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THE COLLISION OF PURE PLANE GRAVITATIONAL AND ELECTROMAGNETIC SHOCK WAVES

by

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Contents

Abstract

By utilizing the Newman-Penrose tetrad formalism, we deduce the exact set of non-linear partial differential equations which describe the collision of gravitational and electromagnetic waves. Unfortunately, this set of equations is too difficult to solve using today's techniques and therefore several simplifications are adopted.

The first simplification treats the electromagnetic field as a test field and hence ignores its stress-energy. Both the exact and weak gravitational metric are used. In both cases, we deduce explicit expressions describing the effect of the gravitational wave on the electromagnetic field. Later, the validity of these expressions is discussed.

The second simplification uses a power-series approach which, although it does not give an extensive solution to the exact problem, does give several properties of the exact solution in the vicinity of the initial interaction. In particular, the applicability of the Lichnerowicz conditions is discussed.

We find that the gravitational wave changes the Petrov type of the electromagnetic field and can even reverse its direction of propagation. Also, observers may experience focusing of, and/or energy transfer to the electromagnetic field.

Finally, although the effects described above are explicit, we find that they are too small for experimental application to the detector of gravitational waves.

Statement

This thesis contains no material accepted or submitted for the award of any degree, and to the best of my knowledge and belief contains no material previously written or published by any other person except where due reference is made in the text.

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Notation and Conventions

The metric tensor is denoted $g_{\mu\nu}$. Its signature is + - - -. Tensor indicies μ,ν,ρ range and sum over 0,1,2,3. Spinor indicies A,B,... range and sum over 0,1. Where small roman letters appear their sum always ranges over the values 2,3. Symmetrization is denoted by round brackets.

$$A_{(\mu}B_{\nu)} = \tfrac{1}{2}(A_{\mu}B_{\nu} + A_{\nu}B_{\mu})$$

Antisymmetrization is denoted by square brackets

$$A_{[\mu}B_{\nu]} = \tfrac{1}{2}(A_{\mu}B_{\nu} - A_{\nu}B_{\mu})$$

Partial derivatives are denoted by

$$A_{,\mu} \text{ or } A_{\mu}$$

Covariant derivatives are denoted by

$$A_{;\mu}$$

Introduction

Although the set of equations required to describe the collision of two electromagnetic fields in General Relativity have been deduced and shown to be well posed (8), present day techniques are too primitive to solve the system.

In the work below, we derive this set in Section 3 and then proceed to simplify them in order to derive a solution. The simplification adopted is to treat the electromagnetic field as a test field and hence ignore its stress energy. Although this appears to be a rather drastic move, it does provide explicit expressions for the effect of a gravitational wave on an electromagnetic field.

We find in Section 4 that the Petrov type of the electromagnetic field changes and, for a given observer, the effects have a reasonably direct physical consequence. This observer detects changes in the electromagnetic Poynting Vector even to the extent of reversing its direction. Thus, the electromagnetic field energy is apparently redistributed and/or supplemented by the gravitational field.

This behaviour would indicate the possibility of using an electromagnetic field to detect gravitational radiation by observing the changes in the field. However, even with the stronger electromagnetic coupling, we show in Section 6 that this approach is of little use.

In Section 6 we also discuss the validity of ignoring the electromagnetic field energy-stress and find, for a given observer, that there is a reasonable range of validity. However, this task is complicated by the ambiguity in defining the energy density of a gravitational wave.

Another approach the systems of equations which are difficult to solve explicitly is to use a power series and study the properties of the solution near the initial interaction (12). We adopt this approach in particular to study the effects on the continuity conditions of the metric along the shock fronts of the two waves. We see in Section 5 that the Lichnerowicz conditions are suitable for all such interactions except when two shock electromagnetic waves collide. We also find that the previous simplification can apparently be refined by solving Maxwell's equations in the colliding gravitational wave metric. Since the problem we are to consider is, in effect, the classical equivalent of a graviton-photon interaction, and further, since ignoring the energy-stress of the electromagnetic field is in line with the approach often adopted when studying quantum fields in curved space time, the results deduced could be of use in considering the quantized problem. To do this, we would require to quantize the electromagnetic and (weak) gravitational fields in the Newman-Penrose null tetrad. This apparently has not yet been done. It may well be a useful next step after this thesis.

The Newman-Penrose Formalism

Throughout this thesis, the Newman-Penrose Spin Co-efficient formalism(1) will be extensively used.

Since this formalism is relatively well known, only a statement of the definitions and relevant results will be made here. (A full listing of the resulting equations is contained in Appendix 1).

We introduce the tetrad of null vectors l_μ, n_μ, m_μ, $\bar{m}_\mu$ where l_μ and n_μ are real and m_μ complex. Their orthogorality properties are

$$l_\mu m^\mu = n_\mu m^\mu = 0 \qquad 1.1)$$

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1$$

$$Z_{m\mu} = (l_\mu,\ n_\mu,\ m_\mu,\ \bar{m}_\mu) \qquad m = 1,\ 2,\ 3,\ 4$$

The tetrad indicies are raised and lowered by flat-space metric

$$\eta_{mn} = \eta^{mn} = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{matrix}$$

and we also have

$$g_{\mu\nu} = Z_{m\mu}\ Z_{n\nu}\ \eta^{mn} = l_\mu\ n_\nu + n_\mu\ l_\nu - m_\mu\ \bar{m}_\nu - \bar{m}_\mu\ m_\nu \qquad 1.2a)$$

$$\eta_{mn} = Z_{m\mu}\ Z_{n\nu}\ g^{\mu\nu} \qquad 1.2b)$$

The complex Ricci rotation coefficients are defined by

$$\gamma_m{}^{np} = Z_{m\mu;\nu}\ Z^{n\mu}\ Z^{p\nu} \qquad 1.3)$$

and by 1.1) have the symmetry

$$\gamma_{mnp} = -\gamma_{nmp}$$

The tetrad components of a tensor $T_{\mu\nu\ldots\rho}$

$$T_{mn\ldots p} = T_{\mu\nu\ldots\rho}\ Z^\mu_m\ Z^\nu_n \ldots Z^\rho_p$$

Intrinsic dervative with respect to a tetrad vector is defined by

$$T_{mn\ldots p;q} = T_{mn\ldots p;\nu}\ Z^\nu_q \qquad 1.4)$$

(N.B. $T_{mn\ldots p}$ is a scalar quality)

Newman and Penrose introduce into 1.4) the notation

$$DT_{mn\ldots p} = T_{mn\ldots p;1} = T_{mn\ldots p;\nu}\, l^{\nu} \qquad 1.5a)$$

$$\Delta T_{mn\ldots p} = T_{mn\ldots p;2} = T_{mn\ldots p;\nu}\, n^{\nu} \qquad 1.5b)$$

$$\delta T_{mn\ldots p} = T_{mn\ldots p;3} = T_{mn\ldots p;\nu}\, m^{\nu} \qquad 1.5c)$$

$$\bar{\delta} T_{mn\ldots p} = T_{mn\ldots p;4} = T_{mn\ldots p;\nu}\, \bar{m}^{\nu} \qquad 1.5d)$$

Utilizing the relationship between tensors and spinors, and in particular between null vectors and spinors, Newman and Penrose introduce spinors o_A and ι_A which satisfy

$$o_A \iota^A = \varepsilon_{AB} o^A \iota^B = -\iota_A o^A = 1 \qquad o_A o^A = \iota_A \iota^A = 0 \qquad 1.6)$$

and

$$l^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, o^A \bar{o}^{\dot{B}} \qquad n^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, \iota^A \bar{\iota}^{\dot{B}}$$

$$m^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, o^A \bar{\iota}^{\dot{B}} \qquad 1.7)$$

where ; ε_{AB} is the Levi Civita symbol with $\varepsilon_{01} = \varepsilon_{\dot{0}\dot{1}} = \varepsilon^{\dot{0}\dot{1}} = \varepsilon^{\dot{0}\dot{1}} = 1$

; the dotted index differentiates between indicies of conjugated spinors and unconjugated spinors.

; $\sigma^{\mu}{}_{A\dot{B}}$ is a hermitian quantity satisfying

$$g_{\mu\nu}\, \sigma^{\mu}{}_{A\dot{B}}\, \sigma^{\nu}{}_{C\dot{D}} = \varepsilon_{AC}\, \varepsilon_{\dot{B}\dot{D}}$$

which sets up a correspondence between spinors and tensors, e.g., the spinor equivalent to the tensor $T_{\mu\nu\ldots\rho}$ is

$$T_{A\dot{B}\,C\dot{D}\ldots E\dot{F}} = T_{\mu\nu\ldots\rho}\, \sigma^{\mu}{}_{A\dot{B}}\, \sigma^{\nu}{}_{C\dot{D}} \ldots \sigma^{\rho}{}_{E\dot{F}} \qquad 1.8a)$$

and inversely

$$T_{\mu\nu\ldots\rho} = T_{A\dot{B}C\dot{D}\ldots E\dot{F}}\, \sigma^{A\dot{B}}_{\mu}\, \sigma^{C\dot{D}}_{\nu} \ldots \sigma^{E\dot{F}}_{\rho} \qquad 1.8b)$$

; Summation of spinor indicies, both dotted and undotted, is over values 1 and 2.

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We also have for the raising and lowering of spinor indicies

$$T^{A} = \varepsilon^{AB}\, T_{B} \qquad T_{B} = T^{A}\, \varepsilon_{AB} \qquad T^{\dot{A}} = \varepsilon^{\dot{A}\dot{B}}\, T_{\dot{B}} \qquad T_{\dot{B}} = T^{\dot{A}}\, \varepsilon_{\dot{A}\dot{B}}$$

Further, from 1.6) we have

$$o_{A}\iota_{B} - \iota_{A}o_{B} = \varepsilon_{AB} \qquad 1.9)$$

(For an extensive introduction to spinors in General Relativity, see (2)). 1.9) is in fact a completeness relation and hence we can take o^{A} and ι^{A} as a basis for the spinor space. Doing this and adopting the notation

$$\zeta_{o}^{A} = o^{A} \; (=\delta_{o}^{A}) \qquad \zeta_{1}^{A} = \iota^{A} \; (=\delta_{1}^{A})$$

$$\bar{\zeta}_{\dot{o}}^{\dot{A}} = \bar{o}^{\dot{A}} \; (=\delta_{\dot{o}}^{\dot{A}}) \qquad \bar{\zeta}_{\dot{1}}^{\dot{A}} = \bar{\iota}_{\dot{1}}^{\dot{A}} \; (=\delta_{\dot{1}}^{\dot{A}}) \qquad 1.10)$$

the dyad components of a spinor, given by

$$T_{ab\dot{c}} = T_{ABC}\, \zeta_{a}^{A}\, \zeta_{b}^{B}\, \bar{\zeta}_{\dot{c}}^{\dot{C}} \qquad 1.11)$$

are identical to the spinor components.

It should be noted that although the lower case indicies behave in the same algebraic way as the upper case indicies, there are not involved in covariant differentiation.

Using 1.9) and 1.10) we see that

$$1^{\mu} = \sigma^{\mu}_{A\dot{B}}\, \zeta_{o}^{A}\, \bar{\zeta}_{\dot{o}}^{\dot{B}} = \sigma^{\mu}_{o\dot{o}}$$

$$n^{\mu} = \sigma^{\mu}_{1\dot{1}} \qquad m^{\mu} = \sigma^{\mu}_{o\dot{1}} \qquad \bar{m}^{\mu} = \sigma^{\mu}_{1\dot{o}}$$

To represent covariant derivitives of spinors in this dyad we need to introduce the spinor affine connection $\Gamma^{B}{}_{A\mu}$ via the covariant derivative $T_{A;\mu}$ of a spinor T_{A}

$$T_{A;\mu} = T_{A;\mu} - T_{B}\, \Gamma^{B}{}_{A\mu} \qquad 1.12)$$

where $\Gamma^{B}{}_{A\mu}$ is fixed by the requirement that $\sigma^{\mu}{}_{AB;\nu} = \varepsilon_{AB;\nu} = 0 = \varepsilon AB;v = 0$

Using 1.12) on the dyad components ζ_{aB} $(= \delta_{a}^{A}\, \varepsilon_{AB} = \varepsilon_{aB})$ we have

$$\zeta_{aA;\mu} = \varepsilon_{aB,\mu} - \zeta_{ac}\, \Gamma^{C}{}_{A\mu} = \zeta_{a}{}^{C}\, \Gamma_{cA\mu} = \Gamma_{aA\mu}$$

Hence

$$\Gamma_{abc\dot{d}} = \zeta_{aA;\mu}\, \zeta^{A}_{b}\, \sigma^{\mu}{}_{c\dot{d}} \qquad 1.13)$$

Also, to be consistent with 1.5), we can write

$$T_{a\ldots}{}^{b\ldots}{}_{,\mu}\, \sigma^{\mu}{}_{c\dot{d}} \equiv \partial_{c\dot{d}}\, T_{a\ldots}{}^{b\ldots}$$

where the intrinsic derivatives become

$$D \equiv \partial_{0\dot{0}} \qquad \Delta \equiv \partial_{1\dot{1}} \qquad \delta \equiv \partial_{0\dot{1}} \qquad \delta \equiv \partial_{1\dot{0}} \qquad 1.14)$$

With the above theory and notation, we can write out explicit dyad component expressions for the complex Ricci rotation coefficients and various other relationships which involve them and their background (curved) Riemanian manifold. We have the following relationships

$$\gamma_{mpn} \leftrightarrow \Gamma_{abc\dot{d}}$$

This is seen by the following

$$\gamma_{mpn} = Z_{m\mu;\nu}\, Z^{\mu}{}_{p}\, Z^{\nu}{}_{n} \rightarrow (\sigma_{\mu A\dot{B}}\, \zeta^{A}_{a}\, \bar{\zeta}^{\dot{B}}_{\dot{b}})_{;\nu}\, \sigma^{\mu}{}_{C\dot{D}}\, \zeta^{C}_{c}\, \bar{\zeta}^{\dot{D}}_{\dot{d}}\, \sigma^{\nu}{}_{EF}\, \zeta^{E}_{e}\, \bar{\zeta}^{\dot{F}}_{\dot{f}}$$

$$= \sigma_{\mu A\dot{B}}\, (\zeta_{a}{}^{A}{}_{;\nu}\, \bar{\zeta}^{\dot{B}}_{\dot{b}} + \zeta_{a}{}^{A}\, \bar{\zeta}_{\dot{b}}{}^{\dot{B}}{}_{;\nu})\, \sigma^{\mu}{}_{C\dot{D}}\, \zeta^{C}_{c}\, \bar{\zeta}^{\dot{D}}_{\dot{d}}\, \sigma^{\nu}{}_{E\dot{F}}\, \zeta^{E}_{e}\, \bar{\zeta}^{\dot{F}}_{\dot{f}}$$

$$= \varepsilon_{AC}\, \varepsilon_{\dot{B}\dot{D}}\, (\Gamma_{a}{}^{A}{}_{e\dot{f}}\, \bar{\zeta}^{\dot{B}}_{\dot{b}} + \zeta^{A}_{a}\, \bar{\Gamma}_{\dot{b}}{}^{\dot{B}}{}_{\dot{f}e})\, \zeta^{C}_{c}\, \bar{\zeta}^{\dot{D}}_{\dot{d}}$$

$$= \Gamma_{ace\dot{f}}\, \varepsilon_{\dot{b}\dot{d}} + \bar{\Gamma}_{\dot{b}\dot{d}\dot{f}e}\, \varepsilon_{ac} \qquad 1.15)$$

Similarly

$$T_{;m;n} - T_{;n;m} = T^{;\ell}\, (\gamma_{m\ell n} - \gamma_{n\ell m}) \rightarrow \{\partial_{a\dot{b}}\, \partial_{c\dot{d}} - \partial_{c\dot{d}}\, \partial_{a\dot{b}}\}\, T \qquad 1.16)$$

$$= \{\varepsilon^{pq}\, (\Gamma_{pac\dot{d}}\, \partial_{q\dot{b}} - \Gamma_{pca\dot{b}}\, \partial_{q\dot{d}}) + \varepsilon^{\dot{r}\dot{s}}\, (\bar{\Gamma}_{\dot{r}\dot{b}\dot{d}c}\, \partial_{a\dot{s}} - \bar{\Gamma}_{\dot{r}\dot{d}\dot{b}a}\, \partial_{c\dot{s}})\}\, T$$

and

$$\gamma^{mnp;q} - \gamma^{mnq;p} = \gamma_{\ell}{}^{mp}\, \gamma^{\ell nq} - \gamma_{\ell}{}^{mq}\, \gamma^{\ell np} + \gamma^{mn\ell}\, (\gamma_{\ell}{}^{qp} - \gamma_{\ell}{}^{pq})$$

$$+ R^{mnpq} \rightarrow \partial_{f\dot{e}}\, \Gamma_{acd\dot{b}} - \partial_{d\dot{b}}\, \Gamma_{acf\dot{e}} = \varepsilon^{pq}\, \{\Gamma_{apd\dot{b}}\, \Gamma_{qcf\dot{e}} + \Gamma_{acp\dot{b}}\, \Gamma_{qdf\dot{e}}$$

$$- \Gamma_{apf\dot{e}}\, \Gamma_{qcd\dot{b}} - \Gamma_{acp\dot{e}}\, \Gamma_{qfd\dot{b}}\} + \varepsilon^{\dot{r}\dot{s}}\, \{\Gamma_{acd\dot{r}}\, \bar{\Gamma}_{\dot{s}\dot{b}\dot{e}f} - \Gamma_{acf\dot{r}}\, \Gamma_{\dot{s}\dot{e}\dot{b}d}\}$$

$$+ \Psi_{acdf}\, \varepsilon_{\dot{e}\dot{b}} + \Lambda\varepsilon_{\dot{e}\dot{b}}\, (\varepsilon_{cd}\, \varepsilon_{af} + \varepsilon_{ad}\, \varepsilon_{cf}) + \Phi_{ac\dot{b}\dot{e}}\, \varepsilon_{fd} \qquad 1.17)$$

where Ψ_{ABCD} corresponds to the Weyl tensor, $\Phi_{ab\dot{c}\dot{d}}$ corresponds to the trace free part of the Ricci tensor and Λ corresponds to the Ricci scalar.

Corresponding equations can also be written out for the Bianchi identities but they will not be reproduced here. (See (1) or (2)). Newman and Penrose adopted the following notation for the spin coefficients $\Gamma_{abc\dot{d}}$

$\Gamma_{abc\dot{d}} =$ 1.18)

cd \ ab	00	01 or 10	11
$0\dot{0}$	κ	ε	π
$1\dot{0}$	ρ	α	λ
$0\dot{1}$	σ	β	μ
$1\dot{1}$	τ	γ	ν

Using 1.14) and 1.18), we can re-write 1.15), 1.16) and 1.17). The resulting equations are quite long and are therefore re-produced in Appendix 1.

Maxwell's Equations in the Newman Penrose Formalism

Maxwell's field equations for the antisymmetric electromagnetic field tensor $F_{\mu\nu}$ written as tensor equations take the form

$$F^{\mu\nu}{}_{;\nu} = 0 \qquad 1.19a)$$

$$F_{[\mu\nu;\rho]} = 0 \qquad 1.19b)$$

Any antisymmetric rank two tensor $F^{\mu\nu}$ when written in spinor form takes on the appearance (2)

$$F_{\mu\nu} \to F_{A\dot{B}C\dot{D}} = \phi_{AC}\,\varepsilon_{\dot{B}\dot{D}} + \bar{\phi}_{\dot{B}\dot{D}}\,\varepsilon_{AB} \qquad 1.20a)$$

where, using $F_{\mu\nu} = -F_{\mu\nu}$,

$$\phi_{AB} \equiv -\tfrac{1}{2} F_{A}{}^{\dot{C}}{}_{B\dot{C}} = -\tfrac{1}{2} F_{B}{}^{\dot{C}}{}_{A\dot{C}} = \phi_{BA}$$

is a symmetric two-spinor.

Taking the dual of $F_{\mu\nu}$

$$^{*}F_{\mu\nu} = \tfrac{1}{2}\,\varepsilon_{\mu\nu}{}^{\sigma\rho}\,F_{\sigma\rho}$$

its spinor equivalent is

$$ {}^{*}F_{\mu\nu} \rightarrow i(\bar{\phi}_{\dot{B}\dot{D}}\ \varepsilon_{AC} - \phi_{AC}\ \varepsilon_{\dot{B}\dot{D}}) $$

Thus, defining the antisymmetric tensor

$$ \hat{F}_{\mu\nu} = \tfrac{1}{2}\ (F_{\mu\nu} + i{}^{*}F_{\mu\nu}) \qquad 1.21) $$

we have spinor equivalent

$$ \hat{F}_{\mu\nu} \rightarrow \phi_{AC}\ \varepsilon_{\dot{B}\dot{D}} \qquad 1.20b) $$

Furthermore, both 1.19a) and 1.19b) are incorporated in the field equation for $\hat{F}_{\mu\nu}$

$$ \hat{F}^{\mu\nu}{}_{;\nu} = 0 $$

which has spinor equivalent

$$ \partial^{A\dot{B}}\ \phi_{AC} = 0 \qquad 1.22) $$

Adopting spinors o^A and ι^A as a basis in spinor space enables us to re-write 1.22) in the Newman-Penrose formalism. The resulting equation is

$$ \partial_{1\dot{b}}\ \phi_{oa} - \partial_{o\dot{b}}\ \phi_{1a} = \phi_{oa}\ (\Gamma_{o11\dot{b}} - \Gamma_{11o\dot{b}}) + \phi_{1a}\ (\Gamma_{1oo\dot{b}} - \Gamma_{oo1\dot{b}}) $$
$$ + \phi_{oo}\ \Gamma_{a11\dot{b}} - \phi_{o1}\ (\Gamma_{a1o\dot{b}} + \Gamma_{ao1\dot{b}}) + \phi_{11}\ \Gamma_{aoo\dot{b}} $$

which in turn gives

$$ D\phi_1 - \bar{\delta}\phi_o = (\pi - 2\alpha)\ \phi_o + 2\rho\ \phi_1 - \kappa\phi_2 \qquad a) $$

$$ D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\ \phi_1 + (\rho - 2\varepsilon)\phi_2 \qquad b) $$

$$ \delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\ \phi_0 + 2\tau\ \phi_1 - \sigma\phi_2 \qquad c) \qquad 1.23) $$

$$ \delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\ \phi_1 + (\tau - 2\beta)\ \phi_2 \qquad d) $$

where

$$ \phi_o = F_{\mu\nu}\ l^{\mu} m^{\nu} = \phi_{oo} $$

$$ \phi_1 = \tfrac{1}{2}F_{\mu\nu}\ (l^{\mu}\ n^{\nu} + \bar{m}^{\mu}\ m^{\nu}) = \phi_{o1} \qquad 1.24) $$

$$ \phi_2 = F_{\mu\nu}\ \bar{m}^{\mu}\ n^{\nu} = \phi_{11} $$

Noting that from 1.21)

$$\hat{F}_{\mu\nu} = \phi_{AC}\, \varepsilon_{\dot{B}\dot{D}}\, \sigma^{A\dot{B}}{}_{\mu}\, \sigma^{C\dot{D}}{}_{\nu}$$

$$= \phi_{AC}\, \varepsilon_{\dot{B}\dot{D}}\, \varepsilon^{AE}\, \varepsilon^{\dot{B}\dot{F}}\, \sigma_{E\dot{F}\mu}\, \varepsilon^{CG}\, \varepsilon^{\dot{D}\dot{H}}\, \sigma_{G\dot{H}\nu}$$

Using 1.9 and 1.10) we have $\varepsilon^{AC} = \zeta^{A}_{a}\, \zeta^{Ca}$ which when substituted into the above gives

$$= \varepsilon^{ac}\, \varepsilon^{ef}\, \varepsilon^{\dot{b}\dot{d}}\, \phi_{ae}\, \sigma_{c\dot{b}\mu}\, \sigma_{f\dot{d}\nu}$$

$$= 2(\phi_0\, \bar{m}_{[\mu}\, n_{\nu]} + \phi_1\, (n_{[\mu}\, l_{\nu]} + m_{[\mu}\, \bar{m}_{\nu]}) + \phi_2\, l_{[\mu}\, m_{\nu]}) \qquad 1.25)$$

The field invariants of the electromagnetic field tensor $F_{\mu\nu}$ will be of interest to us. These invariants are

$$\tfrac{1}{2}F_{\mu\nu}\, F^{\mu\nu} = \hat{B}^2 - \hat{E}^2 \qquad a)$$

$$\tfrac{1}{4}F_{\mu\nu}\, {}^{*}F^{\mu\nu} = \hat{E} \cdot \hat{B} \qquad b) \qquad 1.26)$$

where $\hat{E}$ and $\hat{B}$ are the electric and magnetic field vectors respectively. In terms of $\hat{F}^{\mu\nu}$, the invariants are given by

$$\tfrac{1}{2}F_{\mu\nu}\, F^{\mu\nu} = \mathrm{Re}\{\hat{F}_{\mu\nu}\, \hat{F}^{\mu\nu}\} = \mathrm{Re}\{2\phi_0\, \phi_2 - 4\phi_1^2\} \qquad a)$$

$$\tfrac{1}{4}F_{\mu\nu}\, {}^{*}F^{\mu\nu} = \tfrac{1}{2}\mathrm{Im}\,\{\hat{F}_{\mu\nu}\, \hat{F}^{\mu\nu}\} = \mathrm{Im}\,\{\phi_0\, \phi_2 - 2\phi_1^2\}. \qquad b) \qquad 1.27)$$

From 1.21), the electromagnetic field tensor can be represented by a symmetric two spinor ϕ_{AB}. Let T^A be an arbitary one spinor; we can form the polynonial in T^1 and T^2 given by

$$P(T) = \phi_{AB}\, T^A\, T^B$$

which may be factorized

$$= (\alpha_A\, T^A)\, (\beta_B\, T^B)$$

Since T^A is arbitary and ϕ_{AB} symmetric we therefore have

$$\phi_{AB} = \alpha_{(A}\beta_{B)} \qquad 1.28)$$

The spinors α_A and β_B are called principal spinors and are unique up to complex scalar factor. Each spinor uniquely corresponds to a null real vector via the $\sigma^{\mu}{}_{A\dot{B}}$, i.e., there exists real null vectors.

$$k^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, \alpha^{A}\, \bar{\alpha}^{\dot{B}} \qquad j^{\mu} = \sigma^{\mu}{}_{A\dot{B}}\, \beta^{A}\, \bar{\beta}^{\dot{B}}$$

Hence decomposition 1.28) allows classification of electromagnetic fields corresponding to the number of unique real null vectors of the field. These null vectors are referred to as null directions of the field and the classification is called the Petrov Classification of the field.

