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A note on Navier-Stokes equations with non-orthogonal coordinates

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Abstract

There are many fluid flow problems involving geometries for which a non-orthogonal curvilinear coordinate system may be the most suitable. To the authors' knowledge, the Navier-Stokes equations for an incompressible fluid formulated in terms of an arbitrary non-orthogonal curvilinear coordinate system have not been given explicitly in the literature in the simplified form obtained herein. The specific novelty in the equations derived here is the use of the general Laplacian in arbitrary non-orthogonal curvilinear coordinates and the simplification arising from a Ricci identity for Christoffel symbols of the second kind for flat space. Evidently however, the derived equations must be consistent with the various general forms given previously by others. The general equations derived here admit the well known formulae for cylindrical and spherical polars and, for purposes of illustration, the procedure is presented for spherical polar coordinates. Further, the procedure is illustrated for a non-orthogonal helical coordinate system. For slow flow for which the inertial terms may be neglected, we give the harmonic equation for the pressure function, and the corresponding equation if the inertial effects are included. We also note the general stress boundary conditions for a free surface with surface tension. For completeness, the equations for a compressible flow are derived in an appendix.

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1. Introduction

There are many fluid flows involving curved geometries which are motivated by flows in rivers and pipes, and for which a natural coordinate description might involve the use of non-orthogonal curvilinear coordinates. In the analysis of helical pipe

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flow, researchers either devise special orthogonal coordinate systems from which to analyse the fluid flow [5, 7, 14, 16] or devise approximations for which the first-order estimate (zero torsion limit) is obtained using an orthogonal coordinate system [10]; see also the discussion of [14, 15]. Aris [1] provides all the essential tensorial development to present the general Navier-Stokes equations, and yet does not provide the final simplified form. Wang [15] comes very close to the final equations, giving all the terms, but does not make two essential final simplifications. To the authors' knowledge, the full Navier-Stokes equations for an incompressible fluid expressed in terms of arbitrary non-orthogonal coordinates, seem not to be available in the literature in their simplest form, and our purpose here is to present a concise derivation of these equations. These constitute an advance on those given previously in the sense that they include the Laplacian operator in general curvilinear coordinates and they are based upon the condition (51) for Christoffel symbols, which has not been exploited in all other representations.

In Section 2 we outline the standard formulation for arbitrary non-orthogonal curvilinear coordinates (x^1, x^2, x^3). In Section 3 we define the rate-of-strain tensor and its relation to the Cauchy stress tensor, the condition for incompressibility and the standard momentum equations. In Section 4 we state the general Navier-Stokes equations for an incompressible fluid, and briefly comment on the harmonic equation for the pressure function for slow viscous flow and the corresponding equation when inertial effects are not neglected. Further, the general stress boundary conditions for a free surface with surface tension are briefly noted.

A brief derivation of the key result needed for the general Navier-Stokes equations is presented in Appendix A. The final equations agree with the standard expressions that are known for cylindrical and spherical polars, and some of the details for spherical polar coordinates are briefly included in Section 5 of the paper for the sake of completeness. Section 6 gives the Navier-Stokes and incompressible continuity equations in non-orthogonal helical coordinates, obtained using the general formulation presented herein. These equations agree with the simplified helically-symmetric forms used for modelling of flow in helically-wound channels [8, 2]. In Appendix B we derive, for low Reynolds number flow, the harmonic equation for the pressure function and the corresponding equation when the inertial terms are included. Appendix C summarises the key equations for compressible Navier-Stokes fluids.

2. Mathematical preliminaries

Since the general equations necessarily involve tensors, and since in the literature different velocity components are used (contravariant, covariant and physical), in order to avoid confusion, we first spell out carefully the conventions employed here. Let x^i $(i = 1, 2, 3)$ denote any non-orthogonal curvilinear coordinate system, with metric tensor g_{ij}, conjugate metric tensor g^{ij} and Christoffel symbols of the second kind Γ^i_{jk}. If (x, y, z) denote the usual rectangular Cartesian coordinates, with position vector $\boldsymbol{r}$ given by

$$\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \tag{1}$$

where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote the usual unit vectors, then for the general non-orthogonal coordinates (x^1, x^2, x^3), the base vectors are defined by (see, for example, [4, pp. 430–434])

$$\mathbf{e}_i = \frac{\partial \boldsymbol{r}}{\partial x^i} \quad (i = 1, 2, 3). \tag{2}$$

As usual we have for the line element

$$(ds)^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial x^i} \cdot \frac{\partial \boldsymbol{r}}{\partial x^j} dx^i dx^j = \mathbf{e}_i \cdot \mathbf{e}_j dx^i dx^j, \tag{3}$$

and therefore $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$, where here and throughout we use the Einstein summation convention for a repeated index unless stated otherwise.

