QCD, Gauge Fixing, and the Gribov Problem

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The standard techniques of gauge-fixing, such as covariant gauge fixing, are entirely adequate for the purposes of studies of perturbative QCD. However, they fail in the nonperturbative regime due to the presence of Gribov copies. These copies arise because standard local gauge fixing methods do not completely fix the gauge. Known Gribov-copy-free gauges, such as Laplacian gauge, are manifestly non-local. These issues are examined and the implications of non-local gauge-fixing for ghost fields, BRST invariance, and the proof of renormalizability of QCD are considered.

1. GAUGE FIXING

The naive Lagrangian density of QCD is

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_f \bar{q}_f(i\not{D} - m_f)q_f, \qquad (1)$$

where the index f corresponds to the quark flavours. The naive Lagrangian is neither gauge-fixed nor renormalized, however it is invariant under local $SU(3)_c$ gauge transformations $g(x)$. For arbitrary, small $\omega^a(x)$ we have

$$g(x) \equiv \exp\left\{-ig_s\left(\frac{\lambda^a}{2}\right)\omega^a(x)\right\} \in SU(3), \qquad (2)$$

where the $\lambda^a/2 \equiv t^a$ are the generators of the gauge group $SU(3)$ and the index a runs over the eight generator labels $a = 1, 2, ..., 8$.

Consider some gauge-invariant Green's function (for the time being we shall concern ourselves only with gluons)

$$\langle\Omega|\; T(\hat{O}[A])\; |\Omega\rangle = \frac{\int \mathcal{D}A\; O[A]\; e^{iS[A]}}{\int \mathcal{D}A\; e^{iS[A]}}\,, \qquad (3)$$

where $O[A]$ is some gauge-independent quantity depending on the gauge field, $A_\mu(x)$. We see that the gauge-independence of $O[A]$ and $S[A]$ gives rise to an infinite quantity in both the numerator and denominator, which must be eliminated by gauge-fixing. The Minkowski-space Green's functions are defined as the Wick-rotated versions of the Euclidean ones.

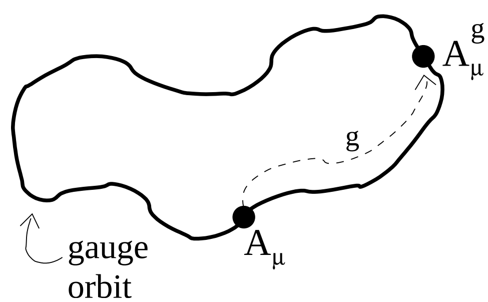

Figure 1. Illustration of the gauge orbit containing A_μ and indicating the effect of acting on A_μ with the gauge transformation g. The action $S[A]$ is constant around the orbit.

The gauge orbit for some configuration A_μ is defined to be the set of all gauge-equivalent configurations. Each point A_μ^g on the gauge orbit is obtained by acting upon A_μ with the gauge transformation g. By definition the action, $S[A]$, is gauge invariant and so all configurations on the gauge orbit have the same action, e.g., see the illustration in Fig. 1.

The integral over the gauge fields can be written as the integral over a full set of gauge-inequivalent (i.e., gauge-fixed) configurations, $\int \mathcal{D}A^{\rm g.f.}$, and an integral over the gauge group $\int \mathcal{D}g$. In other words, $\int \mathcal{D}A^{\rm g.f.}$ is an integral *over* *the set of all possible gauge orbits* and $\int \mathcal{D}g$ is an integral *around the gauge orbits*. Thus we can write

$$\int \mathcal{D}A \equiv \int \mathcal{D}A^{\rm g.f.} \int \mathcal{D}g. \tag{4}$$

To make integrals such as those in the numerator and denominator of Eq. (3) finite and also to study gauge-dependent quantities in a meaningful way, we need to eliminate this integral around the gauge orbit, $\int \mathcal{D}g$.