For a symmetric n-spinor there are at most n unique null directions, thus, from 1.28), there are at most two unique null directions for the electromagnetic field.

Hence the Petrov Classifications for the electromagnetic field are

Multiplicity of null directions	Classification of Field	Form of ϕ_{AB}
[1]	General	$\phi_{AB} = \alpha_{(A}\beta_{B)}$
[2]	Null	$\phi_{AB} = \alpha_{(A}\alpha_{B)}$

It can be seen that for a null electromagnetic field there exists a real null vector k^{μ} such that

$$k^{\mu}\, \hat{F}_{\mu\nu} = 0$$

Thus, by 1.25) for any null electromagnetic field there exists a null tetrad in which $\hat{F}_{\mu\nu}$ has the form

$$\hat{F}_{\mu\nu} = 2\phi_{0}\, \bar{m}_{[\mu}\, n_{\nu]} \qquad \text{or} \qquad \hat{F}_{\mu\nu} = 2\phi_{2}\, l_{[\mu}\, m_{\nu]} \qquad 1.29)$$

where the principal null direction is n^{ν} and l^{ν} respectively. For a general electromagnetic field we must have at least either ϕ_0 and ϕ_2 non-zero or ϕ_1 non-zero. By choosing a particular basis, we may eliminate ϕ_1 from the expression for a general electromagnetic field. Using 1.7), 1.20a) and the symmetry and antisymmetry of ϕ_{AB} and ε_{AB} respectively, ϕ_1 in 1.24) can be expressed as

$$\phi_1 = \phi_{AC}\, (o^{A}\, \iota^{C} + \iota^{A}\, o^{C}) = \phi_{AC}\, o^{A}\, \iota^{C}$$

The only other condition o^{A} and ι^{C} must satisfy for the above decomposition of $F_{\mu\nu}$ is the equivalent of 1.6) i.e.

$$o^A \iota^B - \iota^A o^B = \varepsilon^{AB}$$

Thus, if we use the equation

$$\phi_{AC}\, o^A \iota^C = 0$$

to specify our basis, rather than 1.10), the ϕ_1 term in 1.23) and 1.24) will disappear.

thus either $\Psi_{\mu\nu} = o$, i.e. no discontinuity, or the surface S is null.

It can be shown that such a null surface is a characteristic surface (2). Using the fact that $\Psi_{\mu\nu}$ is an antisymmetric four-dimensional matrix which must have even rank (this can be shown by direct calculation of the determinental rank for the various cases) and also by 2.2) the rank cannot be four, (assuming $F_{\mu\nu}$ is discontinuous), $\psi_{\mu\nu}$ can be written as

$$\Psi_{\mu\nu} = k_{[\mu} \ell_{\nu]}$$

where k_μ and ℓ_ν are linearly independent defined (but not uniquely) on S. From 2.2a) k_μ, ℓ_μ and u_μ are linearly dependent and we can set $\ell_\mu = \psi u_\mu$. Since u_μ is null 2.2b) shows that k_μ is spacelike. Thus we have

$$\Psi_{\mu\nu} = \Psi\, k_{[\mu} u_{\nu]}$$
$$u_\mu u^\mu = o = u_\mu k^\mu \qquad 2.3)$$

We can immediately see from 2.3) that if we set $f_{\mu\nu} = o$ in 2.1), then $F_{\mu\nu}$ is null in terms of the Petrov classification and further has the simple decomposition 1.29) in terms of the null tetrad introduced above.

Discontinuous or Shock Gravitational Radiation Fields

Just as the solution $F_{\mu\nu}$ to Maxwell's equation has characteristics, the solutions to Einstein's equations may have characteristics. Einstein's equations are second order hyperbolic non-linear partial differential equations in the components of the metric and hence possess charecterist: surfaces which may be interpreted as wave fronts across which the metric and its derivatives are discontinuous (4).

The problems of physically interpreting these apparently "radiative" properties of gravitational fields has been discussed in other works (2), (5), (6). Briefly, the problems stem from the fact that, unlike Maxwell's equations, Einstein's equations are non-linear and locally a gravitational field can be transformed away. Thus, we cannot ascribe an "energy content" to a gravitational wave.

Discontinuous or "Shock" Electromagnetic Fields

Maxwell's equations are first order partial differential equations in the antisymmetric tensor $F_{\mu\nu}$. The solutions to such equations admit "characteristic surfaces" (3), (4). These surfaces have physical significance as wave fronts across which solutions can have discontinuities (4). Thus electromagnetic fields can exhibit a discontinuous behaviour across such characteristic surfaces. However, we shall see that such surfaces have particular properties.

We adopt the approach used in (2). Let the surface S: $u(x^\alpha)$ be a hypersurface in flat space time. Let $F_{\mu\nu}$ have the discontinuous form

$$F_{\mu\nu} = f_{\mu\nu} + \Psi_{\mu\nu}\,\theta(u) \qquad 2.1)$$

where $f_{\mu\nu}$ and $\Psi_{\mu\nu}$ are C^0 and piecewise C^1 and $\theta(u)$ is the usual unit step function.

Inserting 2.1) into Maxwell's equations gives

$$F_{\mu\nu}{}^{;\nu} = f_{\mu\nu}{}^{;\nu} + \Psi_{\mu\nu}{}^{;\nu}\,\theta(u) + \Psi_{\mu\nu}\,\delta(u)\,u^\nu$$

$$F_{[\mu\nu;\rho]} = f_{[\mu\nu;\rho]} + \Psi_{[\mu\nu;\rho]}\,\Theta(u) + \Psi_{[\mu\nu}\,u_{\rho]}\,\delta(u)$$

where $u_\nu \equiv u_{,\nu}$, $u^\nu = u_\mu g^{\nu\mu}$ and $\delta(u)$ is the Dirac delta distribution.

Maxwell's equations must hold on and near S, hence the following results

$$\text{for } u \to o^- \qquad f_{\mu\nu}{}^{;\nu} = 0 \quad \text{and } f_{[\mu\nu;\rho]} = 0$$

$$\text{for } u \to o^+ \qquad \Psi_{\mu\nu}{}^{;\nu} = o \text{ and } \Psi_{[\mu\nu;\rho]} = o$$

and hence on S we must have

$$\Psi_{\mu\nu}u^\nu = o \qquad a)$$

and 2.2)

$$\Psi_{[\mu\nu}u_{\rho]} = o \qquad b)$$

Contracting the second of these with u^ρ gives

$$\Psi_{\mu\nu}\,u_\rho\,u^\rho = o$$

To deduce some mathematical results on the likely properties of gravitational radiation, we cannot always use electromagnetic radiation as an analogy. However, we can begin in a similar way when considering gravitational radiation wave-fronts. For heuristic reasons, the type of discontinuity of the metric allowed in General Relativity is restricted to shock discontinuities in the second derivative of the metric. Mathematically this is represented by $g_{\mu\nu}$ and $g_{\mu\nu;\zeta}$ being continuous whereas $g_{\mu\nu;\zeta\sigma}$ may be discontinuous across some hypersurface S: $u(\mathbf{x}^{\mu}) = o$.

Physically, this restriction corresponds to allowing nothing more than shock discontinuities in the curvative tensor $R_{u\nu\zeta\sigma}$. Allowing discontinuities in $g_{\mu\nu}$ and $g_{\mu\nu,\zeta}$ would entail delta discontinuities in $R_{\mu\nu\zeta\sigma}$ which would be difficult to interpret physically.

This restriction on the type of allowable discontinuity is called the Lichnerowicz condition (7). To study the implications of the Lichnerowicz condition we use the method adopted by Pirani (2).

Let Δ denote the discontinuity across hypersurface S.

Thus we can write

$$\Delta(g_{\mu\nu,\zeta\sigma}) = \chi_{\mu\nu\zeta\sigma} = \chi_{\mu\nu\sigma\zeta} \qquad 2.4)$$

and we have assumed that

$$\Delta(g_{\mu\nu}) = \Delta(g_{\mu\nu,\zeta}) = o$$

Considering neighbouring points $\mathbf{x}^{\mu}$ and $x^{\mu} + d\mathbf{x}^{\mu}$ on S and the values of $g_{\mu\nu,\zeta}$ either side of S we have

$$g_{\mu\nu,\zeta}(\mathbf{x} + d\mathbf{x}) = g_{\mu\nu,\zeta}(\mathbf{x}) + g_{\mu\nu,\zeta\sigma}(x)\, d\mathbf{x}^{\sigma}$$

thus for the discontinuities

$$\Delta(g_{\mu\nu,\zeta}(\mathbf{x} + d\mathbf{x})) = \Delta(g_{\mu\nu,\zeta}(\mathbf{x})) + \Delta(g_{\mu\nu,\zeta\sigma}(\mathbf{x}))\, d\mathbf{x}^{\sigma}$$

By the Lichnerowicz conditions

$$\Delta(g_{\mu\nu,\zeta\sigma})\, d\mathbf{x}^{\sigma} = \chi_{\mu\nu\zeta\sigma}\, d\mathbf{x}^{\sigma} = o$$

for all dx^σ lying in hypersurface S. For such dx^σ we have

$$u_{,\nu}dx^\nu = 0$$

thus we must have

$$\chi_{\mu\nu\zeta\sigma} = \chi_{\mu\nu\zeta}u_\sigma$$

for some $\chi_{\mu\nu\zeta\sigma}$. By symmetry of 2.4) we have

$$\chi_{\mu\nu\zeta\sigma} = \chi_{\mu\nu}\, u_\zeta\, u_\sigma \qquad 2.5)$$

Furthermore, using the Ricci identity

$$R_{\mu\nu\zeta\sigma} = 2\,\partial_{[\sigma}\, g_{\zeta]\,[\mu,\nu]} + \text{terms continuous across S}$$

therefore

$$\Delta R_{\mu\nu\zeta\sigma} = 2\, u_{[\sigma}\, \chi_{\zeta]\,[\mu}\, u_{\nu]} \qquad 2.6$$

Since $R_{\mu\nu} = 0$ in a vacuum and hence $\Delta R_{\mu\nu} = 0$ we have

$$2g^{\nu\sigma}\, u_{[\sigma}\, \chi_{\zeta]\,[\mu}\, u_{\nu]} = \tfrac{1}{2}(\chi_{\zeta\mu}\, u^\nu\, u_\nu - \chi_{\nu\,\mu} u_\zeta u^\nu - \chi_{\zeta\nu}\, u^\nu u_\mu$$

$$+ \chi^\nu{}_\nu\, u_\zeta u_\mu) = 0 \qquad 2.7)$$

Multiplying this by $u_\sigma u_\kappa$ and antisymmetrizing on $[\mu\sigma]$ and $[\kappa\zeta]$ gives

$$u_{[\kappa}\chi_{\zeta]\,[\mu}u_{\sigma]}\, u_\nu u^\nu = 0$$

Comparing this with 2.6) shows that if $R_{\mu\nu\zeta\sigma}$ is discontinuous across hypersurface S then S is null.

So, if $g_{\mu\nu,\zeta\sigma}$ suffers a discontinuity across surface S then we can write for S: $u(x^\mu) = 0$

$$g_{\mu\nu,\zeta\sigma} = \overset{o}{g}_{\mu\nu,\zeta\sigma} + \chi_{\mu\nu}\, u_\zeta u_\sigma\, \theta(u) \qquad 2.8)$$

where $\overset{o}{g}_{\mu\nu,\zeta\sigma}$ and $\chi_{\mu\nu}$ are continuous.

Although the Lichnerowicz conditions are widely accepted as the continuity conditions to be imposed on the metric when considering a Cauchy problem, P. Bell and P. Szekeres have shown (8) that there are circumstances where they must be relaxed.

Their approach was to apply the Lichnerowicz conditions to an electromagnetic field with two colliding shock fronts. One may expect the most general form for such a field to be

$$F_{\mu\nu} = f_{\mu\nu} + \Psi_{\mu\nu}\,\theta(u) + \Phi_{\mu\nu}\,\theta(v) + k_{\mu\nu}\,\theta(u)\,\theta(v) \qquad 2.9)$$

However substituting this form into Maxwell's equations gives at:

$$u = o \qquad \psi_{\mu\nu} + k_{\mu\nu}\,\theta(v) = \psi\, e_{[\mu}u_{\nu]}$$

$$v = o \qquad \phi_{\mu\nu} + k_{\mu\nu}\,\theta(u) = \phi\, f_{[\mu}v_{\nu]}$$

where e_μ and f_μ are spacelike vectors orthogonal to u_μ and v_μ respectively. By using the transformation

$$e^\mu \rightarrow e^\mu + \alpha u^\mu \text{ and } f^\mu \rightarrow f^\mu + \beta v^\mu$$

where

$$\alpha = \frac{-e^\mu v_\mu}{u^\nu v_\nu} \qquad \text{and} \qquad \beta = -\frac{f^\mu u_\mu}{u^\nu v_\nu}$$

we get

$$e_\mu u^\mu = e_\mu v^\mu = f_\mu v^\mu = f_\mu u^\mu = o$$

This transformation is well defined since $u_\mu v^\mu \neq o$, the reason for this being that u_μ and v_μ are not proportional to each other by assumption.

The continuity of $\psi_{\mu\nu}$, $\phi_{\mu\nu}$, ψ, ϕ and $k_{\mu\nu}$ imply that at $u = v = o$, $k_{\mu\nu} = o$ and hence doesn't contribute to the shock front. Thus, we can set $k_{\mu\nu} = o$ throughout the region $u>o$, $v>o$.

Before applying the Lichnerowicz conditions to the electromagnetic field, we must note that the hypersurface S at which the shock discontinuity occurs for such a field is null by way of the arguments in the previous section rather than those of this section since for such a field $\Delta R_{\mu\nu} \neq o$. However, the decomposition 2.9) (or 2.5)) is still valid since we did not assum $\Delta R_{\mu\nu} = o$ for that equation.

Hence, for a shock electromagnetic field $\Delta R_{\mu\nu} = 0$ but since u_μ is still null we can write in place of 2.7)

$$\Delta R_{\mu\zeta} = -u_{(\zeta}\, \chi_{\mu)\nu}\, u^\nu + \tfrac{1}{2}\chi^\nu{}_\nu u_\zeta u_\sigma \qquad 2.10)$$

Just as the $k_{\mu\nu}\ \theta(u)\ \theta(v)$ term in 2.9) has $k_{\mu\nu} = o$ it follows analogously that for a double shock fronted electromagnetic field, the Ricci tensor can only have the form

$$R_{\mu\zeta} = R^o{}_{\mu\zeta} + [-u_{(\zeta}\, \chi_{\mu)\nu}\, u^\nu + \tfrac{1}{2}\chi^\nu{}_\nu\, u_\zeta u_\sigma\,]\,\theta(u)$$
$$+ [-v_{(\zeta}\, \Xi_{\mu)\nu}\, v^\nu + \tfrac{1}{2}\Xi^\nu{}_\nu\, v_\zeta v_\mu\,]\,\theta(v) \qquad 2.11)$$

where $R^o{}_{\mu\nu}$, $\chi_{\mu\nu}$ and $\Xi_{\mu\nu}$ are all continuous (i.e. there are no terms of the form $k_{\mu\nu\zeta\sigma}\ \theta(u)\ \theta(v)$). However, Einstein's field equations for such an electromagnetic field are

$$R_{\mu\nu} = \kappa T_{\mu\nu} = -\frac{\kappa}{4\pi}\,(F_{\mu\sigma}\, F_\nu{}^\sigma - \tfrac{1}{4}g_{\mu\nu}\, F_{\sigma\zeta}\, F^{\sigma\zeta})$$

which, upon the use of 2.9) (with $k_{\mu\nu} = o$), will give

$$R_{\mu\nu} = R^o{}_{\mu\nu} + \frac{\kappa}{4\pi}\,\Psi(f_{\sigma(\mu}(e_{\nu)}\, u^\sigma - u_{\nu)}\, e^\sigma) + \tfrac{1}{4}\Psi\, u_\mu u_\nu$$
$$+ \tfrac{1}{2}g_{\mu\nu}\, f_{\sigma\zeta}\, e^\sigma\, u^\zeta)\ \theta(u) + \frac{\kappa}{4\pi}\,\Phi(f_{\sigma(\mu}\ (e_{\nu)}\, v^\sigma - v_{\nu)}\, e^{\sigma)} + \tfrac{1}{4}\Phi\, v_\mu\, v_\nu$$
$$+ \tfrac{1}{2}g_{\mu\nu}\, f_{o\zeta}\, f^\sigma\, v^\zeta)\ \theta(v) + \frac{\kappa}{8\pi}\,\Phi\Psi(e_{(\mu}\, f_{\nu)}\, u_\sigma v^\sigma + u_{(\mu}\, v_{\nu)}\, e_\sigma\, f^\sigma$$
$$- \tfrac{1}{2}g_{\mu\nu}\, e_\sigma\, f^\sigma\, u_\zeta\, v^\zeta)\ \theta(u)\ \theta(v) \qquad 2.12)$$

Comparing 2.11) and 2.12) we see that for colliding electromagnetic fields, the coefficient of the $\theta(u)\ \theta(u)$ term must be zero to satisfy the Lichnerowicz conditions. However, that coefficient in no way need be zero in general. This is seen by contracting it with e^μ giving coefficient

$$-\tfrac{1}{2}\Phi\Psi\, u_\sigma v^\sigma\, f_\nu$$

which is zero only if either or both of the shock fronts vanishes.

Thus, Bell and Szekeres (8) suggested a relaxation of the Lichnerowicz conditions to allow $g_{\mu\nu,\zeta}$ to have shock discontinuities across a hypersurface S: $u(x^{\mu}) = o$ i.e.

$$g_{\mu\nu,\zeta} = g^{o}{}_{\mu\nu,\zeta} + \chi_{\mu\nu\zeta}\ \theta(u) \qquad 2.13)$$

where $g^{o}{}_{\mu\nu,\zeta}$ and $\chi_{\mu\nu\zeta}$ are continuous. Using an approach similar to that which gave 2.5) yields

$$\chi_{\mu\nu\zeta} = \chi_{\mu\nu}\ u_{\zeta}$$

The second derivatives of the metric will therefore be

$$g_{\mu\nu,\zeta\sigma} = g^{1}{}_{\mu\nu,\zeta\sigma} + \chi_{\mu\nu}\ u_{\zeta}\ u_{\sigma}\ \delta(u)$$

where $g^{1}{}_{\mu\nu,\zeta\sigma}$ is piecewise continuous. This gives the Riemann tensor the form

$$R_{\mu\nu\zeta\sigma} = \overset{o}{R}_{\mu\nu\zeta\sigma} + 2u_{[\mu}\chi_{\nu]\ [\zeta}u_{\sigma]}\ \delta(u) \qquad 2.14)$$

where $\overset{o}{R}_{\mu\nu\zeta\sigma}$ is piecewise continuous.

As we saw above, $R_{\mu\nu}$ can have no worse than a shock discontinuity for colliding electromagnetic shock waves since $T_{\mu\nu}$ has only such discontinuities. Therefore, we require that the delta discontinuity term in 2.14) must disappear upon contraction over ν and σ. This gives

$$u^{\sigma}\chi_{\sigma(\zeta}u_{\mu)} = \tfrac{1}{2}\chi^{\sigma}{}_{\sigma}u_{\zeta}u_{\mu} \qquad 2.15)$$

This equality is satisfied by enforcing the O'Brien-Synge conditions on the metric (9). These conditions may be written in the form

$$g^{oi}(\Delta g_{ij,o}) = o$$

$$g^{ij}(\Delta g_{ij,o}) = o$$

where $\Delta g_{\mu\nu\zeta}$ represents the discontinuity across the null hypersurface whose equation is x^{o} = constant. Using 2.13) above and the resulting form for $\chi_{\mu\nu\zeta}$ we find the O'Brien-Synge conditions for the field under consideration are

$$\chi_{\mu\nu} u^{\nu} = \tfrac{1}{2}\chi_{\nu}^{\ \nu} u_{\mu}$$

which is seen to satisfy 2.15). Therefore, although for colliding plane gravitational waves the Lichnerowicz conditions are suitable (12) for colliding plane electromagnetic waves, we must adopt the weaker O'Brien-Synge conditions.

Furthermore, we see from 2.14) that the collision of two electromagnetic shock waves will give rise to two impulse waves propagating along shock fronts u = constant and v = constant.

From the analysis above, we also see that these impulse waves cannot be transformed away.

The Exact Gravitational Wave Metric

Due to the non-linearity and non-tensorial nature of Einstein's field equations, there is no simple method of constructing gravitational wave solutions (2). General gravitational wave solutions cannot be generated by superposition as with electromagnetic waves. As stated above, however, analogy with electromagnetic waves can be utilized to some extent.

The concept of "electromagnetic radiation" entails a transfer of energy, by the field, at the velocity of light. An example of "electromagnetic radiation" is given by the plane electromagnetic wave which is described by an electromagnetic tensor of the form

$$F_{\mu\nu} = T_{\mu\nu} \exp(ix^{\sigma} k_{\sigma})$$

where $T_{\mu\nu}$ and k_{μ} are independent of the co-ordinates. From Maxwell's equations we immediately have

$$F_{\mu\nu} k^{\nu} = 0 \qquad F_{[\mu\nu} k_{\sigma]} = 0$$

and from these we have

$$k_{\mu} k^{\mu} = 0$$

Thus the plane electromagnetic field is a null field by the classification introduced above.