To make precise our ideas relating to non-physical and physical components of velocity and acceleration, we observe that the actual velocity vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} = u^i \mathbf{e}_i, \tag{4}$$

where the last equality, in which $u^i = dx^i/dt$, is readily obtained from the chain rule and (2), while the actual acceleration vector $\boldsymbol{a}$ is defined by

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{dt} = \frac{du^i}{dt}\mathbf{e}_i + u^i \frac{\partial \mathbf{e}_i}{\partial x^j}\frac{dx^j}{dt} = \left(\frac{du^k}{dt} + \Gamma^k_{ij} u^i u^j\right)\mathbf{e}_k, \tag{5}$$

where we have used the standard formula (see for example, [4, pp. 439])

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma^k_{ij}\mathbf{e}_k. \tag{6}$$

In order to make comparisons of the equations derived here with existing formulae, it is important to note that generally the base vectors $\mathbf{e}_i$ are not unit vectors and, therefore (for example) the physical components of velocity which are generally employed in the literature are deduced from the equation

$$\boldsymbol{u} = \sqrt{g_{ii}} u^i \frac{\mathbf{e}_i}{\sqrt{g_{ii}}}, \tag{7}$$

since $|\mathbf{e}_i|^2 = \mathbf{e}_i \cdot \mathbf{e}_i = g_{ii}$ (no summation) and therefore $\mathbf{e}_i / \sqrt{g_{ii}}$ (no summation) are the appropriate unit vectors, while $\sqrt{g_{ii}} u^i$ (again no summation) denote the physical components of the velocity (v^1, v^2, v^3), that is

$$v^1 = \sqrt{g_{11}}\, u^1, \quad v^2 = \sqrt{g_{22}}\, u^2, \quad v^3 = \sqrt{g_{33}}\, u^3. \tag{8}$$

3. Rate-of-strain and Cauchy stress tensors

The rate-of-strain tensor d^{ij} is defined by

$$d^{ij} = \frac{1}{2}\left(g^{ik} u^j_{;k} + g^{jk} u^i_{;k}\right), \tag{9}$$

where the semi-colon throughout denotes partial covariant differentiation, in this case with respect to x^k. For a general curvilinear coordinate system (x^1, x^2, x^3) the partial covariant derivative takes proper account of the coordinate dependence of the base vectors through the Christoffel symbols and equation (6) so that the partial covariant derivatives in (9) are given explicitily by

$$u^i_{;k} = \frac{\partial u^i}{\partial x^k} + \Gamma^i_{jk} u^j. \tag{10}$$

As described in Appendix A, for an incompressible fluid the trace of $d^i_j = g_{jk} d^{ik}$ vanishes giving the incompressibility condition

$$d^i_i = \frac{\partial u^i}{\partial x^i} + \Gamma^i_{ij} u^j = 0, \tag{11}$$

which on using the standard formula of tensor calculus (see for example, [4, pp. 441] or [12, pp. 28]))

$$\Gamma^i_{ij} = \frac{1}{2g} \frac{\partial g}{\partial x^j}, \tag{12}$$

simplifies to become

$$\frac{\partial}{\partial x^i} (\sqrt{g}\, u^i) = 0, \tag{13}$$

where $g = |g_{ij}|$ denotes the determinant of the metric tensor. This latter equation means that the velocity vector is divergence free.

The stress-rate-of-strain relations for a Newtonian fluid are given by

$$t^{ij} = -p g^{ij} + 2\mu d^{ij}, \tag{14}$$

where t^{ij} denotes the Cauchy stress tensor, g^{ij} is the conjugate metric tensor, p is the arbitrary hydrostatic pressure and μ is the viscosity, assumed constant. Conservation of momentum gives

$$\rho a^j = \rho \left(\frac{du^j}{dt} + \Gamma^j_{ik} u^i u^k \right) = t^{ij}_{;i} + \rho f^j, \tag{15}$$

where ρ denotes the constant density, $\mathbf{f}$ denotes an external body force per unit mass, and the partial covariant differentiation denoted by the semi-colon may be written

$$t^{ij}_{;k} = \frac{\partial t^{ij}}{\partial x^k} + \Gamma^i_{k\ell} t^{\ell j} + \Gamma^j_{k\ell} t^{i\ell}. \tag{16}$$

From the stress-rate-of-strain relations and the conservation of momentum, we have

$$\rho \left(\frac{du^j}{dt} + \Gamma^j_{ik} u^i u^k \right) = -\frac{\partial p}{\partial x^i} g^{ij} + 2\mu\, d^{ij}_{;i} + \rho f^j. \tag{17}$$

A detailed expression for $d^{ij}_{;i}$ is derived in Appendix A.

