

Principal Bundles and the Dixmier Douady Class

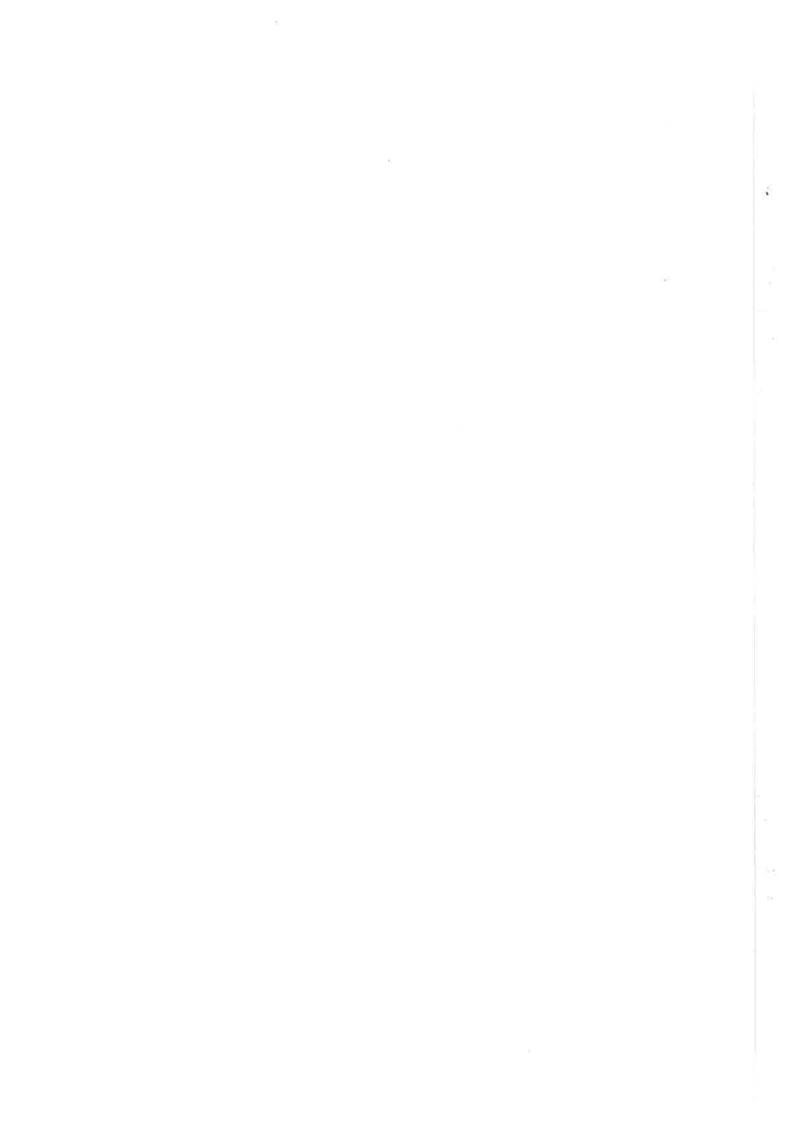
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Contents

ACKNOWLEDGEMENTS	1
STATEMENT	1
Chapter 1. PRINCIPAL BUNDLES: DEFINITIONS AND EXAMPLES	3
1. PRELIMINARIES OF BUNDLE THEORY	3
2. THE HOMOTOPY CATEGORY	9
3. CLASSIFYING SPACES OF HOMOTOPY COFUNCTORS	11
4. CHARACTERISTIC CLASSES	15
5. COHOMOLOGY OF $BS^1 \sim PU$	17
Chapter 2. REDUCTION OF STRUCTURE GROUP	19
1. GENERAL THEORY	19
2. METRICS ON VECTOR BUNDLES	27
3. SEMI-DIRECT PRODUCTS WITH DISCRETE GROUPS	27
4. ALMOST COMPLEX STRUCTURES	28
5. THE CHERN CLASSES	29
Chapter 3. EXTENSION OF STRUCTURE GROUP	31
1. TERMINOLOGY AND GENERAL SETTING	31
2. ASSUMPTIONS	32
3. NON-ABELIAN COHOMOLOGY	33
4. CLASSIFYING SPACES	33
5. SPECTRAL SEQUENCE APPROACH	36
6. THREE "IDENTICAL" CHARACTERISTIC CLASSES	37
7. $Spin(n)$ STRUCTURES ON $SO(n)$ BUNDLES	39
8. CIRCLE EXTENSIONS — THE DIXMIER DOUADY CLASS	39
9. THE FINITE PROJECTIVE UNITARIES	43
10. $Spin^{C}(n)$ STRUCTURES $(n > 2)$.	44
11. STRING STRUCTURES	44
12. QUICK REVIEW OF THE TECHNICAL ASPECTS OF LOOP SPACES, LOOP GROUP	S
AND LOOP BUNDLES	45

1	3. KILLINGBACK'S RESULT	47
1	4. THE RESTRICTED UNITARIES	50
1	5. U_{res} AS A CLASSIFYING SPACE	52
1	6. THE DIXMIER DOUADY CLASS AND THE SECOND CHERN CLASS	53
1	7. CONCLUDING REMARKS	56
BIE	LIOGRAPHY	57

iv

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CONTENTS

ABSTRACT. This thesis exploits the power of the categorical approach to homotopy theory to produce a summary of the theory of principal bundles. A systematic consideration of the problems of the reduction and the extension of structure group is therefore possible and a variety of techniques in each case are explored and related to one another. These techniques are each applied to show the relation between the reduction and the extension of the structure group of a principal bundle and the vanishing of familiar characteristic classes. Of particular interest is the discovery of a systematic approach to the Dixmier–Douady class for string structures in the case of both continuous and differential loops. Finally, I relate the theory of principal bundles with the restricted unitary group, U_{res} , as structure group to reduced K-theory, demonstrating a link between the second Chern class of a bundle in reduced K-theory and the Dixmier–Douady class of the corresponding principal U_{res} -bundle.

STATEMENT

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STATEMENT

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to a copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

August 28th 1996.