Principal Bundles and the Dixmier Douady Class

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ABSTRACT. This thesis exploits the power of the categorical approach to homotopy theory to produce a summary of the theory of principal bundles. A systematic consideration of the problems of the reduction and the extension of structure group is therefore possible and a variety of techniques in each case are explored and related to one another. These techniques are each applied to show the relation between the reduction and the extension of the structure group of a principal bundle and the vanishing of familiar characteristic classes. Of particular interest is the discovery of a systematic approach to the Dixmier–Douady class for string structures in the case of both continuous and differential loops. Finally, I relate the theory of principal bundles with the restricted unitary group, U_{res}, as structure group to reduced K-theory, demonstrating a link between the second Chern class of a bundle in reduced K-theory and the Dixmier-Douady class of the corresponding principal U_{res}–bundle.

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STATEMENT

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to a copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

August 28th
1996.

CHAPTER 1

PRINCIPAL BUNDLES: DEFINITIONS AND EXAMPLES

1. PRELIMINARIES OF BUNDLE THEORY

Early in the development of the theory of fibre bundles it was realised that the set of transformations allowed on the fibre is crucial. For example (see Steenrod 1951 pg 4) the twisted torus is a genuinely twisted circle bundle over the circle if one allows oneself the use of only the identity and 180 degree rotation as operations on the fibre. But the twisted torus is homeomorphic to its straight sibling and, in the context of the full rotation group of the circle, is just a trivial bundle. By the time Husemoller wrote "Fibre Bundles" in 1966 the importance of the structure group was such that general fibre bundles were defined as bundles associated to a principal bundle, a bundle who's fibre and structure group are identical. This thesis will therefore be exclusively concerned with the theory of principal bundles.

Definition 1.1 (PRINCIPAL BUNDLE). A numerable, locally trivial, principal fibre bundle

$$P(M, G, \pi_P)$$

with structure group and fibre G a topological group, base space M and projection π_P is a topological space P, called the total space, and a continuous, surjective map

$$\pi_P : P \to M$$

such that

1) there is a free, right continuous action of G on P and M is the quotient of this action, $M = P/G$

2) the fibre over $x \in M$, $P_x := \pi_P^{-1}(x)$ is homeomorphic to G for all $x \in M$

3) there is a numerable cover $\{U_\alpha, \alpha \in A\}$ of M and G-equivariant homeomorphisms h_α such that

$$h_\alpha : P|U_\alpha := \pi_p^{-1}(U_\alpha) \to U_\alpha \times G$$

$$p \mapsto (\pi_P(p), h_{\alpha,\pi_p}(p))$$

where $h_{\alpha,x} : P_x \to G$ is itself a G-equivariant homeomorphism. The collection of pairs $\{(U_\alpha, h_\alpha) : \alpha \in A\}$ is called a trivialisation of P.

4) for all $x \in U_{\alpha,\beta}$ the map

$$g_{\alpha,\beta}(x) := h_{\alpha,x} \circ h_{\beta,x}^{-1} : G \to G$$

is a homeomorphism of G and may be identified with an element of G via the inclusion, $G \hookrightarrow Homeo(G)$, of left multiplication. Furthermore, we require that the map

$$g_{\alpha,\beta} : U_{\alpha,\beta} \to G$$

$$x \mapsto g_{\alpha,\beta}(x)$$

be continuous. These maps, $g_{\alpha,\beta}$ are called the transition functions of $P(M, G, \pi_P)$.

NOTE 1.1. The condition of numerability on the cover in 3 is important but largely technical. A cover is numerable provided it subordinates a locally finite partition of unity. Hausdorff spaces are paracompact if and only if every cover is numerable. Husemoller (p 48) summarises these details. All references to principal bundles will now be to numerable principal bundles.

NOTE 1.2. The condition of local triviality, 2, is crucial to the theory of bundles I shall develop. It is satisfied in a wide variety of examples but local triviality cannot be assumed, especially in the case of infinite dimensional groups. Henceforth all references to fibre bundles, principal and otherwise, are to locally trivial fibre bundles.

NOTE 1.3. G-equivariant trivialisations correspond uniquely to local sections of P. That is maps

$$s_\alpha : U_\alpha \hookrightarrow P \quad \pi_P \circ s_\alpha = id_{U_\alpha}.$$

One sets $h_\alpha(x) := p/s_\alpha(x)$ where $p/s_\alpha(x)$ is the unique element of G such that $p = s_\alpha(x).g$. In this case it is easy to see that $g_{\alpha,\beta} = s_\beta(x)/s_\alpha(x)$. I shall write $\{s_\alpha, h_\alpha, U_\alpha\}$, $\{s_\alpha, U_\alpha\}$ or $\{h_\alpha, U_\alpha\}$ depending on emphasis.

NOTE 1.4. One may define a smooth principal bundle in the case where P, M and G are all manifolds. In this case G is required to be a Lie group and the maps π_P, h_α, $h_{\alpha,x}$, $g_{\alpha,\beta}$ and $g_{\alpha,\beta}(x)$ are all smooth.

NOTE 1.5. When it is unimportant I shall drop π_P from the notation and refer to the principal G–bundle $P(M, G)$.

NOTE 1.6. One of the most important examples of principal bundles and one that will occupy the majority of this thesis can arise when H is a closed subgroup of a Lie group, G. In this case right multiplication by elements of H gives a free, continuous, right action of H on G and G/H is a Hausdorff space. This gives us all we need for a principal H–bundle except local triviality which need not hold (See Borel p 35 for an example). So far as I am aware, the question of when $\pi : G \to G/H$ defines a locally trivial principal H-bundle has not yet been resolved in general. So long as $\pi : G \to G/H$ has local sections local triviality is assured. It is a theorem of Gleason (1950) that this holds when H is compact and it is a theorem of Michael (1970) that this is also the case when G is a Banach Lie group (a smooth manifold modelled on Banach spaces for which multiplication and taking inverses are smooth operations) and H is a closed Banach Lie subgroup of G. In these cases at least, G is a principal H–bundle over the homogeneous space G/H, $G = G(G/H, H)$.

For completeness and for later use I now give Husemoller's general definition of a fibre bundle.

DEFINITION 1.2. (LOCALLY TRIVIAL FIBRE BUNDLE) Let $P(M, G, \pi_P)$ be a principal G–bundle and let G act non–degenerately and continuously on the left of the topological space F. Then G acts on the right of $P \times F$ by

$$(p, f).g := (p.g, g^{-1}.f)$$

The topological quotient space of $P \times F$ modulo this G–action, $Q := (P \times F)/G$, and the projection

$$\pi_Q : Q \longrightarrow M$$

$$[(p, f)] \mapsto \pi_P(p)$$

together form the F–bundle associated to the principal G–bundle, P.

NOTE 1.7. By way of example, a vector bundle $V(M, F^n, U(F, n))$ is the F^n–bundle associated to its frame bundle.

NOTE 1.8. If F is itself a principal G–bundle, $F(N, G)$ then one can take the associated left action of G on P, $g.p := p \circ g^{-1}$, and construct the space

$$Q = (P \times F)/G.$$

Q is both an F–bundle over M and a P–bundle over N. One has the following diagram of commuting projections. The fibre of each projection is indicated in the middle of the map.

$$\begin{array}{ccccc} F & \xleftarrow{P} & F \times P & \xrightarrow{F} & P \\ {\scriptstyle G}\downarrow & & {\scriptstyle G}\downarrow & & {\scriptstyle G}\downarrow \\ N & \xleftarrow{P} & (F \times P)/G & \xrightarrow{F} & M \end{array}$$

I will exploit this, and other examples of multiple bundle structures on the same space, later on.

Returning to principal bundles, its now time to make precise what is meant by a map which preserves principal bundle structure.