2. GRIBOV COPIES AND THE FADDEEV-POPOV DETERMINANT

Any gauge-fixing procedure defines a surface in gauge-field configuration space. Fig. 2 is a depiction of these surfaces represented as dashed lines intersecting the gauge orbits within this configuration space. Of course, in general, the gauge orbits are hypersurfaces and so are the gauge-fixing surfaces. Any gauge-fixing surface must, by definition, only intersect the gauge orbits at distinct isolated points in configuration space. For this reason, it is sufficient to use lines for the simple illustration of the concepts here. An ideal (or complete) gauge-fixing condition, $F[A] = 0$, defines a surface that intersects each gauge orbit once and only once and by convention contains the trivial configuration $A_\mu = 0$. A non-ideal gauge-fixing condition, $F'[A] = 0$, defines a surface or surfaces which intersect the gauge orbit more than once. These multiple intersections of the non-ideal gauge fixing surface(s) with the gauge orbit are referred to as Gribov copies[1–6] . Lorentz gauge ($\partial_\mu A^\mu(x) = 0$) for example, has many Gribov copies per gauge orbit. By definition an ideal gauge fixing is free from Gribov copies. We refer to the ideal gauge-fixing surface $F[A] = 0$ as the Fundamental Modular Region (FMR) for that gauge choice. Typically the gauge fixing condition depends on a space-time coordinate, (e.g., Lorentz gauge, axial gauge, *etc.*), and so we write the gauge fixing condition more generally as $F([A]; x) = 0$.

Let us denote one arbitrary gauge configuration per gauge orbit as A^0_μ and let this correspond to the "origin" of gauge configurations on that gauge orbit, i.e., to $g = 0$ on that orbit. Then each

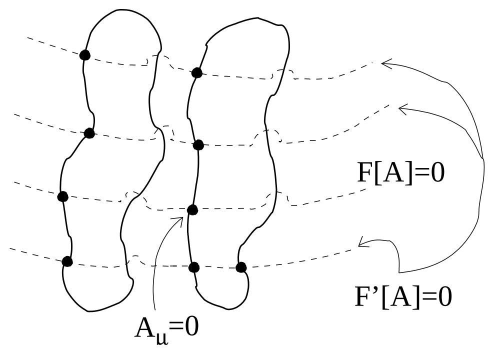

Figure 2. Ideal, $F[A]$, and non-ideal, $F'[A]$, gauge-fixing.

gauge orbit can be labelled by A^0_μ and the set of all such A^0_μ is equivalent to one particular, complete specification of the gauge. Under a gauge transformation, g, we move from the origin of the gauge orbit to the configuration, A^g_μ, where by definition $A^0_\mu \xrightarrow{g} A^g_\mu = gA^0_\mu g^\dagger - (i/g_s)(\partial_\mu g)g^\dagger$. Let us denote for each gauge orbit the gauge transformation, $\tilde{g} \equiv \tilde{g}[A^0]$, as the transformation which takes us from the origin of that orbit, A^0_μ, to the configuration, $A^{\rm g.f.}_\mu$, which lies on the ideal gauge-fixed surface specified by $F([A]; x) = 0$. In other words, we have $F([A]; x)|_{A^{\tilde{g}}} = 0$ for $A^{\tilde{g}} \equiv A^{\rm g.f.}_\mu \in$ FMR.

The *inverse Faddeev-Popov determinant* is defined as the integral over the gauge group of the gauge-fixing condition, i.e.,

$$\Delta_F^{-1}[A^{\rm g.f.}] \equiv \int \mathcal{D}g\ \delta[F[A]]$$
$$= \int \mathcal{D}g\ \delta(g - \tilde{g}) \left| \det \left(\frac{\delta F([A]; x)}{\delta g(y)} \right) \right|^{-1} . \tag{5}$$

Let us define the matrix $M_F[A]$ as

$$M_F([A]; x, y)^{ab} \equiv \frac{\delta F^a([A]; x)}{\delta g^b(y)} . \tag{6}$$

Then the *Faddeev-Popov determinant* for an arbitrary configuration A_μ can be defined as $\Delta_F[A] \equiv |\det M_F[A]|$. (The reason for the name is now clear). Note that we have consistency, since $\Delta_F^{-1}[A^{\rm g.f.}] \equiv \Delta_F^{-1}[A^{\tilde{g}}] = \int \mathcal{D}g\ \delta(g - \tilde{g}) \Delta_F^{-1}[A]$.

