)$ is the dual of $C_0(G)$. This theorem is the converse of the simple observation that

$$f \to \int_G f \, d\mu$$

is a BLF for any $\mu \in M(G)$.

Another important converse is the *Radon-Nikodym Theorem*:

corresponding to μ_a in the Lebesgue decomposition of μ w.r.t. m is $f \in L^1(m)$ s.t.

$$\mu_a(E) = \int_E f \, d\mu \qquad E \in \mathcal{B}$$

and
$$\|\mu_a\| = \int_G |f| dm = \|f\|_1 .$$

This is the converse of the observation that for $f \in L^1(m)$,

$$\mu(E) = \int_E f \, dm$$

defines a measure $\mu << m$.

2.3.2 THE ALGEBRA M(G)

2.3.2.1 Convolution

The addition and scalar multiplication of measures entailed by the RRT are the obvious ones:

$$(\mu+\lambda)(E) = \mu(E) + \lambda(E), \quad (c\mu)(E) = c.\mu(E)$$

with the norm

$$\|\mu\| = |\mu|(G)$$

as above. To make M(G) a Banach algebra we need a suitable "multiplication" of measures and we introduce *convolution* for this, via *product measures*: define

$$\mu \times \lambda\ (E{\times}F) = \mu(E).\lambda(F)$$

for each "rectangle" $E \times F$ E,F Borel.

There is a unique regular extension of this set function to a measure on $G \times G$ we call the product $\mu \times \lambda$. Then the *convolution* of μ and λ, $\mu * \lambda$, is defined as the unique measure guaranteed by the RRT s.t.

$$\int f\ d\mu{*}\lambda = \iint f(t+\tau)d\mu(t)d\lambda(\tau)$$

for $f \in C_0(G)$.

This is equivalent to

$$\mu{*}\lambda(E) = \int \mu(E-\tau)d\lambda(\tau)$$

and

$$\mu{*}\lambda(E) = \mu \times \lambda(\{(x,y) \in G^2 | x-y \in E\}) .$$

Convolution is commutative and associative, the unit mass at zero, $\delta(0)$, is a unit and thus M(G) is a Banach algebra.

2.3.2.2 Md(G) etc.

The discrete measures in $M(G)$, $Md(G)$, form a subalgebra of $M(G)$ while the continuous form an ideal. The measures absolutely continuous with respect to Lebesgue measure form an ideal isomorphic to $L^1(G)$. An *L-subalgebra* of $M(G)$ is a subalgebra which is closed (w.r.t. the total variation norm) and contains μ whenever it contains ν and $\mu \ll \nu$, $\mu \in M(G)$.

2.3.3 CHOQUET'S THEOREM

This is a very general result we shall appeal to at a crucial point in the argument.

2.3.3.1 Representing measures

If X is a nonempty convex compact subset of a locally convex topological vector space E, and μ is a probability measure on X, then $x \in X$ is *represented by* μ if

$$f(x) = \int_X f \, d\mu \quad \text{for every continuous linear}$$

functional f on E.

2.3.3.2 The theorem of Choquet

If X is a metrizable compact convex subset of a locally convex space E and $x \in X$, then there is a probability measure μ on X which represents x and is concentrated on the extreme points of X.

2.4 FOURIER TRANSFORMS

2.4.1 $B(\Gamma)$, THE FOURIER TRANSFORMS OF $M(G)$

2.4.1.1 Haar measure

On any LCA group G there is a nontrivial *translation-invariant* positive measure m, that is s.t.

$$m(E+x) = mE$$

for each $x \in G$ and Borel E, $E+x = \{a+x | a \in E\}$. This measure, called the *Haar measure*, is unique up to a positive constant multiplier. The standard normalisation is $m(G) = 1$ for compact G, $m(\{x\}) = 1$, $x \in G$ for discrete G (except when G is finite). We use $\int_G f(x)dx$ to mean integration w.r.t. Haar measure.

2.4.1.2 $\hat{\mu}$

For each $\gamma \in \Gamma$ there is a nonzero complex homomorphism of $M(G)$ defined by

$$\mu \to \hat{\mu}(\gamma)$$
$$= \int_G (-x,\gamma)d\mu(x).$$

The function $\hat{\mu}$ defined thus on Γ is the *Fourier-Stieltjes transform of* μ.

The set $\{\hat{\mu} | \mu \in M(G)\}$ is called $B(\Gamma)$. Absolutely continuous μ (w.r.t. Haar measure) correspond to $f \in L^1(G)$, and the set of $\hat{\mu}$ (i.e. $\hat{f}$) for these is called $A(\Gamma)$. The characters Γ exhaust the complex homomorphisms of $L^1(G)$, but not of $M(G)$; $\hat{f}$ is also the Gelfand transform of γ, but the Gelfand transform of μ in general extends the Fourier-Stieltjes transform.

2.4.1.3 Eberlein's criterion

This is a test for membership of $B(\Gamma)$: $\phi \in B(\Gamma)$ and $\|\phi\| \leq A$ $\Leftrightarrow$ ϕ is continuous and for every trigonometric polynomial f on G of the form

$$f(x) = \sum_{i=1}^{n} c_i (x,\gamma_i),$$

$$\left| \sum_{i=1}^{n} c_i \; \phi(\gamma_i) \right| \leq A\|f\|_\infty .$$

The Bochner theorem [FAG, 19] is another such criterion, but we shall not use it.

2.4.2 TRANSFORMS ON SUBGROUPS AND QUOTIENT GROUPS

We need to be able to "lift" transforms from subgroups or quotient groups.

2.4.2.1 The annihilator

If H is a closed subgroup of LCA G, the annihilator of H, $H^\perp$, is the set of $\gamma \in \Gamma$ s.t. $(h,\gamma) = 1$ for all $h \in H$. Trivially, $H^\perp$ is the dual group of G/H, while $\Gamma/H^\perp$ is the dual group of H; H is the annihilator of $H^\perp$ and the continuous characters on H are precisely the restrictions of those on G. (c.f. RUDIN [FAG, 35]).

2.4.2.2 Two theorems

(1) $\mu \in M(G)$ is concentrated on H, a closed subgroup of G, $\Leftrightarrow$ $\hat{\mu}$ is constant on cosets of $H^\perp$.

(2) The functions in $B(H^\perp)$ are precisely the restrictions to $H^\perp$ of the functions in $B(\Gamma)$. (c.f. RUDIN [FAG, 53]).

(The latter result is proved using the canonical homomorphism

$$\varphi: G \to G/H$$

to induce a homomorphism

$$\pi : M(G) \to M(G/H) \qquad \text{via}$$

$$f \to \int_G f(\varphi(x))d\mu(x)$$

for $f \in C_0(G/H)$ and the RRT.)

2.5 THE MAXIMAL IDEAL SPACE M(G) AS A SPACE OF GENERALIZED CHARACTERS

2.5.1 COMPLEX HOMOMORPHISMS AS GENERALIZED CHARACTERS

2.5.1.1 Generalized characters

For some Banach algebras, A, the set of complex homomorphisms, that is the space of maximal ideals, is simply identified. For example, $\Delta(L^1(G)) = \Gamma$. For M(G), however, the maximal ideal space is considerably more complicated. "Very curious" homomorphisms can be exhibited (c.f. HEWITT and KAKUTANI [1964, p 489]); Γ is only a small part of $\Delta(M(G))$.

Nevertheless, analytic structure in $\Delta(M(G))$ is shown to exist through its representation as a space of *generalized characters*; doing so, we shall find that the closure of Γ in Δ includes homomorphisms sufficient for our purposes.

The (slightly modified) theorem of SREIDER is that each complex homomorphism of an L-subalgebra N of M(G), corresponds to a generalized character of N, that is an element

$$\chi = (\chi_\mu) \in \prod_{\mu \in N} L^\infty(\mu)$$

such that

GC1	$\mu << \nu \Rightarrow \chi_\mu = \chi_\nu$	(μ a.e.)
GC2	$\chi_{\mu * \nu}(x+y) = \chi_\mu(x).\chi_\nu(y)$	($\mu \times \nu$ a.e.)
GC3	$\sup\{\|\chi_\mu\|_\infty \mid \mu \in N\} > 0$.	

Each such generalized character produces a complex homomorphism on N by

$$\mu \to \int \chi_\mu \, d\mu = \chi(\mu) = \hat{\mu}(\chi)$$

and every homomorphism arises like this.

The Gelfand topology on $\Delta(N)$ coincides with the product topology derived from the $\sigma(L^{\infty}(\mu), L^{1}(\mu))$ topology on each factor.

2.5.2 CALCULUS OF GENERALIZED CHARACTERS

2.5.2.1 Operations in $\Delta(N)$

A number of useful operations can be defined in $\Delta(N)$ in virtue of the Sreider representation:

(A) for $\chi, \xi \in \Delta(N)$, $\chi.\xi$ is defined by

$$(\chi.\xi)_{\mu} = \chi_{\mu}.\xi_{\mu} \qquad \mu \in N .$$

(B) for $\chi \in \Delta(N)$, $\mu \in N$ we define $\chi.\mu$ as the element of N absolutely continuous with respect to μ whose Radon-Nikodym derivative is χ_{μ}.

(C) the conjugate $\bar{\chi}$ of χ is given by

$$(\bar{\chi})_{\mu} = (\overline{\chi_{\mu}}) \qquad \mu \in N.$$

(D) $|\chi|$, the absolute value of χ, by

$$|\chi|_{\mu} = |\chi_{\mu}| \qquad \mu \in N.$$

(E) χ^{0}, the polar part of χ, by

$$\chi_{\mu}^{0}(x) = \chi_{\mu}(x)/|\chi_{\mu}(x)| \quad \text{if} \quad \chi_{\mu}(x) \neq 0$$

$$= 0 \quad \text{otherwise} \qquad x \in G.$$

(F) Calling χ positive if $\chi_{\mu} \geq 0$ $\mu \in N$, we define χ^{z} for $z \in C$, $\mathcal{Re}(z) > 0$ by

$$(\chi^{z})_{\mu} = (\chi_{\mu})^{z} \qquad \mu \in N.$$

(G) Finally, if N is *self-adjoint*, i.e. $\mu \in N \Rightarrow \tilde{\mu} \in N$ we define $\tilde{\chi}$ by $\tilde{\mu}(\chi) = \overline{\mu(\tilde{\chi})}$, i.e.

$$\tilde{\chi}_{\mu}(x) = \overline{\chi_{\tilde{\mu}}(-x)} \qquad \mu \in N,\ x \in G.$$

(The consistency conditions GC1-3 are readily verified for all these objects.) Obviously,

$$\chi\cdot\bar{\chi} = |\chi|^2, \; \chi = |\chi|\cdot\chi^0, \; |\chi^0|^2 = |\chi^0|,$$

and $\chi\tilde{\chi}$ is symmetric for any χ, that is fixed under $\tilde{}$.