In a vacuum, the Weyl tensor satisfies the Bianchi identities which may be written as

$$C_{\mu\nu\zeta\sigma}{}^{;\sigma} = 0 \qquad C_{\mu\nu[\zeta\sigma;\lambda]} = 0$$

which are similar to Maxwell's equations. This suggests that plane wave gravitational radiation should correspond to the Weyl tensor satisfying equations of the form

$$C_{\mu\nu\zeta\sigma} k^{\sigma} = 0 \qquad C_{\mu\nu[\zeta\sigma} k_{\lambda]} = 0$$

for some null vector k_{μ}.

These equations correspond to gravitational field which is null, or Petrov type N, (2), (5), which mathematically means that the Weyl spinor

Ψ_{ABCD} introduced above can be written in the form

$$\Psi_{ABCD} = \Psi\alpha_{(A}\alpha_{B}\alpha_{C}\alpha_{D)}$$

where $k^{\mu} = \sigma^{\mu}_{A\dot{B}}\, \alpha^{A}\, \bar{\alpha}^{\dot{B}}$ (c.f. The decomposition of the electromagnetic field spinor ϕ_{AB} on page 10).

Exact solutions of this nature which admit at least one null Killing vector are called plane-fronted gravitational waves (2) (5) (10). In (10) it is shown that the metric for such an exact solution can be mapped into the form

$$ds^2 = 2dudr - U(u,y^2,y^3)du^2 - (dy^2)^2 - (dy^3)^2 \qquad 3.1)$$

A special case of this metric is the plane-wave which has five Killing vectors. For such a gravitational wave the function $U(u,y^2,y^3)$ has the form (10)

$$U = f(u)\ ((y^2)^2 - (y^3)^2) + 2h(u)y^2y^3 \qquad 3.2)$$

where f(u) and h(u) are arbitary functions.

If $h(u) \equiv 0$, the plane wave is said to have constant polarization.

The plane wave metric 3.1) and 3.2) can be transformed into the more convenient Rosen form (11)

$$ds^2 = 2e^{-M}\, dudv - g_{ij}\, dx^i\, dx^j \qquad (i,j = 2,3) \qquad 3.3)$$

where M(u) and $g_{ij}(u)$ are arbitary functions.

For the case $h(u) \equiv 0$ this transformation is obtained by setting (12)

$$r = v + \tfrac{1}{2}(y^2)^2\ e^{2P}\ \dot{P} + \tfrac{1}{2}(y^3)^2\ e^{2Q}\ \dot{Q}$$

$$x^2 = y^2e^{P} \qquad x^3 = y^3e^{Q}$$

where $P = P(u)$, $Q = Q(u)$, $\dot{} \equiv \frac{d}{du}$ and P and Q satisfy

$$-(\ddot{P} + \dot{P}^2) = (\ddot{Q} + \dot{Q}^2) = f(u)$$

The metric 3.1) then takes the form

$$ds^2 = dudv - (e^{2P}\ (dx^2)^2 + e^{2Q}(dx^3)^2\ ) \qquad 3.4)$$

In this case g_{ij} is diagonal ; in fact g_{ij} can always be diagonalized if the wave has constant polarization. There are several properties of the Rosen form 3.3) worth discussing. Firstly, the symmetries exhibited by the plane wave are more obvious with the metric in this form. There are five Killing vectors for the metric 3.4). By definition a Killing vector T^μ satisfies the equation

$$T_{(\mu;\nu)} = 0$$

which can be written as

$$g_{\mu\nu,\sigma}\, T^\sigma + T^\sigma{}_{,\mu} g_{\sigma\nu} + T^\sigma{}_{,\nu}\, g_{\mu\sigma} = 0$$

With the metric $g_{\mu\nu}$ given by 3.4), we get the following partial differential equations for T^μ

$$T^1{}_{,0} = 0 \quad ; \quad T^0{}_{,1} = 0 \qquad \text{a)}$$

$$T^0{}_{,0} + T^1{}_{,1} = 0 \qquad \text{b)}$$

$$T^1{}_{,2} - e^{2P}\, T^2{}_{,0} = 0 \quad ; \quad T^1{}_{,3} - e^{2Q}\, T^3{}_{,0} = 0 \qquad \text{c)}$$

$$T^0{}_{,2} - e^{2P}\, T^2{}_{,1} = 0 \quad ; \quad T^0{}_{,3} - e^{2Q}\, T^3{}_{,1} = 0 \qquad \text{d)} \qquad 3.5)$$

$$\dot{P}\, T^0 + T^2{}_{,2} = 0 \quad ; \quad \dot{Q}\, T^0 + T^3{}_{,3} = 0 \qquad \text{e)}$$

$$T^3{}_{,2}\, e^{2Q} + T^2{}_{,3}\, e^{2P} = 0 \qquad \text{f)}$$

From 3.5a) and b) we have

$$T^0{}_{,00} = T^1{}_{,11} = 0$$

Thus

$$T^0 = uf(x,y) \qquad T^1 = vd(x,y)$$

for some $f(x,y)$ and $d(x,y)$

From 3.5c) and a) and

$$(e^{2P}\, T^2{}_{,0})_{,0} = 0 = (e^{2Q}\, T^3{}_{,0})_{,0}$$

giving

$$T^2 = h(v,x,y)\ \left(\int e^{-2P}\, du + g(v,x,y)\right)$$

and similarly

$$T^3 = j(v,x,y)\ \left(\int e^{-2Q}\,du + k(v,x,y)\right)$$

Now using 3.5d) we find

$$u\frac{\partial f}{\partial x} = e^{2P}\left[\frac{\partial h}{\partial v}\left(\int e^{-2P}\,du + g\right) + h\frac{\partial g}{\partial v}\right]$$

however, in this equation, the right hand side is a function of v whereas the left hand side is not. The only way to satisfy this is to set:

$$f(x,y) = 0 \quad , \quad g = g(x,y) \quad , \quad h = h(x,y)$$

Thus, $T^0 = 0$, which immediately requires, by 3.5e)

$$h = h(y) \quad , \quad g = g(y) \quad \text{and} \quad j = j(x) \quad , \quad k = k(x)$$

which in turn, using 3.5f), requires that h, g, j and k are arbitary constants. Hence we can write

$$T^0 = 0$$

$$T^2 = h'\int e^{-2P}\,du + g'$$

$$T^3 = j'\int e^{-2Q}\,du + k'$$

Since $T^0 = 0$, $T^1 = d(x,y)$ and using 3.5c) again, we find

$$T^1 = h'x + j'y + l'$$

where l' is a constant

This gives the following five Killing vectors for the plane gravitational wave

$${}_1T^{\mu} = \begin{bmatrix} 0 \\ x \\ \int e^{-2P}du \\ 0 \end{bmatrix} \quad {}_2T^{\mu} = \begin{bmatrix} 0 \\ y \\ \int e^{-2Q}du \\ 0 \end{bmatrix} \quad {}_3T^{\mu} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$${}_4T^{\mu} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad {}_5T^{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad 3.6)$$

Of this set of Killing vectors, we will be mostly interested in the last two. The transformations corresponding to these vectors are

$$\underset{1}{T^{\mu}} \; ; \; v = v' + \alpha x' + \tfrac{1}{2}\alpha^2 \int e^{-2P}\, du \qquad x = x' + \alpha \int e^{-2P}\, du$$

$$\underset{2}{T^{\mu}} \; ; \; v = v' + \beta y' + \tfrac{1}{2}\beta^2 \int e^{-2Q}\, du \qquad y = y' + \beta \int e^{-2Q}\, du$$

$$\underset{3}{T^{\mu}} \; ; \; v = v' + \gamma$$

$$\underset{4}{T^{\mu}} \; ; \; x = x' + \delta$$

$$\underset{5}{T^{\mu}} \; ; \; y = y' + \varepsilon$$

where α, β, γ, δ, ε are parameters.

Another advantage of the Rosen form is that it allows discontinuities in the curvative tensor while still satisfying the Lichnerowicz conditions stated above. The metric given in 3.1) and 3.2), however, cannot accomplish this.

We can see the reason for its failure to satisfy the Lichnerowicz conditions with discontinuous curvature is due to the presence of the product terms $(y^2)^2$, $(y^3)^2$ and $(y^2)(y^3)$. These terms result in f(u) and h(u) appearing in the Riemann tensor (i.e. in $g_{\mu\nu,\sigma\zeta}$ terms) and hence for this quantity to be discontinuous, so must f(u) and h(u). However, this cuts directly across the Lichnerowicz conditions.

Now, if we use the Rosen form 3.3) for the metric, this problem doesn't arise. This advantage of the Rosen form is of particular merit when we deal with sandwich gravitational waves below.

A third advantage of the Rosen form is that both u and v are null co-ordinates thus in the collision problems (represented by Figure 1) considered below both waves can be represented in the same co-ordinate patch (6), (12).

A disadvantage of the Rosen form, when compared to the form 3.1) is that 3.1) can completely cover the space time manifold (i.e. is geodesically complete) (10) whereas there is a situation in which the Rosen form suffers a co-ordinate singularity, as we shall see for the sandwich gravitational wave (14) considered below.

The Co-ordinate System and Field Equations for Gravitational Plane Waves

The properties and results deduced in this section, concerning the co-ordinate system are effectively those given in (12) which are produced in more detail in (6).

The metric 3.3) is particularly suitable to be represented by the use of a double-null co-ordinate system. In this system we consider two families of distinct null hypersurfaces given by

$$u = u(x'^{\mu}) = C_1 \qquad v = v(x'^{\mu}) = C_2 \qquad (C_1 \text{ and } C_2 \text{ constants})$$

which intersect in spacelike two-surfaces. We may choose one of these spacelike two surfaces (say $u = v = 0$) and co-ordinatize it by choosing suitable co-ordinates x^2 and x^3. These co-ordinates can be extended to the whole manifold by appropriate transportation. Finally, we can adopt the co-ordinate x^0 for the family $u(x'^{\mu}) = C_1$ and x^1 for the family $v(x'^{\mu}) = C_2$. Since the two families of null hypersurfaces are distinct we can define a pair of null vectors l_{μ} and n_{μ} by

$$\psi l_{\mu} = u_{,\mu} \qquad \phi n_{\mu} = v_{,\mu} \qquad 3.7)$$

requiring

$$l_{\mu} n^{\mu} = 1 \qquad \text{i.e.} \qquad \phi\psi = g^{\mu\nu} u_{,\mu} v_{,\nu}$$

Using the co-ordinates adopted above we have

$$l_{\mu} = (\psi^{-1},0,0,0) \qquad n_{\mu} = (0,\phi^{-1},0,0) \qquad 3.8)$$

To utilize the Newman-Penrose formalism, we require a null tetrad l_{μ}, n_{μ}, m_{μ}, $\bar{m}_{\mu}$ which satisfies the relationships 1.1). It follows that we require

$$l^{\mu} = (0,\phi,Y^2,Y^3) \qquad n^{\mu} = (\psi,0,X^2,X^3) \qquad m^{\mu} = (0,0,\xi^2,\xi^3) \quad 3.9)$$

where ϕ, ψ, Y^i and X^i are real valued functions and ξ^i are complex-valued functions of the co-ordinates. We also require m^{μ} to satisfy $m^{\mu}\bar{m}_{\mu} = -1$, $m^{\mu}l_{\mu} = 0$ and $m^{\mu}n_{\mu} = 0$.

Since we are dealing with the plane wave interaction depicted in Figure 1, in which the metric, in the various regions of the figure, is either a function of u or v or both u and v, but never x^2 and/or x^3 (12) we see that all the functions in 3.9) are, at most, functions of u and v. Thus, we shall adopt the approach of (12) rather than that of (6).

With this specialization the tetrad has the following five co-ordinate freedoms

(i) Scale transformation

$$l'^{\mu} = A l^{\mu} \qquad n'^{\mu} = A^{-1} n^{\mu} \qquad A = A(u,v) \tag{3.10}$$

Under this transformation, scale functions ψ and ϕ transform as:

$$\psi' = A^{-1}\psi \qquad \phi' = A\phi \tag{3.11}$$

(ii) Spatial rotations

$$m'^{\mu} = e^{ic}\, m^{\mu} \qquad c = c(u,v) \tag{3.12}$$

The ξ^i transform as

$$\xi'^{j} = e^{ic}\, \xi^{j}$$

(iii) Relabelling of null hypersurfaces

$$u' = f(u) \qquad v' = g(v) \tag{3.13}$$

This induces on ψ and ϕ the transformations

$$\psi' = \psi\frac{df}{du} \qquad \phi' = \phi\frac{dg}{dv}$$

(iv) Spatial co-ordinate transformations

$$u' = u \qquad v' = v \qquad x'^{i} = x^{i} + f^{i}(u,v) \tag{3.14}$$

These induce transformations

$$Y'^{i} = Y^{i} + \phi\frac{\partial f^{i}}{\partial v} \qquad X'^{i} = X^{i} + \psi\frac{\partial f^{i}}{\partial u}$$

(v) Linear co-ordinate transformations

$$x^{i} = a^{i}{}_{j}\, x^{j} \qquad a^{i}{}_{j} = \text{constants} \tag{3.15}$$

We can now introduce the Newman-Penrose formalism and equations referred to in Chapter 1 and presented in extensio in Appendix I.

Since the metric and hence the null tetrad involve components which are functions of only u and v, the spin co-efficients will be likewise. Further, the intrinsic derivatives D, Δ, δ and $\bar{\delta}$ simplify to, using 3.9)

$$D \equiv l^{\mu} \frac{\partial}{\partial x^{\mu}} = \phi \frac{\partial}{\partial v} \qquad \Delta \equiv n^{\mu} \frac{\partial}{\partial x^{\mu}} = \psi \frac{\partial}{\partial v} \qquad \delta \equiv m^{\mu} \frac{\partial}{\partial x^{\mu}} = 0$$

Using this in the commutation relations A1.2) gives;

from A.1.2d) $\quad \mu = \bar{\mu} \qquad \rho = \bar{\rho} \qquad \alpha = \bar{\beta}$

A1.2c) $\quad \nu = 0 \qquad \bar{\tau} = 2\alpha$

A1.2b) $\quad \bar{\tau} = \pi \qquad \kappa = 0$

A1.2a) $\quad D\psi = -(\varepsilon + \bar{\varepsilon})\psi \qquad$ 3.16)

$\Delta\phi = (\gamma + \bar{\gamma})\phi \qquad$ 3.17)

By considering the commutation relations componentwise A1.2) gives

from A1.2a) $\quad \Delta Y^{i} - DX^{i} = (\gamma + \bar{\gamma})Y^{i} + (\varepsilon + \bar{\varepsilon})X^{i} - 4\bar{\alpha}\,\bar{\xi}^{i} - 4\alpha\,\xi^{i} \qquad$ 3.18)

A1.2b) $\quad D\xi^{i} = \sigma\bar{\xi}^{i} + (\rho + \varepsilon - \bar{\varepsilon})\xi^{i} \qquad$ 3.19)

A1.2c) $\quad \Delta\xi^{i} = -\bar{\lambda}\bar{\xi}^{i} - (\mu + \bar{\gamma} - \gamma)\xi^{i} \qquad$ 3.20)

Following (12), we can deduce a set of scale invariant spin co-efficients (i.e. spin co-efficients which are invariant under a scale transformation given by 3.10) above.

From A1.1) we see that under such a scale transformation the spin co-efficients transform as follows

$\rho' = A\rho \qquad \sigma' = A\sigma \qquad \mu' = A^{-1}\mu$

$\lambda' = A^{-1}\lambda \qquad \pi' = \pi \qquad \tau' = \tau$

$\varepsilon' = A\varepsilon + \frac{1}{2}DA \qquad$ from which we see that

$(\varepsilon - \bar{\varepsilon})' = A(\varepsilon - \bar{\varepsilon})$

$\gamma' = A^{-1}\gamma - \frac{1}{2}\Delta A^{-1} \qquad$ which similarly implies

$(\gamma - \bar{\gamma})' = A^{-1}(\gamma - \bar{\gamma})$

$\alpha' = \alpha \qquad \beta' = \beta$

The dyad components of the Weyl, Ricci and Maxwell spinors transform as

$$\Psi'_0 = A^2\,\Psi_0 \qquad \Psi'_1 = A\Psi_1 \qquad \Psi'_2 = \Psi_2$$

$$\Psi'_3 = A^{-1}\,\Psi_3 \qquad \Psi'_4 = A^{-2}\,\Psi_4$$

$$\Phi'_{00} = A^2\Phi_{00} \qquad \Phi'_{01} = A\Phi_{01} \qquad \Phi'_{11} = \Phi_{11}$$

$$\Phi'_{02} = \Phi_{02} \qquad \Phi'_{12} = A^{-1}\Phi_{12} \qquad \Phi'_{22} = A^{-2}\Phi_{22}$$

$$\phi'_0 = A\phi_0 \qquad \phi'_1 = \phi_1 \qquad \phi'_2 = A^{-1}\phi_2$$

Finally the intrinsic differential operators transform as

$$D' = AD \qquad \Delta' = A^{-1}\,\Delta$$

From 3.11) we can define a set of scale invariant quantities as follows:

$$\rho^0 = \rho\phi^{-1} \qquad \sigma^0 = \sigma\phi^{-1} \qquad \mu^0 = \mu\psi^{-1}$$

$$\lambda^0 = \lambda\psi^{-1}$$

$$E^0 = i(\varepsilon - \bar{\varepsilon})\phi^{-1} \qquad G^0 = i(\gamma - \bar{\gamma})\psi^{-1}$$

$$\Psi^0_0 = \Psi_0\phi^{-2} \qquad \Psi^0_1 = \Psi_1\phi^{-1}$$

$$\Psi^0_3 = \Psi_3\Phi^{-1} \qquad \Psi^0_4 = \Psi_4\psi^{-2}$$

$$\Phi^0_{00} = \Phi_{00}\phi^{-2} \qquad \Phi^0_{01} = \Phi_{01}\phi^{-1}$$

$$\Phi^0_{12} = \Phi_{12}\psi^{-1} \qquad \Phi_{22}{}^0 = \Phi_{22}\psi^{-2}$$

$$\phi^0_0 = \phi_0\phi^{-1} \qquad \phi^0_2 = \phi_2\psi^{-1}$$

$$D^0 = \phi^{-1}D = \frac{\partial}{\partial v} \qquad \Delta^0 = \psi^{-1}\Delta = \frac{\partial}{\partial u}$$

We can now proceed to write the commutation relations and field equations in a scale invariant form.

Since 3.16) and 3.17) can be written in the form

$$\begin{aligned} \varepsilon + \bar{\varepsilon} &= -\phi(\ln\psi)_{,v} \\ \gamma + \bar{\gamma} &= \psi(\ln\phi)_{,u} \end{aligned} \qquad 3.21)$$

we see that the real parts of ε and γ cannot be written in a scale invariant form.

However, these quantities do not appear explicitly in any of the other field equations. We can regard these equations as expressing $\varepsilon + \bar{\varepsilon}$ and $\gamma + \bar{\gamma}$ in terms of ϕ and ψ.

To write 3.18) in a scale invariant form, we firstly note that the ξ^i are automatically scale invariant. Thus using 3.21) we find

$$Y^{0i}{}_{,u} - X^{0i}{}_{,u} = -4e^{-M}\,(\bar{\alpha}\,\bar{\xi}^i + \alpha\xi^i) \qquad 3.22)$$

where $Y^{0i} = \phi^{-1}\,Y^i \qquad X^{0i} = \psi^{-1}\,X^i$ and

$$M = \ln(\phi\psi) \qquad 3.23)$$

are all scale invariant.

From 3.22) and the co-ordinate freedom encapsulated in 3.14), we see that $\alpha = 0$ is a necessary and sufficient condition for there to exist a spatial co-ordinate transformation 3.14) which makes X^i and Y^i simultaneously zero.

The remaining 3.19) and 3.20) become

$$\xi^i{}_{,v} = \sigma^0\bar{\xi}^i + (\rho^0 + iE^0)\xi^i \qquad 3.24)$$

$$\xi^i{}_{,u} = -\bar{\lambda}^0\,\bar{\xi}^i - (\mu^0 - iG^0)\xi^i \qquad 3.25)$$

The field equations presented in A1.3a)-r) reduce to

a) (reduces to) $\rho^0{}_{,v} = (\rho^0)^2 - \rho^0 M_{,v} + \sigma^0\,\bar{\sigma}^0 + \Phi^0_{00}$ a)

q) $\rho^0{}_{,u} = -(\mu^0\rho^0 + \sigma^0\lambda^0) - 4e^{-M}\,\alpha\bar{\alpha} - e^{-M}\Psi_2$ b)

h) $\mu^0{}_{,v} = \rho^0\mu^0 + \sigma^0\lambda^0 - 4e^{-M}\,\alpha\bar{\alpha} + e^{-M}\,\Psi_2$ c)

n) $\mu^0{}_{,u} = -(\mu^0)^2 - \mu^0 M_{,u} - \lambda^0\bar{\lambda}^0 + \Phi^0_{22}$ d)

b) $\sigma^0{}_{,v} = \sigma^0(2\rho^0 - M_{,v} + 2iE) + \Psi^0_0$ e)

p) $\sigma^0{}_{,u} = (2iG^0 - \mu^0)\sigma^0 - \bar{\lambda}^0\rho^0 - 4e^{-M}\,\bar{\alpha}^2$ 3.26)

$- e^{-M}\,\Phi_{02}$ f)

g) $\lambda^0{}_{,v} = \lambda^0(\rho^0 - 2iE^0) + \bar{\sigma}^0\mu^0 + 4e^{-M}\,\alpha^2$

$+ e^{-M}\,\Phi_{20}$ g)

j) $\lambda^0{}_{,u} = -\lambda^0(2\mu^0 + M_{,u} + 2iG^0) - \Psi^0_4$ h)

d) $\alpha_{,v} = \alpha(3\rho^0 - iE^0) + \bar{\sigma}^0\bar{\alpha} + \Phi^0_{01}$ i)

r) $\alpha_{,u} = -\alpha(\mu^0 + iG^0) - 3\lambda^0\bar{\alpha} - \Psi^0_3$ j) 3.26)

f) $e^{-M}(\Psi_2 + \Phi_{11}) = -12e^{-M}\alpha\bar{\alpha} + \tfrac{1}{2}i(G^0{}_{,v} - E^0{}_{,u})$

$+ \tfrac{1}{2} M_{uv}$ k)

k) $\Psi^0_1 - \Phi_{0}{}^{0}_{1} = 2(\rho^0\bar{\alpha} - \sigma^0\alpha)$ l)

l) $e^{-M}(\Psi_2 - \Phi_{11}) = \mu^0\rho^0 - \lambda^0\sigma^0$ m)

m) $\Psi^0_3 - \Phi_{2}{}^{0}_{1} = 2(\mu^0\alpha - \lambda^0\bar{\alpha})$ n)

Maxwell's equations 1.23) a) to d) reduce to

a) (reduce to) $\phi_{1,v} = 2\rho^0\,\phi_1$ a)

d) $\phi_{1,u} = -2\mu^0\,\phi_1$ b) 3.27)

c) $\phi_0{}^0{}_{,u} = -(\mu^0 - iG^0)\phi_0{}^0 - 4e^{-M}\bar{\alpha}\,\phi_1 + \sigma^0\phi_2{}^0$ c)

b) $\phi_2{}^0{}_{,v} = -\lambda^0\phi^0_0 + 4e^{-M}\alpha\,\phi_1 + (\rho^0 - iE^0)\,\phi_2{}^0$ d)

From the definition of the scale invariant components of the Ricci spinor and Maxwell spinor and using (1)

$\Phi_{ab} = k\phi_a\,\bar{\phi}_b$ (k a constant)

for the electromagnetic field, we see that the scale invariant Ricci components for the electromagnetic field are

$\Phi^0_{00} = k\phi^0_0\,\bar{\phi}^0_0$ $\Phi^0_{01} = k\phi^0_0\,\bar{\phi}_1$ $e^{-M}\,\Phi_{02} = k\phi^0_0\,\bar{\phi}_2{}^0$

$\Phi_{11} = k\phi_1\bar{\phi}_1$ $\Phi^0_{12} = k\phi_1\,\bar{\phi}^0_2$ $\Phi_{2}{}^{0}_{2} = k\phi_2{}^0\,\bar{\phi}_2{}^0$ 3.28)

where $k = \dfrac{G}{8c^4} = \dfrac{\kappa}{64\pi}$

To proceed from 3.25), 3.26) and 3.27) to a meaningful set of field equations, we can avoid an unnecessary complication by using several features of the system we are studying.