We have $1 = \int \mathcal{D}g\ \Delta_F[A]\ \delta[F[A]]$ by definition and hence

$$\int \mathcal{D}A^{\rm g.f.} \equiv \int \mathcal{D}A^{\rm g.f.}\int \mathcal{D}g\ \Delta_F[A]\ \delta[F[A]] = \int \mathcal{D}A\ \Delta_F[A]\ \delta[F[A]] \quad (7)$$

Since for an ideal gauge-fixing there is one and only one $\tilde{g}$ per gauge orbit, such that $F([A];x)|_{\tilde{g}} = 0$, then $|\det M_F[A]|$ is non-zero on the FMR. It follows that if there is at least one smooth path between any two configurations in the FMR and since the determinant cannot be zero on the FMR, then it cannot change sign on the FMR. The *Gribov horizon* is defined to by those configurations with $\det M_F[A] = 0$ which lie closest to the FMR. By definition the determinant can change sign on or outside this horizon. Clearly, the FMR is contained within the Gribov horizon and for an ideal gauge fixing, since the sign of the determinant cannot change, we can replace $|\det M_F|$ with $\det M_F$, [i.e., the overall sign of the functional integral is normalized away in Eq. (3)].

These results are generalizations of results from ordinary calculus, where

$$\left|\det\left(\frac{\partial f_i}{\partial x_j}\right)\right|^{-1}_{\vec{f}=0} = \int dx_1 \cdots dx_n\, \delta^{(n)}(\vec{f}(\vec{x})) \quad (8)$$

and if there is one and only one $\vec{x}$ which is a solution of $\vec{f}(\vec{x}) = 0$ then the matrix $M_{ij} \equiv \partial f_i/\partial x_j$ is invertible (i.e., non-singular) on the hypersurface $\vec{f}(\vec{x}) = 0$ and hence $\det M \neq 0$.

3. GENERALIZED FADDEEV-POPOV TECHNIQUE

Let us now assume that we have a family of *ideal* gauge fixings $F([A];x) = f([A];x) - c(x)$ for any Lorentz scalar $c(x)$ and for $f([A];x)$ being some Lorentz scalar function, (e.g., $\partial^\mu A_\mu(x)$ or $n^\mu A_\mu(x)$ or similar or any nonlocal generalizations of these). Therefore, using the fact that we remain in the FMR and can drop the modulus on the determinant, we have

$$\int \mathcal{D}A^{\rm g.f.} = \int \mathcal{D}A\ \det M_F[A]\ \delta[f[A] - c]\,. \quad (9)$$

Since $c(x)$ is an arbitrary function, we can define a new "gauge" as the Gaussian weighted average over $c(x)$, i.e.,

$$\begin{aligned}\int \mathcal{D}A^{\rm g.f.} &\propto \int \mathcal{D}c\ \exp\left\{-\frac{i}{2\xi}\int d^4x c(x)^2\right\} \\ &\quad\times \int \mathcal{D}A\ \det M_F[A]\ \delta[f[A]-c] \\ &\propto \int \mathcal{D}A\ \det M_F[A] \\ &\quad\times \exp\left\{-\frac{i}{2\xi}\int d^4x f([A];x)^2\right\} \\ &\propto \int \mathcal{D}A\mathcal{D}\chi\mathcal{D}\bar{\chi}\ \exp\left\{-i\int d^4x d^4y \right. \\ &\quad\left.\times\ \bar{\chi}(x) M_F([A];x,y)\chi(y)\right\} \\ &\quad\times \exp\left\{-\frac{i}{2\xi}\int d^4x f([A];x)^2\right\} \end{aligned} \quad (10)$$

where we have introduced the anti-commuting ghost fields χ and $\bar{\chi}$. Note that this kind of ideal gauge fixing does not choose just one gauge configuration on the gauge orbit, but rather is some Gaussian weighted average over gauge fields on the gauge orbit. We then obtain

$$\langle\Omega|\ T(\hat{O}[...])\ |\Omega\rangle = \frac{\int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}\chi\mathcal{D}\bar{\chi}\ O[...]\ e^{iS_\xi[...]}}{\int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}\chi\mathcal{D}\bar{\chi}\ e^{iS_\xi[...]}}\,, \quad (11)$$

where

$$\begin{aligned} S_\xi[q,\bar{q},A,\chi,\bar{\chi}] &= \int d^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} \right. \\ &\quad\left. -\frac{1}{2\xi}\left(f([A];x)\right)^2 + \sum_f \bar{q}_f(i\not{D} - m_f)q_f\right] \\ &\quad + \int d^4x d^4y\ \bar{\chi}(x)M_F([A];x,y)\chi(y)\,. \end{aligned} \quad (12)$$