2.5.3 CONVERGENCE OF GENERALIZED CHARACTERS

2.5.3.1 The closure of Γ on $\Delta(N)$

Naturally $\Gamma \subseteq \Delta(M(G))$; we are particularly interested in Γ^-, the closure of Γ in $\Delta(M(G))$ because the property of a *continuous* operating function F that

$$F(\hat{\mu}(\gamma)) = \hat{\nu}(\gamma) \qquad \text{for } \gamma \in \Gamma$$

carries over in view of the continuity of $\hat{\mu}$ and $\hat{\nu}$ to any $\chi \in \Gamma^-$

$$F(\hat{\mu}(\chi)) = \hat{\nu}(\chi).$$

Actually in the present work we are only interested in very simple $\chi \in \Gamma^-$, namely those for which χ_μ is a constant function $\underline{c}$ for the μ we investigate. We shall need to determine

$$C(\mu) = \{c \in C | \exists \chi \in \Gamma^- \text{ s.t. } \chi_\mu = \underline{c}\}$$

and below we give a convergence criterion of Johnson's which applies for the μ we consider.

2.5.3.2 Johnson's criterion [JOHNSON 1968, p. 291]

Suppose that μ is a measure on G given as an *infinite* convolution of measures with finite support (this is the kind of measure we are concerned with below) and let S be the (not necessarily closed) countable subgroup generated by the union of those supports.

Then: if there is a net χ_α in $G^\wedge$ such that $\hat{\mu}(\chi_\alpha) \to a$ and $\chi_\alpha(s) \to 1$ for all $s \in S$, then $a \in C(\mu)$.

PART 3 : FUNCTIONS THAT OPERATE ON CERTAIN BERNOULLI CONVOLUTIONS

3.1 MEASURES DISCUSSED BELOW

3.1.1 THE CLASSES BT, $B\Delta_p$, $B\pi\infty$, $B_1\pi_k$, $B_2\pi_k$

3.1.2 CONVERGENCE OF MEASURES DISCUSSED

3.2 OUTLINE OF THE ARGUMENT

3.3 $C(\mu)$ FOR THE MEASURES DISCUSSED

3.4 APPLYING CHOQUET'S THEOREM

3.5 $G = T$

3.6 $G = \Delta_p$

3.7 $G = \prod_{j=1}^{\infty} Z_{n_j}$

3.8 MAIN RESULT

3.1 THE MEASURES DISCUSSED BELOW

3.1.1 THE CLASSES BT, $B\Delta_p$, $B\pi^\infty$, $B_1\pi_k$, $B_2\pi_k$.

3.1.1.1 Nature of the measures

Applying the structure theorems of 2.1.2 for LCA groups will enable us to direct our attention mainly at three kinds of group:

T, Δ_p and $\prod_{j=1}^{\infty} Z_{m_j}$, $m_j \in Z$.

For the third class we shall have to treat specially the case where $m_j < M$ for all j. Otherwise, we obtain measures forcing analyticity on any operating function from *Bernoulli convolutions*, that is, measures of the form

$$\mu = \mathop{*}_{k=1}^{\infty} \tfrac{1}{2}(\delta(g_k) + \delta(-g_k))$$

for sequences $(g_k) \subseteq G$ satisfying suitable conditions (stated below). In the infinite product *bounded* case, we introduce more complicated measures

$$\mu = \mathop{*}_{r=1}^{\infty} \mu_r$$

with

$$\mu_r = a_r\,\delta(0) + (1-a_r)[\tfrac{1}{2}\,\delta(1_r) + \tfrac{1}{2}\delta(-1_r)]$$

where $1_r = (0,0,\ldots,1,0,0,\ldots)$ has 1 in the r^{th} factor as its only nonzero component. $\{a_r\}$ is chosen dense in $[-1,1]$.

3.1.1.2 Conditions on (g_k)

The constraints imposed on (g_k) are as follows.

When $G = T$,

$$g_k = (n_1 n_2 \ldots n_k)^{-1}$$

where $n_k \in Z$, $n_k \geq 2$, and $\sup_k n_k = \infty$.

When $G = \Delta_p$,

$$g_k = p^{n_k}$$

where $n_k \in Z$, $n_k \geq 2$, $\sup_k (n_k - n_{k-1}) = \infty$ and $n_k > n_{k-1}$ for all k.

When $G = \prod_{k=1}^{\infty} Z_{n_k}$, $\sup_k n_k = \infty$,

$$g_k = (0,0,\ldots 0,t_k,0\ldots)$$

where t_k is the only nonzero coordinate and

$$\sup_k (\text{order } t_k) = \infty .$$

When $G = \prod_{r=1}^{m} Z_k$,

$$g_k = (0,\ldots 0,t_k,0,\ldots,0)$$

Measures of the kind indicated on T form the class BT; those on Δ_p, $B\Delta_p$; those on $\prod_{k=1}^{\infty} Z_{n_k}$, $\sup_k n_k = \infty$, $B\pi\infty$; the class $B_1\pi_k$ are those formed as in $B\pi\infty$ but with $n_j = k$ for each j; and the class $B_2\pi_k$, those measures of the form $*\mu_r$ indicated in 3.1.1.1.

3.1.1.3 Convergence of the infinite convolutions

Criteria for convergence are provided by Pakshirajan's generalisation of Kolmogorov's three series theorem [1963]. Defining X_k as a random variable on G with equal chances of taking the values g_k and $-g_k$, the requirements are the convergence of

(1) $\sum_{k=1}^{\infty} P(\omega | X_k(\omega) \notin N)$,

where N is an arbitrary but fixed compact neighbourhood of 0.

(2) $\sum_{k=1}^{\infty}$ E log $\chi(X_n)$ for any $\chi \in \Gamma$

(3) $\sum_{k=1}^{\infty}$ var log $\chi(X_k)$ for any $\chi \in \Gamma$.

Since for our infinite product groups the infinite convolutions of measures are merely infinite product measures, we need only to verify these conditions for BT, $B\Delta_p$.

For $\mu \in BT$, the series (1) has only finitely many nonzero terms and obviously converges; all terms of series (2) are zero since our μ are symmetric; while if $\chi(x) = \exp 2\pi icx$ var log $\chi(X_k) = -2\pi^2 g_k{}^2 c^2$ so that the criterion is $\sum g_k{}^2 < \infty$, which our condition obviously ensures.

When $G = \Delta_p$, the first and second series converge as for T, while var log $\chi(X_k) = 0$ for k sufficiently large, since $\chi(x) = \exp 2\pi iep^{-L}x$ and $g_k \to \infty$, so that the third series converges too.

3.2 OUTLINE OF THE ARGUMENT

The basic idea (Moran's) is to invoke Choquet's theorem. To do so, it is necessary to show that extreme points in the set of operating functions K_μ on a given measure μ of the class discussed must be functions of the form x^t, t a nonnegative integer. Choquet's theorem then produces the desired conclusion. By using the machinery of generalized characters we are able to show that an extreme point must be of the form x^t, $t \in R$, $t \geq 0$. But to fix t more precisely we treat each kind of group separately. This is done in sections 3.5-3.7. In section 3.3 which follows, we identify $C(\mu)$ for the measures of interest. Then in section 3.4 we show how this knowledge can be applied to give extreme points of K_μ as x^t. Then we turn to the four cases T, Δ_p, $\prod_{k=1}^{\infty} Z_{n_k}$, $\sup_k n_k = \infty$; and $\prod_{j=1}^{m} Z_k$.

3.3 $C(\mu)$ FOR THE MEASURES DISCUSSED

3.3.1 LEMMA

For $\mu \in BT$, $B\Delta_p$ or $B\pi^\infty$, $C(\mu) \supseteq [-1,1]$, while for $\mu \in B_2\pi_k$, $C(\mu) \subseteq [m_\mu,1]$ where

$$m_\mu = \cos \frac{\pi}{k} [\frac{k}{2}].$$

Proof

We adapt the method of Brown and Moran [BMA, 83]. In each case we can define an auxiliary measure ν on R with the following properties

(1) ν is not a point measure, in fact is continuous.

(2) $\|\nu\| = 1$.

(3) $\hat{\nu}(\theta) \in C(\mu)$ for any $\theta \in (0,\frac{1}{2})$.

(4) $\hat{\nu}(0) = 1$.

Now since $\hat{\nu}$ is positive-definite, by the formula 32.4(v) of Hewitt and Ross [AHA II p. 255],

$$|\hat{\nu}(xy)-\hat{\nu}(x)\hat{\nu}(y)|^2 \leq [1-|\hat{\nu}(x^{-1})|]^2[1-|\hat{\nu}(y)|]^2$$

so that were $|\hat{\nu}(y)| = 1$ throughout $(0,\frac{1}{2})$ we would find $|\hat{\nu}(x)| = 1$ everywhere and ν a point measure. So there is a $\theta \in (0,\frac{1}{2})$ with $\hat{\nu}(\theta) = r < 1$ and so by the continuity of $\hat{\nu}$, $(r,1] \subseteq C(\mu)$. If we show $m_\mu \in C(\mu)$ then the required result will follow from the closure of $C(\mu)$ under ordinary multiplication.

Proof that $C(\mu) \subseteq [-1,1]$ for $\mu \in B\Delta_p$

$$\mu = \mathop{*}_{k=1}^{\infty} \tfrac{1}{2}[\delta(p^{n_k}) + \delta(-p^{n_k})] \qquad (1)$$

where

$$n_k \in Z \text{ and } \sup_k(n_k - n_{k-1}) = \infty \qquad (2)$$

for $k = 1,2,3,...$ choose $k(j)$ so that

$$n_{k(j)} - n_{k(j)-1} > j \qquad (3)$$

and define

$$\nu_j = \mathop{*}_{k=k(j-1)+1}^{k(j)} \tfrac{1}{2}[\delta(p^{n_k - n_{k(j)}}) + \delta(-p^{n_k - n_{k(j)}})]$$

as a measure on R. In fact ν_j is a positive measure on $[-1,1]$ with $\|\nu_j\| = 1$.

Obviously there is a subsequence of $\{\nu_j\}$ with a $\sigma(M(R),C(R))$ limit, say ν, and we can assume without loss of generality that $\nu_j \to \nu$. Note that $\nu \neq \delta(0)$.