DEFINITION 1.3. (PRINCIPAL BUNDLE MORPHISMS) Let H be a topological subgroup of a topological group G and let $Q(N, H)$ and $P(M, G)$ be principal bundles. A bundle morphism from Q to P is a continuous map,

$$\phi : Q \to P$$

such that

1. it takes fibres, Q_x, of Q homeomorphically into their image which is contained in a fibre, P_y, of P

$$\phi_x := \phi|Q_x : Q_x \to \phi(Q_x) \in P_y$$

for some $y \in M$.

2. it commutes with the action of H on Q

$$\phi(x.h) = \phi(x).h$$

for all $x \in N$ and $h \in H$. Such a ϕ induces a continuous map

$$\phi' : N \to M, \quad \phi'(x) = y.$$

NOTE 1.9. An injective bundle morphism is called a bundle injection. If it is also bijective and its inverse is a bundle map then it is called a bundle isomorphism. In this case ϕ' is a homeomorphism from N to M.

NOTE 1.10. A principal G–bundle is called trivial if and only if it is bundle isomorphic to $M \times G$. Note 1.3 entails that a principal G–bundle is trivial if and only if it has a section defined over all of its base space, i.e. a global section.

NOTE 1.11. The class of principal G–bundles together with bundle morphisms as maps forms a category, $Bun(G)$. A given topological space, M, defines the subcategory of all principal G–bundles over M, $Bun_M(G)$. It is of note that this latter category is a groupoid (every morphism is an isomorphism). (See Husemoller pg 42.) The relation defined by bundle isomorphism is an equivalence relation on $Bun(G)$ and $Bun_M(G)$. In what follows we shall be interested in isomorphism classes of principal G–bundles and in the new categories, $Bun(G)/\sim$ and $Bun_M(G)/\sim$. Elements of $Bun_M(G)/\sim$, isomorphism classes of bundles, shall be denoted $< P >$ where P is a principal G–bundle. Note that $Bun_M(G)/\sim$ is a category with only identity maps since $Bun_M(G)$ is a groupoid. I now move to the first solution to classifying the elements of $Bun_M(G)/\sim$.

PROPOSITION 1.1. *(TRANSITION FUNCTIONS AND NON-ABELIAN COHOMOLOGY) Let $P(M,G)$ be a principal G–bundle.*

1) The transitions functions of P define a cocycle in $Z(M,\underline{G})$ — the group of continuous Cech one-cocyles with coefficients in $\underline{G}$.

2) Given an element, $\xi \in Z(M,\underline{G})$ one may construct a principal G–bundle whose transition functions are ξ.

3) Two principal G–bundles, $P_1(M,G)$ and $P_2(M,G)$ with transition functions ξ_1 and ξ_2 are bundle isomorphic if and only if ξ_1 and ξ_2 are in the same class of $H^1(M,\underline{G})$ — the first Cech cohomology set of M with coefficients in $\underline{G}$. We write

$$[P_i] = [\xi_i] \quad i = 1,2.$$

$$< P_1 >=< P_2 > \text{ if and only if } [P_1] = [P_2]$$

NOTE 1.12. This is a standard result, for a proof the reader is referred to Husemoller (pg 61) for the details beyond the following sketch. Let $g_{\alpha,\beta}$ be transition functions for $P(M,G)$. Then

$$\begin{aligned}\delta_{\alpha,\beta,\gamma} &= g_{\alpha,\beta} \circ g_{\beta,\gamma} \circ g_{\gamma,\alpha} \\ &= h_\alpha \circ h_\beta^{-1} \circ h_\beta \circ h_\gamma^{-1} \circ h_\gamma \circ h_\alpha^{-1} \\ &= id_{(U_{\alpha,\beta,\gamma})} \text{ for all } \alpha,\beta,\gamma.\end{aligned}$$

This proves part 1. Part 3 is more subtle. By taking $\{U_\alpha\}$ to be a common refinement of covers which trivialise P_1 and P_2 we may choose $[P_i] = [\{g_{i\alpha,\beta}, U_\alpha\}]$ to be transition functions derived from the trivalisations $\{s_{i\alpha}, U_\alpha\}$ with $i = 1, 2$. Then $[P_1] = [P_2]$ means there are continuous maps

$$f_\alpha : U_\alpha \to G$$

such that

$$g_{1\alpha,\beta} = f_\alpha \circ g_{2\alpha,\beta} \circ f_\alpha^{-1}$$

One now sets

$$\phi : P_1 \to P_2$$

$$s_{1\alpha}(x).g \mapsto s_{2\alpha}(x).g.f_\alpha(x)$$

for all $g \in G$ and verifies that ϕ is independent of the choice of α (hence well defined on all of M) and that ϕ is a bundle isomorphism.

NOTE 1.13. The group of cocycles, $Z(M,\underline{G})$, and the set $H^1(M,\underline{G})$ are part of the general theory of cohomology with coefficients in (a possibly) non-abelian group. The presheaf used is the presheaf of continuous functions from M to G. Note that elements of $H^1(M,\underline{G})$ are defined as equivalence classes over refinements of the cover of M chosen. If we take P_1 and P_2 to be the same bundle in part three of the proposition we see that the cover chosen for a trivialisation does not affect the cohomology class of the resulting transition functions. Part 3 also gives us a one to one correspondence between isomorphism classes of principal G–bundles and the cohomology class of their transition functions

$$Bun_M(G)/\sim \longleftrightarrow H^1(M,\underline{G})$$

$$< P > \longleftrightarrow [P].$$

Given a set of transition functions, $\{g_{\alpha,\beta}, U_\alpha\}$ one can construct the corresponding principal G–bundle by starting with the disjoint union of locally trivial bundles,

$$X := \amalg_\alpha (U_\alpha \times G).$$

Then one defines an equivalence relation on X, $(x,g) \sim (x',g')$ if and only if $x = x'$ and there are α and β such that $g = g_{\alpha,\beta}(x).g'$ as well as a G–action on $P := X/\sim$ by

$$[(x,g')].g = [(x,g'.g)].$$

It is easy to verify that this G–action is well-defined and that $P(P/G,G)$ is a principal G–bundle with $\{g_{\alpha,\beta}, U_\alpha\}$ as a set of transition functions.

NOTE 1.14. The homotopy invariance of Cech cohomology suggests an important theorem of bundle theory. Given two homotopic maps

$$f_i : N \to M \quad (i = 1,2)$$

and

$$[\xi] \in H^1(M,\underline{G}) \quad \xi = \{g_{\alpha,\beta}, U_\alpha\}$$

we know that the pullback cocycles

$$f_i^*(\xi) = \{g_{\alpha,\beta} \circ f_i, f_i^{-1}(U_\alpha)\}$$

define the same class in $H^1(M,\underline{G})$,

$$f_1^*[\xi] := [f_1^*(\xi)] = [f_2^*(\xi)] := f_2^*[\xi].$$

it remains only to define the pullback of a principal G–bundle to transport this result to $Bun(G)/\sim$.

DEFINITION 1.4. (THE PULLBACK BUNDLE) Let $P(M,G)$ be a principal G–bundle and let $f : N \to M$ be a continuous map. The pullback of P over N, $f^*P(N,G)$, is defined to be the set of pairs $(x,p)_f$ where $x \in N$ and $p \in P_{f(x)}$. We define the projection, π , and G–action of f^*P by

$$\pi((x,p)_f) = x, \quad (x,p)_f.g = (x,p.g)_f \quad \text{for all } x \in N,\ p \in P \text{ and } g \in G.$$

There is a map

$$f' : f^*P \to P$$
$$(x,p)_f \mapsto p.$$

f^*P has the topology that it inherits as a subspace of $N \times P$. One verifies that the projection and G–action defined above are continuous and that f' is a bundle map.