4. General incompressible Navier-Stokes equations

From equation (17) and the expressions (52) for $d^{ij}_{;i}$ the general Navier-Stokes equations for an incompressible fluid in an arbitrary non-orthogonal curvilinear coordinate system are

$$\rho\left(\frac{du^j}{dt}+\Gamma^j_{ik}u^iu^k\right)=-\frac{\partial p}{\partial x^i}g^{ij}+\mu\left\{\nabla^2u^j+2g^{ik}\Gamma^j_{i\ell}\frac{\partial u^\ell}{\partial x^k}+g^{ik}\frac{\partial\Gamma^j_{ik}}{\partial x^\ell}u^\ell\right\}$$
$$+\rho f^j, \tag{18}$$

where ∇^2 denotes the usual Laplacian operator which in general curvilinear coordinates is ([12, pp. 32])

$$\nabla^2=g^{ij}\left(\frac{\partial^2}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial}{\partial x^k}\right), \tag{19}$$

noting that for any scalar ϕ, $\nabla^2\phi$ arises from the partial covariant derivatives

$$\left(g^{ij}\frac{\partial\phi}{\partial x^j}\right)_{;i}=\left(g^{ij}\phi_{;j}\right)_{;i}=g^{ij}\phi_{;i;j} \tag{20}$$

since the partial covariant derivative of the metric tensor vanishes. The new element of these equations in general curvilinear coordinates comprises both the explicit introduction of the general Laplacian above and the consequent simplification of the general equations making use of the identity given in Appendix A (51) for the Christoffel symbols. However, of course the same equations must clearly be implicit in all other general descriptions such as, for example, those given by Aris [1] or Wang [15]. For example, for the term in curly brackets in (18), Wang [15] gives the expression

$$g^{ik}u^j_{;ik}=g^{ik}\left\{\frac{\partial^2u^j}{\partial x^i\partial x^k}+\Gamma^j_{im}\frac{\partial u^m}{\partial x^k}+\Gamma^j_{km}\frac{\partial u^m}{\partial x^i}-\Gamma^m_{ik}\frac{\partial u^j}{\partial x^m}\right.$$
$$\left.+\left(\frac{\partial\Gamma^j_{im}}{\partial x^k}+\Gamma^j_{nk}\Gamma^n_{im}-\Gamma^n_{ik}\Gamma^j_{mn}\right)u^m\right\} \tag{21}$$

which is entirely in accord with (48) of Appendix A, noting that two terms of (48) cancel. Aris [1, p. 182] identifies the left-hand side of this equation as the critical contribution in the Navier-Stokes equations, but does not provide the general expression in terms of Christoffel symbols. Our claim is that the novel elements stated above appear here for the first time. As far as the authors are aware, the explicit use of the general Laplacian and the identity (51) for the Christoffel symbols leading to the general Navier-Stokes equations in the form given above have not been given previously in the literature. We further comment that the terms in the curly brackets of (18) give rise to the known expressions for the Navier-Stokes equations in cylindrical and spherical polar coordinates and that the time derivative on the left-hand side is the usual material time derivative, namely

$$\frac{du^j}{dt} = \frac{\partial u^j}{\partial t} + u^i \frac{\partial u^j}{\partial x^i}, \tag{22}$$

where the partial time derivative denotes differentiation with respect to time at a point fixed in space.

From a practical perspective, for a given non-orthogonal coordinate system it is simpler to obtain (18) in terms of non-physical components of velocity u^j $(j = 1, 2, 3)$, then make the transformations (8), namely

$$u^1 = \frac{v^1}{\sqrt{g_{11}}}, \quad u^2 = \frac{v^2}{\sqrt{g_{22}}}, \quad u^3 = \frac{v^3}{\sqrt{g_{33}}}, \tag{23}$$

to deduce the Navier-Stokes equations in terms of physical velocity components v^j $(j = 1, 2, 3)$.

In Appendix B we note that for low Reynolds number flow, for which we may neglect the inertia, the pressure function satisfies the harmonic equation

$$\nabla^2 p = 0, \tag{24}$$

with the Laplacian operator defined by (19). If the inertial effects are included, then the equation corresponding to (24) becomes

$$\left(\left(pg^{ij} + \rho u^i u^j\right)_{;i}\right)_{;j} = 0, \tag{25}$$

which cannot be written concisely in terms of the Laplacian but nevertheless applies for all time dependent incompressible viscous flows in the absence of body forces.

Finally in this section, we note the general boundary conditions on a free surface with surface tension. We suppose that $S(x^1, x^2, x^3) = 0$ represents a free surface with outward drawn unit normal $\mathbf{n}$ based upon the gradient

$$\nabla S = g^{ij} \frac{\partial S}{\partial x^i} \mathbf{e}_j, \tag{26}$$

so that $\mathbf{n} = \nabla S / |\nabla S|$. On the free surface of the fluid the stress vector is given by

$$\mathbf{t} = t^i \mathbf{e}_i = t^{ij} n_j \mathbf{e}_i, \tag{27}$$

and the standard boundary condition that the stress vector be aligned along the normal vector, giving zero tangential components of the stress vector and the normal stress vector component balanced by surface tension γ, namely $\mathbf{t} = -\gamma\kappa\mathbf{n}$, becomes

$$(t^{ij} + \gamma\kappa g^{ij}) \frac{\partial S}{\partial x^j} = 0, \tag{28}$$

where κ denotes the mean curvature (see for example [12, pp. 76–77]).