Now let $\theta \in (0,\tfrac{1}{2})$ and for $j=1,2,3,...$ choose

$$\theta_j = K_j^{-1}.p^{+n_{k(j)} - n_{k(j-1)}} \qquad K_j \in N$$

so that $\theta_j \to \theta$.

Consider the sequence χ_j of characters on Δ_p corresponding to

$$\theta_j \; p^{-n_{k(j)}} = K_j^{-1} \; p^{-n_{k(j-1)}} .$$

For each $p^e \in \Delta$,

$$\begin{aligned} \langle p^e, \chi_j \rangle &= \exp 2\pi i \; \theta_j \; p^{e - n_{k(j)}} \\ &= \exp 2\pi i \; p^e \; K_j^{-1} \; p^{-n_{k(j-1)}} \\ &= (\exp 2\pi i \; K_j^{-1} \; p^{-n_{k(j-1)}})^{p^e} \\ &\to 1 \quad \text{as} \quad j \to \infty \end{aligned}$$

so this is true for all $d \in D$.

Now $\hat{\mu}(\chi_j) = \prod_{k=1}^{\infty} \cos 2\pi\, \theta_j\, p^{n_k} p^{-n_{k(j)}}$

while $\hat{\nu}_j(\theta_j) = \prod_{k=k(j-1)+1}^{k(j)} \cos 2\pi\, \theta_j\, p^{n_k - n_{k(j)}}$

so
$$|\hat{\nu}_j(\theta_j) - \hat{\mu}(\chi_j)| = |\hat{\nu}_j\, \theta_j|.|1 - \prod_{k=1}^{k(j-1)} \cos 2\pi\, \theta_j\, p^{n_k - n_{k(j)}}|$$
$$\leq |\hat{\nu}_j(\theta_j)|.|1 - \prod_{k=1}^{k(j-1)} \cos 2\pi\, K_j^{-1}\, p^{n_k - n_{k(j-1)}}|$$
$$\to 0 \quad \text{as} \quad j \to \infty$$

since the product certainly exceeds the $k(j-1)$th partial product of $\frac{\sin k_j^{-1}}{k_j}$ which converges to 1.

Since $\theta_j \to \theta$, $\exp 2\pi i\, \theta_j x$ converges on $[-1,1]$ to $\exp 2\pi i\, \theta x$, as $j \to \infty$

So
$$|\hat{\nu}_j(\theta_j) - \hat{\nu}(\theta)| \leq |\hat{\nu}_j(\theta_j) - \hat{\nu}_j(\theta)| + |\hat{\nu}_j(\theta) - \hat{\nu}(\theta)|$$
$$\leq \sup_{-1 \leq x \leq 1} |\exp 2\pi i\, \theta_j x - \exp 2\pi i\, \theta x| + |\hat{\nu}_j(\theta) - \hat{\nu}(\theta)|$$
$$\to 0 \quad \text{as} \quad j \to \infty .$$

So $\hat{\mu}(\chi_j) \to \hat{\nu}(\theta)$ so by Johnson's criterion $\hat{\nu}(\theta) \in C(\mu)$. A similar treatment of $G = T$ is given by Brown and Moran [BMA]. Simple versions could be given for $B\pi^{\infty}$ and $B_2\pi_k$, though obvious direct arguments suffice anyway.

3.4 APPLYING CHOQUET'S THEOREM

What makes possible a fruitful application of Choquet's theorem for the measures in question is this result, adapted from Moran [ISCM2 p. 6].

3.4.1 LEMMA

Let μ be a hermitian probability measure on G and suppose $C(\mu) \supset [m_\mu, 1]$. If F is continuous on $[-1,1]$ and operates on μ with $F \circ \hat{\mu} = \hat{\nu}$ then there exists a bounded holomorphic function $\tilde{F}$ on the slit disc $US = \{|z| \le 1\} \backslash (-\infty, m_\mu)$ such that

$$\tilde{F}|_{(0,1)} = F \quad \text{and} \quad \|\tilde{F}\|_\infty \le \|\nu\|$$

and $\tilde{F}$ operates on μ with $\tilde{F} \circ \hat{\mu} = \hat{\omega}$.

Proof

Choose $\xi_m \in \Gamma^-$ so that $(\xi_m)_\mu = \underline{e^{-2^{-m}}}$ and make $\xi_m \ge 0$ (by replacing (ξ_m) by $(\xi_{m+1} . \bar{\xi}_{m+1})$ if need be). For $z \in US$, let

$$G_m(z) = \hat{\nu}(\xi_m^{2^m \log z}).$$

Then $\sup_{z \in US} |G_m(z)| \le \|\nu\|$.

So (G_m) is a normal family of holomorphic functions on US. Let x be $\exp(-p2^{-q})$ with p and q positive integers. For $m \ge q$

$$G_m(x) = \hat{\nu}(\xi_m^{p\,2^{m-q}}) = F(\hat{\mu}(\xi_m^{p\,2^{m-q}}))$$

since $\xi_m^{p\,2^{m-q}} \in \Gamma^-$. But $\hat{\mu}(\xi_m^{p\,2^{m-q}}) = x$ so that $G_m(x) = F(x)$.

So $\lim_{m \to \infty} G_m(x) = F(x)$ for all $x \in (0,1)$ of the form stated. Therefore by Vitali's theorem, G_m converges uniformly on compact subsets of US to a bounded holomorphic G on US. Clearly G coincides with F on $(0,1)$.

Now $F \circ \hat{\mu}(\xi_1^k e(\gamma)) = \hat{\nu}(\xi_1^k e(\gamma))$ for $k=1,2,3,\dots$ and γ for which $\hat{\mu}(\gamma) = \hat{\mu}(e(\gamma)) \in [0,1]$. $\quad (e(\gamma) = e^{2\pi i\gamma})$

For each such γ,

$$z \to F \circ \hat{\mu}(\xi_1^z(e(\gamma)) \quad \text{and} \quad z \to \hat{\nu}(\xi_1^z e(\gamma))$$

are bounded, holomorphic on the right half plane coinciding on the positive integers, so by Carlson's theorem they coincide everywhere.

Now let $z \to 0$ along the real axis. Then $\xi_1^z \to \xi_1^0$ where $(\xi_1)_\mu^0 = 1$ μ a.e. and $(\xi_1^0)_\nu$ may take both the values 0 and 1. Let $\omega = (\xi_1^0)_\nu.\nu$.

Then

$$\tilde{F} \circ \hat{\mu}(e(\gamma)) = \lim_{t\to 0} F \circ \hat{\mu}(\xi_1^t e(\gamma))$$

$$= \lim_{t\to 0} \hat{\nu}(\xi_1^t e(\gamma)) = \hat{\omega}(e(\gamma)).$$

Thus $\tilde{F}$ operates on μ and $\tilde{F} \circ \mu = \omega = (\xi_1^0)_\nu.\nu$.

□

3.4.2 PREPARATION FOR THE APPLICATION OF CHOQUET'S THEOREM

Let $\theta = \{\gamma | \hat{\mu}(\gamma) \neq 0,\pm 1\}$, $B(\theta)$ be the restrictions of the transforms $B(\Gamma)$ under the quotient space norm.

Let K_μ be the set of all functions F continuous on $(m_\mu,0) \cup (0,1)$ which operate on μ, under the topology of uniform convergence on compact subsets of $(m_\mu,0) \cup (0,1)$, and $\|F \circ \hat{\mu}\|_{B(\theta)} \le 1$.

Lemma

K_μ is a compact, convex set.

Proof

(a) Convexity. If $\alpha_1+\alpha_2 = 1$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, and F_1 and F_2 operate on μ then

$$(\alpha_1F_1+\alpha_2F_2)(\hat{\mu}(\gamma)) = \alpha_1F_1(\hat{\mu}(\gamma))+\alpha_2F_2(\hat{\mu}(\gamma))$$

$$= \alpha_1\hat{\nu}_1(\gamma)+\alpha_2\hat{\nu}_2(\gamma)$$

$$= (\alpha_1\nu_1+\alpha_2\nu_2)\hat{}(\gamma) .$$

So that $\alpha_1F_1+\alpha_2F_2$ also operates, and $\|F \circ \hat{\mu}\|_{B(\theta)} \leq \alpha_1+\alpha_2 = 1$.

(b) Compactness. It is enough to show that any sequence (F_n) of elements of K_μ has a convergent subsequence, since the topology of K_μ is metrizable.

Each F_n extends (uniquely) to a holomorphic $\tilde{F}_n$ on US and $\sup\{|\tilde{F}_n(z)| : z \in US, n = 1,2,\ldots\} \leq \|\nu\|$ so that $(\tilde{F}_n)$ is a normal family and has a convergent subsequence. Restricting to $[m_\mu,1]$ we obtain a continuous F such that $F_{n'} \to F$ uniformly on compact sets. For each n we have ν_n s.t. $F_n \circ \hat{\mu} = \hat{\nu}_n|_\theta$ and $\|\nu_n\| \leq 1 + \frac{1}{n}$.

Then for some subsequence $\nu_{n'} \to \nu$ in the weak topology and $F \circ \hat{\mu} = \hat{\nu}|_\theta$ while $\|F \circ \hat{\mu}\|_{B(\theta)} \leq \|\nu\| \leq 1$ so $F \in K_\mu$.

Next we characterise the extreme points as functions of the form Cx^t, $C \in \mathbb{C}$, $t \geq 0$. Most of the rest of the work after that concerns specifying possible values of t.

3.4.3 LEMMA (Moran)

Let F be an extreme point of K_μ with $F \circ \hat{\mu} = \hat{\nu}$. If $\xi \in \Gamma^-$ $\xi \geq 0$ and ξ_μ is a nonzero constant, then ξ_ν is constant almost everywhere.

Corollary

$F(x) = Cx^t$ where $C \in \mathbb{C}$, $t \geq 0$ and $x \in [m_\mu, 1]$.

Proof

Let $\xi_\mu = \underline{\beta}$, $d = (\log \beta)^{-1}$ and assume ξ_ν not constant a.e. We show that F cannot be an extreme point by exhibiting it as a nontrivial convex combination of two functions in K_μ. Specifically, let $C \in (0,1)$ be s.t. both

$$A_1 = \{t \mid \xi_\nu(t) < C\} \quad \text{and} \quad A_2 = \{t \mid \xi_\nu(t) \geq C\}$$

have positive $|\nu|$ measure, α_i. Define

$$H_i(z) = \alpha_2^{-1} \int_{A_i} \xi_\nu(t)^{d \log z} \, d\nu(t)$$

for $i=1,2$, $z \in US$.