THEOREM 1.2. *(HOMOTOPY INVARIANCE OF THE ISOMORPHISM CLASS OF THE PULLBACK BUNDLE)*

1) Let P(M,G) be a principal G–bundle and let $f_1, f_2 : N \to M$ be homotopic maps. Then the pullback bundles, f_1^*P and f_2^*P, are isomorphic.

2) Let $P_1(N,G)$ and $P_2(N,G)$ be two principal G–bundles and let ϕ be a G–bundle morphism from P_2 to P_1. Then P_2 is isomorphic to the pullback bundle $\phi'^*(P_1)$ where ϕ' is the map of basespaces induced by ϕ.

PROOF. Part 1: As noted above, cohomology theory says that

$$f_1^*[P] = f_2^*[P]$$

and by 1.1(3) it follows that

$$< f_1^*P > = < f_2^*P > .$$

Part 2: Put

$$\psi : P_2 \to \phi'^*(P_1)$$
$$p \mapsto (x, \phi(p))_{\phi'}$$

for all $p \in P_2$ and $x \in M$ given by $\pi_{P_2}(p) = x$. By definition $\phi(p) \in P_{1\phi'(x)}$ so ψ is well–defined

$$\begin{aligned}\psi(p.g) &= (\pi_{P_2}(p.g), \phi(p.g)) \\ &= (\pi_{P_2}(p), \phi(p).g) \\ &= (x, \phi(p)).g \\ &= \psi(p).g\end{aligned}$$

Hence, leaving the reader to verify continuity, ψ is an isomorphism since it is a G–bundle morphism in $Bun_M(G)$. c.f. Note 1.11. □

2. THE HOMOTOPY CATEGORY

The above result tells us that our investigation of $Bun(G)/\sim$ can be moved into the homotopy category of pointed sets since the isomorphism class of a pullback principal G–bundle is invariant under homotopies of the pullback map. We shall now digress briefly into homotopy theory to continue our investigation of the classification problem for $Bun_M(G)/\sim$.

In order to gain the benefits of the general setting afforded by homotopy theory we need to be precise about the category of sets in which we are working. The largest amenable category is SP_0, the category of pointed, path connected topological spaces with pointed continuous maps. (Note that matters arising due to base points are peripheral to my concerns and notationally distracting so I shall generally suppress base points in my notation but it should be remembered that all spaces and maps in SP_0 and CW_0 are pointed.) Given a continuous map between two pointed topological spaces, $f : X \to Y$, I shall denote the class of pointed spaces homotopic to X and Y by (X) and (Y) and the homotopy class of the map f by $[f]$. If $X \in (Y)$ then I shall write $X \simeq Y$ for "X is homotopic to Y". We write HSP_0 for the category $SP_0/\simeq$ of homotopy classes of pointed spaces and homotopy classes of pointed continuous maps.

Husemoller (1966) demonstrates the important classification results for principal G–bundles in the full generality of the category of pointed, topological spaces, SP_0 and it would be foolish to overlook the generality of this result. However, it is much easier and sufficient for many of the purposes of this thesis to work in an important subcategory of SP_0. This is the category of pointed, path connected spaces with the homotopy type of a CW–complex, CW_0. CW–complexes are significantly better behaved than general topological spaces. Any map between two CW-complexes whose associated maps on the homotopy groups are all isomorphisms (a weak homotopy equivalence) is, in fact, a homotopy equivalence. Moreover, CW_0 is a large subcategory of SP_0 in at least the following senses. It contains all differentiable manifolds (of infinite and finite dimension), it is closed under the operation of taking continuous loops and CW-complexes can be used to approximate arbitrary spaces in SP_0. That is, for ever space $Y \in SP_0$ there is a space $X \in CW_0$ and a weak homotopy equivalence $f : X \to Y$. If X_1 and X_2 are two CW-approximations to Y then then $X_1 \simeq X_2$. This can be used to define a further equivalence relation, weak homotopy equivalence, on SP_0. $Y_1 \sim Y_2$ if and only if there is an $X \in CW_0$ and weak homotopy equivalences, $f_i : X \to Y_i$ $(i = 1, 2)$. I shall write $((Y_1))$ for the weak homotopy class of Y_1. Note that if $f : Y_1 \to Y_2$ is a weak homotopy equivalence then $Y_1 \sim Y_2$. For a summary and proof of these results about CW-complexes see, for example, Spanier (Ch 7, sections 6 and 8).

From an abstract point of view, algebraic topology can be regarded as the study of functors and cofunctors from SP_0, to some other category of pointed sets, SS which descend to functors on HSP_0. Such (co)functors are called homotopy (co)functors. The algebra comes in when we consider further structures on SS such as when SS is the category of of groups with homomorphisms for maps, GG. The topology comes from using the homotopy category so that spaces of the same homotopy type correspond to the same algebraic group and homotopic maps give rise to the same homomorphisms.

To be specific, let us consider an obvious (co)functor by chosing a topological space, X, in SP_0 and defining the functor and cofunctor

$$\begin{aligned} \phi_X : SP_0 &\to XX \\ Y &\mapsto [X, Y] \\ \phi^X : SP_0 &\to YY \\ Y &\mapsto [Y, X]. \end{aligned}$$

Here $[X, Y]$ denotes the homotopy classes of maps from X to Y. The categories XX and YY are categories of pointed sets, the point (in the latter case) being the homotopy class of the trivial map

$$O : Y \to x_0$$

for x_0 the preferred point of X. The morphisms of X and Y are defined by pre and post compositions with maps $f : Y \to Y'$. For example, if $[g] \in [X, Y]$ then

$$\phi_X(f)([g]) = f \circ g.$$

It is elementary that ϕ^X and ϕ_X descend to $\phi^{(X)}$ and $\phi_{(X)}$ on HSP_0.

An obvious and crucial question for algebraic topology is "Under what conditions on X is ϕ_X (or ϕ^X) a (co)functor into the category of groups?". Spaces for which ϕ_X is such a functor are called co–H spaces and spaces for which ϕ^X is such a cofunctor are called H spaces. (For a look at the preliminary theory of H spaces and co–H spaces the reader is referred to Whitehead pp 116–127). I will mention only some salient points without proof.

CONSISTENCY: If X_1 is a co–H space and X_2 is an H space then the group structures defined on $[X_1, X_2]$ are the same. The homotopy class of the trivial map is always the identity.

PLURALITY: In general, a space may have more than one H or co–H structure defined on it.

TOPOLOGICAL GROUPS: Topological groups are always H spaces under pointwise multiplication of functions.

LOOPS AND SUSPENSIONS: For any spaces X and Y the sets $[\Sigma X, Y]$ and $[X, \Omega_c Y]$ are isomorphic groups. In general, ΣX is always a co–H space where ΣX is the suspension of X

$$\Sigma X := X \times I/\sim$$

$$(x_1, t_1) \sim (x_2, t_2) \quad \text{if and only if} \quad t_1 = t_2 = 0, 1 \quad \text{or} \quad x_1 = x_2 = x_0$$

$\Omega_c X$ is the space of continuous, based loops $\{l : S^1 \to X\}$ equipped with the compact open topology. Since the spheres are all suspensions, $S^q \simeq \Sigma S^{q-1}$, we see another route to demonstrating that the homotopy groups are well–defined and well behaved under homotopy.