We will be concerned with the interaction between particular types of gravitational and electromagnetic waves, and hence we may exploit the various properties of these waves to simplify the tasks at hand. The first simplification arises from the fact that for a purely gravitational wave the Ricci tensor vanishes ($R_{\mu\nu} = 0$). Hence in the equations above, all the corresponding Ricci Spinor terms vanish (i.e. all the Φ^0's are zero) for such a wave.

Secondly, for the electromagnetic wave, we have the Ricci spinor components given by 3.28); where the ϕ^0's are the electromagnetic field spinor components. Thus, if we can set up the interaction such that one of the ϕ^0's vanish throughout the space-time, a further simplification will arise.

In fact, by choosing the electromagnetic waves to be initially null in the sense of the classification on page 10 we can set, in particular, ϕ_1 to zero. If this is done, we see from 3.27 a) and b) and the uniqueness of solutions of partial differential equations that ϕ_1 will vanish throughout the whole space-time.

Thus from 3.28) $\Phi^0_{01} = \Phi_{11} = \Phi^0_{12} = 0$ throughout the whole space-time.

With these results, we see that in both cases, (i.e. gravitational and initially null electromagnetic wave interactions), the partial differential equations 3.26 i) and j) for α become, using 3.26n)

$$\alpha_{,v} = \alpha(3\rho^0 - iE^0) + \bar{\sigma}^0 \bar{\alpha}$$

$$\alpha_{,u} = -\alpha(3\mu^0 + iG^0) - 2\lambda^0\bar{\alpha}$$

For the collision problems we shall be studying, (see Figure 1) α is is initially zero (i.e. Region I in Figure 1 is flat space) hence using the above two equations to evaluate α throughout the space-time, with initial condition $\alpha = 0$, we see that α is zero everywhere.

However, it was noted above that if α vanishes then there exists a transformation 3.14) such that the components Y^i and X^i $(i = 2,3)$ of null vectors l^μ and n^μ respectively in 3.9) can be set to zero. Hence these components can be set to zero throughout the space-time.

With these simplifications, many of the equations in 3.26) simplify somewhat.

Using 3.26) m) with $\Phi_{11} = 0$ on 3.26) b) and c), with $\alpha = 0$, gives

$$\rho^0{}_{,\mu} = -2\mu^0 \rho^0 \qquad \text{a)}$$

$$\mu^0{}_{,v} = 2\mu^0 \rho^0 \qquad \text{b)} \qquad 3.29)$$

Combining 3.26) k) and m)

$$\mu^0\rho^0 - \lambda^0\sigma^0 = \tfrac{1}{2}i(G^0{}_{,v} - E^0{}_{,u}) + \tfrac{1}{2} M_{uv} \qquad \text{c)}$$

Finally, we have for the Weyl spinor components

$$\Psi_1^0 = 0$$

$$\Psi_3^0 = 0$$

Proceeding toward a more meaningful set of field equations we see that since l^μ, n^μ, m^μ, and $\bar{m}^\mu$ form a null tetrad (satisfying 1.1)) we can use 1.2), 3.8), 3.9) and 3.23) to write

$$ds^2 = 2e^{-M} du dv + g_{ij}\, dx^i\, dx^j$$

where $i,j = 2, 3$ and

$$g^{ij} = -(\xi^i \bar{\xi}^j + \bar{\xi}^i \xi^j)$$

Writing $g_{ij}\, dx^i\, dx^j$ in the form (12)

$$-e^{-U}(e^V \cosh W\, (dx^2)^2 + e^{-V} \cosh W\, (dx^3)^2 - 2\sinh W\, dx^2 dx^3)$$

where

$$U = -\ln(\det g_{ij})$$

From 3.29) we have

$$\xi^2 = \frac{1}{\sqrt{2}} e^{\frac{1}{2}(U-V)} (\cosh\tfrac{1}{2}W + i\sinh\tfrac{1}{2}W) \qquad \text{a)}$$

$$\xi^3 = \frac{1}{\sqrt{2}} e^{\frac{1}{2}(U+V)} (\sinh\tfrac{1}{2}W + i\cosh\tfrac{1}{2}W) \qquad \text{b)} \qquad 3.30)$$

Substituting these into 3.24) and 3.25) gives

$$\rho^0 = \tfrac{1}{2}U_v \qquad \mu^0 = -\tfrac{1}{2}U_u$$

$$E^0 = -\tfrac{1}{2}V_v \text{Sinh}W \qquad G^0 = -\tfrac{1}{2}V_u \text{Sinh}W$$

$$\sigma^0 = \tfrac{1}{2}iW_v - \tfrac{1}{2}V_v \text{Cosh}W \qquad \lambda^0 = \tfrac{1}{2}iW_u + \tfrac{1}{2}V_u \text{Cosh}W$$

where subscripts u and v represent partial differentiation taken with respect to that variable. These values may now be substituted into 3.26) and 3.29) to give

3.26 c),b) (gives) $U_{uv} = U_u U_v$ a)

a) $2U_{vv} - U_v^{\,2} + 2U_v M_v = W_v^{\,2} + V_v^{\,2}\text{Cosh}^2W + 4\Phi_{00}$ b)

d) $2U_{uu} - U_u^{\,2} + 2U_u M_u = W_u^{\,2} - V_u^{\,2}\text{Cosh}^2W + 4\Phi_{22}^{\,0}$ c)

e) $\Psi^0_{\,0} = -\tfrac{1}{2}(V_{vv}\text{Cosh}W + 2V_v W_v \text{Sinh}W\; V_v(U_v - M_v)\text{Cosh}W) + \tfrac{1}{2}i(W_{vv} - W_v(U_v - M_v) - V_v^2 \text{Cosh}W\,\text{Sinh}W)$ d)

f),g) $2W_{uv} - U_u W_v - U_v W_u = 2V_u V_v \text{Cosh}W\,\text{Sinh}W - 4e^{-M}\,\text{Im}\{\Phi_{02}\}$ e) 3.31)

f),g) $2V_{uv} - U_u V_v - U_v V_u = -2(V_v W_u + V_u W_v)\tanh W + 4e^{-M}\,\text{Re}\{\Phi_{20}\}$ f)

h) $\Psi^0_{\,4} = -\tfrac{1}{2}(V_{uu}\text{Cosh}W - 2V_u W_u \text{Sinh}W - V_u(U_u - M_u)\text{Cosh}W) - \tfrac{1}{2}i(W_{uu} - W_u(U_u - M_u) - V_u^{\,2}\text{Cosh}W\,\text{Sinh}W)$ g)

k) $e^{-M}\Psi_2 = \tfrac{1}{2}M_{uv} - \tfrac{1}{4}i(V_u W_v - V_v W_u)\text{Cosh}W$ h)

m) $e^{-M}\Psi_2 = -\tfrac{1}{4}U_u U_v + \tfrac{1}{4}V_u V_v \text{Cosh}^2W + \tfrac{1}{4}W_u W_v - \tfrac{1}{4}i(W_v V_u - W_u V_v)\text{Cosh}W$ i)

Similarly, for Maxwell's equations 3.27 we get

3.27) a),b) (gives) $\phi_1 = 0$ a) 3.32)

c) $$\phi^0_{0,u} = \tfrac{1}{2}(U_u - iV_u \text{Sinh}W)\phi^0_0 - \tfrac{1}{2}(V_v \text{Cosh}W - iW_v)\phi^0_2 \qquad \text{b)}$$

d) $$\phi^0_{2,v} = -\tfrac{1}{2}(V_u \text{Cosh}W + iW_u)\phi^0_0 + \tfrac{1}{2}(U_v + iV_v \text{Sinh}W)\phi^0_2 \qquad \text{c)} \qquad 3.32)$$

To describe the interaction depicted in Figure 1 we must solve the above set of equations. This task is somewhat simplified if we re-arrange several of the equations realizing that some equations are integrability conditions.

Upon re-arrangement and using 3.28) where necessary we get the following system of equations

3.31) a) $$U = -\ln(f(u) + g(v)) \qquad \text{a)}$$

b) $$2U_{vv} - U_v^2 + 2U_v M_v = W_v^{\,2} + V_v^{\,2}\text{Cosh}W^2 + 4k\phi^0_0\, \bar{\phi}^0_0 \qquad \text{b)}$$

c) $$2U_{uu} - U_u^{\,2} + 2U_u M_u = W_u^{\,2} + V_u^{\,2}\text{Cosh}^2W + 4k\phi^0_2\, \bar{\phi}^0_2 \qquad \text{c)}$$

e) $$2W_{uv} - U_u W_v - W_u U_v = 2V_u V_u \text{Cosh}W\text{Sinh}W - 2ik(\phi^0_2\, \bar{\phi}^0_0 - \bar{\phi}^0_2\, \phi^0_0) \qquad \text{d)} \qquad 3.33)$$

f) $$2V_{uv} - U_u V_v - V_u U_v = -2(V_u W_v + W_u V_u)\tanh W + 2k(\phi^0_2\, \bar{\phi}^0_0 + \bar{\phi}^0_2\, \phi^0_0) \qquad \text{e)}$$

i),h) $$2M_{uv} + U_u U_v - W_u W_v = V_u V_v \text{Cosh}W \qquad \text{f)}$$

These equations along with 3.32) form the set of equations to be solved. In this set, we can see that 3.32) and 3.33) d) and e) are integrability conditions for 3.33) b) c) and f). Hence, we may attempt to solve 3.32) and 3.33) d) and e) for V, W, ϕ^0_0 and ϕ^0_2 in the interaction region (Region IV in Fig.1) with the initial data properly set. Then M can be found from 3.33) f) by integration and using its initial values on u = 0 and v = 0.

As stated in (8) 3.32) and 3.33) d) and e) are a quasilinear hyperbolic system of equations (see (4) for the definition of such a system) in the two independent variables u and v. Junction conditions require U, V, W and M to be continuous everywhere, ϕ^0_2 to be continuous across v = 0 and ϕ^0_0 to continuous across u = 0.

Therefore if we know the forms of U, V, W, M, ϕ_0^0 and ϕ_2^0 from initial (non-interacting) plane waves in Regions II and III we can deduce U(u,v) throughout Region IV from 3.33)a) and we will know the values of the other functions on the appropriate boundaries. This is all the characteristic initial value data needed to determine V, W, M, ϕ_0^0 and ϕ_2^0 uniquely in Region IV(6). Although this (characteristic) initial value problem can be set, and further some of the properties of the general solution studied (6), the system is too complicated to solve for the general case, with present day partial differential equation theory.

Therefore, the approach to be taken here will be to simplify this system in the following various ways.

1) Make the electromagnetic field a "test field" in that we ignore its gravitational field

2) Since almost all of the experimental tests involving General Relativity utilize the weak field approximation, (13),(23) we could adapt 1) immediately above to a weak field approximation. In the usual weak field approach the gravitational field is linearized which results in mere superposition of waves, we shall slightly modify the weak field assumptions to give a non-trivial interaction.

3) Although the system above cannot be solved exactly we can find an approximate solution for values of u and v small and positive by using a power series expansion (12). This will give us some general properties of the solution in that region.

The Weak Gravitational Wave Metric

An extensive survey of the properties of weak or linearized gravitational field and, in particular, gravitational plane waves is contained in (13). In that reference, the gauge freedom properties of the weak field metric are used to introduce specific co-ordinate systems (e.g. harmonic gauge and transverse traceless or TT gauge). However, since we are particularly interested in the Rosen form for the exact metric, we shall retain this form.

Before proceeding, we must realize that the term "weak field" is rather misleading for the single plane wave field. This is due to the problems encountered in measuring the strength of a gravitational field (15). To measure this quantity we would ideally use some property of the field invariant so that all observers would agree on the strength of the field. At first thought, a scalar invariant deduced from the Riemann tensor appears suitable. (E.g. R or $R_{\mu\nu\zeta\sigma} R^{\mu\nu\zeta\sigma}$ etc.). In most cases, such a quantity will be suitable, however for the plane gravitational wave all of these invariants are zero (although $R_{\mu\nu\zeta\sigma}$ is not) and therefore we must measure the field's strength using some other method.

In recognizing this problem Hawking and Ellis (15) suggest using the components of the Riemann curvature tensor. Although in (15) their purpose differs from ours, in that they are studying space time singularities, we have little choice but to follow their suggestion. Resulting from this, because the components $R_{\mu\nu\zeta\sigma}$ of the Riemann tensor are not invariant, we find that different observers will ascribe different strengths to the same gravitational wave, the values depending upon their worldline.

Furthermore, since the plane gravitational wave has no timelike Killing vectors there is no "naturally implied" co-ordinate frame from which we may analyse the wave and ascribe a strength.

However, for the interaction of two plane waves depicted in Figure 1, although the former problem persists (i.e. all scalar invariants are zero) the latter does not. We can see from the symmetry of the interaction that a suitable reference frame is that with its worldline tangent vector pointing vertically up the diagram. Such a worldline is, in fact, a geodesic. Therefore, when we refer to a weak gravitational wave, it shall be with respect to such an observer.

As stated above, for weak field theory, the metric is assumed to have the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad 3.34)$$

where $\eta_{\mu\nu}$ is the flat space metric and $h_{\mu\nu}$ represents a small perturbation from the flat metric. The conditions placed upon the $h_{\mu\nu}$ are usually

$$|h_{\mu\nu}| << 1 \qquad 3.35)$$

$$|h_{\mu\nu,\zeta}| |h_{\sigma\tau,\gamma}| << |h_{\mu\nu,\zeta\gamma}| \qquad 3.36)$$

We shall adopt both these conditions and also we shall adopt a kind of generalization of 3.36) which we shall find useful. Mathematically this condition has the form

$$|h_{,\mu_1 \ldots \mu_n}| |h_{,\nu_1 \ldots \nu_m}| << |h_{,\zeta_1 \ldots \zeta_{mn}}| \qquad 3.37)$$

where h represents any of the $h_{\mu\nu}$ and 3.37) is assumed to hold for all μ_i, ν_j, ζ_k and all m, n, including the case $|h| |h,\mu| << |h,\mu|$.

With these conventions, we may re-formulate the metric and field equations above for the weak field. The metric now takes the form

$$ds^2 = 2(1-M)dudv - [(1-U+V)(dx^2)^2 + (1-U-V)(dx^3)^2$$
$$- 2Wdx^2dx^3] \qquad 3.38)$$

where the moduli of M, U, V and W are all much smaller than unity.

We have used

$$e^F = \sum_{n=0}^{\infty} \frac{1}{n!} F^n$$

and neglected all but terms of lowest order. Since the metric 3.38) is of the same form as the exact metric, all of Einstein's field equations apply as before, with the result that we can proceed to deduce the scale invariant Newman-Penrose spin co-efficients and equations.

Using a similar approach, we see from 3.38) and 3.29) we can write

$$\xi^2 = \frac{1}{\sqrt{2}} (1 + \tfrac{1}{2}(U-V) + \frac{iW)}{2}$$

$$\xi^3 = \frac{1}{\sqrt{2}} (i(1 + \tfrac{1}{2}(U+V)) + \frac{W}{2})$$

Substituting these into 3.24) and 3.25) gives

$$\rho^0 = \tfrac{1}{2}U_v \qquad \mu^0 = -\tfrac{1}{2}U_u$$

$$E^0 = G^0 = 0$$

$$\sigma^0 = \tfrac{1}{2}i\, W_v - \tfrac{1}{2}V_v \qquad \lambda^0 = \tfrac{1}{2}i\, W_u + \tfrac{1}{2}V_u$$

Furthermore, 3.26) becomes

$$U = f(u) + g(v) \qquad \text{a)}$$

$$U_{vv} = 2k\phi_0^0\, \bar{\phi}_0^0 \qquad \text{b)}$$

$$U_{uu} = 2k\phi_2^0\, \bar{\phi}_2^0 \qquad \text{c)} \qquad 3.39)$$

$$W_{uv} = -ik(\phi_2{}^0\bar{\phi}_0^0 - \bar{\phi}_2{}^0\phi_0^0) \qquad \text{d)}$$

$$V_{uv} = k(\phi_2^0\, \bar{\phi}_0^0 + \bar{\phi}_2^0\, \phi_0^0) \qquad \text{e)}$$

$$M_{uv} = 0 \qquad \text{f)}$$

where $k = \dfrac{G}{8c^4} = \dfrac{\kappa}{64\pi}$

Also, Maxwell's equations 3.32) will give

$$\phi_1 = 0 \qquad \phi^0_{0,u} = 0 \qquad \phi^0_{2,v} = 0$$

These equations indicate that the two waves pass through each other without interaction; i.e. superposition. However, this occurs because we have neglected all but the lowest order terms. We can "induce" an interaction by adopting 3.37) on Maxwell's equations and ignoring all but the lowest two orders in h. In doing this, we must split Maxwell's equations away from the system 3.39) since they are now quite different in their meaning.

Using this approach gives Maxwell's equations as

$$\phi_1 = 0 \qquad \text{a)}$$

$$\phi^0_{0,u} = \tfrac{1}{2}U_u\, \phi^0_0 - \tfrac{1}{2}(V_u - iW_u)\, \phi^0_2 \qquad \text{b)} \quad 3.40)$$

$$\phi^0_{2,v} = -\tfrac{1}{2}(V_u + iW_u)\, \phi^0_0 + U_v\, \phi^0_2 \qquad \text{c)}$$

We again treat the electromagnetic field as a test field and consider 3.39) merely as a set of equations describing a weak gravitational wave. (Obviously since the electromagnetic field is a test field, the ϕ_0^0 and ϕ_2^0 terms are set to zero in 3.39)).

If we had attempted to apply this "second order" approximation to the full system, we would have resulted with a set of equations just as intractable as the full theory equations.

Before proceeding any further, we should recognize a possible disadvantage of using weak field theory. This criticism of weak field theory was developed by Synge (14). Essentially Synge's argument states that Einstein's field equations can be read from left to right or vice versa. Therefore, we may ascribe an energy momentum distribution ($T_{\mu\nu}$, the energy momentum tensor) and find the corresponding space-time metric, or we may start with the metric and deduce the corresponding energy momentum tensor. Therefore what may be an approximate metric for one given source is, in fact, an exact solution for that source described by direct calculation of the energy momentum tensor.

Although we are dealing with source-free fields, the essentials of Synge's argument can still be applied. The weak field metric we have utilized above may, in fact, represent an exact solution of a different case of gravitational radiation. The form of the weak field wave certainly satisfies the required conditions for it to represent gravitational radiation (5), (6), (11), however, it does not share the same properties as the exact gravitational wave metric.

The best example of this is related to the non-periodicity of exact pure gravitational plane waves deduced by Synge (14). This property is easily seen from 3.33)b) or c). For a non-interacting plane wave, the metric is a function of only one variable; u say. Therefore, of the equations in 3.33) only c) will survive.

For a non-interacting wave, $M = M(u)$ and therefore using the co-ordinate transform

$$u' = \int e^{-2M(u)} du$$

in the Rosen metric A2.1) we can eliminate the M_u terms from the field equations. This leaves 3.33)c) as

$$U_{uu} = U_u^{\ 2} + W_u^{\ 2} + V_u^{\ 2} \cosh^2 W$$

and therefore U cannot be periodic since $U_{uu} > 0$. Also, the only component of the Weyl spinor which is non-zero is $\Psi_4^{\ 0}$ which, from 3.31)g), is

$$\Psi_4^0 = -\tfrac{1}{2}(V_{uu} \cosh W + 2V_u W_u \sinh W - V_u U_u \cosh W)$$

$$-\tfrac{1}{2} i (W_{uu} - W_u V_u - V_u^{\ 2} \cosh W \sinh W)$$

From these two equations we see that for a gravitational wave which is not trivial (i.e. trivial being U = V = W = constant) we have

$$U_{uu} \not\equiv 0 \qquad 3.41)$$

We can see that the significant aspect of these equations is that U, V and W are coupled by 3.33)c) in such a way that the condition 3.41) is forced upon us for any such non-trivial gravitational wave. However, in the case of the weak pure gravitational wave, the weak field equations give no coupling between U, V and W; we have from 3.39)c)

$$U_{uu} \equiv 0$$

in contrast to 3.41).

The Sandwich Wave

The sandwich wave is a particularly interesting plane wave. It is, in effect, a finitely thick layer of non-zero curvature propagating through space-time bounded on both sides by null hyperplanes in flat space-time. The gravitational sandwich wave was introduced by Bondi, Pirani and Sachs (18) and its properties were further discussed by Synge (14). In both these articles, reference was made to the co-ordinate singularities which occur. This aspect of these waves was later shown by Penrose (17) to be related to the focusing property of such waves.

We shall consider this property with respect to the collision interactions we are studying. To do this, it is useful to briefly review the properties of the plane sandwich wave using the Rosen form for the metric. The appropriate equations for this task are, for the pure gravitational field, with M set to zero;

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2 \operatorname{Cosh}^2 W \qquad \text{a)}$$

$$\Psi_4^0 = -\tfrac{1}{2}(V_{uu}\operatorname{Cosh}W + 2V_u W_u \operatorname{Sinh}W - V_u U_u \operatorname{Cosh}W) \qquad 3.42)$$

$$+\tfrac{1}{2}i(W_{uu} - W_u U_u - V_u^2 \operatorname{Cosh}W \operatorname{Sinh}W) \qquad \text{b)}$$

These equations 3.42) must be satisfied throughout the space-time manifold or, at least, throughout the co-ordinate patch for which they are written.

In the initial flat Region I (Figure 2) we can set U, V and W to zero, thus trivially satisfying 3.42). In the non-flat Region II, we see from 3.42) that U(u) increases hence $U(u) > 0$ within this region.

Obviously we also have $\Psi_4^0 \neq 0$

Moving into the second flat Region III we must now have $\Psi_4^0 = 0$. This gives rise to the equations

$$V_{uu}\operatorname{Cosh}W + 2V_u W_u \operatorname{Sinh}W - V_u U_u \operatorname{Cosh}W = 0 \qquad \text{a)}$$

$$3.43)$$

$$W_{uu} - W_u U_u - V_u^2 \operatorname{Cosh}W \operatorname{Sinh}W = 0 \qquad \text{b)}$$

From 3.43) a) we have

$$V_u = Ce^V \operatorname{Sech}^2 W$$

where C = constant. Placing this into 3.42) a) and using

$$2U_{uu} - U_u^2 = -4(e^{-U/2})_{uu} e^{U/2}$$

gives

$$(e^{-U/2})_{uu} = -\tfrac{1}{4}e^{-\frac{U}{2}}(W_u^2 + e^{2U} \operatorname{Sech}^2 W) < 0 \qquad 3.44)$$

Since U is continuous throughout the co-ordinate patch, $(e^{-\frac{U}{2}})$ is also continuous. Further, its initial value, i.e. in Region I, is unity and $(e^{-U/2})_u$ is initially zero.

Adopting the Lichnerowicz conditions, we see from 3.42)a) that U_{uu} is continuous and hence U_{uu} diverges if and only if U_u does. Now V_{uu} and W_{uu} can, at most, have finite step discontinuities, therefore we can see by inspecting 3.42) that if Ψ_4^0 diverges, so must U_{uu}.