5. Spherical polar coordinates (r, θ, ϕ) as an example

We may readily confirm that (18) gives rise to standard formulae for the case of cylindrical and spherical polar coordinates. By way of a brief illustration, for standard spherical polar coordinates (r, θ, ϕ), the non-zero components of the metric tensors and the Christoffel symbols are

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta, \tag{29a}$$

$$g^{11} = 1, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2 \sin^2 \theta}, \tag{29b}$$

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r \sin^2 \theta, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \tag{29c}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \tag{29d}$$

while the Laplacian ∇^2 becomes

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{30}$$

On making the transformations to physical velocity components

$$v^1 = u^1, \quad v^2 = ru^2, \quad v^3 = r\sin\theta\, u^3, \tag{31}$$

the critical new terms of (18) which appear in the curly brackets can be shown to become, for $j = 1,\ 2$ and 3, respectively,

$$\begin{aligned} \nabla^2 u^1 + 2g^{22}\Gamma^1_{22}\frac{\partial u^2}{\partial x^2} + 2g^{33}\Gamma^1_{33}\frac{\partial u^3}{\partial x^3} + g^{22}\frac{\partial \Gamma^1_{22}}{\partial x^1}u^1 + g^{33}\frac{\partial \Gamma^1_{33}}{\partial x^1}u^1 + g^{33}\frac{\partial \Gamma^1_{33}}{\partial x^2}u^2 \\ = \nabla^2 v^1 - \frac{2}{r^2}\frac{\partial v^2}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v^3}{\partial\phi} - \frac{2}{r^2}v^1 - \frac{2\cot\theta}{r^2}v^2, \end{aligned} \tag{32a}$$

$$\begin{aligned} \nabla^2 u^2 + 2g^{33}\Gamma^2_{33}\frac{\partial u^3}{\partial x^3} + 2g^{11}\Gamma^2_{12}\frac{\partial u^2}{\partial x^1} + 2g^{22}\Gamma^2_{21}\frac{\partial u^1}{\partial x^2} + g^{33}\frac{\partial \Gamma^2_{33}}{\partial x^2}u^2 \\ = \frac{1}{r}\left[\nabla^2 v^2 + \frac{2}{r^2}\frac{\partial v^1}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v^3}{\partial\phi} - \frac{1}{r^2\sin^2\theta}v^2\right], \end{aligned} \tag{32b}$$

$$\begin{aligned} \nabla^2 u^3 + 2g^{22}\Gamma^3_{23}\frac{\partial u^3}{\partial x^2} + 2g^{33}\Gamma^3_{32}\frac{\partial u^2}{\partial x^3} + 2g^{11}\Gamma^3_{13}\frac{\partial u^3}{\partial x^1} + 2g^{33}\Gamma^3_{31}\frac{\partial u^1}{\partial x^3} \\ = \frac{1}{r\sin\theta}\left[\nabla^2 v^3 + \frac{2}{r^2\sin\theta}\frac{\partial v^1}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v^2}{\partial\phi} - \frac{1}{r^2\sin^2\theta}v^3\right], \end{aligned} \tag{32c}$$

while, for $j = 1,\ 2$ and 3, the terms in brackets on the left-hand side of (18) lead to

$$\begin{aligned}
&\frac{\partial u^1}{\partial t} + u^1\frac{\partial u^1}{\partial x^1} + u^2\frac{\partial u^1}{\partial x^2} + u^3\frac{\partial u^1}{\partial x^3} + \Gamma^1_{22}u^2u^2 + \Gamma^1_{33}u^3u^3 \\
&\quad = \frac{\partial v^1}{\partial t} + v^1\frac{\partial v^1}{\partial r} + \frac{v^2}{r}\frac{\partial v^1}{\partial \theta} + \frac{v^3}{r\sin\theta}\frac{\partial v^1}{\partial \phi} - \frac{v^2v^2}{r} - \frac{v^3v^3}{r},
\end{aligned} \tag{33a}$$

$$\begin{aligned}
&\frac{\partial u^2}{\partial t} + u^1\frac{\partial u^2}{\partial x^1} + u^2\frac{\partial u^2}{\partial x^2} + u^3\frac{\partial u^2}{\partial x^3} + 2\Gamma^2_{12}u^1u^2 + \Gamma^2_{33}u^3u^3 \\
&\quad = \frac{1}{r}\left(\frac{\partial v^2}{\partial t} + v^1\frac{\partial v^2}{\partial r} + \frac{v^2}{r}\frac{\partial v^2}{\partial \theta} + \frac{v^3}{r\sin\theta}\frac{\partial v^2}{\partial \phi} + \frac{v^1v^2}{r} - \frac{\cot\theta\, v^3v^3}{r}\right),
\end{aligned} \tag{33b}$$

$$\begin{aligned}
&\frac{\partial u^3}{\partial t} + u^1\frac{\partial u^3}{\partial x^1} + u^2\frac{\partial u^3}{\partial x^2} + u^3\frac{\partial u^3}{\partial x^3} + 2\Gamma^3_{13}u^1u^3 + 2\Gamma^3_{23}u^2u^3, \\
&\quad = \frac{1}{r\sin\theta}\left(\frac{\partial v^3}{\partial t} + v^1\frac{\partial v^3}{\partial r} + \frac{v^2}{r}\frac{\partial v^3}{\partial \theta} + \frac{v^3}{r\sin\theta}\frac{\partial v^3}{\partial \phi} + \frac{v^1v^3}{r} + \frac{\cot\theta\, v^2v^3}{r}\right).
\end{aligned} \tag{33c}$$