3. CLASSIFYING SPACES OF HOMOTOPY COFUNCTORS

A classifying space for a homotopy cofuntor $\phi : SP_0 \to SS$ is a space $B \in SP_0$ such that $\phi = \phi^B$ and of course all spaces in the homotopy class of B have this property. An element $u \in \phi(B)$ is called universal for ϕ if and only if for every space $X \in SP_0$ and every element $v \in \phi(X)$ there is a unique homotopy class of maps, $[f_{v,B}]$ with $f_{v,B} : X \to B$ such that

$$v = \phi(f_{v,B})(u).$$

A universal element defines a one to one correspondence between elements $\phi(X)$ and $[X, B]$ for all spaces, $X \in SP_0$ via

$$v \longleftrightarrow (f_{v,B}).$$

It is elementary to show that for any pair of classifying spaces with universal elements, $\{B_1, u_1\}$ and $\{B_2, u_2\}$, that the maps f_{u_1,B_2} and f_{u_2,B_1} define an homotopy between B_1 and B_2. Thus, the classifying space for any cofunctor, if it exists, is uniquely defined up to homotopy.

Everything said above about classifying spaces could have been restricted to CW_0 and it is remarkable (see Spanier Ch 7 section 7) that in this case every homotopy cofunctor, ϕ, from CW_0 into a category of pointed sets has a classifying space, $B_{CW_0} \in CW_0$ and a universal element, $u_{CW_0} \in \phi(B_{CW_0})$ (note that my definition of universal element differs from Spanier's but the content of some of his results is that the two definitions are equivalent). There are two homotopy cofuntors of particular interest to us.

3.1. CASE 1 - ABELIAN CECH COHOMOLOGY.

$$\phi : CW_0 \to GG, \quad X \to H^q(X, A)$$

where A is an abelian group. In this case the classifying spaces are the Eilenberg–MacLane spaces, $K(A, q)$. It is an important theorem of algebraic topology that the Eilenberg–MacLane spaces are completely classified up to weak homotopy by the property

$$\pi_p(K(A,q)) = \begin{cases} 0 & \text{if } p \neq q, \\ A & \text{if } p = q. \end{cases}$$

Eilenberg-MacLane spaces are thus uniquely classified up to homotopy equivalence in CW_0. Eilenberg-MacLane spaces play a crucial role in obstruction theory and attempts to decompose spaces in the homotopy category. Since ϕ is a cofunctor into the category of groups all Eilenberg–MacLane spaces are H–spaces and it can be shown that they all possess a unique H structure. Some examples which can be verified by computing their homotopy groups are

(a): S^1 is a $K(\mathbb{Z},1)$)

(b): CP^∞, infinite complex projective space in the topology induced from the norm topology on $\mathcal{H}$, an infinite dimensional Hilbert space, is a $K(\mathbb{Z},2)$.

(c): $U(\mathcal{H})$, is contractible in both the uniform norm (Kuiper (1950)) and strong operator (Dixmier and Douady (1963)) topologies so that the uniform norm projective unitaries, $PU(\mathcal{H})_u$ and the strong operator projective unitaries, $PU(\mathcal{H})_{so}$ are both $K(\mathbb{Z},2)$'s.

(c): $L(\mathbb{Z}_n,\infty)$, infinite Lens spaces are $K(\mathbb{Z}_n,1)$'s.

3.2. CASE 2: NON-ABELIAN CECH COHOMOLOGY/ISOMORPHISM CLASSES OF PRINCIPAL G–BUNDLES.

$$\phi(X) := H^1(M,\underline{G}) = Bun_M(G)/\sim$$

Theorem 1.2 states that ϕ is a homotopy cofunctor on SP_0 and we therefore know that it has a classifying space and a universal bundle when restricted to CW_0. However, Milnor's construction (see, for example, Husemoller (1966)) furnishes a universal bundle, $EG(BG,G)$ over all SP_0. This is a principal G–bundle, $EG(BG,G)$ such that for all spaces $M \in SP_0$ and bundles $P \in Bun_M(G)$ there is a map,

$$f : M \to BG,$$

defined uniquely up to homotopy such that

$$< P >=< f^*EG > .$$

Thus we have a second answer to the classifying problem for isomorphism classes of principal G–bundles over a space M.

$$Bun_M(G)/\sim = [M,BG] = H^1(M,\underline{G})$$

The map f is called the classifying map for P and from now on I shall write, where desirable, $P(M,G,f)$ for a principal G–bundle with classifying map f. Note that we now have a topological way to answer the primarily algebraic question, under what circumstances is $H^1(M,\underline{G})$ a group for all M? This is so if and only if BG is an H–space.

The existence of classifying spaces and universal bundles for principal G–bundles is an initially surprising and important theorem of bundle theory. Typically, however, classifying spaces are complicated topological spaces and, aside from Milnor's construction, it can be difficult to find concrete examples of universal bundles or classifying spaces. However, when one restricts oneself to CW_0 one can make better progress. Let $E_{CW_0}G(B_{CW_0}G, G)$ denote any bundle which is universal for the homotopy cofunctor $H^1(\_,\underline{G})$ restricted to CW_0. Note that though there is a $B_{CW_0}G \in CW_0$, $B_{CW_0}G$ need not be a CW–complex. It is a theorem that a principal G–bundle, $P(M,G)$, is a CW_0–universal bundle if and only if its total space is weakly contractible. That is, $\pi_q(P) = 0$ for all q. In general, one can define the notion of an n–CW_0–universal principal G–bundle, $E^n_{CW_0}G(B^n_{CW_0}G, G)$, as one which acts as a classifying space for all CW-complexes of dimension n or less. So that if M is a CW-complex and $\dim M \leq n$

$$Bun(M,G)/\sim \quad = \quad [M, B^n_{CW_0}G]$$

It is a theorem that a principal G–bundle $P(M,G)$ is n–universal if and only if it is n–connected. That is

$$\pi_q(P) = 0 \quad \text{for all } q \leq n.$$

One can let $n \to \infty$ and attain the result on all CW_0. (For proofs of these results see Husemoller pp 53 – 57 and Steenrod pp 99 - 101). The contractibility of the total spaces of CW_0–universal bundles means that the classifying map, f, of $E_{CW_0}G(B_{CW_0}G, G, f)$ is a weak homotopy equivalence. Thus $B_{CW_0}G$ defines a weak homotopy class of spaces with the classifying map of any CW_0–universal bundle over a CW-complex realising the CW_0-approximation of it's image. An interesting example of this situation arises with realisation of $E_{CW_0}S^1$ as $U(\mathcal{H})(PU(\mathcal{H})_u, S^1)$. $PU(\mathcal{H})_u$ is in CW_0 since it is a Banach Lie group and in particular an infinite dimensional manifold. It is known, however, that $PU(\mathcal{H})_{so}$ is not homotopic to $PU(\mathcal{H})_u$ and thus $U(\mathcal{H})_{so}(PU(\mathcal{H})_{so}, S^1)$ is a CW_0–universal bundle with its base space in SP_0 but not in CW_0. $PU(\mathcal{H})_u$ is thus a CW_0-approximation of $PU(\mathcal{H})_{so}$ The identity map $id : U_u \to U_{so}$ defines a continuous S^1–bundle morphism between $U_u(\mathcal{H})(PU_u(\mathcal{H}), S^1)$ and $U(\mathcal{H})_{so}(PU(\mathcal{H})_{so}, S^1)$. The associated commutative diagram of homotopy groups gives that $id : PU(\mathcal{H})_u \to PU(\mathcal{H})_{so}$ is a weak homotopy equivalence).

The Gauss map first suggested the existence of universal bundles. Let's consider a principal bundle, P(M,SO(n)), of the classical group SO(n) ($n > 1$) over a closed manifold M. These bundles are in bijective correspondence with oriented R^n bundles over M. The former being the frame bundles of the latter. Consider the oriented Steiffel manifold

$$V(n,N) := \{(v,W) : v \in W,\ W \text{ an } n-\dim \text{ oriented subspace of } R^N\}.$$

$V(n,N)$ defines a principal–SO(n) bundle over the oriented Grassmanian

$$G(n,N) := \{W : W \text{ is an } n - \dim \text{ oriented subspace of } R^N\}$$

and one can verify that it is n–universal for large N. We now embed P in R^N for some large N and define

$$f : M \to G(n,N)$$

$$x \mapsto P_x.$$

One can see without too much difficulty that f is the classifying map for P. If we let $N \to \infty$ then $V(n, \infty)$ is contractible and the infinite, oriented Grassmanian $G(n) = G(n, \infty)$ is a CW_0–classifying space for $SO(n)$.