Furthermore, 3.43) becomes meaningless since they contain divergent quantities namely U_u. Referring back to 3.42) we then see that U_u diverges before the second flat region, there can be no such region since Ψ_4^0 cannot converge back to a finite value. Because Ψ_4^0 diverges, the singularity so produced is a real singularity and covers the rest of the manifold after its initial divergence.

From this, we see that, using the Lichnerowicz conditions, we cannot have a sandwich wave which contains a real singularity. The plane wave must extend to infinity.

However, if we adopt the O'Brien-Synge conditions then although U_{uu} can, at most, have finite step discontinuities, V_{uu} and W_{uu} can have delta discontinuities. Therefore, we can have real (delta) singularities occurring in a sandwich wave.

We can apply a similar argument as above to the behaviour of U_{uu} and U_u to the O'Brien-Synge conditions. From this we see that only "delta singularities" can occur in a sandwich wave under the O'Brien-Synge conditions.

With these considerations, we can assume that for a sandwich wave, U is finite at u = u', the "far-end" of the wave. Therefore, $(e^{-U/2})$ is finite (and less than unity by 3.42)). From 3.44), $e^{-U/2}$ must go to zero in Region III and hence U must diverge in that region. So, again, we have a singularity however, this is a co-ordinate singularity since Ψ_4^0 is zero in Region III.

This is the singularity referred to by Penrose (17) and corresponds to the focusing properties of such plane gravitational waves. The properties of these co-ordinate singularities are found by solving 3.43)a) and 3.42). In doing so, we simplify this task by assuming the waves are linearly polarized (i.e. set W = 0).

$$2U_{uu} - U_u^{\,2} - V_u^{\,2} = 0 \qquad \text{a)}$$
$$\qquad\qquad\qquad\qquad 3.45)$$
$$V_{uu} - V_u U_u = 0 \qquad \text{b)}$$

As found in (18), there are several possible solutions. The first is trivial, i.e. U = constant, V = constant, and since we are in Region III we have, from above, $U \neq 0$.

Secondly, setting V_u = constant gives, using $U = -2\ln Y$ in 3.45)a);

$$U(u) = -2\ln(au + b)$$

Applying the Lichnerowicz conditions (both these and the O'Brien Synge conditions are equivalent in this case) we find

$$a = -\tfrac{1}{2}\, e^{\frac{-U(u')}{2}}\, U_u(u') \qquad b = e^{\frac{-U(u')}{2}}\left(1 + \tfrac{1}{2}U_u(u')\, u'\right)$$

Since $U_{uu} > 0$ and $U_u = 0$ at $u = 0$, $U_u > 0$ for $u > 0$. Therefore, a and b will have opposite signs, hence the resultant metric

$$ds^2 = du\, dv - (au + b)^2 \left((dx^2)^2 + (dx^3)^2\right)$$

is singular.

Alternatively, we can combine 3.45) a) and b) and set $(U + V) = \phi$ giving

$$2\phi_{uu} - (\phi_u)^2 = 0$$

This is done in Appendix II and the existence of unavoidable co-ordinate singularities is shown.

The Electromagnetic Test Field Case with Exact Wave Metric

The set of equations for this case is given in 3.32) and is reproduced here with the appropriate changes

$$\phi^0_{0,u} = \tfrac{1}{2}(U_u - iV_u \text{ Sinh}W)\phi_0 \qquad \text{a)}$$

$$\phi^0_{2,v} = -\tfrac{1}{2}(V_u \text{ Cosh}W + iW_u)\phi_0 \qquad \text{b) } 4.1)$$

$$2U_{uu} - U_u^{\ 2} = W_u^{\ 2} + V_u^{\ 2} \text{ Cosh}^2 W \qquad \text{c)}$$

where the initial conditions are

$$\phi^0_0 = \phi^0_0(v) \quad \text{at} \quad u = 0 \qquad \phi_2^{\ 0} = 0 \quad \text{at } u = 0$$

$$U = V = W = M = 0 \quad \text{at } u = 0$$

and U, V, W are all functions of u only. Since the gravitational field of the electromagnetic shockwave is neglected (since it is a test field), there are no ϕ terms in the gravitational wave field equation. Also, as far as the gravitational wave is concerned, it is propagating through flat space and hence there are no v dependent terms in 4.1)c). For the same reason, the terms in 4.1)a) and b) there are no terms involving the derivatives of U, V or W with respect to v and M = 0. The characteristic initial value problem depicted by 4.1) is easily solved for ϕ^0_0 and ϕ^0_2 in the interaction region. From 4.1)a)

$$(\ln\phi^0_0)_u = \tfrac{1}{2}(U_u - iV_u \text{ Sinh}W)$$

therefore

$$\phi^0_0 = A(v)e^{\tfrac{1}{2}(U - i\int_0^u V_{u'} \text{Sinh}W du')}$$

From the initial conditions we have

$$\phi^0_0(u,v) = \phi^0_0(v)\Big|_{u=0} e^{\tfrac{1}{2}(U-i\int_0^u V_u{}' \text{Sinh}W\, du')} \qquad 4.2)$$

Placing this into 4.1)b), with the initial conditions, gives

$$\phi^0_2(u,v) = -\tfrac{1}{2}(V_u \text{Cosh}W + iW_u)e^{\tfrac{1}{2}(U-i\int_0^u V_u{}' \text{Sinh}W du')}$$

$$\times \int_0^v \phi^0_0(v')\Big|_{u=0} dv' \qquad 4.3)$$

Therefore, in Region IV (the interaction region) we have, from 1.25)

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + i^*F_{\mu\nu} = 2e^{\frac{1}{2}(U - i\int_0^u V_{u'} \sinh W \, du')} [\, \phi_0^0(v)\,|_{u=0}\ \bar{m}_{[\mu}v_{\nu]}$$

$$-\tfrac{1}{2}(V_u \cosh W + iW_u)\int_0^v \phi_0^0(v')\,|_{u=0} dv'\ u_{[\mu}m_{\nu]}] \qquad 4.4)$$

(Since $M = 0$, we have $l_\mu = u_{,\mu}$ and $n_\mu = v_{,\mu}$)

From inspecting 4.4), we see that the resulting electromagnetic field is not null (and therefore not a plane wave). We can see that there has been a fundamental change in the field. This fact is reflected by the change in the field invariants $F_{\mu\nu}F^{\mu\nu}(=2(\hat{B}^2 - \hat{E}^2))$ and $F_{\mu\nu}{*F}^{\mu\nu}(=4\,\hat{E}\cdot\hat{B})$ where $\hat{E} = (E^1\ E^2\ E^3)$ and $\hat{B} = (B^1\ B^2\ B^3)$ are the components of the electric and magnetic fields, respectively, in an orthonormal tetrad for an observer with a timelike tangent vector.

Given $\hat{F}_{\mu\nu}$ we have

$$\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = 2(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}{*F}^{\mu\nu}) \qquad 4.5)$$

Since we can set the initial value of $\phi_0^0(v)$ to be real, we can calculate these invariants for the interacting electromagnetic field directly from 4.4).

For the initial electromagnetic field (Region II), from 1.25) with only ϕ_0^0 not zero, we see that both the invariants are initially zero.

However, in Region IV, substituting 4.4) into 4.5), we find

$$\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = 2(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}{*F}^{\mu\nu})$$

$$= -2e^{(U - i\int_0^u V_u' \sinh W du')}(V_u \cosh W + iW_u)\ \phi_0^0(v)\,|_{u=0}\int_0^v \phi_0^0(v')\,|_{u=0} dv'$$

Which, in turn, gives

$$F_{\mu\nu} F^{\mu\nu} = -e^{U}(V_u \mathrm{Cos}(\int_0^u V_{u'} \mathrm{Sinh}W du') \mathrm{Cosh}W + W_u \mathrm{Sin}(\int_0^u V_{u'} \mathrm{Sinh}W du')$$

$$\times (\phi_0^0(v) \int_0^v \phi_0^0(v')dv')$$

$$= -e^{U}(2U_{uu} - U_u^{\ 2})^{\frac{1}{2}} \mathrm{Sin}(\int_0^u V_{u'} \mathrm{Sinh}W du' + \tan^{-1} \frac{(V_u \mathrm{Cosh}W))}{W_u}$$

$$\times (\phi_0^0(v) \int_0^v \phi_0^0(v')dv') \qquad 4.6)$$

and similarly

$$F_{\mu\nu} {}^*F^{\mu\nu} = -e^{U}(W_u \mathrm{Cos}(\int_0^u V_{u'} \mathrm{Sinh}W du') - V_u \mathrm{Sin}(\int_0^u V_{u'} \mathrm{Sinh}W du')\mathrm{Cosh}W)$$

$$\times (\phi_0^0(v) \int_0^v \phi_0^0(v')dv')$$

$$= - e^{U}(2U_{uu} - U_u^{\ 2})^{\frac{1}{2}} \mathrm{Sin}(\int_0^u V_{u'} \mathrm{Sinh}W du' - \tan^{-1} \frac{(V_u \mathrm{Cosh}W))}{W_u}$$

$$\times (\phi_0^0(v) \int_0^v \phi_0^0(v')dv') \qquad 4.7)$$

where we have used a trigonometric identity and 4.1)c). Since $\phi_0^0(v)$ is a function of v only the "$|_u=0$" is understood. From 4.6) and 4.7) we can show that the electromagnetic field must become non-null. For the field to stay null we would require both 4.6) and 4.7) to be zero. However we shall show that this can only happen if the gravitational wave vanishes.

For both 4.6) and 4.7) to be zero, for a non-trivial electromagnetic field, we can require either $(2U_{uu} - U_u^{\ 2})$ to vanish or the two Sine terms to vanish.

If $2U_{uu} - U_u^{\ 2} = 0$, from 4.1)c), the gravitational wave disappears. If the Sine terms are zero throughout the interaction region then we require

$$\int_0^u V_{u'} \,\mathrm{Sinh}W du' + \tan^{-1}\left(\frac{V_u \mathrm{Cosh}W}{W_u}\right) = 2\pi n$$

$$\int_0^u V_{u'} \,\mathrm{Sinh}W du' - \tan^{-1}\left(\frac{V_u \mathrm{Cosh}W}{W_u}\right) = 2\pi m$$

where n and m are integers. From these equations we have

$$\int_0^u V_{u'} \,\mathrm{Sinh}W du' = (m + n)\pi$$

giving $V_u = 0$ (hence V = constant) or W = 0

Also,

$$\frac{V_u \mathrm{Cosh}W}{W_u} = \tan (n - m)\pi = 0$$

giving V = constant. In this case, from 4.1)c) we get for the field equation of the gravitational wave

$$2U_{uu} - U_u^{\,2} = 0$$

which has a solution of the form $U = \ln(au+b)$ which, in turn, gives a flat-space metric (i.e. $R_{\mu\nu\zeta\sigma} = 0$). Therefore again the gravitational wave disappears since $\Psi^0_4 = 0$. From this, we see that if the gravitational wave is non-trivial, then the resultant electromagnetic field in Region IV must be non-null for all observers.

A quantity of significant importance in experimental physics and microwave engineering is the Poynting Vector (19) defined by

$$\hat{P} = \frac{1}{4\pi} \hat{E} \times \hat{B} \qquad 4.8)$$

Although the Poynting Vector is not an invariant, for a given observer it represents the energy flow per unit area of electromagnetic field wave front, and is an experimentally measureable quantity.

Consider a geodesic observer in the plane gravitational wave space-time given above.

If the motion is restricted to the plane x^2 = constant, x^3 = constant, since the only non-zero Christoffel Symbols are Γ^1_{22}, Γ^1_{33}, Γ^1_{23}, Γ^2_{20}, Γ^2_{23}, Γ^3_{30}, Γ^3_{23}, we see that the path of the observer is given by

$$u = As + u_o \qquad v = Bs + v_o \qquad 2AB = 1$$

By using a co-ordinate transformation $u' = ku$, $v' = k^{-1}v$ (k a constant) we can make $A = B = 1/\sqrt{2}$. Therefore, we lose no generality by considering observers whose worldline has the unit tangent vector.

$$e^{\mu}(o) \equiv \frac{dx^{\mu}}{ds} = \frac{1}{\sqrt{2}}(u^{\mu} + v^{\mu}) \qquad \text{a) 4.9)}$$

We can construct the orthonormal tetrad field

$$e^{\mu}(1) = \frac{1}{\sqrt{2}}(u^{\mu} - v^{\mu}) \qquad \text{b)}$$

4.9)

$$e^{\mu}(2) = \frac{1}{\sqrt{2}}(m^{\mu} + \bar{m}^{\mu}) \qquad \text{c)}$$

$$e^{\mu}(3) = \frac{1}{i\sqrt{2}}(m^{\mu} - \bar{m}^{\mu}) \qquad \text{d}$$

with respect to which we can calculate the electric and magnetic field strengths.

Using this tetrad and 4.4) with $\phi^0_0(v)$ set to be real initially, we see that for Region II we have

$$F = \tfrac{1}{4}\begin{bmatrix} 0 & 0 & \phi^0_0(v) & 0 \\ 0 & 0 & -\phi^0_0(v) & 0 \\ -\phi^0_0(v) & \phi^0_0(v) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

that is $E_2 = -\phi^0_0(v) \quad B_3 = -\phi^0_0(v)$

From this, the Poynting Vector is

$$\underset{\wedge}{P} = \frac{1}{64\pi}\,\phi^0_0(v)^2\,\underset{\wedge}{e}(1) \qquad 4.10)$$

where $\underset{\wedge}{e}(1)$ represents the space part of $e^{\mu}(1)$ as a vector in three-space.

For the electromagnetic field in the interacting Region IV we use the same procedure, however since the calculations are much more involved we shall include several steps. Further, it will be convenient to simplify the notation in the following way: let

$$A(u) = \int_0^u V_u{}' \mathrm{Sinh} W du'$$

$$B(u) = V_u \mathrm{Cosh} W \qquad 4.11)$$

With these, we have from 4.4 and 4.9)

$$F_{\mu\nu} + i{*}F_{\mu\nu} = e^{U/2}(\mathrm{Cos}A(u) - i\mathrm{Sin}A(u)\ [\phi_0^0(v)\ (e_{[\mu}(2) - ie_{[\mu}(3))$$

$$(e_{\nu]}(0) - e_{\nu]}(1)) - \tfrac{1}{2}(B(u) + iW_u) \int \phi_0^0(v')dv'\ (e_{[\mu}(0) + e_{[\mu}(1)$$

$$(e_{\nu]}(2) + ie_{\nu]}(3))\ ]$$

Since all the functions in this expression are real, to get $F_{\mu\nu}$ we only need separate out that part of the expression with no "i" term involved. Considering the first term on the right hand side, we have

$$e^{U/2}\ \phi_0^0(v)\ (\mathrm{Cos}A(u) - i\mathrm{Sin}A(u))\ [\,e_{[\mu}(2)\ (e_{\nu]}(0) - e_{\nu]}(1)$$

$$- ie_{[\mu}(3)\ (e_{\nu]}(0) - e_{\nu]}(1)\ ]$$

The real part of this is

$$e^{U/2}\ \phi_0^0(u)\ [\mathrm{Cos}A(u)\ e_{[\mu}(2)\ (e_{\nu]}(0) - e_{\nu]}(1)) - \mathrm{Sin}A(u)\ e_{[\mu}(3)$$

$$(e_{\nu]}(0) - e_{\nu]}(1)\,] \qquad 4.12)$$

Considering the second term on the right hand side

$$-\tfrac{1}{2}e^{U/2}\ (\mathrm{Cos}A(u) - i\mathrm{Sin}A(u))\ (B(u) + iW_u) \int_0^v \phi_0^0(v')dv'$$

$$[(e_{[\mu}(0) + e_{[\mu}(1))\, e_{\nu]}(2) + i(e_{[\mu}(0) + e_{[\mu}(1))\, e_{\nu]}(3)]$$

which gives

$$-\tfrac{1}{2}e^{U/2}\ (B(u)\ \mathrm{Cos}A(u) + W_u \mathrm{Sin}A(u) + i(W_u \mathrm{Cos}A(u) - B(u)\mathrm{Sin}A(u)))\ldots$$

this part of the expression can be written as

$$-\tfrac{1}{2}e^{U/2}\,(B(u)^2 + W_u{}^2)^{\frac{1}{2}}\,(\mathrm{Sin}(A(u) + \tan^{-1}\frac{B(u)}{W_u}) + i\mathrm{Sin}(A(u)$$

$$- \tan^{-1}\frac{B(u)}{W_u}))\ldots$$

With this and using 4.11) and 4.1) we get for the real part of this term

$$-\tfrac{1}{2}e^{U/2}\,(2U_{uu} - U_u^2)^{\frac{1}{2}}\int_0^v \phi_0^0(v')dv'\;\;[\,\mathrm{Sin}(A(u) + \tan^{-1}\frac{B(u)}{W_u})$$

$$(e_{[\mu}(0) + e_{[\mu}(1))\;e_{\nu]}(2) - \mathrm{Sin}(A(u) - \tan^{-1}\frac{B(u)}{W_u})$$

$$e_{[\mu}(0) + e_{[\mu}(1))\;e_{\nu]}(3)] \qquad 4.13)$$

Combining 4.12) and 4.13) and taking the components in the given observers tetrad 4.9), we get

$$F = \begin{bmatrix} 0 & 0 & -E_2 & -E_3 \\ 0 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & 0 \\ E_3 & B_2 & 0 & 0 \end{bmatrix}$$

where

$$E_2 = -\frac{e^{U/2}}{4}\,[\,\phi_0^0(v)\;\mathrm{Cos}A(u) + \tfrac{1}{2}\int_0^v \phi_0^0(v')dv'\;(2U_{uu} - U_u{}^2)^{\frac{1}{2}}$$

$$\mathrm{Sin}(A(u) + \tan^{-1}\frac{B(u)}{W_u})\,]$$

$$E_3 = \frac{e^{U/2}}{4}\,[\phi_0^0(v)\;\mathrm{Sin}A(u) + \tfrac{1}{2}\int_0^v \phi_0^0(v')dv'\;(2U_{uu} - U_u{}^2)^{\frac{1}{2}}$$

$$\mathrm{Sin}(A(u) - \tan^{-1}\frac{B(u)}{W_u})] \qquad 4.14)$$

$$B_2 = \frac{e^{U/2}}{4}\,[-\phi_0^0(v)\;\mathrm{Sin}A(u) + \tfrac{1}{2}\int_0^v \phi_0^0(v')dv'\;(2U_{uu} - U_u{}^2)^{\frac{1}{2}}$$

$$\mathrm{Sin}(A(u) - \tan^{-1}\frac{B(u)}{W_u})]$$

$$B_3 = \frac{e^{U/2}}{4}\,[-\phi_0^0(v)\;\mathrm{Cos}A(u) + \tfrac{1}{2}\int_0^v \phi_0^0(v')dv'\;(2U_{uu} - U_u{}^2)^{\frac{1}{2}}$$

$$\mathrm{Sin}(A(u) + \tan^{-1}\frac{B(u)}{W_u})]$$

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From this and 4.8), the Poynting Vector is seen to be

$$\hat{P} = \frac{1}{64\pi}(E_2 B_3 - B_2 E_3)\,\hat{e}(1)$$

$$= \frac{e^U}{64\pi}\ \phi_0^0(v)^2 - \tfrac{1}{4}\left(\int_0^v \phi_0^0(v')d'\right)^2 (2U_{uu} - U_u{}^2)\left(\mathrm{Sin}^2\!\left(A(u) + \tan^{-1}\frac{B(u)}{W_u}\right)\right.$$

$$\left. + \mathrm{Sin}^2\!\left(A(u) - \tan^{-1}\frac{B(u)}{W_u}\right)\right) \qquad 4.15)$$

where A(u) and B(u) are given by 4.11).

In considering the physical implications of 4.6), 4.7) and 4.15) we must always be aware of the simplification adopted in deducing these equations. We have effectively introduced a distinction between "electromagnetic field" energy and "gravitational field" energy. In reality this is completely artificial. However, for the present we shall persist with this discrimination and consider the results which follow.

From 4.6) and 4.7) we have for all observers

$$\hat{B}^2 - \hat{E}^2 = -\tfrac{1}{2}e^U(2U_{uu} - V_u{}^2)^{\frac{1}{2}}\ \mathrm{Sin}\ \int_0^u V_u{}'\ \mathrm{Sinh}W du' + \tan^{-1}\frac{(V_u \mathrm{Cosh}W))}{W_u}$$

$$\times\ \phi_0^0(v)\int_0^v \phi_0^0(v')dv' \qquad 4.16)$$

$$\hat{E}\bullet\hat{B} = |\hat{E}||\hat{B}|\mathrm{Cos}\theta = -\tfrac{1}{4}e^U(2U_{uu} - U_u{}^2)^{\frac{1}{2}}\mathrm{Sin}\left(\int_0^u V_u{}'\ \mathrm{Sinh}W du' - \tan^{-1}\right.$$

$$\left.\frac{(V_u \mathrm{Cosh}W))}{W_u} \times\ \phi_0^0(v)\int_0^v \phi_0^0(v')dv'\right) \qquad 4.17)$$

where θ is the observed angle between $\hat{E}$ and $\hat{B}$. For a non-trivial gravitational wave, we can see that there must be some modulation of the electromagnetic field quantities. Further, since the quantities above are invariants, all observers will detect this modulation.

Turning to the Poynting Vector, for the observer introduced above, we have 4.15). From this, assuming $\phi_0^0(v)$ is a simple sinusoid, we can see that the magnitude of the Poynting Vector almost inevitably eventually increases.

This is primarily due to the e^U term since $U_{uu}>0$ by 4.1)c). As a result of this, there will be detected, by our observer, an ultimately increasing energy flow in the electromagnetic field. This may be the result of either or both of two possible mechanisms. Either the gravitational field is supplying energy in some way to the electromagnetic field or the gravitational field is focusing the electromagnetic field. The former has been the subject to some studies along the view of graviton-photon conversion (20), (21), (22). The latter is effectively the phenomenon discussed by Penrose in (17). We shall see an example of this below.

We may also consider the direction of propagation of the average Poynting Vector as this indicates the direction of electromagnetic energy flow. Before the interaction, this Vector, for a simple sinusoid such as

$$\phi_0^0(v) = \phi \mathrm{Sin} v$$

has the form

$$\underset{\wedge}{P}_{ave} = \frac{1}{128\pi} \phi^2 \, \underset{\wedge}{e}(1) \qquad 4.18)$$

and we can see the energy in the electromagnetic field propagates in the same direction as the field.

However, in Region IV, from 4.15)

$$\underset{\wedge}{P}_{ave} = \frac{e^U}{128\pi} \phi^2 \left[1 - \tfrac{1}{4}(2U_{uu} - U_u{}^2)^{\frac{1}{2}} \left(\mathrm{Sin}^2(A(u) + \tan^{-1}\frac{B(u)}{W_u}) + \mathrm{Sin}^2(A(u) - \tan^{-1}\frac{B(u)}{W_u})\right)\right] \underset{\wedge}{e}(1) \qquad 4.$$

The magnitude of $\underset{\wedge}{P}_{ave}$ will behave similarly to the magnitude of the Poynting Vector. However, the direction of $\underset{\wedge}{P}_{ave}$ can change, depending upon the gravitational field quantities.

Since it is $\underset{\wedge}{P}_{ave}$ which is experimentally measured, we can determine from 4.19) what our observer may experience as they progress up their world-line through the interaction Region IV.

The average Poynting Vector may change direction several times, alternating between that of the electromagnetic field and gravitational field. An example of what such an observer may detect is given in Figure 3. Physically, the observer will experience changes in the average intensity of the electromagnetic field. At points such as A on their worldline, there will be a maximum in the electromagnetic field strength. At points such as B, there will be a minimum. This type of behaviour of the interacting field would point more toward the gravitational field redistributing the energy of the electromagnetic field to give maxima and minima.