These are clearly both defined and holomorphic on US. Now if F were an extreme point, $\|\nu\| = 1$ so $\alpha_1 + \alpha_2 = 1$ and

$$\alpha_1 H_1 + \alpha_2 H_2 = \tilde{F},$$

the holomorphic extension of F to US guaranteed by Lemma 3.4.1. We conclude the proof by showing H_i operates on μ, $i=1,2$. In fact

$$H_i \circ \hat{\mu}|_\theta = \hat{\nu}_i|_\theta \qquad i=1,2,$$

where $\nu_i(B) = \alpha_i^{-1} \nu(A_i \cap B)$ for Borel sets B.

To show this, fix $\gamma \in \theta$ and consider

$$\alpha_1 H_1(\hat{\mu}(\gamma.\xi^m)) + \alpha_2 H_2(\hat{\mu}(\gamma.\xi^m)) = \tilde{F}(\hat{\mu}(\gamma.\xi^m))$$

$$= \hat{\nu}(\gamma.\xi^m) = \alpha_1 \hat{\nu}_1(\gamma.\xi^m) + \alpha_2 \hat{\nu}_2(\gamma.\xi^m)$$

(The first equality holds by the continuity of H_i and $\hat{\mu}$

since $\xi^m \in \Gamma^-$.)

Now we may replace m by a complex z, $Re(z) > 0$ to obtain two bounded holomorphic functions

$\alpha_1 H_1(\hat{\mu}(\gamma.\xi^z)) + \alpha_2 H_2(\hat{\mu}(\gamma.\xi^z))$ and $\alpha_1\hat{\nu}_1(\gamma.\xi^z) + \alpha_2\hat{\nu}_2(\gamma.\xi^z)$

which coincide on the positive integers. So by Carlson's theorem they coincide. So for $Re(z) \geq 0$

$$\alpha_1(H_1(\hat{\mu}(\gamma.\xi^z)-\hat{\nu}_1(\gamma.\xi^z)) = -\alpha_2(H_2(\hat{\mu}(\gamma.\xi^z)-\hat{\nu}_2(\gamma.\xi^z)) .$$

Now let $z = Re(z) \to \infty$ and consider LHS. Comparing with C^x (C as above),

$$|C^{-x}\alpha_1(H_1(\hat{\mu}(\gamma.\xi^z))-\hat{\nu}_1(\gamma.\xi^z))|$$

$$= C^{-x}\left|\int(\xi_\nu(t)^{d\log\hat{\mu}(\gamma)}-\gamma)\xi_\nu(t)^t\, d\nu_1(t)\right|$$

$$\leq \int|\xi_\nu(t)^{d\log\hat{\mu}(\gamma)}-\gamma|\left(\frac{\xi_\nu(t)}{C}\right)^x d\nu_1(t)$$

which $\to 0$ as $x \to \infty$ because ν_1 is concentrated on A_1 where $\xi_\nu(t) < C$. So the RHS of the equation tends to zero uniformly in $y = Im(z)$ as $x \to \infty$. This can only happen if $H_2(\hat{\mu}(\gamma.\xi^z)) = \hat{\nu}_2(\gamma.\xi^z)$ for all z with $Re(z) > 0$, for the following reason:

Let λ be the measure $(\xi_\nu^{d\log\hat{\mu}(\gamma)}-\gamma).\nu_2$ and define $\varphi(t) = \log(\xi_\nu(t)/C)$.

Then ϕ is ν_2-measurable and maps the set A_2 (on which ν_2 is concentrated) into R^+. Let ρ be the measure on R^+ induced by ϕ from λ so

$$\int f\circ\phi(t)d\lambda(t) = \int f(t)d\rho(t),$$

for $f \in C(R^+)$. Then

$$\int (\xi_\nu^{d \log \hat{\mu}(\gamma)} - \gamma)\left(\frac{\xi\nu(t)}{C}\right)^z d\nu_2(t) = \int e^{zt} d\rho(t).$$

Since supp ρ is compact $L(z) = \int e^{zt} d\rho(t)$ is entire, and since supp $\rho \subseteq [0,\infty)$, $L(z_k)$ is unbounded only for sequences with $\mathrm{Re}(z_k) \to \infty$. On the other hand

$$L(z) = C^{-x} \alpha_2 H_2(\hat{\mu}(\gamma.\xi^z)) - \hat{\nu}_2(\gamma.\xi^z) \to 0$$

as $\mathrm{Re}(z) \to \infty$ from the above argument, so by Liouville's theorem $L \equiv 0$.

So $H_i(\hat{\mu}(\gamma.\xi^z)) = \hat{\nu}_i(\gamma.\xi^z)$ for $\mathrm{Re}(z) > 0$, $i=1,2$.

Letting $z \to 0$ along the real axis we obtain

$$H_2(\hat{\mu}(\gamma)) = \hat{\nu}_2(\gamma) \quad \text{and so} \quad H_1(\hat{\mu}(\gamma)) = \hat{\nu}_1(\gamma).$$

So we have $\xi_{\nu_1} = \underline{\beta}$, $\xi_{\nu_2} = \underline{\theta}$ say. Then for any positive integer n,

$$\tilde{F}(\beta^n) = \hat{\nu}(\xi^n) = \theta^n \hat{\nu}(e(0)) = C\theta^n .$$

So by Carlson's theorem we may substitute z for n ($\mathrm{Re}(z) > 0$). Letting $z = d^{-1} \log x$ we get

$$F(x) = \tilde{F}(\beta^z) = C\theta^z = Cx^t$$

where $t = d^{-1} \log \theta \geq 0$ (since $\xi \geq 0$). □

In the next 3 sections we investigate what values of t are actually possible, the aim being to exclude non-integers. This turns out to be possible for $\mu \in BT$, $B\Delta_p$, $B\pi^\infty$, $B_2\pi_k$ but not $B_1\pi_k$.

3.5 $G = \mathbb{T}$

3.5.1 THEOREM

If $\mu \in B\mathbb{T}$ and F is a continuous function in $[-1,1]$ which operates on μ, then

$$F(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{with} \quad \sum_{n=0}^{\infty} |b_n| < \infty$$

$$\text{for} \quad x \in [-1,1] .$$

Proof

It is proved below that $|z|^t$ can only operate on μ if $t \in 2\mathbb{Z}$, so by Choquet's theorem any even continuous function on $[-1,1]$ which operates on μ has the form

$$F(x) = \sum_{n=0}^{\infty} b_{2n} x^{2n} \qquad \sum_{n=0}^{\infty} |b_{2n}| < \infty .$$

Now if G is an odd continuous function on $[-1,1]$ operating on μ,

$$x \to xG(x)$$

is even and so

$$G(x) = \sum_{n=1}^{\infty} b_{2n+1} x^{2n+1} , \quad \sum_{n=0}^{\infty} |b_{2n+1}| < \infty .$$

Finally since $\frac{1}{2}(F(x)+F(-x))$ and $\frac{1}{2}(F(x)-F(-x))$ are even and odd respectively, the theorem will follow from the demonstration that $F(-x)$ operates on μ if $F(x)$ does:

Let $\xi \in \Gamma^{-}$ have $\xi_\mu = -1$.

Then $F(-\hat{\mu}(\gamma)) = F(\hat{\mu}(\xi.e(\gamma))$

$= \hat{\nu}(\xi.e(\gamma))$

$= (\xi_\nu.\nu)\hat{}(\gamma) \qquad \gamma \in \Gamma$

so that $F(-x)$ does indeed operate.

3.5.2 LEMMA

If $\mu \in BT$ $|x|^t$ only operates on μ if $t \in 2Z$.

Proof

μ has the form

$$\mu = \mathop{*}_{n=1}^{\infty} \tfrac{1}{2}(\delta(-d_n^{-1}) + \delta(d_n^{-1}))$$

where $d_n = p_1 \cdot p_2 \cdot \ldots \cdot p_n$

with p_n a sequence of integers $\to \infty$.

Suppose $|x|^t$ does operate for some $t \notin 2Z$ so that

$$\hat{\nu}(m) = \prod_{j=1}^{\infty} |\cos 2\pi m d_j^{-1}|^t \qquad m \in Z$$

is the transform of some probability measure ν on T. We show that no such ν exists by showing its norm exceeds all bounds. There are 9 steps: the basic idea is to construct suitable approximating polynomials $Q_{n,s}$ of large norm by multiplying together polynomials of norm exceeding a constant greater than one approximating to each convolution factor.

We argue as follows:-

1. Let $f(x) = |\cos 2\pi x|^t$, and fix $N = d_{n_0}$ so that $\sum_{|j| \le N} |\hat{f}(j)| > 1 + 15\delta/16$.

2. Define μ_n so that $\|\mu_n\| > 1 + 7\delta/8$, $|\mu_n|(A_n) > 1 + 7\delta/8$.
 $(A_n = [-Nd_n^{-1}, Nd_n^{-1}] \cap \operatorname{supp}\{d_n^{-1}\})$

3. Define P_n so that $\int P_n(d_{n-1}\mu)d\mu_n(u) > 1 + 3\delta/4$ for $n \ge n_0$.

4. Show that $\int P_n(d_{n-1}u)P_{n+1}(d_n u)d\mu_n * \mu_{n+1} > (1 + \delta/2)^2$ for $n \ge n_0$,

using

5. $\left(\int P_n(d_{n-1}(u+v))d\mu_n(u) - \int P_n(d_{n-1}u)d\mu_n(u)\right) \to 0$ as $n \to \infty$.

6. Define $Q_{n,s}(x) = \prod_{j=n}^{n+s} P_n(d_{n-1}u)$, $\tau_{n,s} = \mu_n * \ldots * \mu_{n+s}$,

$\omega_n = \mu_1 * \ldots * \mu_n$ and show $\int Q_{n,s}(u)d\tau_{n,s}(u) > (1 + \delta/2)^s$

for $s \in Z$, $s \geq 1$ if n is sufficiently large.

7. Show $\int Q_{n,s}(u)d\omega_{n+s}(u) = \int Q_{n,s}(u)d\tau_{n,s}(u)$, $\sup_n \| Q_{n,s} \|_{A(T)} < \infty$.

8. Show $\inf_m \{ \prod_{j=n+s+1}^{\infty} |\cos 2\pi m d_j^{-1}|^t : Q^\wedge_{n,s}(m) \neq 0\} \to 1$ as $n \to \infty$.

9. Using 7 and 8, show $\|\nu\| > \int Q_{n,s}(u)d\nu(u) \geq (1 + \delta/2)^s - 1$

for n sufficiently large.