Another way, this time in the complex case, to discover the infinite Grassmanian as a CW_0-classifying space is via Kuiper's contractibility result for $U(\mathcal{H})_u$. We embed $U(n)$ in $U(\mathcal{H})_u$ by splitting off the first n vectors from an orthonormal basis for $\mathcal{H}$. $\mathcal{H} := C^n \oplus \mathcal{H}'$. Then $U(n)$ is the closed subgroup of $U(H)_u$ whose elements fix every vector in $\mathcal{H}'$ and $U(\mathcal{H}')_u$ is the closed subgroup whose elements fix every vector in C^n. Note that $U(\mathcal{H}')_u$ is contractible, that the obvious right actions of these subgroups on $U(\mathcal{H})_u$ commute and that $U(\mathcal{H}')_u$ is a closed sub-Banach Lie group of $U(\mathcal{H})_u$. Thus $U(\mathcal{H})_u(U(\mathcal{H})_u/U(\mathcal{H}')_u, U(\mathcal{H}')_u)$ is a principal $U(\mathcal{H}')_u$–bundle (see Note 1.6) and so $E = (U(\mathcal{H})_u/U(\mathcal{H}')_u)$ is contractible. This entails that $E(E/U(n), U(n))$ is a CW_0–universal bundle (see Note 1.6 and remember that $U(n)$ is a compact Lie group). We have recovered the Grassmanian as a CW_0-classifying space for $U(n)$ since $G(n) = E/U(n)$. Now, the Peter Weyl Theorem states that every compact Lie group, G, can be embedded in $U(n)$ for some n so the inclusion $G \hookrightarrow U(n)$ induces a free right action of G on E. Thus, so long as the inclusion is closed, $E(E/G, G)$ is a CW_0–universal bundle for the compact Lie group G.

Although it is often the case there is no general guarantee that one can find a closed embedding of a topological group or even a Lie group in $U(\mathcal{H})_u$ (or even $U(\mathcal{H})_{so}$ which would could suffice since $U(\mathcal{H})_{so}$ is also contractible). A general construction for the classifying space of any topological group was provided by Milnor. For a full account the reader is referred to Husemoller (Ch 4 section 11). In this construction we see the importance of the condition of numerability in the definition of a principal G–bundle. One starts by defining the infinite–join of G as the set of formal sums

$$X = \{(t, g) = \sum_{i=1}^{\infty}(t_i, g_i)\}$$

where $t_i \in [0, 1]$, $g_i \in G$, only finitely many t_i are non-zero and $\sum_{i=1}^{\infty} t_i = 1$. Then one places an equivalence relation on X by

$$(t, g) \sim (t', g') \quad \text{if and only if } (t = t' = 0) \text{ or } (t = t' \text{ and } g = g').$$

and sets

$$EG := X/\sim .$$

To place a topology on EG the following maps are used,

$$t_j : EG \to [0, 1]$$

$$\sum(t_i, g_i) \mapsto t_j$$

$$g_j : t_j^{-1}((0, 1]) \to G$$

$$\sum(t_i, g_i) \mapsto g_j.$$

EG is given the coarsest topology which makes all the g_j and t_j continuous. With respect to this topology, the G–action

$$\sum(t_i, g_i).g := \sum(t_i, g_i.g)$$

is continuous. One can show that EG is thus the total space of a principal G–bundle, $EG(BG, G)$ where $BG = EG/G$ and one proves that EG is a universal bundle for SP_0. I shall merely outline the key step in defining the classifying map in this proof. Given a principal G–bundle, P(M,G), a trivialisation (h_α, U_α) with transition functions $\{g_{\alpha,\beta}, U_\alpha\}$ and subordinate partition of unity $\{\rho_\alpha, U_\alpha\}$, define the classifying map for P with $x \in U_{\alpha_0}$ by

$$f_{\alpha_0} : M \to BG$$
$$x \mapsto [\sum_\alpha(\rho_\alpha(x), g_{\alpha,\alpha_0}(x))]$$

(here, [ ] denotes equivalence classes of the G–action in E/G). The map is independent of the choice of α_0 since, if one chooses another α_i with $x \in U_\alpha$ then

$$\begin{aligned} f_{\alpha_1}(x) &= [\sum_\alpha(\rho_\alpha(x), g_{\alpha,\alpha_1}(x))] \\ &= [\sum_\alpha(\rho_\alpha(x), g_{\alpha,\alpha_0}(x).g_{\alpha_0,\alpha_1}(x))] \\ &= [\sum_\alpha(\rho_\alpha(x), g_{\alpha,\alpha_0}(x))] \\ &= f_{\alpha_0}(x) \end{aligned}$$

4. CHARACTERISTIC CLASSES

We have seen that principal G–bundles are classified by the first cohomology class defined by their transition functions. In general, principal G–bundles give rise to a number of classes in the integral cohomology of their base spaces which contain information about the bundle and are cofunctorial. Such cohomology classses are called characteristic classes of the bundle. More precisely we have the following.

DEFINITION 1.5. (CHARACTERISTIC CLASSES) A characteristic class, $c(P)$, of a principal G–bundle, $P(M, G)$ is an element of the the cohomology of the base space of the bundle which is cofunctorial on SP_0. Precisely, a characteristic class is a map between the cofunctors $H^1(\_,\underline{G})$ and $H^n(\_,\underline{A})$, where A is an abelian group, usually the integers, $c(P) \in H^n(M, \underline{A})$. If $f : N \to M$ then $c(f^*P) = f^*(c(P))$.

NOTE 1.15. We shall encounter many characteristic classes in the next two chapters and discover some of the information they hold in terms of the possibility of the reduction and extension of the structure group of P.

NOTE 1.16. Since characteristic classes are cofunctorial, they are determined by their value on the universal bundle, EG. Hence, if $P = P(M, G, f)$ then

$$c(P) = c(f^*P) = f^*(c(EG)).$$

We shall call $c(EG)$ the universal characteristic class of c or simply the universal class.

The cohomology groups of the classifying space BG thus completely determine the characteristic classes of principal G–bundles. Since there is not, in general, an exact cohomology sequence for fibrations as there is for homotopy more demanding and subtle methods have to be used to investigate the relations between the cohomology groups of total space, fibre space and base space. The most important of these methods is that of spectral sequences to which I now briefly digress.

DEFINITION 1.6. (SPECTRAL SEQUENCES) For a complete treatment the reader is referred to Chapter 14 of Bott and Tu. Given a bi–graded algebra, $E = \sum_{p,q}(E^{p,q})$, with grading horizontal filtration, E^n and differential D such that

1. $E^n := \sum_{p\geq n,\ q\geq 0}(E^{p,q})$
2. $F^n := \sum_{p+q=n}(E^{p,q})$
3. $D^n := D|F^n : F^n \to F^{n+1}$,

a spectral sequence for the pair $\{E, D\}$ is a sequence of bi–graded differential algebras $\{E_r, d_r\}$ such that

$$E_{r+1} = H(E_r, d_r)$$

and

$$\{E_r, d_r\} \to GH(E, D) \quad \text{as} \quad r \to \infty.$$

Here, H denotes taking cohomology with respect to d_r and G denotes the natural horizontal grading induced on $H(E, D)$ via the filtration E^n.

$$GH(E, D) = \sum_{n=1}^{\infty} HE^n/HE^{n+1}$$

THEOREM 1.3. *(LERAY SPECTRAL THEOREM) Let $P(M, F)$ be a locally trivial fibre bundle and A an abelian group. Then there is a spectral sequence of bigraded algebras, $\{E_r^{(p,q)}(P), d_r\}$ converging to the natural horizontal grading of the total Cech cohomology of P, $GH^*(P, A)$. The second term in the spectral sequence is derived from the cohomologies of the fibre and base space via*

$$E_2^{p,q}(P) = H^p(M, H^q(\underline{F}))$$

where $H^q(\underline{F})$ denotes a presheaf which is locally the constant sheaf $H^q(F, A)$.