It is now straight-forward to obtain the standard formulae given, for example, by Goldstein [6, pp. 103–105], Ramsey [11, pp. 371–374] and Batchelor [3, pp. 600–603]. For example, for $j = 2$, we have

$$\rho\left(\frac{\partial v^2}{\partial t} + v^1\frac{\partial v^2}{\partial r} + \frac{v^2}{r}\frac{\partial v^2}{\partial \theta} + \frac{v^3}{r\sin\theta}\frac{\partial v^2}{\partial \phi} + \frac{v^1v^2}{r} - \frac{\cot\theta\, v^3v^3}{r}\right) \tag{34}$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 v^2 + \frac{2}{r^2}\frac{\partial v^1}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v^3}{\partial \phi} - \frac{v^2}{r^2\sin^2\theta}\right] + \rho r f^2. \tag{35}$$

6. Non-orthogonal helical coordinates

To model inviscid flow in the coiled cochlea, Manoussaki and Chadwick [9] employ the non-orthogonal helical coordinate system (β, r, z) defined by

$$\boldsymbol{r}(\beta, r, z) = r\cos\beta\,\hat{\mathbf{i}} + r\sin\beta\,\hat{\mathbf{j}} + (P\beta + z)\,\hat{\mathbf{k}}, \tag{36}$$

where P is a constant. The same coordinate system is used to model helically-symmetric viscous flow in a helically-wound channel [8, 2], and the derivation of the helically-symmetric Navier-Stokes equations is given in an appendix of [8]. As an illustrative example of (18) for a non-orthogonal coordinate system, we here write the full Navier-Stokes equations in this helical coordinate system, which, to the authors' knowledge, have not been previously written down.

Let $(\beta, r, z) \equiv (x^1, x^2, x^3)$. Defining, for convenience, $\Lambda(r) = P/r$ and $\Upsilon(r) = 1 + P^2/r^2 = 1 + \Lambda^2$, the non-zero components of the metric tensors and the Christoffel symbols are

$$g_{11} = r^2\Upsilon, \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{13} = g_{31} = r\Lambda, \tag{37a}$$

$$g^{11} = \frac{1}{r^2}, \quad g^{22} = 1, \quad g^{33} = \Upsilon, \quad g^{13} = g^{31} = -\frac{\Lambda}{r}, \tag{37b}$$

$$\Gamma^2_{11} = -r, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{r}, \quad \Gamma^3_{12} = \Gamma^3_{21} = -\Lambda, \tag{37c}$$

while the physical velocity components are given by

$$v^1 = (r\sqrt{\Upsilon})u^1, \quad v^2 = u^2, \quad v^3 = u^3. \tag{38}$$

Then, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \Upsilon\frac{\partial^2}{\partial z^2} - \frac{2\Lambda}{r}\frac{\partial^2}{\partial\beta\partial z}, \tag{39}$$

and the Navier-Stokes equations are

$$\rho\left(\frac{\partial u^1}{\partial t} + u^1\frac{\partial u^1}{\partial\beta} + u^2\frac{\partial u^1}{\partial r} + u^3\frac{\partial u^1}{\partial z} + \frac{2u^1u^2}{r}\right) = -\frac{1}{r^2}\frac{\partial p}{\partial\beta} + \frac{\Lambda}{r}\frac{\partial p}{\partial z}$$
$$+\mu\left\{\nabla^2 u^1 + \frac{2}{r}\left(\frac{\partial u^1}{\partial r} + \frac{1}{r^2}\frac{\partial u^2}{\partial\beta} - \frac{\Lambda}{r}\frac{\partial u^2}{\partial z}\right)\right\} + \rho f^1, \tag{40a}$$

$$\rho\left(\frac{\partial u^2}{\partial t} + u^1\frac{\partial u^2}{\partial\beta} + u^2\frac{\partial u^2}{\partial r} + u^3\frac{\partial u^2}{\partial z} - ru^1u^1\right) = -\frac{\partial p}{\partial r}$$
$$+\mu\left\{\nabla^2 u^2 - \frac{2}{r}\frac{\partial u^1}{\partial\beta} + 2\Lambda\frac{\partial u^1}{\partial z} - \frac{u^2}{r^2}\right\} + \rho f^2, \tag{40b}$$

$$\rho\left(\frac{\partial u^3}{\partial t} + u^1\frac{\partial u^3}{\partial\beta} + u^2\frac{\partial u^3}{\partial r} + u^3\frac{\partial u^3}{\partial z} - 2\Lambda u^1u^2\right) = -\Upsilon\frac{\partial p}{\partial z} + \frac{\Lambda}{r}\frac{\partial p}{\partial\beta}$$
$$+\mu\left\{\nabla^2 u^3 - \frac{2\Lambda}{r^2}\frac{\partial u^2}{\partial\beta} + \frac{2\Lambda^2}{r}\frac{\partial u^2}{\partial z} - 2\Lambda\frac{\partial u^1}{\partial r}\right\} + \rho f^3. \tag{40c}$$