3.5.2.1 $\sum_{|n| \leq N} |\hat{f}(n)| > 1 + 15\delta/16$

Let $f(x) = |\cos 2\pi x|^t$. $f \in A(T)$, since f is of bounded variation and in Lip(T) (cf. Zygmund [TS, 1 p 241]),

$$\sum_{j \in Z} |\hat{f}(j)| < \infty \qquad (1)$$

Since cos is even, $\hat{f}(-j) = \hat{f}(j)$, so

$$\sum_{j \in Z} \hat{f}(j) = \hat{f}(0) = 1 \qquad (2)$$

whereas explicit computations show that $\hat{f}(j) < 0$ for some values of j.

$$\int_T e^{2\pi i j x} |\cos 2\pi x|^t \, dx = 2\int_{-1/2}^{1/2} e^{2\pi i j x} |\cos 2\pi x|^t \, dx \quad \text{if} \quad j \in 2Z$$

$$= \frac{2\pi\Gamma(t+1)}{2^t \Gamma(t+j/2)\Gamma(t-j/2)} \quad \text{[IT, 138, \#19a]}$$

$$< 0 \quad \text{if} \quad t-j/2 \in \bigcup_{k=0}^{\infty} (-2k+1,-2k).$$

Thus precisely when $t \notin 2Z$ there are j for which $\hat{f}(j) < 0$ such as $-[t/2] - 1$.

So $$\|f\|_{A(T)} = \sum_{j \in Z} |\hat{f}(j)| = 1 + \delta > 1. \qquad (3)$$

Select n_0 and $N = d_{n_0}$ so that

$$\sum_{|j| \leq N} |\hat{f}(j)| \geq 1 + 15\delta/16. \qquad (4)$$

Let $A_n = [-Nd_n^{-1}, Nd_n^{-1}] \cap gp\{d_n^{-1}\}$.

Define μ_n on $gp\{d_n^{-1}\}$ by

$$\mu_n\{kd_n^{-1}\} = \sum_{j\in Z} \hat{f}(k+jd_n) \qquad k = 1,2,\ldots,d_n \tag{5}$$

Then
$$\hat{\mu}_n(k) = |\cos 2\pi k d_n^{-1}|^t . \tag{6}$$

3.5.2.2 $\|\mu_n\| > 1 + 7\delta/8$ and $|\mu_n|(A_n) > 1 + 7\delta/8$

We assume from now on that n is such that $d_n \geq 2N$.

Then

$$\|\mu_n\| = \sum_{g\in gp\{d_n^{-1}\}} |\mu\{g\}|$$

$$= \sum_{k=1}^{d_n} \Big|\sum_{j\in Z} \hat{f}(k+jd_n)\Big|$$

$$= \sum_k \Big|\hat{f}(k) + \sum_{\substack{j\in Z\\ j\neq 0}} \hat{f}(k+jd_n)\Big|$$

$$\geq \sum_k \Big(|\hat{f}(k)| - \Big|\sum_{\substack{j\in Z\\ j\neq 0}} \hat{f}(k+jd_n)\Big|\Big)$$

$$> 1 + 15\delta/16 - \delta/16 = 1 + 7\delta/8$$

since $\displaystyle\sum_k \Big|\sum_{j\neq 0} \hat{f}(k+jd_n)\Big| \leq \sum_{|j|>N} |\hat{f}(j)| < \delta/16.$

And similarly

$$|\mu_n|(A_n) = \sum_{|j|\leq N} |\hat{f}(j) + \hat{f}(j+d_n) + \ldots|$$

$$\geq \sum_{|j|\leq N} |\hat{f}(j)| - \sum_{|j|\leq N} |\hat{f}(j+d_n) + \ldots|$$

$$> 1 + 15\delta/16 - \delta/16 = 1 + 7\delta/8 .$$

Thus

$$\|\mu_n\| > 1 + 7\delta/8 \quad\text{and}\quad |\mu_n|(A_n) > 1 + 7\delta/8 . \tag{7}$$

3.5.2.3 $\int P_n(d_{n-1}u)d\mu_n(u) > 1 + 3\delta/4$

(In what follows $[x]$ denotes the integer part of x, $\{x\} = x - [x]$.)

Select a trigonometric polynomial

$$P(x) = \sum_{|k| \le K} a_k \exp 2\pi ikx \tag{8}$$

on T with $\|P\|_\infty = 1$ and

$$\int P(x)d\mu_{n_0}(x) > 1 + 3\delta/4 \tag{9}$$

This is possible because $\|\mu_{n_0}\| > 1 + 3\delta/4$ and the trigonometric polynomials are dense in $C_0(T)$. For each $n \ge n_0$ s.t. $d_n d_{n-1}^{-1} \ge 2N$, let

$$P_n(x) = \sum_{|k| \le K} a_k \exp 2\pi ik \left[\frac{d_n}{Nd_{n-1}}\right] x\ . \tag{10}$$

Then we also have

$$\int P_n(d_{n-1}u)d\mu_n(u) > 1 + \delta/2 \tag{11}$$

for

$$\int P_n(d_{n-1}u)d\mu_n(u) - \int P(u)d\mu_{n_0}(u) \tag{12}$$

$$= \sum_{|k| \le K} a_k [f(d_{n-1}\left[\frac{d_n}{Nd_{n-1}}\right] kd_n^{-1}) - f(kd_{n_0}^{-1})]\ .$$

Now the difference between the two arguments of f is

$$d_{n-1}\left[\frac{d_n}{Nd_{n-1}}\right] kd_n^{-1} - kd_{n_0}^{-1}$$

$$= kd_{n-1}d_n^{-1}[d_n d_{n-1}^{-1}N - \{d_n d_{n-1}^{-1}N\} - d_n d_{n-1}^{-1}N]$$

$$= k(d_{n-1}d_n^{-1})\{d_n d_{n-1}^{-1}N\}\ . \tag{13}$$

Since $|k| \le K$ (fixed) and $\{a\} \in [0,1)$, the difference can be made as small as desired if n is made large enough. Therefore by the uniform continuity of f, so can the difference between the two integrals (13).

3.5.2.4 $\int P_n(d_{n-1}u)P_{n+1}(d_n u)d\mu_n * \mu_{n+1}(u) > (1 + \delta/2)^2$

The integral in question is

$$J = \iint P_n(d_{n-1}(u+v))P_{n+1}(d_n(u+v))d\mu_n(u)d\mu_{n+1}(v)$$

$$= \int [\int P_n(d_{n-1}(u+v))d\mu_n(u+v)]P_{n+1}(d_n v)d\mu_{n+1}(v)$$

since $d_n\mu \in Z$ for $\mu \in gp\{d_n^{-1}\}$

and P_n is 1-periodic for all n

$$> \int_{A_{n+1}} I(v)P_{n+1}(d_n v)d\mu_{n+1}(v) - \delta/8 ,$$ $I(v)$ being the inner integral

since $|\int_{A'_{n+1}}| \leq |\mu_{n+1}|(A'_{n+1}) < \delta/8$. ($A'_{n+1}$ complement of A_{n+1})

Using the fact that $|I(v) - \int P(d_{n-1}u)d\mu_n(u)| \to 0$ as $n \to \infty$ (3.5.2.5 below), we see for n large enough

$$J > (1 + 3\delta/4)\int_{A_{n+1}} P_{n+1}(d_n v)d\mu_{n+1}(v) - \delta/8$$

$$\geq (1 + 3\delta/4)(1 + 5\delta/8) - \delta/8$$ using 3.5.2.2 and 3.5.2.3

$$> (1 + \delta/2)^2 .$$

3.5.2.5 $(I(v) - \int P_n(d_{n-1}u)d\mu_n(u)) \to 0$ as $n \to \infty$

To show this, consider the difference

$$|\int [P_n(d_{n-1}(u+v)) - P_n(d_{n-1}u)]d\mu_n(u)|$$

$$= |\int [\sum_{|k|\leq K} a_k(\exp(2\pi ikd_{n-1}\left[\frac{d_n}{Nd_{n-1}}\right](u+v))-\exp(2\pi ikd_{n-1}\left[\frac{d_n}{Nd_{n-1}}\right]u))]d\mu_n(u)|$$

$$\leq \int \Big(\sum_{|k| \leq K} |a_n| \, |\exp(2\pi i k d_{n-1} \left[\frac{d_n}{Nd_{n-1}}\right] v - 1|\Big) d|\mu_n|(u)$$

$$\leq \|\mu_n\| \sum_{|k| \leq K} |a_n| \, |2\pi i k d_{n-1} \left[\frac{d_n}{Nd_{n-1}}\right] j d_{n+1}^{-1}|$$

where $v = jd_{n+1}^{-1}$, $|j| \leq N$

since $|e^{i\theta} - 1| \leq |\theta|$.

Since $|j| \leq N$, each term involving j is less than

$$2\pi K d_{n-1} \left[\frac{d_n}{Nd_{n-1}}\right] N d_{n+1}^{-1}$$

$$= 2\pi K d_{n-1} N d_{n+1}^{-1} \left(\frac{d_n}{Nd_{n-1}} - \left\{\frac{d_n}{Nd_{n-1}}\right\} \right)$$

$$= 2\pi K \frac{d_n}{d_{n+1}} - 2\pi K N \frac{d_{n-1}}{d_{n+1}} \left\{\frac{d_n}{Nd_{n-1}}\right\}$$

and each of these terms is arbitrarily small for n sufficiently large.

3.5.2.6 $\underline{\int Q_{n,s}(u) d\tau_{n,s}(u) > (1 + \delta/2)^s}$

Define

$$Q_{n,s}(u) = \prod_{j=n}^{n+s} P_j(d_{j-1} u)$$

and

$$\tau_{n,s} = \mu_n * \ldots * \mu_{n+s}$$

$$\omega_n = \mu_1 * \mu_2 * \ldots * \mu_n .$$

Then $\quad I = \int Q_{n,s}(u) d\tau_{n,s}(u)$

$$= \int \Big(\int P_n d_{n-1}(u+v) d\mu_n(u) \Big) Q_{n+1,s-1}(v) d\mu_{n+1} * \ldots * \mu_{n+s}(v)$$

since $d_j \mu \in Z$ for $j \geq n$.

As in 3.5.2.5, $\int P_n(d_{n-1}(u+v)) d\mu_n(u) \to \int P_n(d_{n-1} u) d\mu_n(u)$

so

$$I > (1 + \delta/2)\int_{A_{n+s}} Q_{n+1,\,s-1}(v)d\mu_{n+1}*\ldots*d\mu_{n+s}(v) - \delta/8$$

$$\geq (1 + \delta/2)(1 + \delta/2)^{s-1} - \delta/8$$

$$> (1 + \delta/2)^{s}$$

by induction on $s \geq 1$, for n sufficiently large.