NOTE 1.17. The spectral sequence is defined by taking a good cover, $\{U_\alpha\}$ for M and lifting it via the projection, π , of the bundle to P. One then defines the first graded, algebra of the sequence to be

$$E^0_{(p,q)}(P) := C^p(M, C^q(\underline{F}))$$

This is the group of Cech p–cocycles defined over the cover $\{\pi^{-1}(U_\alpha)\}$ of P, taking values in the presheaf of Cech q–cocyles on P. Since $\{U_\alpha\}$ is a good cover, $\pi^{-1}(U_\alpha)$ is homotopic to F for all α and hence, under the homotopy invariance of Cech cohomology, we arrive at the cohomology groups of the fibre, F, after applying d_1.

NOTE 1.18. Calculation is much more tractable if the presheaf $H^q(\underline{F})$ is globally constant so that we have standard cohomology with the coefficient group $H^q(F, A)$. This is the case if the action of $\pi(B)$ on the cohomology of the fibres is trivial. Clearly this is the case if $\pi(B) = 0$. A second way this can occur is in the crucial hypothesis of the Leray–Hirsch spectral theorem, namely, that there be global cocycles on in $H^q(P, A)$ which restrict to generate $H^q(F, A)$ for all q. In this case the Leray–Hirsch theorem states that,

$$GH^q(P, A) = \sum_{i+j=q} H^i(M, A) \otimes H^j(F, A).$$

NOTE 1.19. One can easily verify that all trivial bundles, $P = M \times F$, satisfy the Leray–Hirsch hypothesis and hence we derive a version of the Kunneth formula.

NOTE 1.20. The total Cech cohomology of any space M, $H^*(M, A)$ is a graded algebra under the multiplication defined by the cup product (See Bott and Tu for a definition). It is of note that even though each d_r is a graded algebra homomorphism, this does not guarantee that $H^*(P, A)$ and $GH(E_\infty, d_\infty)$ are isomorphic as graded algebras.

NOTE 1.21. A morphism between two spectral sequences, $\{E_r, d_r\}$ and $\{F_r, d_r\}$ is a collection of algebra homomorphisms

$$f_r : E_r \to F_r$$

which satisfy

1. $f_r(E_r^{(p,q)}) \in F_r^{(p,q)}$
2. $f_r(d_r^E(x)) = d_r^F(f_r(x))$.

When $\phi : Q(N, G) \to P(M, G)$ is a principal bundle morphism, there is a pullback morphism of spectral sequences from $E_r(P)$ to $E_r(Q)$.

$$\phi_r^* : E_r(P) \to E_r(Q).$$

Let $x \in E_0^{p,q}(P) = C^p(P, C^q(P, A))$. x is just a Cech p–cycle on P with values in the presheaf $C^q(P, A)$ and $\phi^*(x)$ is defined to be the pullback of x, $\phi^*(x) \in C^p(Q, C^q(Q, A)) = E_0^{p,q}(Q)$. One can show that ϕ^* commutes with each d_r and so can be defined for all r.

NOTE 1.22. The machinery of spectral sequences is not restricted in its application to principal bundles or even fibre bundles. All of the above can be carried over to fibrations and fibration maps.

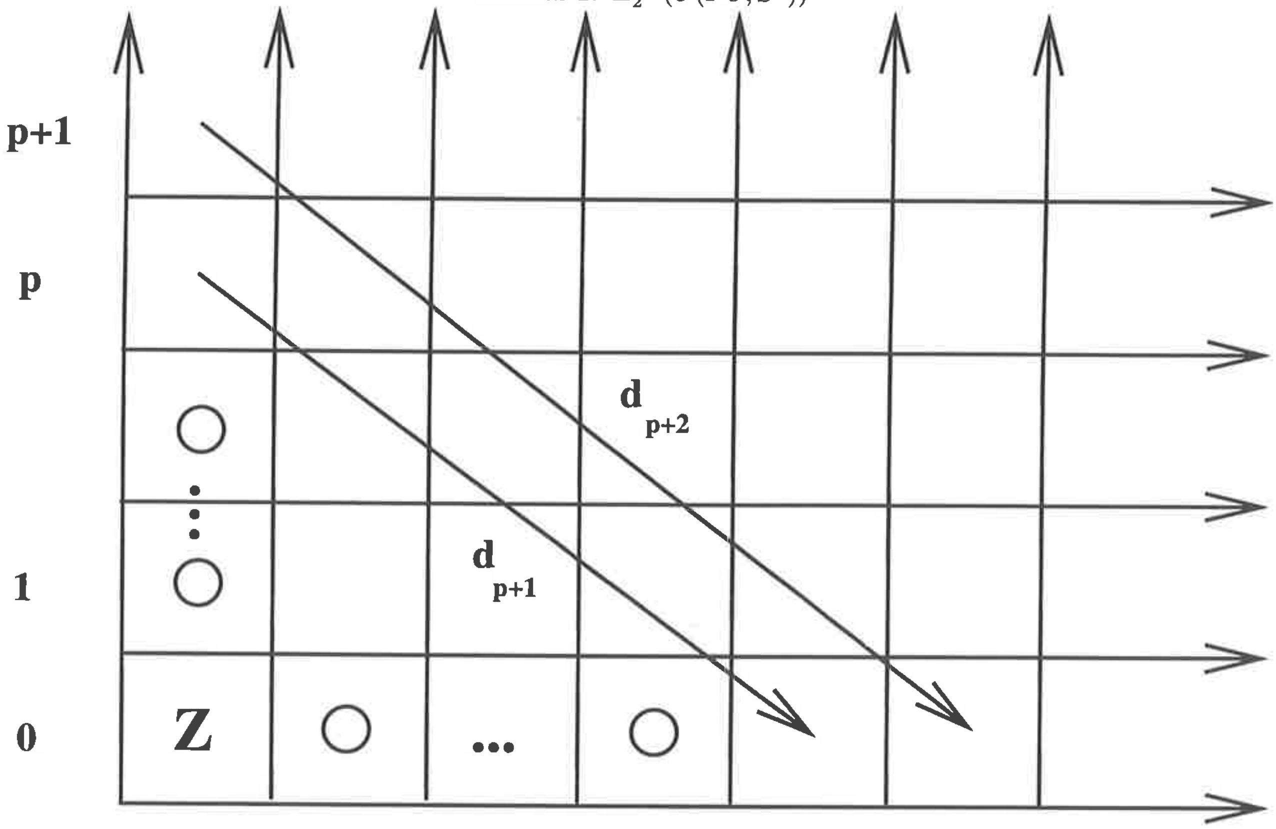
5. COHOMOLOGY OF $BS^1 \sim PU$

As an example I shall calculate the cohomology of BS^1 which I realise as the infinite projective unitaries, PU. We saw above that $PU \simeq K(\mathbb{Z}, 2)$ and hence is simply connected. Now we apply the Leray spectral sequence to the full unitary group, U, of Hilbert space, $U = U(PU, S^1)$.