However, we can see from 4.19) the possibility of there being a persistent increase in the electromagnetic field strength. By arguments of energy conservation alone, the observer would have to conclude that there is some external source which is providing the extra energy. The only source available is the gravitational wave, hence there is the possibility of an energy transfer between the fields.

Unfortunately, due to the difficulty and ambiguity of defining gravitational field energy, one cannot make precise quantitative statements about this apparent phenomenon. A further result of the interaction under consideration is the apparent "dragging" of the electromagnetic field in the direction of propagation of the gravitational wave. This is revealed by the possible change in direction of the average Poynting Vector. Since this Vector indicates the direction of propagation of energy of the electromagnetic field, its change in direction due to the gravitational field quantities indicates the electromagnetic field has reversed its direction of propagation to correspond with that of the gravitational field.

We could use for the electromagnetic field a sandwich field such that

$\phi^0_0(v) = 0$ for $v>\hat{v}$ but $\int_0^{\hat{v}} \phi^0_0(v)\,dv \neq 0$ where $v = \hat{v}>0$ corresponds

to the "far edge" of the field. In this case, from 4.4) we have for $v>\hat{v}$

$$\hat{F}_{\mu\nu} = -e^{-\frac{1}{2}(U-\int_0^u V_{u'}\,\mathrm{Sinh}W du')}(V_u \mathrm{Cosh}W + iW_u)\int_0^v \phi_0^0(v')dv' u_{[\mu}m_{\nu]} \qquad 4.20)$$

Although the initial electromagnetic field $\phi_0^0(v) = 0$ for $v>\hat{v}$ there is still some kind of "echo field" present which is propagating in the direction of the gravitational wave. Further, since the v dependent term in 4.20) is constant for $v>\hat{v}$, any variation of this field is due to the gravitational wave only.

From 4.6) and 4.7) we see that this "echo field" is null. We can easily deduce the field components, for our observer, from 4.20) as we did previously. Further, the average Poynting Vector for a sinusoidal $\phi_0^0(u)$ is

$$\underset{\wedge}{P}_{ave} = \frac{-e^U}{512\pi}\phi^2(2U_{uu} - U_u)^2\left(\mathrm{Sin}^2\left(A(u) + \tan^{-1}\frac{B(u)}{W_u}\right)\right.$$

$$\left. + \mathrm{Sin}^2\left(A(u) - \tan^{-1}\frac{B(u)}{W_u}\right)\right)\underset{\wedge}{e}(1) \qquad 4.21)$$

We can see that $\underset{\wedge}{P}_{ave}$ cannot change direction.

The Sandwich Wave Colliding with the Test Field

The equations describing this interaction are a particular case of 4.1) above, in which we have set W = 0 throughout the space-time and, although U and V must obey only 4.1)c) in the non-flat region, when we consider the second flat region (u>u' in Figure 2), U and V have the explicit forms listed in Appendix II.

With W = 0 4.4) becomes

$$\hat{F}_{\mu\nu} = 2e^{U/2}[\phi_0^0(v)\,\bar{m}_{[\mu}v_{\nu]} - \tfrac{1}{2}V_u\int_0^v \phi_0^0(v')\,dv'\,u_{[\mu}m_{\nu]}] \qquad 4.22)$$

From the list of explicit forms for U and V given in Appendix II, we can immediately see that there is the possibility that the electromagnetic wave will become null again after the interaction with the sandwich wave. This is the case V = constant for u>u'. However, for the other cases, the wave will become non-null.

For convenience, we shall consider the case when

$$U(u) = -\ln(au+b) \qquad V(u) = \ln(au+b) \qquad \text{for } u>u'$$

where $a \neq 0$ and $b \neq 0$ with $a<0$ and $b>0$.

Using 4.22) with these gives

$$\hat{F}_{\mu\nu} = 2 \frac{1}{(au+b)^{\frac{1}{2}}} \left[\phi_0^0(v)\big|_{u=0} \bar{m}_{[\mu}v_{\nu]} - \tfrac{1}{2} \frac{a}{(au+b)} \int_0^v \phi_0^0(v')\, dv'\, u_{[\mu}m_{\nu]} \right] \qquad 4.23)$$

The focusing effect of the sandwich gravitational wave discussed above is immediately apparent, through the presence of the $(au+b)^{-1}$ terms.

Turning to the invariants, from 4.16)

$$\hat{B}^2 - \hat{E}^2 = \tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} = -\tfrac{1}{2} \frac{a}{(au+b)} \phi_0^0(v)\big|_{u=0} \int_0^v \phi_0^0(v')\big|_{u=0}\, dv'$$

and 4.17)

$$\hat{E} \cdot \hat{B} = \tfrac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu} = \tfrac{1}{4} \frac{a}{(au+b)} \phi_0^0(u)\big|_{u=0} \int_0^v \phi_0^0(v')\big|_{u=0}\, dv'$$

From these equations, we see that although the region on the far side of the gravitational wave is flat, the electromagnetic wave remains non-null (i.e. non-plane).

From 4.11) and 4.15) we see that the Poynting Vector, in the flat region $u>u'$, becomes,

$$\hat{P} = \frac{1}{64\pi(au+b)} \left[\phi_0^{02}(v) - \tfrac{1}{2} \frac{a^2}{(au+b)^2} \left(\int_0^v \phi_0^0(v')\, dv' \right)^2 \right] \hat{e}(1)$$

Therefore, again setting $\phi_0^0(v) = \phi \sin v$, we get

$$\hat{P}_{ave} = \frac{1}{128\pi(au+b)} \phi^2 \left[1 - \tfrac{1}{2} \frac{a^2}{(au+b)^2} \right] \hat{e}(1)$$

We can now evaluate what our observer sees. Since $(au+b)$ goes to zero for some value of $u>u'$, although the average Poynting Vector may initially have a direction corresponding to the direction of the electromagnetic field, as the observer moves up their worldline they will see the average Poynting Vector change direction so that it would ultimately point in the observed direction of propagation of the gravitational wave.

Further, the magnitude of the average Poynting Vector would diverge at u such that au+b = 0. At the value of u for which $P_{\hat{a}ve}$ is zero, the observer would see a local inflexion in the energy of the electromagnetic field. Then, as $P_{\hat{a}ve}$ diverges, the observer would experience the focusing of the electromagnetic field referred to above. Notice that an effect of the interaction is to reverse the observed direction of propagation of electromagnetic field.

If we had used a sandwich electromagnetic field as well, we would have, from 4.21)

$$P_{\hat{a}ve} = \frac{-1}{256\pi} \quad \phi^2 \quad \frac{a^2}{(au+b)^3} \quad \hat{e}(1)$$

Although $P_{\hat{a}ve}$ obviously diverges for these cases, there is little doubt that it is due to the focusing effect of the sandwich gravitational wave and not due to any form of energy transfer between the two waves. Since sandwich plane waves have infinite total energy content, these divergences in $P_{\hat{a}ve}$ offer no problem in interpreting their physical meaning.

The Electromagnetic Test Field Case with Weak Wave Metric

In experimental physics, the weak field limit of General Relativity is usually used to predict results of experiments and tests of the Theory. Similarly, the weak gravitational wave field is commonly used when discussing gravitational wave detectors. Examples of this are (13), (20), (21), (23).

The above results are easily adopted to the weak wave metric. This being so, below we effectively quote the results and give a brief analysis. The appropriate equations are, from 3.39) and 3.40) using a test field,

$$\phi^0{}_{0,u} = \tfrac{1}{2}U_u \, \phi_0 \qquad \text{a)}$$

$$\phi^0{}_{2,v} = -\tfrac{1}{2}(V_u + iW_u) \, \phi_0 \qquad \text{b) } 4.24)$$

$$U_{uu} = 0 \qquad \text{c)}$$

V and W can be arbitary but must satisfy the limits on their magnitude given by 3.35) and 3.36). Since the solution for U is of the form au+b, which will ultimately fail to satisfy 3.35) we shall set a = 0 and $|b| \ll 1$.

Then this set of equations gives

$$\hat{F}_{\mu\nu} = 2(\phi_0^0(v)\ \bar{m}_{[\mu}v_{\nu]} - \tfrac{1}{2}(V_u + iW_u)\int_0^v \phi_0^0(v')dv'\ u_{[\mu}m_{\nu]}) \qquad 4.25)$$

which in turn gives

$$F_{\mu\nu}F^{\mu\nu} = 2(\hat{B}^2 - \hat{E}^2) = -\phi_0^0(v)\int_0^v \phi_0^0(v')dv'\ V_u \qquad 4.26)$$

$$F_{\mu\nu}{}^*F^{\mu\nu} = 4(\hat{E}\cdot\hat{B}) = -\phi_0^0(v)\int_0^v \phi_0^0(v')du'\ W_u \qquad 4.27)$$

Since, for the weak field, we can write

$$\Psi_4^0 = -\tfrac{1}{2}(V_{uu} + iW_{uu})$$

we see that the electromagnetic field must become non-null.

Adopting the same observer as before to calculate an observed Poynting Vector, we get

$$E_2 = -\tfrac{1}{4}\phi_0^0(v) - \tfrac{1}{8}V_u\int_0^v \phi_0^0(v')dv'$$

$$E_3 = \tfrac{1}{8}W_u\int_0^v \phi_0^0(v')dv' \qquad 4.28)$$

$$B_2 = \tfrac{1}{8}W_u\int_0^v \phi_0^0(v')dv'$$

$$B_3 = -\tfrac{1}{4}\phi_0^0(v) + \tfrac{1}{8}V_u\int_0^v \phi_0^0(v')dv'$$

From this

$$\hat{P} = \frac{1}{64\pi}\left[\phi_0^0(v)^2 - \tfrac{1}{4}(V_u^{\ 2} + W_u^{\ 2})\left(\int_0^v \phi_0^0(v')dv'\right)^2\right]\hat{e}(1) \qquad 4.29)$$

and for a sinusoidal $\phi_0^0(v)$

$$\hat{P}_{ave} = \frac{1}{128\pi}\ \phi^2\ [1 - \tfrac{1}{4}(V_u^{\ 2} + W_u^{\ 2})]\ \hat{e}(1) \qquad 4.30)$$

From these results we see the phenomena occurring for the weak gravitational wave metric are very similar to those occurring with the exact metric. However there are several differences, the most obvious being that the field quantities and invariants cannot diverge. Where as before the gravitational field quantities occurred in rather complicated functions, in the weak field case their appearance is somewhat simple and direct.

Further, due to the conditions on the magnitudes of the gravitational field quantities, in several cases their effect would be difficult to measure. For example in 4.30), the $\frac{1}{4}(V_u^2 + W_u^2)$ term would most likely be so much smaller than unity as to be negligible.

Despite this, we can see that the phenomena of "dragging" the electromagnetic field in the direction of the gravitational field and the echo field for a sandwich electromagnetic field are still present although their effects may be small.

Finally, we can see from 4.26), 4.27) and 4.28) that although the Poynting Vector may not be discernably affected by the interaction, the field components and invariants most likely are detectably affected. These results could possibly imply experimental techniques to detect and analyse gravitational waves. We shall discuss this below.

Approximate Solution for the Exact Equations

The set of equations 3.32), 3.33) although well-posed are not exactly solvable by modern techniques. However we can extract some information about the solution of the system by adopting a power series approach which will give approximate solutions in the regions around u = 0 and v = 0.

This technique was used by Szekeres, (12) (24), to deduce some properties of the similar collision of plane gravitational waves.

In using this method, we must remain aware of the continuity conditions we are imposing on the metric. As shown in a previous section, the Lichnerowicz conditions are too strong for the interaction we are considering and hence we must adopt the O'Brien-Synge conditions. Although this is so for the interacting waves, it is not necessary for the non-interacting Regions II and III. In these regions we can, and shall, adopt the Lichnerowicz conditions. Even so doing, we find the O'Brien-Synge conditions forced upon us in the interaction Region IV.

We shall follow the method adopted in (12) and further since the interaction we are primarily interested in (which has $\phi_2(u) = 0$ in Region II) is just the general interaction with a particular initial condition, we shall consider the general case.

From 3.33)a)

$$U = -\ln(f(u) + g(v))$$

which will give U(u,v) throughout the space-time. Further U, V, W, M, ϕ_0 and ϕ_2 must satisfy the system 3.33) throughout the space-time. With the metric given by A2.1), we can write the following forms for the various functions in the various Regions.

Region I (Minkovski space-time)

$$f(u) = g(v) = \tfrac{1}{2} \qquad M = V = W = \phi_0 = \phi_2 = 0$$

Region II (the u-wave)

$$g = \tfrac{1}{2} \quad f = f(u) \quad M = M(u) \quad V = V(u) \quad W = W(u)$$

$$\phi_0 = 0 \quad \phi_2 = \phi_2(u)$$

$\Psi^0_4 = \Psi^0_4(u)$ all other Ψ^0's zero

Region III (the v-wave)

$$g = g(v) \quad f = \tfrac{1}{2} \quad M = M(v) \quad V = V(v) \quad W = W(v)$$

$$\phi_0 = \phi_0(v) \quad \phi_2 = 0$$

$\Psi^0_0 = \Psi^0_0(v)$ all other Ψ^0's zero

For the non-interacting waves we adopt the Lichnerowicz conditions. Hence the metric and its first derivatives are continuous across the boundaries between Regions I and II and Regions I and III. Consider the I-II boundary ($u = 0$ $v<0$). We have

$$f = \tfrac{1}{2} \quad M = V = W = f_u = M_u = V_u = W_u = \phi_0 = 0$$

$$\phi_2(u) = \phi_2(0)$$

Note that $\phi_2(u)$ has a step discontinuity across this boundary. From 3.33)c) we have on the boundary

$$\frac{1}{f^2} f_u^{\,2} - \frac{1}{f} f_{uu} = 2k\phi_2\bar{\phi}_2$$

But with the boundary values of f and f_u we get

$$f_{uu} = -\,k\phi_2\bar{\phi}_2 \qquad 5.1)$$

Differentiating 3.33)c) gives

$$f_{uuu} = -k(\phi_2\bar{\phi}_2)_u \qquad 5.2)$$

and again gives

$$U_{uuuu} - U_{uu}^{\,2} + 2M_{uu}\,U_{uu} = (W_{uu}^{\,2} + V_{uu}^{\,2}) + 2k(\phi_2\bar{\phi}_2)_{uu}$$

which in turn gives

$$f_{uuuu} = -[\tfrac{1}{2}(V_{uu}^{\,2} + W_{uu}^{\,2}) + 2kM_{uu}\,\phi_2\bar{\phi}_2 + 4k^2(\phi_2\bar{\phi}_2)^2 + k(\phi_2\bar{\phi}_2)_{uu}] \qquad 5.3)$$

We see from 5.1) to 5.3) that the Lichnerowicz conditions are satisfied and provide expressions relating the various derivatives. Note that Ψ_4^0 may have a discontinuity on this boundary.

Assuming a power series at u = 0 (as in (12)) we find that for u>0

$$\phi_2 = a_0 + a_1 u + a_2 u^2 + \ldots \quad \text{a)}$$

$$V = b_1 u^2 + \ldots \quad \text{b)}$$

$$W = c_1 u^2 + \ldots \quad \text{c)} \quad 5.4)$$

$$M = d_1 u^2 + \ldots \quad \text{d)}$$

$$f = \tfrac{1}{2} + e_1 u^2 + e_2 u^3 + e_3 u^4 + \ldots \quad \text{e)}$$

where

$$e_1 = \frac{-k}{2} a_0 \bar{a}_0 \qquad e_2 = \frac{-k}{6}(a_0 \bar{a}_1 + a_1 \bar{a}_0)$$

$$e_3 = -\frac{1}{12}[b_1{}^2 + c_1{}^2 + 2k\, d a_0 \bar{a}_0 + 2k^2 (a_0 \bar{a}_0)^2 + k(a_1 \bar{a}_1 + a_2 \bar{a}_0 + \bar{a}_2 a_0)]$$

and

$$\Psi_4^0 = -b_1 - ic_1 + \ldots \quad 5.5)$$

Similarly, for the I-III boundary v = 0 u<0 we have

$$\phi_0 = h_0 + h_1 v + h_2 v^2 + \ldots \quad \text{a)}$$

$$V = j_1 v^2 + \ldots \quad \text{b)}$$

$$W = l_1 v^2 + \ldots \quad \text{c)} \quad 5.6)$$

$$M = m_1 v^2 + \ldots \quad \text{d)}$$

$$g = \tfrac{1}{2} + n_1 v^2 + n_2 v^3 + n_3 v^4 + \ldots \quad \text{e)}$$

where

$$n_1 = \frac{-kh_0\bar{h}_0}{2} \qquad n_2 = -\frac{k}{6}(h_0\bar{h}_1 + h_1\bar{h}_0)$$

$$n = -\frac{1}{12}[j_1{}^2 + l_1{}^2 + 2km_1h_0\bar{h}_0 + 2k^2(h_0\bar{h}_0)^2$$

$$+ k(h_1\bar{h}_1 + h_2\bar{h}_0 + \bar{h}_1h_0)]$$

and

$$\Psi_0^0 = -j_1 + il_1 + \ldots \qquad 5.7)$$

To now study the forms of the functions in the interaction Region IV, we must relax the continuity conditions to the O'Brien-Synge conditions (i.e. the metric is continuous but its first derivatives can have step discontinuities). Therefore, we have the following conditions. U, V, M and W are continuous everywhere. Further, since we are disallowing shock waves of the form 2.9), ϕ_2 is continuous across $v = 0$ and ϕ_0 across $u = 0$. Therefore given these functions in Regions II and III, by the continuity of U, f(u) and g(v) will have the same form in Region IV as they have in Regions II and III respectively. We know V, W and M along $u = 0$ and $v = 0$ as well as $\phi_2(u)$ along $v = 0$ and $\phi_0(v)$ along $u = 0$.

With these conditions, we can use 3.33) to deduce the leading terms of these functions within a small region close to the origin with $u>0$ and $v>0$. In deducing these terms, we must be mindful of the fact that since the functions U, V, W etcetera appear in several equations the higher power terms in the power series will begin to "feed back". That is, the expression for, say, V will depend on terms in the expression for ϕ_0, however terms in ϕ_0 will also depend on terms in V. This "feed back" occurs only for terms with powers of v and u above a certain value. Although, theoretically, one could deduce the full power series expansion, for convenience we shall not go above the "non-feed back" limit.

Reviewing the system 3.32), .33) we recall that we may attempt to solve 3.32) and 3.33)d) and e) for V, W, ϕ_0^0, ϕ_2^0 and then find M from 3.33)f). U will be given directly by 3.33)a).

We shall deduce the power series expansion for W(u,v) in some detail and since the expansions for the other functions are found similarly we shall quote those results.

Since the expansions are made near u = v = 0, we can utilize the Taylor expansions for the logarithmic and hyperbolic functions.

These give

$$U_u \simeq - f_u \qquad U_v \simeq - g_v$$

$$\text{Cosh}W \simeq 1 + \frac{W^2}{2!} \qquad \text{Sinh}W \simeq W + \frac{W^3}{3!}$$

$$\tanh W \simeq W - \frac{W^3}{3}$$

Viewing 3.33)d) we see, using the expansions above, that the lowest order terms are of order two, for $U_u W_v$ and $U_v W_u$, order four for $2V_u V_v$ CoshW SinhW and zero for $2ik(\phi_2\bar{\phi}_0 - \bar{\phi}_2\phi_0)$. Therefore the lowest order in W will be two and when inserted back into 3.33)d) will contribute an order of one. The only contribution of this type will be from the $2ik(\phi_2\bar{\phi}_0 - \bar{\phi}_2\phi_0)$ term. Therefore, we can take the series for W to the fourth order provided we make allowance for this "feed back".

We can do this by writing

$$W_v = 2l_1 v + 3l_2 v^2 + \ldots \qquad -2ik(a_0\bar{h}_0 - \bar{a}_0 h_0)u + \ldots$$

$$W_u = 2c_1 u + 3c_2 u^2 + \ldots \qquad -2ik(a_0\bar{h}_0 - \bar{a}_0 h_0)v + \ldots$$

We should write similar equations for V_u and V_v to be inserted in 3.33)d), however, the term involving these quantities only contributes to W above order six, and therefore will not enter our calculations.

Inserting the appropriate power series to 3.33)d) gives the leading terms for W in Region IV.

$$W = c_1u^2 + \ldots \quad + l_1v^2 + \ldots \quad -ik\{(a_0\bar{h}_0 - \bar{a}_0h_0)uv$$

$$+\tfrac{1}{2}(a_1\bar{h}_0 - \bar{a}_1h_0)u^2v + \tfrac{1}{2}(a_0\bar{h}_1 - \bar{a}_0h_1)uv^2 + \tfrac{1}{3}[(a_2\bar{h}_0 - \bar{a}_2h_0)$$

$$-2(a_0\bar{h}_0 - \bar{a}_0h_0)e_1]\ u^3v + \tfrac{1}{3}[(a_0\bar{h}_2 - \bar{a}_0h_2) - 2(a_0\bar{h}_0 - \bar{a}_0h_0)n_1]uv^3\}$$

$$-\tfrac{1}{2}(l_1e_1 + c_1n_1 + \frac{ik}{2}(a_1\bar{h}_1 - \bar{a}_1h_1))\ u^2v^2 + \ldots \qquad 5.8)$$

Repeating the same procedure for V gives, in Region IV,

$$V = b_1u^2 + \ldots + j_1v^2 + \ldots + k\{(a_0\bar{h}_0 + \bar{a}_0h_0)uv + \tfrac{1}{2}(a_1\bar{h}_0 + \bar{a}_1h_0)u^2v$$

$$\tfrac{1}{2}(a_0\bar{h}_1 + \bar{a}_0h_1)uv^2 + \tfrac{1}{3}[(a_2\bar{h}_0 + \bar{a}_2h_0) - 2(a_0\bar{h}_0 + \bar{a}_0h_0)e_1]u^3v$$

$$+ \tfrac{1}{3}[(a_0\bar{h}_2 + \bar{a}_0h_2) - 2(a_0\bar{h}_0 + \bar{a}_0h_0)n_1]\ uv^3 + \}$$

$$-\tfrac{1}{2}(b_1n_1 + j_1e_1 - \tfrac{1}{2}k(a_1\bar{h}_1 + \bar{a}_1h_1))u^2\ v^2 + \ldots \qquad 5.9)$$

Turning now to ϕ_0^0 and ϕ_2^0 we see by arguments similar to the above that there is no "feeding back" for powers of two or less in the expansions for these quantities. Using 5.8) and 5.9) with 5.6) and 5.4) in 3.32)b) we result with the following for ϕ_0^0 in Region IV.

$$\phi_0^0 = h_0 + h_1v + h_2v^2 + \ldots - h_0(ka_0\bar{a}_0 + \tfrac{1}{2}e_1)u^2$$

$$- a_0(j_1 - il_1)uv + \ldots \qquad 5.10)$$

and for ϕ_2^0

$$\phi_2^0 = a_0 + a_1u + a_2u^2 + \ldots -a_0(kh_0\bar{h}_0 + \tfrac{1}{2}n_1)v^2$$

$$- h_0(b_1 + ic_1)uv + \ldots \qquad 5.11)$$

Finally, from 3.33)f)

$$M = d_1u^2 + (d_2 + \frac{1}{3}k(b_1B_{00} - ic_1A_{00}))u^3$$

$$+ m_1v^2 + (m_2 + \frac{1}{3}k(j_1B_{00} - il_1A_{00}))v^3 + \ldots$$

$$+ \tfrac{1}{2}(c_1l_1 + b_1j_1 - e_1n_1 + \tfrac{1}{4}k(B_{00}^2 - A_{00}^2))u^2v^2 \qquad 5.12)$$

where $A_{00} = a_0\bar{h}_0 - \bar{a}_0h_0$ and $B_{00}\ a_0\bar{h}_0 + \bar{a}_0h_0$

Before calculating the Weyl spinor components, we shall consider the continuity properties of the metric and its derivatives. In doing this we must be aware of the shock nature of the electromagnetic waves. Rigourously speaking, in the expansions for ϕ_0^0 and ϕ_2^0 we should have in place of each h_i, $h_i\theta(v)$ and in place of each a_i, $a_i\theta(u)$ respectively. To evaluate the derivatives of the metric, we must make these insertions and calculate the derivaties accordingly. Therefore, from 5.8) we can write

$$W_u = 2c_1u + \ldots - ik\theta(u)\theta(v)\{(a_0\bar{h}_0 - \bar{a}_0h_0)v + (a_1\bar{h}_0 - \bar{a}_1h_0)uv$$

$$+ \tfrac{1}{2}(a_0\bar{h}_1 - \bar{a}_0h_1)v^2 + \ldots\} + \ldots$$

$$W_v = 2l_1v + \ldots - ik\theta(u)\theta(v)\{(a_0\bar{h}_0 - \bar{a}_0h_0)u + \tfrac{1}{2}(a_1\bar{h}_0 - \bar{a}_1h_0)u^2$$

$$+ (a_0\bar{h}_1 - \bar{a}_0h_1)uv + \ldots\} + \ldots$$

and similarly for V_u and V_v. In deducing these derivatives we have used

$$\delta(x)x = 0$$

From these expressions, we see that both W_u and V_u have step discontinuities at $u = 0$ similarly W_v and V_v have step discontinuities at $v = 0$. We have

$$W_u\big|_{u=0} = -ik(a_0\bar{h}_0 - \bar{a}_0h_0)\ \theta(u)\ \theta(v)v$$

$$W_v\big|_{v=0} = -ik(a_0\bar{h}_0 - \bar{a}_0h_0)\ \theta(v)\ \theta(u)u$$

$$V_u\big|_{u=0} = k(a_0\bar{h}_0 + \bar{a}_0h_0)\ \theta(u)\ \theta(v)v$$

$$V_v\big|_{v=0} = k(a_0\bar{h}_0 + \bar{a}_0h_0)\ \theta(v)\ \theta(u)u$$

As a result of these equations, the Lichnerowicz conditions are inapplicable to this interaction, however the O'Brien-Synge conditions still apply.