3.5.2.7 $\underline{\int Q_{n,s}\,d\omega_{n+s} = \int Q_{n,s}\,d\tau_{n,s}\ ;\ \sup_n \| Q_{n,s} \|_{A(T)} < \infty}$

These are so because

$$\int Q_{n,s}\,d\omega_{n+s} = \iint Q_{n,s}(u+v)d\omega_{n-1}(u)d\tau_{n,s}(v)$$

$$= \int Q_{n,s}(v)\left(\int 1 d\omega_{n-1}(u)\right)d\tau_{n,s}(v)$$

since ω_{n-1} concentrates on $gp\{d_{n-1}^{-1}\}$,

and $Q_{n,s}$ is d_{n-1}^{-1} - periodic, because

N divides d_n.

$$= \int Q_{n,s}(v)d\tau_{n,s}(v)$$

($\int 1 d\mu_j = 1$ for all j since $\hat{f}$ is odd).

Furthermore $\| Q_{n,s} \|_{A(T)} = \sum_m \hat{Q}_{n,s}(m)$ is the sum of the absolute values of the coefficients of $Q_{n,s}$ which for fixed s is certainly smaller than $(2K+1)^s\,[\max_{k \in K} a_k]^s < \infty$, independent of n. Finally $\| Q_{n,s} \|_\infty \leq 1$ since each factor of Q is so bounded.

3.5.2.8 $\underline{\inf_s \{ \prod_{j=n+s+1}^{\infty} |\cos 2\pi m d_j^{-1}|^t \ |\hat{Q}_{n,s}(m) \neq 0\} \to 1 \quad \text{as} \quad n \to \infty}$

To establish this it is enough to show that for any given J, if n is large enough

$$\prod_{j=n+s+1}^{\infty} |\cos 2\pi m d_j^{-1}|^t > \prod_{j=J}^{\infty} |\cos 2\pi . 2^{-j}|^t$$

since the R.H.S. can be made arbitrarily near 1 if J is big enough.

To do so, consider the largest possible value of m:

$$d_{n-2}K\left[\frac{d_{n-1}}{Nd_{n-2}}\right] + d_{n-1}K\left[\frac{d_n}{Nd_{n-1}}\right] + \ldots + d_{n+s-1}K\left[\frac{d_{n+s}}{Nd_{n+s-1}}\right]$$

$$< (s+1)KN^{-1}d_{n+s} \quad .$$

So the first factor of $\prod_{j=n+s+1}^{\infty} |\cos 2\pi m d_j^{-1}|^t$ is at least $|\cos 2\pi(s+1)KN^{-1}d_{n+s}d_{n+s+1}^{-1}|^t$ which is larger than $|\cos 2\pi 2^{-J}|^t$, no matter how large J, if n is large enough. And subsequent factors are larger than corresponding factors of $\prod_{j=J}^{\infty} |\cos 2\pi 2^{-j}|^t$ since $d_j \geq 2$ for all j. (We assume n large enough that all arguments are smaller than 2π.)

3.5.2.9 $\underline{\int Q_{n,s}(u)d\nu(u) > (1 + \delta/2)^s - 1}$

For $\int Q_{n,s}(u)d\nu(u)$

$$= \sum_m Q\hat{}_{n,s}(m)\left(\prod_{j=1}^{\infty} |\cos 2\pi m d_j^{-1}|^t\right)$$

$$\geq \sum_m Q\hat{}_{n,s}(m)\left(\prod_{j=1}^{n+s} |\cos 2\pi m d_j^{-1}|^t (1-\varepsilon)\right)$$

for any small ε, by 3.5.2.8

$$= \sum_m Q\hat{}_{n,s}(m) \prod_{j=1}^{n+s} |\cos 2\pi m d_j^{-1}|^t - \varepsilon \sum_m Q\hat{}_{n,s}(m)$$

$$\geq \sum_m Q\hat{}_{n,s}(m) \prod_{j=1}^{n+s} |\cos 2\pi m d^{-1}|^t - 1$$

if ε is chosen smaller than $(\sup_n \| Q_{n,s} \|_{A(T)})^{-1}$ as it may from 3.5.2.7,

so for sufficiently large n,

$$\| \nu \| \geq \int Q_{n,s}(u)d\nu(u) \geq \int Q_{n,s}(u)d\omega_{n+s}(u) - 1, \quad \text{by the above}$$

$$\geq (1 + \delta/2)^s - 1, \quad \text{from 3.5.2.6}$$

for any $s \in Z$, $s > 0$

so that no such ν exists.

This argument is a revision and expansion of that given in Moran [ICSM2, p. 21] in which it appears that the *same* trigonometric polynomial is invoked for each n at 3.5.2.3 (10). This seems difficult to secure.

3.6 $G = \Delta_p$

3.6.1 THEOREM

If $\mu \in B\Delta_p$ and F is a continuous function on $[-1,1]$ which operates on μ, then

$$F(x) = \sum_{j=0}^{\infty} b_j x^j \qquad \text{with} \qquad \sum_{j=0}^{\infty} |b_j| < \infty$$

$$\text{for } x \in [-1,1].$$

Proof

The only change needed to the proof 3.5.1 is to replace the lemma 3.5.2 with the corresponding result for Δ_p proved below, lemma 3.6.2. Γ becomes $Z(p^\infty)$ rather than Z but that does not affect the argument.

3.6.2 LEMMA

If $\mu \in B\Delta_p$, $|x|^t$ operates on μ only if $t \in 2Z$.

Proof

The argument is like that for T but some details are different. μ has the form

$$\mu = \mathop{*}_{n=1}^{\infty} \tfrac{1}{2}(\delta(-p^{e_n}) + \delta(p^{e_n}))$$

where (e_n) is a sequence of integers such that $\sup_n (e_n - e_{n-1}) = \infty$ and $e_n > e_{n-1}$ for all n.

Suppose $|x|^t$ operates on μ for some $t \notin 2Z$, so that

$$\begin{aligned} \hat{\nu}(mp^{-M}) &= |\hat{\mu}(mp^{-M})|^t \\ &= \prod_{j=1}^{\infty} |\cos 2\pi m p^{e_j - M}|^t \end{aligned}$$

$$\text{for } mp^{-M} \in Z(p^\infty)$$

is the transform of some probability measure ν on Δ_p. We show that ν does not exist by showing its norm exceeds all bounds, through nine steps:

1. Let $f(x) = |\cos 2\pi x|^t$ and fix $N = p^{e_{n_0}}$ so that

$$\sum_{|j| \leq N} |\hat{f}(j)| \geq 1 + 15\delta/16 \quad ;$$

2. Define μ_n so that $\|\mu_n\| > 1 + 7\delta/8$ and

$$|\mu_n^{\prime}|(V_n) > 1 + 7\delta/8 \qquad n \geq n_0 \;;$$

3. Define P_n so that $\int P_n(p^{-e_{n-1}}u)d\mu_n(u) > 1 + 3\delta/4$,

for all sufficiently large n;

4. Show $\int P_n(p^{-e_{n-1}}u)P_{n+1}(p^{-e_n}u)d\mu_n * \mu_{n+1}(u) > (1 + \delta/2)^2$,

for all sufficiently large n,

using

5. $\left(\int P_{n+1}(p^{-e_n}(u+v))d\mu_{n+1}(v) - \int P_{n+1}(p^{-e_n}v)d\mu_{n+1}(v)\right) \to 0$,

as $n \to \infty$;

6. Define $Q_{n,s}(x) = \prod_{j=n}^{n+s} P_j(p^{-e_{j-1}}x)$, $\tau_{n,s} = \mu_n * \mu_{n+1} * \ldots * \mu_{n+s}$

and show $\int Q_{n,s}(u)d\tau_{n,s}(u) > (1 + \delta/2)^s$,

$s \in Z$ for all sufficiently large n;

7. Show $\sup_n \|Q_{n,s}\|_{A(\Delta_p)} < \infty$;

8. Show $\inf_{(m,M)} \{ \prod_{j=1}^{n-1} |\cos 2\pi m p^{-e_j - M}|^t : \hat{Q}_{n,s}(mp^{-M}) \neq 0\}$

$\to 1$ as $n \to \infty$;

and

9. using 7 and 8 show

$$\|\nu\| \geq \int Q_{n,s}(u)d\nu(u) > (1 + \delta/2)^s - 1$$

for all sufficiently large n.

(The sense of "sufficiently large n" is made precise in the detailed argument to follow.)

3.6.2.1 $\sum_{|j|\leq N} |\hat{f}(j)| > 1 + 15\delta/16$

f is still the same function as in 3.5.2.1 in $A(T)$, so no change is required except to ensure $N = p^{e_{n_0}}$. Let $V_n = \{jp^{e_n} | j = -N,-N+1,\ldots,N-1,N\} \subseteq \Delta_p$.

3.6.2.2 $\|\mu_n\| > 1 + 7\delta/8$ and $|\mu_n|(V_n) > 1 + 7\delta/8$

Define μ_n on $p^{e_n}\Delta_p$ by

$$\mu_n\{zp^{e_n}\} = \hat{f}(z) \quad \text{for all } z \in Z .$$

Notice that

$$\hat{\mu}_n(mp^{-M}) = \int \exp 2\pi imup^{-M} d\mu_n(u)$$

$$= \sum_{z\in Z} \exp(2\pi imzp^{e_n-M})\mu_n\{zp^{e_n}\}$$

$$= \sum_{z\in Z} \exp(2\pi imzp^{e_n-M})\hat{f}(z)$$

$$= f(mp^{e_n-M}) = |\cos 2\pi mp^{e_n-M}|^t .$$

Assume $p^{e_n} > 2N$

$$\|\mu_n\| = \sum_{z\in Z} |\mu_n\{zp^{e_n}\}|$$

$$= \sum_{z\in Z} |\hat{f}(z)| = 1 + \delta > 1 + 7\delta/8$$

while

$$|\mu_n|(V_n) = \sum_{|j|\le N} |\hat{f}(j)| > 1 + 15\delta/16 > 1 + 7\delta/8$$

so

$$|\mu_n|(V_n') < \delta/8 .$$

3.6.2.3 $\int P_n(p^{-e_{n-1}}u)d\mu_n(u) > 1 + 3\delta/4$

Choose a trigonometric polynomial on Δ_p

$$P(x) = \sum_{\lambda=1}^{\Lambda} \sum_{\tau=0}^{p^\lambda-1} a_{\lambda,\tau} \exp 2\pi i\tau p^{-\lambda}x$$

so that $\|P\|_\infty = 1$ and $a_{\lambda,\tau} = 0$ if $(\tau,p^\lambda) \ne 1$ and

$$\sum_{\lambda,\tau} a_{\lambda,\tau} f(\tau p^{e_{n_0}-\lambda}) = \int P(u)d\mu_{n_0}(u) > 1 + 3\delta/4 .$$

(This way of writing $P(x)$ is useful in 3.6.2.8 below.)