We see in Figure 1 that $d_r = 0$ for $r > 2$ and hence, since the total space is contractible, that each d_2 is an isomorphism. Thus setting e to be a generator of $H^1(S^1, \mathbb{Z})$,

$$d_2(e) = c_1$$

FIGURE 1. $E_2^{p,q}(U(PU, S^1))$



and we deduce

$$H^*(PU, \mathbb{Z}) = [c_1].$$

Here c_1 is called the first Chern class of the universal bundle $U(PU, S^1)$ and $[c_1]$ denotes the polynomial ring generated by c_1.

CHAPTER 2

REDUCTION OF STRUCTURE GROUP

1. GENERAL THEORY

The structure group of a general fibre bundle contains important information about the bundle. It is therefore important to know when one can alter the structure group, reduce or extend it, without in some sense altering the bundle. This chapter is concerned with the former situation and in a sense, we shall be investigating when a bundle uses only a subgroup, H, of its structure group, G. My treatment closely follows Husemoller's (Ch 6) but I consider the more general case where H need not be closed in G and the more specific case where $\pi : G \to G/H$ defines a principal H–bundle.

DEFINITION 2.1. (A REDUCTION OF THE STRUCTURE GROUP) We say that the structure group of principal G–bundle, $P(M,G)$ reduces to H if we can choose transition functions, $\{g_{\alpha,\beta}, U_\alpha\}$ for P with range in H for all α, β. A reduction of the structure group of P to H is the class in $H^1(M, \underline{H})$ of such a choice of transition functions.

NOTE 2.1. This definition differs from Husemoller's but I use it since it is more intuitive. Theorem 2.2 will prove the equivalence of our definitions.

NOTE 2.2. If $\phi : P(M,G) \cong P'(M,G)$ is an isomorphism of principal G–bundles and the structure group of P reduces to H for transition functions defined by the trivialisation $\{s_\alpha, U_\alpha\}$ then $\{\phi \circ s_\alpha, U_\alpha\}$ is a trivialisation of P' giving rise to transition functions taking values only in H. Hence the structure group of P reduces to H if and only if the structure group of P' reduces to H for all $P' \in < P >$. In fact, since the complete attention of this thesis is on properties of principal G–bundles invariant over their isomorphism classes I will adopt a sloppiness in notation for isomorphism classes and their representatives.

NOTE 2.3. A reduction of structure group need not be unique in the sense that there may be a number of different choices of transition functions for P in the class $[\{g_{\alpha,\beta}, U_\alpha\}] \in H^1(M, \underline{G})$ which are members of different classes in $H^1(M, \underline{H})$.

LEMMA 2.1. *(THE G/H-FIBRE BUNDLE ASSOCIATED TO A PRINCIPAL G-BUNDLE) Suppose that $P(M,G)$ is a principal G-bundle and H a closed subgroup of G. Then $P/H = P/H(M, G/H, G/S)$ is a fibre bundle with fibre G/H, structure group G/S and transition functions $\{\rho(g_{\alpha,\beta}), U_\alpha\}$. Where S is the closed, normal subgroup*

$$S := \cap_{g \in G}\ gHg^{-1}$$

and ρ is the canonical projection

$$\rho : G \longrightarrow G/S.$$

PROOF. One defines the projection, $\pi_{P/H}$, of $P/H(M, G/H, G/S)$ on any class, $[p]$ of P/H with $p \in P$ by

$$\pi_{P/H}([p]) = \pi_P(p).$$

The fibres of this projection, $\pi_{P/H}^{-1}(x)$, $x \in M$, are clearly homeomorphic to G/H and local triviality arises from the local triviality of P. The alteration of structure group arises as follows. G acts on the left of G/H by

$$g.(g'H) = (g.g')H.$$

Under this action the stabiliser of the coset gH is the group gHg^{-1}. Hence

$$S := \cap_{g \in G}(g^{-1}Hg)$$

is the trivial part of the action of G on G/H and we arrive at the new structure group for P/H which is G/S. □

NOTE 2.4. In the case where H is a normal subgroup of G then G/H is a group and $S = H$ so that P/H is a principal G/H–bundle.

NOTE 2.5. Since S is a normal subgroup of G, P/S is a principal G/S–bundle over M and, in fact, P/H is the G/H–bundle associated to P/S.

$$P/H = (P/S \times G/H)/(G/S)$$

Lemma 2.1 is concerned with the fibre-bundle structure of P/H but in the case where $\pi : G \to G/H$ defines a principal G–bundle (see Note 1.6) it follows that P/H is the base space of $P(P/H, H)$ which is a principal H–bundle (since local sections of $\pi : G \to G/H$ and the local triviality of $P(M, G)$ allow one to define local sections of $\pi_P : P \to P/H$). Thus P has the structure of a bundle of bundles. This situation is nice since when we take $P = EG$ (or even merely $E_{CW_0}G$) to be the universal G–bundle then $EG(EG/H, H)$ is a principal H–bundle with weakly contractible total space and hence a CW_0–universal H–bundle. Thus we have realised $B_{CW_0}H = EG/H$ as a G/H-fibre bundle over BG. We can, however, still make a good deal of progress in cases where $\pi : G \to G/H$ does not define a principal H-bundle. One considers the transition functions of a universal H–bundle $EH(BH, H)$ as taking their values in G and thus they define a principal G–bundle, $B(BH, G, g)$ over BH (equivalently, on sets $B := (EH \times G)/H$). For the question of the reduction of structure group, it is sufficient to consider the classifying map of B, $g : BH \to BG$, in the homotopy category HSP_0 and take $[g]$ as a fibration. The problem, however, is that in this case the homotopy type of the fibre of $[g]$ is not, in general, known.

We are now in a position to state the central theorem of this chapter.

THEOREM 2.2. *(NECESSARY AND SUFFICIENT CONDITIONS FOR THE REDUCTION OF STRUCTURE GROUP OF A BUNDLE) Let $P(M, G, \pi_P, f)$ be a principal G–bundle.*

(I) There is a one to one correspondence between the following.

1) Reductions of the structure group of P to H.

2) Elements of the set $i^{-1}([P]) \subseteq H^1(M, \underline{H})$ where i is the map on first cohomology induced by its namesake,

$$i : H \hookrightarrow G$$

$$i : H^1(M, \underline{H}) \to H^1(M, \underline{G}).$$

3) Isomorphism classes of H-bundles, $< R(M, H) >$, such that there is a bundle inclusion

$$i : R \hookrightarrow P.$$

4) Homotopy classes of maps, $\hat{f} : M \to BH$ such that (with g as defined above) $f \simeq g \circ \hat{f}$.

(II) If H is closed in G then the structure group of P reduces to H if and only if $P/H(M, G/H, G/S, \pi_{P/H})$ has a global section.

(III) If H is normal in G then $i^{-1}([P])$ is non-empty if and only if $\rho([P]) = 0$ where ρ is the map on first cohomology induced by its namesake the canonical projection,

$$\rho : G \longrightarrow G/H$$

$$\rho : H^1(M, \underline{G}) \to H^1(M, \underline{G/H}).$$

(IV) If $\pi : G \to G/H$ defines a principal H–bundle then the structure group of P reduces to H if and only if and only if there is a map $\hat{f}$ such that $f = \pi_{EG/H} \circ \hat{f}$. Moreover, $\pi_{EG/H} : EG/H \to BG$ is the classifying map for $B := (EG \times G)/H$ and, in the case that $M \in CW_0$, the reductions of the structure group of P to H correspond bijectively with homotopy classes of maps $\hat{f}$

PROOF. (I)

($1 \leftrightarrow 2$) 2 is just a restatement of the definition of a reduction of structure group.

($2 \to 3$) If $[\xi] = [\{h_{\alpha,\beta}, U_\alpha\}] \in H^1(M, \underline{H})$ and $i([\xi]) = [P]$ then we may construct bundles R and P' such that $[R] = [\xi]$ and $[P'] = [P]$ following Note 1.13. There is an obvious bundle inclusion of R in P' and, since $P' \cong P$ there is a bundle inclusion $i : R \hookrightarrow P$.