In terms of the physical velocity components these equations become

$$\rho\left\{\frac{\partial v^1}{\partial t} + \frac{v^1}{r\sqrt{\Upsilon}}\frac{\partial v^1}{\partial\beta} + v^2\frac{\partial v^1}{\partial r} + v^3\frac{\partial v^1}{\partial z} + \frac{(\Upsilon^2+\Lambda^2)v^1v^2}{r\Upsilon^2}\right\} =$$
$$-\frac{\sqrt{\Upsilon}}{r}\frac{\partial p}{\partial\beta} + \Lambda\sqrt{\Upsilon}\frac{\partial p}{\partial z} + \mu\left\{\nabla^2 v^1 + \frac{2\Lambda^2}{r\Upsilon}\frac{\partial v^1}{\partial r} - \frac{\left(\Upsilon + 3\Lambda^2\right)v^1}{r^2\Upsilon^2}\right.$$
$$\left.+\frac{2\sqrt{\Upsilon}}{r^2}\frac{\partial v^2}{\partial\beta} - \frac{2\Lambda\sqrt{\Upsilon}}{r}\frac{\partial v^2}{\partial z}\right\} + \rho r\sqrt{\Upsilon}f^1, \tag{41a}$$

$$\rho\left(\frac{\partial v^2}{\partial t} + \frac{v^1}{r\sqrt{\Upsilon}}\frac{\partial v^2}{\partial\beta} + v^2\frac{\partial v^2}{\partial r} + v^3\frac{\partial v^2}{\partial z} - \frac{v^1v^1}{r\Upsilon}\right) = -\frac{\partial p}{\partial r}$$
$$+\mu\left\{\nabla^2 v^2 - \frac{2}{r^2\sqrt{\Upsilon}}\frac{\partial v^1}{\partial\beta} + \frac{2\Lambda}{r\sqrt{\Upsilon}}\frac{\partial v^1}{\partial z} - \frac{v^2}{r^2}\right\} + \rho f^2, \tag{41b}$$

$$\rho\left(\frac{\partial v^3}{\partial t}+\frac{v^1}{r\sqrt{\Upsilon}}\frac{\partial v^3}{\partial \beta}+v^2\frac{\partial v^3}{\partial r}+v^3\frac{\partial v^3}{\partial z}-\frac{2\Lambda v^1 v^2}{r\sqrt{\Upsilon}}\right)=-\Upsilon\frac{\partial p}{\partial z}+\frac{\Lambda}{r}\frac{\partial p}{\partial \beta}$$
$$+\mu\left\{\nabla^2 v^3-\frac{2\Lambda}{r^2}\frac{\partial v^2}{\partial \beta}+\frac{2\Lambda^2}{r}\frac{\partial v^2}{\partial z}-\frac{2\Lambda}{r\sqrt{\Upsilon}}\frac{\partial v^1}{\partial r}+\frac{2\Lambda v^1}{r^2\Upsilon^{3/2}}\right\}+\rho f^3. \tag{41c}$$

For completeness we give the incompressible continuity equation (11) as

$$\frac{\partial u^1}{\partial \beta}+\frac{\partial u^2}{\partial r}+\frac{\partial u^3}{\partial z}+\frac{u^2}{r}=0, \tag{42}$$

leading to

$$\frac{1}{r\sqrt{\Upsilon}}\frac{\partial v^1}{\partial \beta}+\frac{1}{r}\frac{\partial}{\partial r}(rv^2)+\frac{\partial v^3}{\partial z}=0. \tag{43}$$

Equations (40) and (42), or alternatively (41) and (43), together with suitable initial and boundary and conditions, may be used to model unsteady viscous flow in helically-wound channels and ducts. On assuming the flow to be independent of both time t and angular position β, we obtain the steady helically-symmetric fluid flow equations which form the basis of the thin-film models of flow in helically wound channels derived and analysed in [2]. The helically-symmetric Cauchy momentum equations derived in [8], on assuming a constant viscosity μ, also agree with the Navier-Stokes equations derived here and we note that the derivation of the general fluid-flow equations presented in this paper is readily extended for fluid properties that depend on both time and spatial position.

7. Conclusions

We have derived the general incompressible Navier-Stokes equations (18) in terms of an arbitrary non-orthogonal coordinate system. We believe that the final form of (18), involving the generalised Laplacian and the use of the identity (51) of Appendix A, represents important simplifications of the existing forms of the Navier-Stokes equations. If the inertial terms are neglected then the pressure function satisfies $\nabla^2 p = 0$, while (25) gives the appropriate generalisation if the inertial terms are included. Equation (25) is deduced by contracted partial covariant differentiation of the general Navier-Stokes equations (18), and is interesting because it applies for every time-dependent viscous flow, assuming only incompressibility and the absence of external body forces. The derivations of (18) and (25) hinge on correctly commuting partial covariant derivatives and exploiting the incompressibility constraint (11) in the form $u^i_{;i} = 0$, where the semi-colon denotes the partial covariant derivative, in this case with respect to x^i. Interested readers are referred to Appendix C for the compressible Navier-Stokes equations.