Now for all n such that $e_n - e_{n-1} > \Lambda - e_{n_0}$ define

$$P_n(x) = \sum_{\lambda,\tau} a_{\lambda,\tau} \exp 2\pi i\tau x p^{-\lambda-e_n+e_{n-1}+e_{n_0}}$$

so that we have

$$\begin{aligned}\int P_n(p^{-e_{n-1}}u)d\mu_n(u) &= \sum_{\lambda,\tau} a_{\lambda,\tau}\hat{\mu}_n(\tau p^{-\lambda-e_n+e_{n_0}}) \\ &= \sum_{\lambda,\tau} a_{\lambda,\tau} f(\tau p^{e_n}.p^{-\lambda-e_n+e_{n_0}}) \\ &= \sum_{\lambda,\tau} a_{\lambda,\tau} f(\tau p^{e_{n_0}-\lambda}) \\ &= \int P(u)d\mu_{n_0}(u)\end{aligned}$$

so that $\int P_n(p^{-e_{n-1}}u)d\mu_n(u) > 1 + 3\delta/4$ too.

3.6.2.4 $\int P_n(p^{-e_{n-1}}u)P_{n+1}(p^{-e_n}u)d\mu_n*\mu_{n+1}(u) > (1 + \delta/2)^2$

The integral in question is

$$J = \iint P_n(p^{-e_{n-1}}(u+v))P_{n+1}(p^{-e_n}(u+v))d\mu_n(u)d\mu_{n+1}(v)$$

$$= \int\left(\int P_{n+1}(p^{-e_n}(u+v))d\mu_{n+1}(v)\right)P_n(p^{-e_{n-1}}u)d\mu_n(u)$$

since for $u \in p^{e_n}\Delta_p$, $v \in p^{e_{n+1}}\Delta_p$, we have

$P_n(p^{-e_{n-1}}(u+v))$

$$= P(p^{e_{n_0}+e_{n-1}-e_n}p^{-e_{n-1}}(ab^{e_n}+bp^{e_{n+1}}))$$

$$= P(ap^{e_{n0}}+bp^{e_{n_0}-e_n+e_{n+1}})$$

$= P(ap^{e_{n0}})$ since $e_{n+1} - e_n > \Lambda - e_{n_0}$ and P is p^{Λ}-periodic

$$= P_n(p^{-e_{n-1}}u)$$

So $J > \int_{V_n} I(u)P_n(p^{-e_{n-1}}u)d\mu_n(u) - \delta/8$, $I(u)$ being the inner integral,

since $\left|\int_{V_n'}\right| \leq |\mu_n|(V_n') < \delta/8$.

So using the fact proved in 3.6.2.5 below that

$$(I(u) - \int P_{n+1}(p^{-e_n}v)d\mu_{n+1}(v)) \to 0 \quad \text{as} \quad n \to \infty,$$

we see

$$J > (1 + 3\delta/4)\int_{V_n} P_n(p^{-e_{n-1}}u)d\mu_n(u) - \delta/8$$

$$> (1 + 3\delta/4)(1 + 5\delta/8) - \delta/8$$

$$> (1 + \delta/2)^2.$$

3.6.2.5 $\underline{(I(u) - \int P_{n+1}(p^{-e_n}v)d\mu_{n+1}(v)) \to 0 \text{ as } n \to \infty}$

To see this, consider the difference

$$\left|\int P_{n+1}(p^{-e_n}(u+v))d\mu_{n+1}(v) - \int P_{n+1}(p^{-e_n}v)d\mu_{n+1}(v)\right|$$

$$= \left|\int \sum_{\lambda,\tau} a_{\lambda,\tau}[\exp(2\pi i p^{-e_n} \cdot p^{e_{n_0}-\lambda+e_n-e_{n+1}}(u+v)) - \exp(2\pi i p^{-e_n}\tau p^{e_{n_0}-\lambda+e_n-e_{n+1}}u)]\right|$$

$$\le \int \sum_{\lambda,\tau} |a_{\lambda,\tau}||\exp 2\pi i\tau p^{e_{n_0}-\lambda-e_{n+1}}u - 1|d|\mu_{n+1}(v)|$$

$$\le \|\mu_{n+1}\| \sum_{\lambda,\tau} |a_{\lambda,\tau}||\tau p^{e_{n_0}-\lambda-e_{n+1}}u| .$$

Now since $u \in V_n$ and so $|u| \le Np^{e_n}$, the last factor is uniformly arbitrarily small for n large enough, and the assertion follows.

3.6.2.6 $\int Q_{n,s}(u)d\tau_{n,s}(u) > (1 + \delta/2)^s$

Define

$$Q_{n,s}(u) = \prod_{j=n}^{n+s} P_j(p^{-e_j-1}u)$$

$$\tau_{n,s} = \mu_n * \mu_{n+1} * \ldots * \mu_{n+s} .$$

Then

$$\int Q_{n,s}(u)d\tau_{n,s}(u)$$

$$= \int\left[\int P_{n+s}(p^{-e_{n+s-1}}(u+v))d\mu_{n+s}(v)\right]Q_{n,s-1}(u)d\tau_{n,s-1}(u)$$

because for $v \in p^{e_{n+s}}\Delta_p$ and $u \in p^{e_{n+s-1}}\Delta_p$,
$Q_{n,s-1}(u+v) = \prod_{j=n}^{n+s-1} P_j(p^{-e_j-1}(u+v))$ and
$P_j(p^{-e_j-1}(u+v)) = P_j(p^{-e_j-1}u)$ for each j as in 3.6.2.4

So $\int Q_{n,s}(u)d\tau_{n,s}(u) > \int_{V_n} I(u)Q_{n,s-1}d\tau_{n,s-1} - \delta/8$

$$> (1 + 3\delta/4)[(1 + \delta/2)^{s-1} - \delta/8] - \delta/8 > (1 + \delta/2)^s$$

by induction on s, for n large enough, and $s \ge 2$ integral.

3.6.2.7 $\underline{\sup_n \| Q_{n,s} \|_{A(\Delta_p)} < \infty}$

As in 3.5.2.7,

$$\| Q_{n,s} \| = \sum_{m,M} Q^{\wedge}_{n,s}(mp^{-M}) < (p^{\Lambda})^s [\max_{\tau,\lambda} a_{\tau,\lambda}]^s < \infty$$

independently of n.

And we still have $\| Q_{n,s} \|_{\infty} \leq 1$ since each factor in $Q_{n,s}$ is so bounded.

3.6.2.8 $\underline{\inf_{m,M} \{ \prod_{j=1}^{n-1} |\cos 2\pi m p^{-e_j - M}|^t : Q^{\wedge}_{n,s}(mp^{-M}) \neq 0 \} \to 1 \text{ as } n \to \infty}$

Examining the frequencies of $Q_{n,s}$, we see that

$$Q^{\wedge}_{n,s}(mp^{-M}) \neq 0 \Leftrightarrow mp^{-M} = \tau_1 p^{e_{n_0} - \lambda_1 - e_n} + \tau_2 p^{e_{n_0} - \lambda_2 - e_{n+1}} + \ldots + \tau_s p^{e_{n_0} - \lambda_s - e_{n+s}}$$

and recalling our conditions on $a_{\tau,\lambda}$, this equation can only be satisfied if $(m,p^M) = 1$ and $p^{\min(-\lambda_j - e_{n+j})} = p^{-M-e_{n_0}}$

and so

$$1 + e_n - e_{n_0} \leq M \leq e_{n+s} + \Lambda - e_{n_0} .$$

So the smallest possible product $\prod_{j=1}^{n-1}$ has n-1 factors each at least

$$|\cos 2\pi p^{e_j - 1 - e_n}|^t \qquad \text{for } j < n .$$

Now if we compare $\prod_{j=n-1}^{j=\infty} |\cos 2\pi p^{e_j - 1 - e_n}|^t$ with $\prod_{j=J}^{\infty} |\cos 2\pi 2^{-j}|^t$ we see that we may make the first factor of the former exceed that of the latter by making n sufficiently large; thereafter the corresponding factors of the first product are larger than those of the second since $p \geq 2$ and $e_j - e_{j-1} \geq 1$, and of course run out after n of them. So since our comparison product exceeds $1-\varepsilon$ for any given ε if J is made large enough, it follows that our assertion is proved.

3.6.2.9 $\|\nu\| > (1+\delta/2)^s - 1$

If n is sufficiently large,

$$\|\nu\| \geq \int Q_{n,s}(u)d\nu(u)$$

$$= \sum_{(m,M)} [Q^\wedge_{n,s}(mp^{-M})(\prod_{j=1}^{\infty} |\cos 2\pi mp^{e_j -M}|^t)]$$

$$= \sum_{(m,M)} [Q^\wedge_{n,s}(mp^{-M}).\prod_{j=1}^{n+s} |\cos 2\pi mp^{e_j -M}|^t . \prod_{j=n+s+1}^{\infty} |\cos 2\pi mp^{e_j -M}|^t]$$

$$= \sum_{(m,M)} [Q^\wedge_{n,s}(mp^{-M}) \prod_{j=1}^{n+s} |\cos 2\pi mp^{e_j -M}|^t]$$

since $Q^\wedge_{n,s}(mp^{-M}) \neq 0$ only if $M \leq e_{n+s} + \Lambda - e_{n_0}$

so that $p^{e_j -M} \in Z$ for $j > n + s$ and so all

factors after $|\cos 2\pi mp^{e_{n+s} -M}|^t$ are 1 for the

terms with $Q^\wedge_{n,s}(mp^{-M}) \neq 0$.

$$\geq \sum_{(m,M)} [Q^\wedge_{n,s}(mp^{-M}).\prod_{j=n}^{n+s} |\cos 2\pi mp^{e_j -M}|^t .(1 - \varepsilon/\sup\|Q\|_{A(T)})]$$

for n sufficiently large, by 3.6.2.8

$$\geq \sum_{(m,M)} [Q^\wedge_{n,s}(mp^{-M}) \prod_{j=n}^{n+s} |\cos 2\pi mp^{e_j -M}|^t] - 1$$

$$\geq (1+\delta/2)^s - 1 \qquad \text{by } 3.6.2.6$$

from which we see that ν does not exist.