4. STANDARD GAUGE FIXING

We can now recover standard gauge fixing schemes as special cases of this generalized form. First consider standard covariant gauge, which we obtain by taking $f([A];x) = \partial_\mu A^\mu(x)$ and by neglecting the fact that this leads to Gribov copies. We need to evaluate $M_F[A]$ in the vicinity of the gauge-fixing surface for this choice:

$$M_F([A];x,y)^{ab} = \frac{\delta F^a([A];x)}{\delta g^b(y)} = \frac{\delta[\partial_\mu A^{a\mu}(x) - c(x)]}{\delta g^b(y)} = \partial^x_\mu \frac{\delta A^{a\mu}(x)}{\delta g^b(y)]} . \quad (13)$$

Under an infinitesimal gauge transformation g we have

$$A^a_\mu(x) \xrightarrow{\text{small } g} (A^g)^a_\mu(x) = A^a_\mu(x) + g_s f^{abc}\omega^b(x)A^c_\mu(x) - \partial_\mu\omega^a(x) + \mathcal{O}(\omega^2) \quad (14)$$

and hence in the neighbourhood of the gauge fixing surface (i.e., for small fluctations along the gauge orbit around $A^{\text{g.f.}}_\mu$), we have

$$M_F([A];x,y)^{ab} = \partial^x_\mu \frac{\delta A^{a\mu}(x)}{\delta\omega^b(y)]}\bigg|_{\omega=0} \quad (15)$$
$$= \partial^x_\mu \left([-\partial^{x\mu}\delta^{ab} + g_s f^{abc} A^{c\mu}(x)] \times \delta^{(4)}(x-y) \right) .$$

We then recover the standard covariant gauge-fixed form of the QCD action

$$S_\xi[q,\bar{q},A,\chi,\bar{\chi}] = \int d^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu(x)\right)^2 + \sum_f \bar{q}_f(i\not{D} - m_f)q_f \right] + (\partial_\mu\bar{\chi}_a)(\partial^\mu\delta^{ab} - g f_{abc}A^\mu_c)\chi_b . \quad (16)$$

However, this gauge fixing has not removed the Gribov copies and so the formal manipulations which lead to this action are not valid. This Lorentz covariant set of naive gauges corresponds to a Gaussian weighted average over generalized Lorentz gauges, where the gauge parameter ξ is the width of the Gaussian distribution over the configurations on the gauge orbit. Setting $\xi = 0$ we see that the width vanishes and we obtain Landau gauge (equivalent to Lorentz gauge, $\partial^\mu A_\mu(x) = 0$). Choosing $\xi = 1$ is referred to as "Feynman gauge" and so on.

We can similarly recover the standard QCD action for axial gauge, where $n_\mu A^\mu(x) = 0$. Proceeding as for the generalized covariant gauge, we first identify $f([A];x) = n_\mu A^\mu(x)$ and obtain the gauge-fixed action

$$S_\xi[q,\bar{q},A] = \int d^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{1}{2\xi}\left(n_\mu A^\mu(x)\right)^2 + \sum_f \bar{q}_f(i\not{D} - m_f)q_f \right] . \quad (17)$$

Taking the "Landau-like" zero-width limit $\xi \to 0$ we select $n_\mu A^\mu(x) = 0$ exactly and recover the usual axial-gauge fixed QCD action. Axial gauge does not involve ghost fields, since in this case

$$M_F([A^{\text{g.f.}}];x,y)^{ab} = n_\mu \frac{\delta A^{a\mu}(x)}{\delta\omega^b(y)]}\bigg|_{\omega=0} = n_\mu \left([-\partial^{x\mu}\delta^{ab}]\delta^{(4)}(x-y) \right) , \quad (18)$$

which is independent of A_μ since $n^\mu A^{\text{g.f.}}_\mu(x) = 0$. In other words, the gauge field does not appear in $M_F[A]$ on the gauge-fixed surface. Unfortunately axial gauge suffers from singularities which lead to significant difficulties when trying to define perturbation theory beyond one loop. A related feature is that axial gauge is not a complete gauge fixing prescription. While there are complete versions of axial gauge on the lattice, these always involve a nonlocal element, or reintroduce Gribov copies at the boundary so as not to destroy the Polyakov loop.