A. Evaluation of $d^{ij}_{;i}$

In this appendix we provide a derivation of the given expression for $d^{ij}_{;i}$. The rate-of-strain tensor d^{ij} is defined by

$$d^{ij} = \frac{1}{2}\left(g^{ik}u^{j}_{;k} + g^{jk}u^{i}_{;k}\right), \tag{44}$$

where the semi-colon denotes partial covariant differentiation with respect to x^k. We need the partial covariant derivative of d^{ij} with respect to x^m, and then subsequently make the contraction $m \to i$. Since the partial covariant derivatives of the metric tensor and its conjugate are zero, we have

$$d^{ij}_{;m} = \frac{1}{2}\left(g^{ik}\left(u^{j}_{;k}\right)_{;m} + g^{jk}\left(u^{i}_{;k}\right)_{;m}\right), \tag{45}$$

and since in flat Euclidean space, we may commute partial covariant differentiation, we see that the second term involves

$$\left(u^{i}_{;k}\right)_{;m} = \left(u^{i}_{;m}\right)_{;k} = \left(\frac{\partial u^i}{\partial x^m} + \Gamma^{i}_{m\ell}u^{\ell}\right)_{;k}, \tag{46}$$

and on contraction this expression vanishes for an incompressible fluid on using equation (11). Thus from (45) we have

$$d^{ij}_{;i} = \frac{1}{2}g^{ik}\left(u^{j}_{;k}\right)_{;i}, \tag{47}$$

where $\left(u^{j}_{;k}\right)_{;i}$ becomes

$$\begin{aligned}\left(\frac{\partial u^j}{\partial x^k} + \Gamma^{j}_{k\ell}u^{\ell}\right)_{;i} = & \frac{\partial^2 u^j}{\partial x^i \partial x^k} + \Gamma^{j}_{i\ell}\frac{\partial u^\ell}{\partial x^k} - \Gamma^{m}_{ki}\frac{\partial u^j}{\partial x^m} + \Gamma^{j}_{k\ell}\left(\frac{\partial u^\ell}{\partial x^i} + \Gamma^{\ell}_{im}u^m\right) \\ & + \left(\frac{\partial \Gamma^{j}_{k\ell}}{\partial x^i} + \Gamma^{j}_{im}\Gamma^{m}_{k\ell} - \Gamma^{n}_{ik}\Gamma^{j}_{n\ell} - \Gamma^{n}_{\ell i}\Gamma^{j}_{kn}\right)u^{\ell},\end{aligned} \tag{48}$$

and the final two terms in each of the two brackets cancel. From (47) we may obtain

$$d^{ij}_{;i} = \frac{1}{2}\left\{\nabla^2 u^j + 2g^{ik}\Gamma^{j}_{i\ell}\frac{\partial u^\ell}{\partial x^k} + g^{ik}\left(\frac{\partial \Gamma^{j}_{k\ell}}{\partial x^i} + \Gamma^{j}_{im}\Gamma^{m}_{k\ell} - \Gamma^{n}_{ik}\Gamma^{j}_{n\ell}\right)u^{\ell}\right\}, \tag{49}$$

where as usual, the generalized ∇^2 is defined by

$$\nabla^2 = g^{ik}\left(\frac{\partial^2}{\partial x^i \partial x^k} - \Gamma^{m}_{ik}\frac{\partial}{\partial x^m}\right). \tag{50}$$

But for flat Euclidean space, a standard result from tensor analysis (see, for example, [12, pp. 49–56] or [13, pp. 88–107]) gives

$$\frac{\partial \Gamma^j_{k\ell}}{\partial x^i} - \frac{\partial \Gamma^j_{ki}}{\partial x^\ell} + \Gamma^j_{im}\Gamma^m_{k\ell} - \Gamma^n_{ik}\Gamma^j_{n\ell} = 0, \tag{51}$$

and therefore equation (49) becomes

$$d^{ij}_{;i} = \frac{1}{2}\left\{\nabla^2 u^j + 2g^{ik}\Gamma^j_{i\ell}\frac{\partial u^\ell}{\partial x^k} + g^{ik}\frac{\partial \Gamma^j_{ki}}{\partial x^\ell}u^\ell\right\}, \tag{52}$$

which gives the expression used in (18). We observe that (47) and (48) are entirely consistent with expressions given by Wang [15], who does not subsequently simplify the expressions using (50) and (51).