This completes the proof of the lemma. (It is interesting to observe the complementary way the "early" and the "late" factors in the infinite products are removed in the two cases of T and Δ_p.)

3.7 $G = \prod_{j=1}^{\infty} Z_{n_j}$

After T and Δ_p, the other groups we need to consider are $\prod_{j=1}^{\infty} Z_{n_j}$. In some ways these are much simpler (because infinite convolutions on them reduce to infinite products), but our results are a little more complicated. The crucial question is whether the n_j are bounded.

3.7.1 $G = \prod_{j=1}^{\infty} Z_{n_j}$, n_j unbounded

We first develop the required information for a measure on a single factor.

3.7.1.1 Lemma

Let $\mu = \tfrac{1}{2}\,\delta(a) + \tfrac{1}{2}\,\delta(-a)$ on Z_{n_k},

$\hat{\nu} = |\hat{\mu}|^t$, $t \geq 0$, $a \neq 0$.

For a given t, if n_k is large enough there are j for which $\nu(j) < 0$, so that $\|\nu\| > 1 + \delta$ for some $\delta > 0$ not depending on n_k.

Proof

$$\nu(j) = 1/k \sum_{s=0}^{k-1} \exp 2\pi ijsk^{-1} |\cos 2\pi sk^{-1}|^t .$$

For a given j and t, as $k \to \infty$

$$\nu(j) \to \int_0^1 \exp 2\pi ixj |\cos 2\pi x|^t \, dm(x)$$ this function being Riemann integrable

$$= \frac{2\pi\Gamma(t+1)}{2^t\,\Gamma(t+j/2)\Gamma(t-j/2)} = \theta \qquad \text{(c.f. 3.5.2.1)}$$

which is negative if $t-j/2 \in \bigcup_{r=0}^{\infty} (-2r+1, 2r)$.

Since $\|\nu\| = \sum_j |\nu(j)|$, obviously $\|\nu\| > 1 + \delta$ where δ may be chosen as near as desired to $|\theta|$.

Theorem

Let $G = \prod_{j=1}^{\infty} Z_{n_j}$, n_j unbounded, $\mu \in B\pi^\infty$.

Then if F continuous on $[-1,1]$ operates on μ,

$$F(x) = \sum_{n=0}^{\infty} b_n x^n \qquad x \in [-1,1]$$

$$\text{with} \quad \sum_{n=0}^{\infty} |b_n| < \infty .$$

Proof

This is a consequence of Lemma 3.7.1.1, the unboundedness of n_j and the usual arguments: since

$$\hat{\nu} = |\hat{\mu}|^t , \quad \hat{\nu}_n = |\hat{\mu}_n|^t \Rightarrow \|\nu\| = \prod_{j=1}^{\infty} \|\nu_j\| ,$$

$\|\nu\|$ is only 1 if $t \in 2N$, since for any other finite t $|\cdot|^t$ will not operate on ν_j for sufficiently large j. So $|\cdot|^t$ operates for $t \in 2N$ and the conclusion follows as before.

3.7.2 $G = \prod_{j=1}^{\infty} Z_k$

If $G = \prod_{j=1}^{\infty} Z_{n_j}$, with n_j bounded then G is a product of groups $\prod_{j=1}^{\infty} Z_k$ and a finite number of factors Z_{n_j} .

So we look at groups of the first kind.

3.7.2.1 Lemma

Let $\mu = a\,\delta(0) + (1-a)[\tfrac{1}{2}\,\delta(1) + \tfrac{1}{2}\,\delta(-1)]$ on Z_k ,

$\hat{\nu} = |\hat{\mu}|^t$, $a \in [0,1]$, $t \geq 0$.

Then if a is near enough to 1, $\nu(1) < 0$, and

$\|\nu\| > 1 + \delta$ for some fixed $\delta > 0$.

Proof

$$\nu(1) = \sum_{s=0}^{k-1} \exp 2\pi i s k^{-1} |a + (1-a) \cos 2\pi s k^{-1}|^t$$

now at $a = 1$, $\nu(1) = 0$.

Further, $\frac{\partial\nu(1)}{\partial a} = t\sum_{s=0}^{k-1} \exp 2\pi isk^{-1}|a+(1-a)\cos 2\pi sk^{-1}|^{t}.(1-\cos 2\pi k^{-1}s)$

(we only consider a near 1)

and $\frac{\partial\nu(1)}{\partial a} > 0$ at $a = 1$ (since the numerically largest terms have the smallest multipliers).

So that for $a < 1$ but near to 1, $\nu(1) < 0$.

3.7.2.2 Theorem

Let $G = \prod_{j=1}^{\infty} Z_k$, $\mu \in B_2\pi_k$.

Then if F continuous on $[m_\mu,1]$ operates on μ iff

$$F(x) = \sum_{n=0}^{\infty} b_n x^n \qquad x \in [m_\mu,1]$$

$$\text{with } \sum_{n=0}^{\infty} |b_n| < \infty .$$

Proof

Since $\{a_n\}$ is dense in $[-1,1]$ there are infinitely many factors with a_j near enough to 1 for the Lemma 3.7.2.1 to apply so that $\|\nu_j\| > 1 + \delta$ and ν can only exist if $t \in ZN$.

The usual arguments then apply, but now we only have the conclusion for $[m_\mu,1]$ because only that interval is in $C(\mu)$: $\{\hat{\mu}(\gamma) : \gamma \in \Gamma\}^-$ is the regular k-gon subset of the unit disc determined by $\{\exp 2\pi isk^{-1}|s=0,\ldots,k-1\}$.

3.8 MAIN RESULT

3.8.1 DEFINITION

We shall say that $\mu \in M(G)$ *has property* Z if F continuous on $[m_\mu, 1]$ only operates on μ if

$$F(x) = \sum_{n=0}^{\infty} b_n x^n, \quad \sum_{n=0}^{\infty} |b_n| < \infty, \quad x \in C(\mu).$$

We have proved that $\mu \in BT$, $\mu \in B\Delta_p$, $\mu \in B_2\pi Zk$ and $\mu \in B\pi Z(\infty)$ all have property Z, but that $\mu \in B_1\pi Z_k$ does not.

3.8.2 LEMMA

Let H be a compact subgroup of G,

$$\pi : m_H * M(G) \to M(G/H)$$

be the isomorphism induced by the canonical homeomorphism $G \to G/H$.

If $\mu \in M(G/H)$ has property Z, so has $\nu = \pi^{-1}\mu \in M(G)$.

Proof

The measure corresponding to μ is actually determined by the continuous linear functional

$$I(f) = \int_{G/H}\left(\int_H f(x+y)dm_H(y)\right)d\mu(X)$$

since the inner integral depends only on the coset X containing x, for each $f \in C_c(G)$.

Clearly $\hat{\nu}(\gamma) = \hat{\mu}(\gamma)$ for $\gamma \in (G/H)\hat{}$

and $\hat{\nu}(\gamma) = 0$ otherwise.

So if F operates on ν, i.e. $F(\hat{\nu}(\gamma)) = \lambda\hat{}(\gamma)$, $\gamma \in PD(G)$ for all $\gamma \in \Gamma$ with $\hat{\nu}(\gamma) \in \text{dom } F$, then $\lambda\hat{}$ restricted to $H^\perp = (G/H)\hat{}$ is a transform, by 1.4.2.3. But this is to say that $F \circ \hat{\mu}$ is a transform and so by hypothesis

$$F(x) = \sum_{n=0}^{\infty} b_n x^n, \quad \sum_{n=0}^{\infty} |b_n| < \infty \quad \text{for} \quad x \in C(\mu).$$

Since $C(\mu) = C(\nu)$, we conclude that ν has property Z.

3.8.3 LEMMA

Let H be a subgroup of G. If $\mu \in M(H)$ has property Z, so has μ considered as a measure on G.

Proof

Denote μ considered as a measure on G by ν. Let F continuous on $[m_\mu,1]$ operate on μ. Since ν is concentrated on H, $\hat{\nu}$ is constant on cosets of $H^\perp$; so $F \circ \hat{\nu}$ being a transform entails $F \circ \hat{\mu}$ is a transform in $B(H^\wedge)$, by Eberlein's criterion, and so by hypothesis

$$F(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{with} \quad \sum_{n=0}^{\infty} |b_n| < \infty \quad x \in C(\mu) = C(\nu)$$

so ν has property Z.

3.8.4 LEMMA

M(R) has measures with property Z.

Proof

Let μ be a measure of the class BT.

Consider μ as a measure on R, i.e. $\mu \in M(R)$. If F operates on μ as a measure on R, so that $F \circ \hat{\mu} = \hat{\nu} \in M^\wedge(R)$, then we apply Eberlein's criterion 2.4.1.3 to see that $\hat{\nu}|_Z \in M^\wedge(T)$ so that F operates on μ as a member of M(T), so F is $\sum b_n x^n$, $\sum |b_n| < \infty$ and μ has property Z for R too.

Eberlein's criterion applies thus:

since $\hat{\nu} \in M^\wedge(R)$, by the criterion if $\|\hat{\nu}\| \leq A$ then $\hat{\nu}$ is continuous on R and

$$\left|\sum_{j=1}^{n} c_j (x,\gamma_j)\right| \leq A\|f\|_\infty \qquad (1)$$

for every trigonometric polynomial $f = \sum_{j=1}^{n} c_j \gamma_j$ on R. Obviously then (1) holds for every trigonometric polynomial on Z, and $\hat{\nu}$ is continuous on Z since every function is so applying the criterion again $\hat{\nu} \in B(Z) = M\hat{\ }(T)$.

3.8.5 MAIN THEOREM

For any non-discrete LCA group G there is a class of measures with property Z.

Proof

The two lemmas preceding reduce the problem to finding a subgroup of G (lemma 3.8.2) or of Γ (lemma 3.8.1 and duality) on which suitable infinite convolutions with property Z can be defined.

The structure theorem for LCA groups reduces the problem to compact abelian groups since any LCA $G = R^n \times F$ with $n > 0$ succumbs to the lemmas $(T = R/Z)$. If G is compact, either Γ has elements of infinite order, or not. In the former case Γ has a subgroup Z and lemma 2 applies, if G is torsion, Γ either has a nontrivial divisible subgroup, or not. If not, Γ has as subgroup a weak product $\oplus \sum_{j=1}^{\infty} Z(m_j)$ (FUCHS [II, 65]) so 3.7 assures us of suitable μ; whereas if Γ has a nontrivial divisible subgroup, being torsion and abelian it must contain a subgroup isomorphic to $Z(p^\infty)$ for some prime p, and so G has a quotient isomorphic to Δ_p and 2.6 completes the proof.

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