5. DISCUSSION AND CONCLUSIONS

There is no known Gribov-copy-free gauge fixing which is a *local* function of $A_\mu(x)$. In other words, such a gauge fixing cannot be expressed as a function of $A_\mu(x)$ and a finite number of its derivatives, i.e., $F([A];x) \neq F(\partial_\mu, A_\mu(x))$ for all x. Hence, the gauge-fixed action, $S_\xi[\cdots]$, in Eq. (12) becomes non-local and gives rise to a nonlocal quantum field theory. Since this action serves as the basis for the proof of the renormalizability of QCD, the proof of asymptotic freedom, local BRS symmetry, and the Schwinger-Dyson equations (to name but a few) the nonlocality of the action leaves us without a reliable basis from which to prove these features of QCD in the nonperturbative context.

It is well-established that QCD is asymptotically free, i.e., it is weak-coupling at large momenta. In the weak coupling limit the functional integral is dominated by small action configurations. As a consequence, momentum-space Green's functions at large momenta will receive their dominant contributions in the path integral from configurations near the trivial gauge orbit, i.e., the orbit containing $A_\mu = 0$, since this orbit minimizes the action. If we use standard gauge fixing, which neglects the fact that Gribov copies are present, then at large momenta $\int \mathcal{D}A$ will be dominated by configurations lying on the gauge-fixed surfaces in the neighbourhood of *each* of the Gribov copies on the trivial orbit. Since for small field fluctuations the Gribov copies cannot be aware of each other, we merely overcount the contribution by a factor equal to the number of copies on the trivial orbit. This overcounting is normalized away by the ratio in Eq. (3) and becomes irrelevant. Thus it is possible to understand why Gribov copies can be neglected at large momenta and why it is sufficient to use standard gauge fixing schemes as the basis for calculations in perturbative QCD. Since renormalizability is an ultraviolet issue, there is no question about the renormalizability of QCD.

Lattice QCD has provided numerical confirmation of asymptotic freedom, so let us now turn our attention to the matter of Gribov copies in lattice QCD. Since the observable $O[A]$ and the action are both gauge-invariant it doesn't matter whether we sample from the FMR of an ideal gauge-fixing or elsewhere on the gauge orbit. The trick is simply to sample *at most once* from each orbit. Since there are an infinite number of gauge orbits (even on the lattice), no finite ensemble will ever sample the same orbit twice. This makes Gribov copies and gauge-fixing irrelevant in the calculation of color-singlet quantities on the lattice.

The calculation of gauge-*dependent* Green's functions on the lattice does require that the gauge be fixed. The standard choice is naive lattice Landau gauge, which selects essentially at random between the Landau gauge Gribov copies for the gauge orbits represented in the ensemble. This means that, while the gauge fixing is well-defined in that there are no Gribov copies, the Landau gauge-fixed configurations are not from a single connected FMR. For this reason lattice studies of gluon and quark propagators are now being extended to Laplacian gauge for comparison. Laplacian gauge is interesting because it is Gribov-copy-free (except on a set of configurations of measure zero) and it reduces to Landau gauge at large momenta. Lattice calculations of the Laplacian gauge and Landau gauge quark and gluon propagators converge at large momenta and hence are consistent with this expectation.

In conclusion, it should be noted that throughout this discussion there has been the implicit assumption that nonperturbative QCD should be defined in such a way that each gauge orbit is represented only once in the functional integral, i.e., that it be defined to have no Gribov copies. This is the definition of nonpertubative QCD implicitly assumed in lattice QCD studies. We have seen that this assumption destroys locality and the BRS invariance of the theory. An equally valid point of view is that locality and BRS symmetry are central to the definition of QCD and must not be sacrificed in the nonpertubative regime, (see, e.g., Ref. [3–6]). This viewpoint impies that Gribov copies are necessarily present, that gauge orbits are multiply represented, and that the definition of nonperturbative QCD must be considered with some care. Since these nonpertubative definitions of QCD appear to be different, establishing which is the one appropriate for the description of the physical world is of considerable importance.

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