We should also include $\theta(u)$ and $\theta(v)$ in the power series for V, W, U and M. However, for these quantities, the first derivatives always give rise to terms of the form $\delta(x)x^n$ and $\theta(x)x^n$, $n \geq 1$.

Therefore step discontinuities appear only on the second derivative and delta discontinuities never appear in the curvature quantities. This is why the Lichnerowicz conditions are suitable for the collision of two plane pure gravitational waves (12). On the other hand for the interaction we are considering, we can see that step discontinuities in the first derivative are unavoidable if a_0 and h_0 are non-zero.

Calculating the leading terms of the Weyl spinor gives

$$\Psi_0^0 = -j_1 + \ldots - \tfrac{1}{2}k\{ B_{00}\ \theta(u)\ \delta(v)u + \tfrac{1}{2} B_{10}\ \theta(u)\ \delta(v)u^2$$

$$+ B_{01}\ u + \ldots \} - 2j_1(n_1 + m_1)v^2 + \ldots$$

$$-2k\ \bar{a}_0 h_0(n_1 + m_1)uv + \ldots$$

$$+ il_1 + \ldots + \tfrac{1}{2}k\{ A_{00}\ \theta(u)\ \delta(v)u + \tfrac{1}{2}A_{10}\ \theta(u)\ \delta(v)u^2$$

$$+ A_{01}\ u + \ldots \} + 2il_1(n_1 + m_1)v^2 + \ldots \qquad 5.13)$$

$$\Psi_4^0 = -b_1 - \ldots -\tfrac{1}{2}k\{B_{00}\ \theta(v)\ \delta(u)v + \tfrac{1}{2}B_{01}\ \theta(v)\ \delta(u)v^2 + B_{10}v + \quad \}$$

$$- 2b_1(e_1 + d_1)u^2$$

$$- 2ka_0\bar{h}_0(e_1 + d_1)uv + \ldots$$

$$- ic_1 - \ldots -\tfrac{1}{2}k\{A_{00}\ \theta(v)\ \delta(u)v + \tfrac{1}{2}A_{01}\ \theta(v)\ \delta(u)v^2 + A_{10}v + \ldots\}$$

$$+ 2ic_1(e_1 + d_1)u^2 + \ldots \qquad 5.14)$$

$$\Psi_2^0 = \{c_1 l_1 + b_1 j_1 - e_1 n_1 + \tfrac{1}{4}k(B_{00}^2 - A_{00}^2) + 2i(j_1 c_1 - b_1 l_1)\}\ uv$$

$$- k(b_1 A_{00} - ic_1 B_{00})u^2 + k(j_1 A_{00} - il_1 B_{00})v^2 + \ldots \qquad 5.15)$$

where $A_{01} = a_0\bar{h}_1 - \bar{a}_0 h_1 \qquad A_{10} = a_1\bar{h}_0 - \bar{a}_1 h_0$

$$B_{01} = a_0\bar{h}_1 + \bar{a}_0 h_1 \qquad B_{10} = a_1\bar{h}_0 + \bar{a}_1 h_0$$

and the unit step and delta functions appear explicitly only to indicate delta discontinuities in the quantities.

For all the other terms, the appropriate step functions are understood. Finally,

$$\Psi_1^0 = \Psi_3^0 = 0$$

To consider the initial effects of the interaction, we consider the equations 5.8) through 5.15). These confirm that the O'Brien-Synge conditions are the appropriate continuity conditions for the metric. Further, we can see that the electromagnetic fields are the reason for the Lichnerowicz conditions failing. This is in line with the considerations above regarding shock fields and continuity conditions. It is interesting to note that although the Lichnerowicz conditions fail when two electromagnetic shock waves collide, they do not fail for an electromagnetic shock-gravitational wave collision. We see from 5.13) and 5.14) that the co-efficients of the delta terms depend on both ϕ_0^0 and ϕ_2^0 being initially zero.

From 5.10) and 5.11), the two electromagnetic components, ϕ_0^0 and ϕ_2^0, will inevitably arise even if one is initially zero. However, if one is initially zero, the consequential effect on the other quantities (U, V etc.) is somewhat less than if they were both initially non-zero. This would imply that another suitable approximation to our initial problem of a gravitational-electromagnetic wave collision would be to solve Maxwell's equations in the solution metric of the collision of two pure gravitational waves.

The two pure gravitational wave collision for linear polarization (W = 0) has been solved by Szekeres (12) and slightly generalized to two plane waves of matched polarization by Panov (16). Thus, we only need insert the resulting forms for V, U etcetera into 4.1) and follow the analysis through as done above. Unfortunately, the explicit solution for V is not easily integrated to give a form which can be easily manipulated.

Finally, 5.10) and 5.11) do not give rise to any terms of the form $k_{\mu\nu}\ \theta(u)\ \theta(v)$ hence our exclusion of such terms can be considered justified since it is consistent with our results.

Conclusions

Although we have been able to give explicit expressions for the effect of a pure gravitational wave when it collides with an electromagnetic field, we have had to make some apparently drastic simplifications. That is, by treating the electromagnetic field as a test field, we have entirely ignored its possible contribution to the space-time curvature. Regarding the power series approach, although it does provide us with one useful result (see below), it certainly does not give an extensive solution throughout the interaction region.

Therefore, we must now consider whether the results deduced are valid and, if so, what restrictions exist on their applicability.

We could adopt the viewpoint that since we have effectively neglected the energy-stress of the electromagnetic field, the results we have deduced would be approximately correct for an interaction in which the energy density of the electromagnetic field is in some way negligible compared to that of the gravitational field.

Unfortunately, the definition of energy-density of a gravitational field is, in the least, somewhat ambiguous. This matter was discussed above.

However, for our purposes, there is a reasonable definition of energy density which we can adopt for the case at hand. The energy-momentum pseudo-tensor $t^{\mu\nu}$ introduced by Landau and Lifshitz (25) is directly anologuous to the energy-momentum tensor of the electromagnetic field (26). Unfortunately, the energy-momentum pseudo-tensor can be transformed away since it depends on Christoffel Symbol quantities and not their derivatives. However, again referring to the observer introduced above, we can make meaningful comparisons between the electromagnetic stess-tensor and the gravitational stress pseudo-tensor.

Therefore, we may write, as the condition which determines the "domain of validity" of the results above, the inequality

$$T_{EM}^{\mu\nu} << t_{GW}^{\mu\nu} \qquad 6.1)$$

where $T_{EM}^{\mu\nu}$ are the components of the energy-momentum stress-tensor of the electromagnetic field and $t_{GW}^{\mu\nu}$ the energy momentum stress pseudo-tensor of the gravitational field and both are evaluated in the given observer's frame. Since $t_{GW}^{\mu\nu}$ can be explicitly deduced for a weak gravitational wave (25) and also since, from 4.14) where the exact metric is used, there is the possibility of $T_{EM}^{\mu\nu}$ not satisfying 6.1) throughout the interaction region, we shall restrict our considerations to the weak field case.

From (25) and using 3.38), since the gravitational wave depends on u only, we have

$$t_{GW}^{01} = \frac{c^2}{16\pi G}\left(\dot{h}_{23}^2 + \tfrac{1}{4}(\dot{h}_{22} + \dot{h}_{33})^2\right)$$

$$= \frac{c^4}{16\pi G}\left(W_u^{\,2} + V_u^{\,2}\right) \qquad 6.2)$$

To get an understanding of the orders of magnitude we are dealing with, we can proceed as follows. By 6.1) we require

$$t_{GW} \simeq \frac{c^4}{16\pi G}\, O(h^2_{,u}) >> T_{EM} \simeq \frac{1}{4\pi}\, O(F^2)$$

where F represents the field components of the electromagnetic field and O(A) indicates "the order of A". Therefore

$$O(F) << \frac{c^2}{4G^{\frac{1}{2}}}\, O(h_{,u})$$

or, using c.g.s. units

$$O(F) << 10^{24}\, O(h_{,u}) \qquad 6.3)$$

We shall set the flux of the gravitational wave to be of the order of 10^{10} ergs/cm^2/sec, which roughly corresponds to flux, experienced at the Earth, of a gravitational wave generated by the conversion of about

$10^{-4}M_{\odot}$ to gravitational radiation in 10^{-5} seconds at the centre of the galaxy (8). From (25) this flux is given by ct^{01}_{GW}. Therefore, from 6.2)

$$10^{10} \simeq \frac{c^5}{16\pi G} (W_u{}^2 + V_u{}^2)$$

which gives, assuming $O(W_u) \simeq O(V_u)$

$$O(W_u) \simeq 10^{-23} \text{ cm}^{-1} \qquad O(V_u) \simeq 10^{-23} \text{ cm}^{-1} \tag{6.4}$$

Thus, from 6.3) we require

$$O(F) << 10^{-1} \text{ gm}^{\frac{1}{2}} \text{ cm}^{-\frac{1}{2}} \text{ sec}^{-1} \tag{6.5}$$

This corresponds to an electromagnetic flux, given by the Poynting Vector, of

$$O(P) = \frac{c}{4\pi} O(F^2) << 10^8 \text{ ergs/cm}^2\text{/sec} \tag{6.6}$$

This is quite a reasonable flux for an electromagnetic field, and certainly well above that used in experimental physics. Therefore, the above analysis will be quite reasonable for experimental purposes.

Having deduced a "domain of validity" which is easily satisfied, we may now consider whether the results given in 4.28) -.30) can be used to detect and/or analyse gravitational radiation.

Placing 6.4) into 4.30) immediately shows the average Poynting Vector is of little use for such a use. However, it is not immediately clear whether or not we can use 4.28) in some way.

Before considering this question in detail, it should be noted that since we have shown that no curvature impulses occur in the collision of pure gravitational and on electromagnetic waves, we are effectively forced to look toward 4.28) to provide a mechanism to detect gravitational waves by this means. Further, although it has been shown that two electromagnetic waves in collision give rise to curvature impulses, in general we would expect the gravitational waves impinging upon the Solar System to be pure.

This is because the processes which produce the fluxes used above require massive objects to participate in the catastrophe (e.g. gravitational capture) and one would expect such objects to be almost electrically neutral. Even if the objects involved were sufficiently charged, the stronger interaction of the resulting electromagnetic field with charge distributions would probably cause it to split off from the gravitational wave. So, looking to 4.28), we need a method by which we can measure the various components of the electromagnetic field. This is provided by (13) on page 72. In brief, since magnetic fields only act upon charges with a non-zero non-parallel velocity, by appropriate choice of a charged particle's velocity we can measure the various components of the field.

Although in (13), all the components of the field are measured, we shall consider measuring E_3 in 4.28) only. This will be sufficient to indicate whether or not we can use 4.28) to detect gravitational waves since to use the magnetic field, we would require the velocity of the charged particle to be close to that of light. Therefore, if we cannot detect the effect of the gravitational wave on E_3, we will be even less able to do so using the magnetic field.

From 6.6) we can assume

$$\frac{c}{4\pi}\, O(\phi_0^{0\,2}) \simeq 10^6 \text{ ergs/cm}^2\text{/sec}$$

that is

$$O(\phi_0^0) \simeq 10^{-2} \text{ gm}^{\frac{1}{2}} \text{ cm}^{-\frac{1}{2}} \text{ sec}^{-1} \qquad 6.7)$$

Now that we have some grasp of the orders of magnitude which we are dealing with, we must consider the behaviour of the quantities. Since $\phi_0^0(v)$ is something we can dictate by the experimental set up, we can leave consideration of this quantity until last and exploit it to maximize the sensitivity of detector.

The quantities W_u and V_u, however, are dictated by the physical event which gives rise to the gravitational radiation. Above, we adopted a radiation flux which corresponded to a conversion of matter to gravitational radiation at the centre of the galaxy. This flux, if it occurs, would correspond to the capture of a small star by a black hole since only such a catastrophe would give rise to such fluxes. Such a process takes the order of 10^{-5} seconds to occur during which there is a significant gravitational radiation burst.

The profile of the curvature tensor components during this burst is, strangely enough, fairly restricted. In (27), it is shown that the components of the curvature tensor must change sign so that the gravitational wave burst only has finite energy. Therefore, the behaviour of W_u and V_u must be oscillatory in some way. From (27), we can expect the Fourier Transform frequency components of the curvature tensor to have a maximum at a frequency of the order $\omega_1 = 2\pi\,\tau^{-1}$ where τ is the duration of the gravitational radiation burst. The bandwidth would be of the order of ω_1 also. To detect the E_3 field, let us use a small charged particle on a spring which can be set into vibratory motion by the E_3 field only. Although the particle, once moving, is acted upon by the magnetic field, if its velocity is small, these magnetic field effects will be negligible. Thus, the equation of motion will be

$$\frac{d^2z}{dt^2} + \frac{\omega_0}{Q}\frac{dz}{dt} + \omega_0{}^2 z = \frac{q}{m} E_3 \qquad 6.8)$$

where E_3 is given by

$$E_3 = -\tfrac{1}{8}\,\omega_t \int_0^T \phi_0^0(t)\; dt \qquad 6.9)$$

since we are using the observer previously adopted and hence there is no spatial change in their position.

For simplicity and since we are primarily interested in orders of magnitude, let us assume that at the observer Ψ_4^0 is such that

$$W_{tt} \simeq A \sin \omega_1 t \qquad V_{tt} \simeq A \sin (\omega_1 t + \alpha)$$

Further, we shall set $\phi_0^0(t)$ to be a sinusoid,

i.e.

$$\phi_0^0 (t) = B \sin (\omega_2 t + \beta)$$

Therefore 6.8) can be written as

$$\frac{d^2z}{dt^2} + \frac{\omega_0}{Q}\frac{dz}{dt} + \omega_0^2 z \simeq \frac{-q}{16M}\frac{AB}{\omega_1\omega_2}\Big[\cos((\omega_1 + \omega_2)\ t + \beta)$$

$$+ \cos((\omega_1 - \omega_2)\ t - \beta)\Big]$$

Setting $\omega_2 << \omega_1$ and tuning the system such that $\omega_0 \simeq \omega_1$, we can write

$$\frac{d^2z}{dt^2} + \frac{\omega_0}{Q}\frac{dz}{dt} + \omega_0^2 z \simeq \frac{-q}{8M}\frac{AB}{\omega_1\omega_2}\cos\omega_1 t$$

This system will give the response (28)

$$Z(t) \simeq C(\omega_1) \cos (\omega_1 t - \eta)$$

where $C(\omega_1) \simeq \dfrac{-q}{8M\omega_1\omega_2}\dfrac{AB}{\left((\omega_0^2 - \omega_1^2)^2 + \dfrac{\omega_1^2\omega_0^2}{Q^2}\right)^{\frac{1}{2}}}$

$$\tan \eta \simeq \frac{\omega_1\omega_0}{Q(\omega_0^2 - \omega_1^2)}$$

If we assume the detector is tuned close to ω_1, we get

$$Z(t) \simeq \frac{-qABQ}{8M\omega_1^3\omega_2}\cos(\omega_1 t - \frac{\pi}{2})$$

For the duration of the pulse, the energy absorbed by the electromagnetic field detector will be

$$E.A._1 = \int_0^{\tau} \frac{q}{M} E_3 \left(\frac{dz}{dt}\right) dt \simeq \frac{\tau^4 q^2 A^2 B^2 Q}{128(2\pi)^3 M \omega_2^2} \qquad 6.10)$$

If we now repeat this experiment, however we use an uncharged mass on a spring directed along the x^2 or x^3 axis, and rely solely on the gravitational fields coupling, we have from the equation of geodisic deviation

$$\frac{dz^2}{dt^2} + \frac{\omega_0}{Q}\frac{dz}{dt} + \omega_0^2 z = -c^2 l R_{0202}$$

$$\simeq -\tfrac{1}{2} l A \operatorname{Sin} \omega_1 t$$

where l is the length of the spring.

Repeating the same analysis for this system gives the energy absorbed by the gravitational wave detector during the pulse is of the order

$$E.A._2 \simeq \frac{m l^2 A^2 Q \tau^2}{16\pi} \qquad 6.11)$$

We can now compare $E.A._1$ and $E.A._2$ to see if the electromagnetic field detector is any better.

We can see that for long bursts, the electromagnetic detector will be far more sensitive, however the most energetic gravitational wave pulses will most likely be the short bursts corresponding to gravitational catastrophes such as collapse into a black hole. Next, the ratio q^2/m will favour highly charged small particles such as electrons. There may be the possibility of looking for a change in atomic electron energy levels or the like to detect gravitational waves. This ratio does give the electromagnetic detector an advantage in that a large massive detector is not required.

Finally, there is the presence of the electromagentic field frequency (ω_2) and amplitude(B) in 6.10). We could, within limits, attempt to manipulate these to improve the detectors sensitivity.

We can see that a stronger field of low frequency is desirable, however the field strength must satisfy the limits deduced above for it.

With the electromagnetic detector, there are several quantities which can be manipulated to improve its sensitivity, where as with the direct gravitational wave detector, there are relatively few.

To compare the sensitivities of the two detectors, let us assume that both have the same mass, 10^6gms, are 10^2cm long. This will give

$$\frac{E.A._1}{E.A._2} \simeq \frac{\tau^2 \; q^2 \; B^2}{64\pi^2 \; M^2 \; l^2 \; \omega_2{}^2} \simeq \frac{q^2 B^2}{\omega_2{}^2} \; 10^{-18}$$

Assuming B satisfies the limitation given above and $\omega_2 \simeq 10^3$Hz we get

$$\frac{E.A._1}{E.A._2} \simeq q^2 \; 10^{-38}$$

Therefore, to make the electromagnetic gravitational wave detector as sensitive as a "Weber type" detector, we would require it to carry a charge in the order of 10^9 Coulombs which is rather large for such an apparatus. We could go to the other extreme and consider the use of an electron as the charged particle, as was suggested above. In this case, the energy absorbed would be

$$E.A._1 \simeq 10^{-69} \text{ergs}$$

This is way too small to be measureable (e.g. the cosmic background radiation photons have energies of the order 10^{-16} ergs).

Thus, we can see that although the analysis utilized above has some validity, it does not help solve the technical problems involved in detecting gravitational radiation. In fact, the numbers deduced above would indicate very little promise for such detectors even in the future.

APPENDIX 1

Using the Newman-Penrose formalism 1.14) has the form

$$k = \gamma_{131} = l_{\mu;\nu} m^{\mu} l^{\nu} \qquad \pi = -\gamma_{241} = -n_{\mu;\nu} \bar{m}^{\mu} l^{\nu}$$

$$\varepsilon = \tfrac{1}{2}(\gamma_{121} - \gamma_{341}) = \tfrac{1}{2}(\ell_{\mu;\nu} n^{\mu} \ell^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} \ell^{\nu})$$

$$\rho = \gamma_{134} = \ell_{\mu;\nu} m^{\mu} \bar{m}^{\nu} \qquad \lambda = -\gamma_{244} = -n_{\mu;\nu} \bar{m}^{\mu} \bar{m}^{\nu}$$

$$\alpha = \tfrac{1}{2}(\gamma_{124} - \gamma_{344}) = \tfrac{1}{2}(\ell_{\mu;\nu} n^{\mu} \bar{m}^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} \bar{m}^{\nu})$$

$$\sigma = \gamma_{133} = \ell_{\mu;\nu} m^{\mu} m^{\nu} \qquad \mu = -\gamma_{243} = -n_{\mu;\nu} \bar{m}^{\mu} m^{\nu} \qquad \text{A1.1)}$$

$$\beta = \tfrac{1}{2}(\gamma_{123} - \gamma_{343}) = \tfrac{1}{2}(\ell_{\mu;\nu} n^{\mu} m^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} m^{\nu})$$

$$\nu = -\gamma_{242} = -n_{\mu;\nu} \bar{m}^{\mu} n^{\nu} \qquad \tau = \gamma_{132} = \ell_{\mu;\nu} m^{\mu} n^{\nu}$$

$$\gamma = \tfrac{1}{2}(\gamma_{122} - \gamma_{342}) = \tfrac{1}{2}(\ell_{\mu;\nu} n^{\mu} n^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} n^{\nu})$$

1.15) has the form

$$(\Delta D - D\Delta)\,T = [\,(\gamma + \bar{\gamma})\,D + (\varepsilon + \bar{\varepsilon})\Delta - (\tau + \bar{\pi}\,)\bar{\delta}$$

$$- (\bar{\tau} + \pi)\delta]\,T \qquad \text{a)}$$

$$(\delta D - D\delta)T = [(\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \varepsilon - \bar{\varepsilon})\delta\,]\,T \qquad \text{b)}$$

A1.2)

$$(\delta\Delta - \Delta\delta)T = [-\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\,\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta\,]\,T \qquad \text{c)}$$

$$(\delta\bar{\delta} - \bar{\delta}\delta)T = [\,(\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta\,]\,T \qquad \text{d)}$$

1.16) has the form

$$D\rho - \bar{\delta}\kappa = (\rho^2 + \sigma\bar{\sigma}) + (\varepsilon + \bar{\varepsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \qquad \text{a)}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho})\,\sigma + (3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \qquad \text{b)}$$

$$D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\varepsilon - \bar{\varepsilon})\tau + (3\gamma + \bar{\gamma})\kappa$$

$$+ \Psi_1 + \Phi_{01} \qquad \text{c)}$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \beta\bar{\varepsilon} - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10} \qquad \text{d)}$$

$$D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \varepsilon)\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon + \Psi_1 \qquad e)$$

$$D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta + (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon$$

$$+ \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11} \qquad f)$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\varepsilon - \bar{\varepsilon})\lambda$$

$$+ \Phi_{20} \qquad g)$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} + (\varepsilon + \bar{\varepsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu\kappa$$

$$+ \Psi_2 + 2\Lambda \qquad h)$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu$$

$$+ \Psi_3 + \Phi_{21} \qquad i)$$

A1.3)