B. Harmonic equation for low Reynolds number and its generalisation to include inertial effects

In this appendix we first show that when the inertia terms are negligible, we may deduce the harmonic equation (24) for the case of no external body forces. From (17) and (47) we have

$$g^{ij}\frac{\partial p}{\partial x^i} = \mu g^{ik}\left(u^j_{;k}\right)_{;i}, \tag{53}$$

so that on taking the partial covariant derivative with respect to x^j, interchanging orders of differentiation, and using the incompressibility condition (11) in the form $u^j_{;j} = 0$, we may deduce

$$g^{ij}\left(\frac{\partial^2 p}{\partial x^i \partial x^j} - \Gamma^m_{ij}\frac{\partial p}{\partial x^m}\right) = 0, \tag{54}$$

namely (24). If the inertial terms are included in the above calculation then, on applying the partial covariant derivative with respect to x^j to the term

$$\begin{aligned}\rho\left(\frac{du^j}{dt} + \Gamma^j_{ik}u^i u^k\right) &= \rho\left(\frac{\partial u^j}{\partial t} + u^i\frac{\partial u^j}{\partial x^i} + \Gamma^j_{ik}u^i u^k\right)\\ &= \rho\left(\frac{\partial u^j}{\partial t} + u^i u^j_{;i}\right),\end{aligned} \tag{55}$$

and using the incompressibility condition, again in the form $u^j_{;j} = 0$, we may deduce that the equation corresponding to (54) that includes inertial effects is

$$\left(\left(p g^{ij} + \rho u^i u^j\right)_{;i}\right)_{;j} = 0, \tag{56}$$

where the density ρ is constant.

C. Extension of results to compressible fluids

In this appendix we briefly note the major equations applying for compressible fluids. The rate of strain tensor defined by (9), on contraction yields

$$\Theta = d^i_i = u^i_{;i}, \tag{57}$$

and conservation of mass gives

$$\frac{d\rho}{dt} + \rho u^i_{;i} = \frac{\partial \rho}{\partial t} + u^i \rho_{;i} + \rho u^i_{;i} = \frac{\partial \rho}{\partial t} + (\rho u^i)_{;i} = 0, \tag{58}$$

where Θ is sometimes referred to as the dilatation and ρ is the non-constant density. The stress-rate-of-strain relations for a compressible fluid become

$$t^{ij} = (-p + \lambda\Theta)g^{ij} + 2\mu d^{ij}, \tag{59}$$

and conservation of momentum yields

$$\rho\left(\frac{\partial u^j}{\partial t} + u^i u^j_{;i}\right) = -\frac{\partial}{\partial x^i}\left(p - \lambda\Theta\right) g^{ij} + 2\mu d^{ij}_{;i} + \rho f^j. \tag{60}$$

For a compressible fluid, in place of (52) we have

$$d^{ij}_{;i} = \frac{1}{2}\left(g^{ik}\left(u^j_{;k}\right)_{;i} + g^{ij}\Theta_{;i}\right), \tag{61}$$

which becomes

$$d^{ij}_{;i} = \frac{1}{2}\left(\nabla^2 u^j + 2g^{ik}\Gamma^j_{i\ell}\frac{\partial u^\ell}{\partial x^k} + g^{ik}\frac{\partial \Gamma^j_{ki}}{\partial x^\ell}u^\ell + g^{ij}\Theta_{;i}\right) \tag{62}$$

and from (60) the Navier-Stokes equations for a compressible fluid become

$$\begin{aligned}\rho\left(\frac{\partial u^j}{\partial t} + u^i u^j_{;i}\right) = &- \frac{\partial}{\partial x^i}\left(p - (\lambda+\mu)\Theta\right) g^{ij} \\ &+ \mu\left(\nabla^2 u^j + 2g^{ik}\Gamma^j_{i\ell}\frac{\partial u^\ell}{\partial x^k} + g^{ik}\frac{\partial \Gamma^j_{ki}}{\partial x^\ell}u^\ell\right) + \rho f^j.\end{aligned} \tag{63}$$

In order to deduce the equation corresponding to (25), we need to add u^j times (58) to the left-hand side of (60) so that the acceleration term becomes

$$a^j = \frac{\partial}{\partial t}(\rho u^j) + (\rho u^i u^j)_{;i}, \tag{64}$$

and then by partial covariant differentiation of (60) with respect to x^j and with $d^{ij}_{;i}$ given by (61) we may obtain

$$\frac{\partial}{\partial t}(\rho u^j)_{;j} + \left((\rho u^i u^j)_{;i}\right)_{;j} = -g^{ij}\left((p - (\lambda+\mu)\Theta)_{;i}\right)_{;j} + \mu g^{ik}\left((u^j_{;j})_{;i}\right)_{;k} \tag{65}$$

assuming no body forces. On using (57) and (58) this equation simplifies to give

$$\frac{\partial^2 \rho}{\partial t^2} = \left(\left((p - (\lambda + 2\mu)\Theta)g^{ij} + \rho u^i u^j\right)_{;i}\right)_{;j} \tag{66}$$

as the appropriate generalisation of (25) for a compressible fluid. This equation can be alternatively written as

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 \left(p - (\lambda + 2\mu)\Theta\right) + \left((\rho u^i u^j)_{;i}\right)_{;j} \tag{67}$$

where, as usual, the Laplacian is defined by (19).

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