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4 \qquad j)$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa$$

$$\Psi_1 + \Phi_{01} \qquad k)$$

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho})$$

$$+ \varepsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda + \Phi_{11} \qquad l)$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta)$$

$$- \Psi_3 + \Phi_{21} \qquad m)$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\nu$$

$$+ \Phi_{22} \qquad n)$$

$$\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \varepsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu)$$

$$+ \alpha\bar{\lambda} + \Phi_{12} \qquad o)$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu}$$

$$+ \Phi_{02} \qquad p)$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho + \nu\kappa$$

$$- \Psi_2 - 2\Lambda \qquad q)$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau+\beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3 \qquad r)$$

where

$$\Psi_o = - C_{1313} = - C_{\alpha\beta\gamma\delta}\, l^\alpha\, m^\beta\, l^\gamma\, m^\delta = \Psi_{0000}$$

$$\Psi_1 = - C_{1213} = - C_{\alpha\beta\gamma\delta}\, l^\alpha\, n^\beta\, l^\gamma\, m^\delta = \Psi_{0001}$$

$$\Psi_2 = -\tfrac{1}{2}(C_{1212} - C_{1234}) = -\tfrac{1}{2}\, C_{\alpha\beta\gamma\delta}(l^\alpha\, n^\beta\, l^\gamma\, n^\delta$$

$$- l^\alpha\, n^\beta\, m^\gamma\, \bar{m}^\delta) = \Psi_{0011} \qquad \text{A1.4)}$$

$$\Psi_3 = C_{1224} = C_{\alpha\beta\gamma\delta}\, l^\alpha\, n^\beta\, n^\gamma\, \bar{m}^\delta = \Psi_{0111}$$

$$\Psi_4 = -C_{2424} = -C_{\alpha\beta\gamma\delta}\, n^\alpha\, \bar{m}^\beta\, n^\gamma\, \bar{m}^\delta = \Psi_{1111}$$

$$\Phi_{00} = -\tfrac{1}{2}R_{11} = \Phi_{00\dot{0}\dot{0}} = \bar{\Phi}_{00}$$

$$\Phi_{11} = -\tfrac{1}{4}(R_{12} + R_{34}) = \Phi_{01\dot{0}\dot{1}}$$

$$\Phi_{01} = -\tfrac{1}{2}R_{13} = \Phi_{00\dot{0}\dot{1}} = \bar{\Phi}_{10}$$

$$\Phi_{12} = -\tfrac{1}{2}R_{23} = \Phi_{01\dot{1}\dot{1}}$$

$$\Phi_{10} = -\tfrac{1}{2}R_{14} = \Phi_{01\dot{0}\dot{0}} = \bar{\Phi}_{01}$$

$$\Phi_{21} = -\tfrac{1}{2}R_{24} = \Phi_{11\dot{0}\dot{1}}$$

$$\Phi_{02} = -\tfrac{1}{2}R_{33} = \Phi_{00\dot{1}\dot{1}} = \bar{\Phi}_{20}$$

$$\Phi_{22} = -\tfrac{1}{2}R_{22} = \Phi_{11\dot{1}\dot{1}}$$

$$\Phi_{20} = -\tfrac{1}{2}R_{44} = \Phi_{11\dot{0}\dot{0}}$$

$$\Lambda = R/24$$

APPENDIX II

The Sandwich Plane Gravitational Wave

Using the Rosen form for the metric of a plane gravitational wave:

$$ds^2 = 2e^{-M}\,dudv - e^{-U}(e^{V}\mathrm{Cosh}W\,(dx^2)^2 + e^{-V}\mathrm{Cosh}W\,(dx^3)^2 - 2\mathrm{Sinh}W\,dx^2\,dx^3) \qquad \text{A2.1)}$$

Where M, V, U and W are all functions of u only and satisfy the field equation, when written in the Newman-Penrose formalism.

$$2U_{uu} - U_u^{\,2} + 2M_uV_u = W_u^{\,2} + V_u^{\,2}\mathrm{Cosh}^2W \qquad \text{A2.2)}$$

The only non-zero component of the Weyl tensor is given by

$$\Psi^0_4 = -\tfrac{1}{2}[V_{uu}\,\mathrm{Cosh}W + 2V_uW_u\mathrm{Sinh}W - V_u(U_u - M_u)\,\mathrm{Cosh}W] + \tfrac{1}{2}i\,[W_{uu} - W_u(U_u - M_u) - V_u^{\,2}\mathrm{Cosh}W\,\mathrm{Sinh}W] \qquad \text{A2.3)}$$

If the wave propagates undisturbed through flat spacetime, we can set $M(u) = 0$ without loss of generality.

To describe the sandwich wave in Figure 2, we require that A2.2) be satisfied throughout the whole space-time patch and that Ψ^0_4 be non-zero in the non-flat region $0<u<u'$ and be identically zero elsewhere. This gives the following set of equations for the flat regions:

$$2U_{uu} - U_u^{\,2} = W_u^{\,2} + V_u^{\,2}\,\mathrm{Cosh}^2W \qquad \text{a)}$$

$$V_{uu}\,\mathrm{Cosh}W + 2V_uW_u\mathrm{Sinh}W - V_uU_u\,\mathrm{Cosh}W = 0 \qquad \text{b) A2.4)}$$

$$W_{uu} - W_uU_u - V_u^{\,2}\mathrm{Cosh}W\,\mathrm{Sinh}W = 0 \qquad \text{c)}$$

For the region $u<0$, we have the trivial solution $U = V = W = 0$. For the region $u>u'$, A2.4) is a characteristic initial value problem where the initial data are provided by the junction conditions at $u = u'$ i.e. U, V and W are continuous. The initial values of U_u, V_u and W_u will be given by U, V and W in the non-flat region if we adopt the Lichnerowicz conditions, but if we adopt the O'Brien-Synge conditions, they would have to be inserted. The solution obtained will determine U, V and W uniquely in the given co-ordinate patch.

Unfortunately the system A2.4) is not amenable to a straightforward solution. However if we assume the $W = 0$ throughout the region $u<u'$, i.e. the sandwich wave is linerarly polarized, then by A2.4)c) $W = 0$ throughout the region $u>u'$ (4). This reduces the system to

$$2U_{uu} - U_u^{\,2} = V_u^{\,2} \qquad \text{a)} \qquad \text{A2.5)}$$

$$V_{uu} - V_u U_u = 0 \qquad \text{b)}$$

This system has a straightforward solution.

Setting

$$X(u) = U(u) - V(u) \qquad \text{A2.6)}$$

and adding A2.5)a) to twice b) gives

$$2(U_{uu} - V_{uu}) - (U_u^{\,2} - 2V_u U_u + V_u^{\,2}) = 0$$

which is

$$2X_{uu} - X_u^{\,2} = 0 \qquad \text{A2.7)}$$

Setting $X(u) = -2\ln Y(u)$, we find

$$X_u = -2Y^{-1} Y_u \qquad X_{uu} = 2Y^{-2} Y_u^{\,2} - 2Y^{-1} Y_{uu}$$

giving from A2.7)

$$Y_{uu} = 0$$

Thus we have

$$X(u) = -2\ln(au + b) \qquad \text{A2.8)}$$

where a and b are constants. This gives rise to several possible solutions. These solutions are found by substituting A2.8) into A2.6) and in turn, this into A2.5)b) giving

$$V_{uu} - V_u\left(\frac{-2a}{(au+b)} + V_u\right) = 0$$

This equation is simplified using the substitution

$$V(u) = -\ln Z(u)$$

giving

$$\frac{d^2Z}{du^2} + \frac{2a}{(au+b)}\frac{dZ}{du} = 0 \qquad \text{A2.9)}$$

Let $A(u) = \dfrac{dZ(u)}{du}$ giving

$$\frac{dA}{du} + \frac{2a}{(au+b)} A = 0$$

which has solution

$$A(u) = C(au + b)^{-2}$$

which in turn gives

$$Z(u) = d(au + b)^{-1} + f$$

where $d = \frac{-c}{a}$

Therefore the solutions for U(u) and V(u) are

$$U(u) = -2\ln(au + b) - \ln(d(au + b)^{-1} + f)$$

$$= -\ln[(au + b)(d + f(au + b))] \qquad \text{a)}$$

$$= -\ln[(au + b)(gu + h)] \qquad \text{A2.10)}$$

$$V(u) = -\ln[(au + b)^{-1}(gu + h)] \qquad \text{b)}$$

where g = af and h = d + fb

The case with a = 0 in A2.8) also satisfies A2.7) however this case gives a different equation in A2.9). In this situation we, instead, have

$$\frac{d^2Z}{du^2} = 0$$

giving

$$U(u) = f - \ln(au + b) \qquad \text{a)}$$

$$\text{A2.11)}$$

$$V(u) = -\ln(au + b) \qquad \text{b)}$$

Therefore, the set of possible solutions to A2.5) are

$$U(u) = V(u) = 0 \qquad \text{a)}$$

$$U(u) = -\ln[(au + b)(gu + h)]$$

$$\text{b)} \qquad \text{A2.12)}$$

$$V(u) = -\ln[(au + b)^{-1}(gu + h)]$$

$$U(u) = f - \ln(au + b)$$

$$\text{c)}$$

$$V(u) = -\ln(au + b)$$

where the values of the constants a, b, ...f are deduced from initial conditions which we shall take as given by the Lichnerowicz conditions. It is worthwhile calculating these constants since their values demonstrate the unavoidability of co-ordinate singularities in the region u>u'.

We can immediately discount A2.12)a) as a solution in the region $u>u'$ since it would require the gravitational wave to vanish.

The presence of the co-ordinate singularity is more obvious using A2.12)c) hence we shall consider this first. It is easily seen that we have $u = u'$

$$f = U(u') - V(u')$$

$$a = -e^{-V(u')} U_u(u') \qquad \text{A2.13)}$$

$$b = e^{-V(u')} (1 + U_u(u') u')$$

Since we have assumed $U(u) = V(u) = 0$ for $u<0$, from A2.5)a) $U_{uu}>0$ for $u>0$ and hence $U_u>0$ for $u>0$ therefore $a<0$ and $b>0$. Further, the value of u for which $au+b = 0$ is seen to be greater than u' and hence the co-ordinate singularity is unavoidable for this solution.

To prove the similar result for the metric A2.12)b) is similar. From A2.12)b) we have

$$a = \tfrac{1}{2}e^{\frac{1}{2}(-U(u') + V(u'))} (-U_u(u') + V_u(u'))$$

$$b = e^{\frac{1}{2}(-U(u') + V(u'))} (1 + \tfrac{1}{2}(U_u(u') - V_u(u')) u')$$

$$g = -\tfrac{1}{2}e^{-\frac{1}{2}(U(u') + V(u'))} (U_u(u') + V_u(u'))$$

$$h = e^{-\frac{1}{2}(U(u') + V(u'))} (1 + \tfrac{1}{2}(U_u(u') + V_u(u') u')$$

From these results we see that the values of u for which $(au+b) = 0$ and $(gu+h) = 0$ are greater than u'. Therefore the co-ordinate singularities are again unavoidable.

The table below sets out the metric and the appropriate co-ordinate transformations to transform away the co-ordinate singularities.

Metric	Transformation
$ds^2 = 2dudv - ((au+b)^2 (dx^2)^2 + (gu+h)^2 (dx^3)^2)$	$u = X^0$ $v = X^1 - \frac{(X^2)^2}{(aX^0+b)} - \frac{(X^3)^2}{(gX^0+h)}$ $x^2 = \frac{X^2}{(aX^0+b)} \quad x^3 = \frac{X^3}{(gX^0+h)}$
$ds^2 = 2dudv - e^{-f}((dx^2)^2 + (au+b)^2 (dx^3)^2)$	$u = X^0$ $v = X^1 - \frac{(X^3)^2}{(aX^0+b)}$ $x^2 = e^f X^2 \quad x^3 = \frac{e^f X^3}{(aX^0+b)}$

The resulting flat metric is

$$ds^2 = 2dX^0\, dX^1 - (dX^2)^2 - (dX^3)^2$$

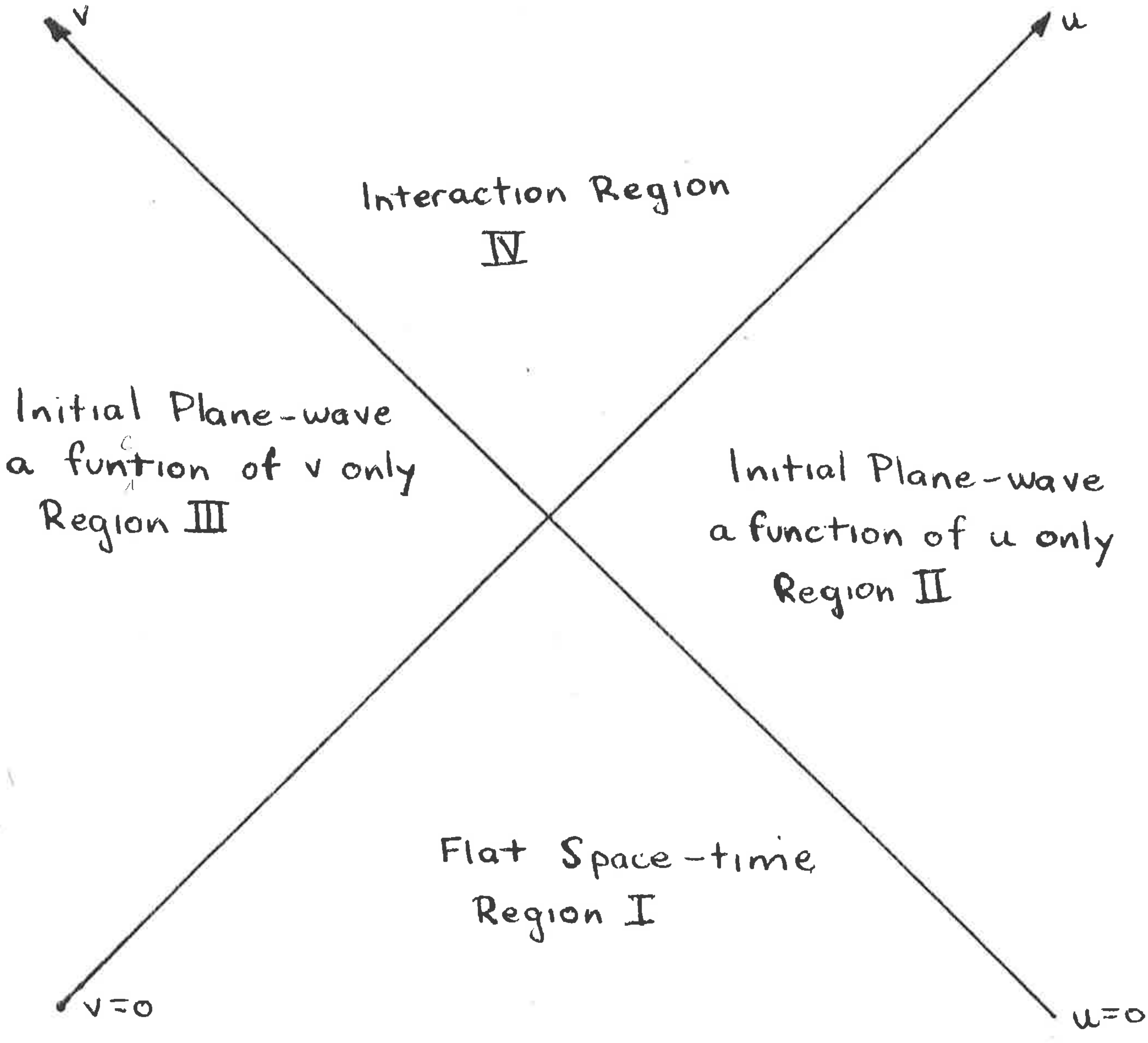

Figure 1

A "head-on" collision between two plane waves as viewed by an observer with worldline $u = kv$ $(k>0)$ x^2 = constant, x^3 = constant. For other timelike observers the collision will not be head-on but at an angle. This more general situation can also be analysed using the present approach. The hypersurfaces u = constant, v = constant are null.

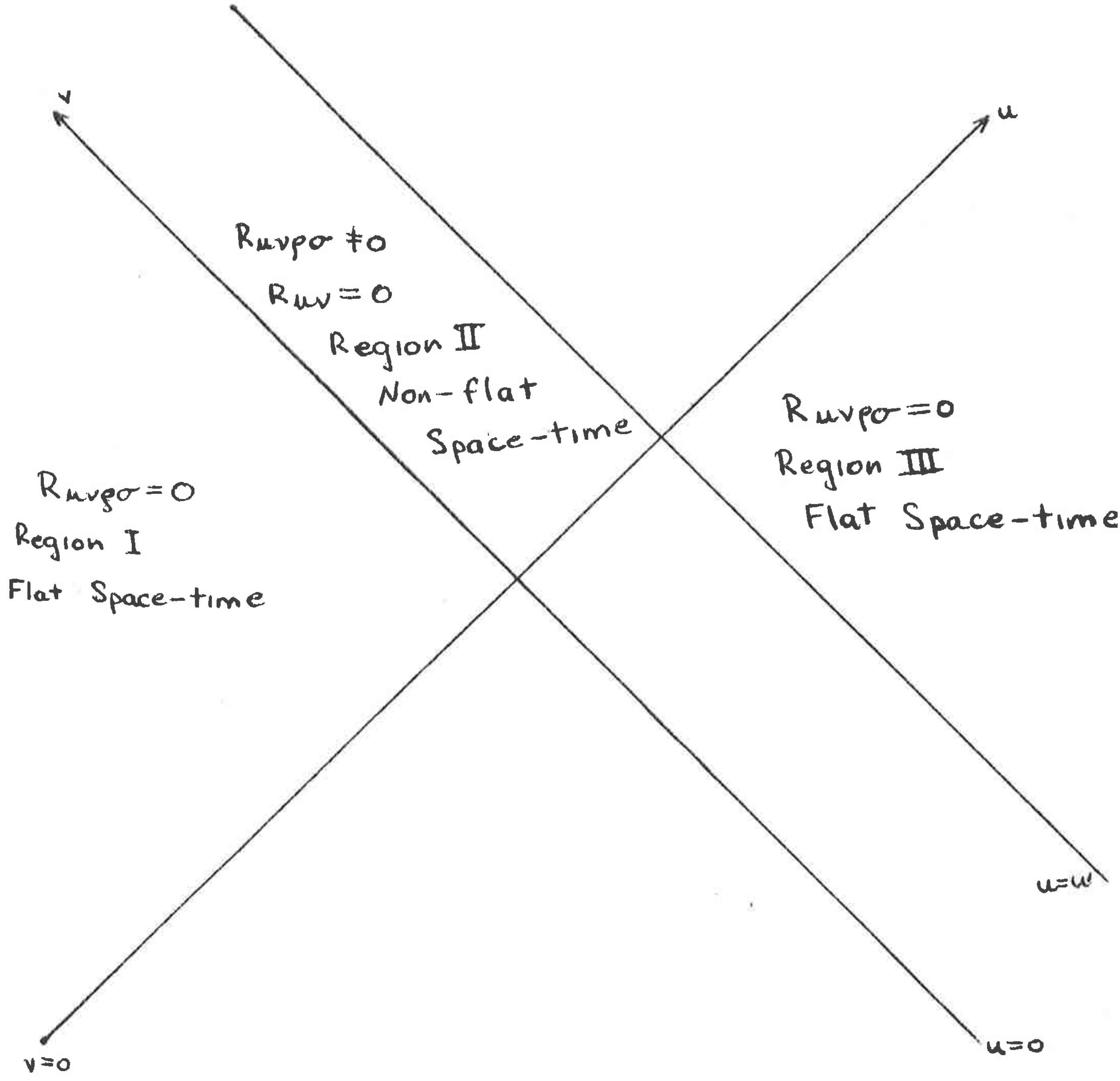

Figure 2

Plane sandwich gravitational wave. The metric is a function of u only.

The non-flat region is bounded by null hypersurfaces u = 0 and u = u'.

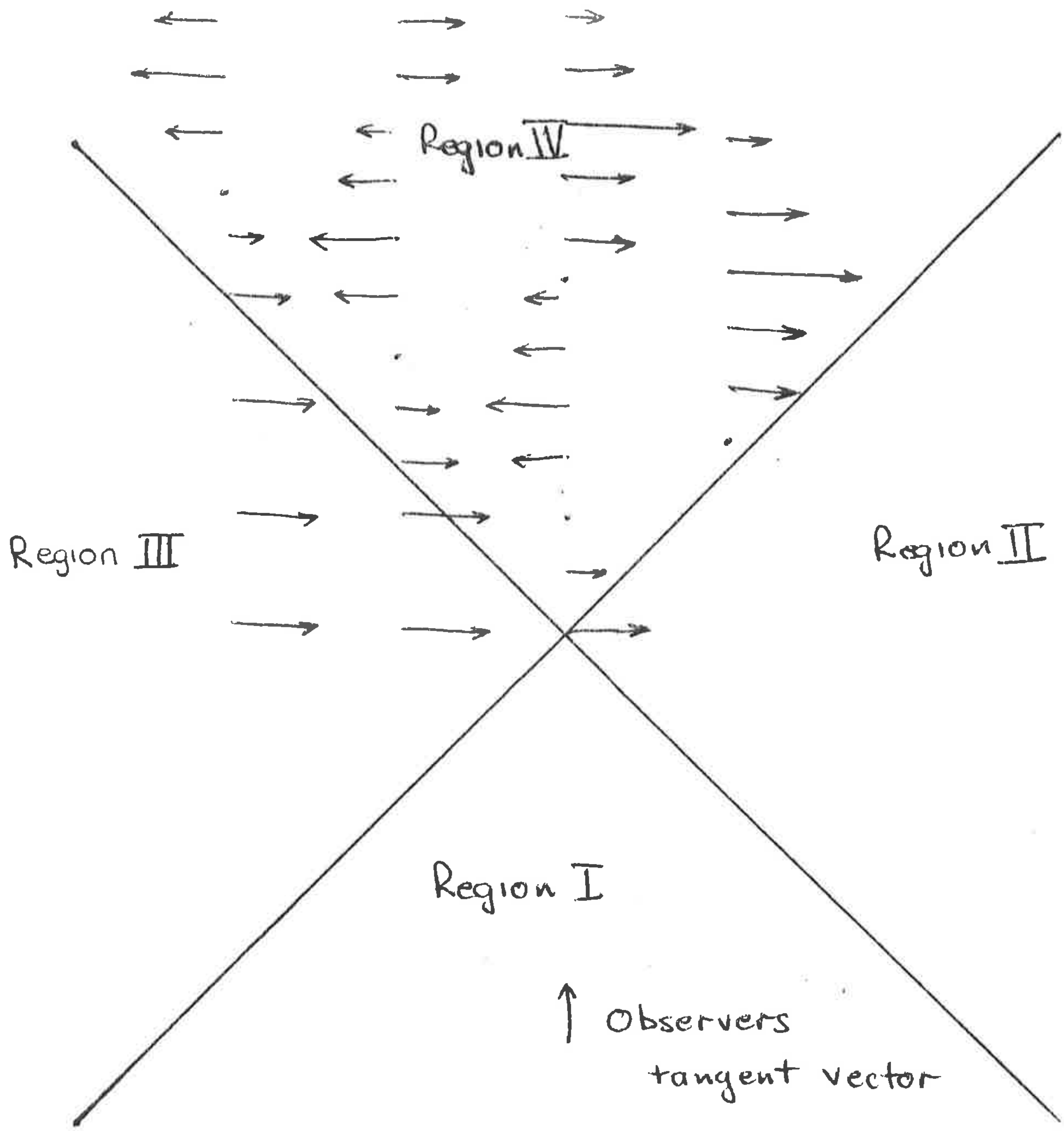

Figure 3

A schematic diagram indicating a possible observed behaviour of the direction and magnitude of $P_{\wedge ave}$ in the head on collision of a gravitational plane wave and "test" electromagnetic plane wave.

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