($2 \leftarrow 3$) Given $i : R \hookrightarrow P$, assume, by refining if necessary, that a trivialisation, $\{t_\alpha, U_\alpha\}$, of R trivialises P also. Let $\{h_{\alpha,\beta}\}$ be the transition functions associated to this trivialisation. Setting $s_\alpha = i \circ t_\alpha$, yields local sections of P whose induced transition functions,

$$g_{\alpha,\beta}(x) = s_\alpha(x)/s_\beta(x) \quad x \in U_{\alpha,\beta}$$

$$= t_\alpha(x)/t_\beta(x)$$

are precisely $\{h_{\alpha,\beta}\}$. Thus $[R] \in i^{-1}([P])$.

$(3 \to 4)$ Given $[EG] \in H^1(BG, \underline{G})$ and $[EH] \in H^1(M, \underline{H})$ we have that $i([EH]) = g^*[EG]$. If there is an H–bundle, $R(M, H, \hat{f})$ and inclusion $i : R \hookrightarrow P$ then we have,

$$\begin{aligned} f^*[EG] &= [P] \\ &= i([R]) \\ &= i(\hat{f}^*[EH]) \\ &= \hat{f}^* i([EH]) \quad \text{(This is elementary)} \\ &= \hat{f}^* g^*[EG] \\ &= (g \circ \hat{f})^*[EG]. \end{aligned}$$

Hence, since EG is a universal bundle, $f \simeq g \circ \hat{f}$.

$(3 \leftarrow 4)$ If there is an $\hat{f}$ such that $f \simeq g \circ \hat{f}$ then setting $R := \hat{f}^* EH$ we have that

$$\begin{aligned} i([R]) &= i(\hat{f}^*[EH]) \\ &= \hat{f}^* i([EH]) \\ &= \hat{f}^* g^*[EG] \\ &= (g \circ \hat{f})^*[EG] \\ &= f^*[EG] \\ &= [P]. \end{aligned}$$

(II) $(\Rightarrow)$ Suppose the structure group of P reduces to H. Let π_H be the canonical projection

$$\pi_H : P \longrightarrow P/H$$

Given local sections, s_α, of $P \to M$ whose associated transition functions, $\{g_{\alpha,\beta}\}$, take values in H, we define a global section of π_P/H by setting

$$s'_\alpha := \pi_H \circ s_\alpha : U_\alpha \to P/H|U_\alpha.$$

The s'_α piece together to yield a global section of $\pi_{P/H}$ since, given $x \in U_{\alpha,\beta}$,

$$\begin{aligned} s'_\beta(x) &= \pi_H(s_\beta(x)) \\ &= \pi_H(s_\alpha(x).g_{\alpha,\beta}(x)) \\ &= \pi_H(s_\alpha(x)) \quad (\text{since } g_{\alpha,\beta}(x) \in H) \\ &= s'_\alpha(x) \end{aligned}$$

$(\Leftarrow)$ If s is a section of $\pi_{P/H} : P/H \to M$ then one can define a map,

$$\psi : P \longrightarrow G/H$$

$$p \mapsto [p]/s(x)$$

where $p \in P_x$ and $[p]/s(x)$ is the coset in G/H such that for all $g \in [p]/s(x), [p] = g.s(x)$. One then sets $R := \psi^{-1}(e_G.H) \subset P$ and verifies that in the subspace topology from P, $R(M, H, \pi_P|R)$ is a principal H–bundle. (For these details, see Husemoller pp 71-2.) Clearly there is a bundle inclusion $i : R \hookrightarrow P$.

(III) If H is normal in G then we have the short exact sequence of Lie groups

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\rho} G/H \longrightarrow 1$$

which, just as with abelian Cech cohomology, induces a long exact sequence in non–abelian cohomology

$$\begin{aligned} &1 \longrightarrow H^0(M,\underline{H}) \xrightarrow{i} H^0(M,\underline{G}) \xrightarrow{\rho} H^0(M,\underline{G/H}) \\ &\xrightarrow{d} H^1(M,\underline{H}) \xrightarrow{i} H^1(M,\underline{G}) \xrightarrow{\rho} H^1(M,\underline{G/H}). \end{aligned} \tag{1}$$

Once again, we exploit the correspondence between $H^1(M,\underline{G})$ and $Bun_M(G)/\sim$ so that $[P]$ is an element of $H^1(M,\underline{G})$. Since 1 is exact, we see that the transition functions of P can be taken to have values in H if and only if $\rho[P] = 0$. In that case, and only that case, there is a class in $H^1(M,\underline{H})$, $[R]$ say, and corresponding H–bundle, $R(M, H)$, for which

$$i[R] = [P] \quad \text{and}$$

$$i : R \hookrightarrow P.$$

(IV) I first show that the structure group of P reduces to H if and only if there is an $\hat{f}$ such that $f = \pi_{EG/H} \circ \hat{f}$. For simplicity I take $P = f^*EG$. It is easy to check that reducing via the action of H factors through pulling back bundles. i.e. $(f^*EG)/H = f^*(EG/H)$. We have the following commutative diagram.

$$\begin{array}{ccc} P & \xrightarrow{f'} & EG \\ \downarrow & & \downarrow \\ P/H & \xrightarrow{f'_{/H}} & EG/H \\ \downarrow & & \downarrow \pi_{EG/H} \\ M & \xrightarrow{f} & BG \end{array}$$

If the structure group of P reduces to H then $P/H \to M$ has a global section, $s : M \hookrightarrow P/H$ and we set

$$\hat{f} := f_{/H} \circ s : M \to BH.$$

$\hat{f}'$ is clearly continuous. The fact that $f = \pi_{EG/H} \circ \hat{f}'$ follows from the commutativity of the diagram.

Conversely, given $\hat{f}$, put $R := \hat{f}^*(EG(EG/H, H))$. Every point in R is given by a pair $(x,p)_{\hat{f}}$ where $x \in M$ and $p \in EG_{\hat{f}}(x)$. The obvious inclusion map is

$$i : R \hookrightarrow P = f^*EG$$

$$(x,p)_{\hat{f}} \mapsto (x,p)_f$$

and it is straight forward to verify that i so defined commutes with the H–action on R and P and is a homeomorphism onto its image.

That $\pi_{EG/H}$ is the classifying map for $B = (EG \times G)/H$ follows from the fact that $\pi_{EG/H} = id_{EG/H} \circ \pi_{EG/H}$. Hence the structure group of $\pi^*_{EG/H} EG(BG, G)$ reduces to H and

$$
\begin{aligned}
[\pi^*_{EG/H} EG(BG,G)] &= i(id^*_{EG/H}[EG(EG/H,H)]) \\
&= i([EG(EG/H,H)] \\
&= [B] \quad \text{(by definition of } B)
\end{aligned}
$$

so that $B = B(EG/H, G, \pi_{EG/H})$.

Finally, since EG/H is a B_{CW_0} if $M \in CW_0$, then $Bun_M(H)/\sim \; = [M, EG/H]$ and the correspondence claimed is evident. □

NOTE 2.6. Extending the sequence 1 is no easy task since it involves defining second and higher degree non–abelian cohomology. In the case that H is central one can certainly extend with another boundary map, d, to $H^2(M, \underline{H})$ whose definition is well known. We shall also see this sequence go one step further in the next chapter but that will require G to be contractible and hence all it's non-abelian cohomology groups can be taken to be singletons. For a full treatment see Frenkel (1957). Another peculiar feature of the sequence is that, even though not all of the sets in it are groups, we may still speak of it's exactness — the image of one map always equals the pre–image of the trivial co-cycle of the next. One also sees that $H^0(M, \underline{G})$ is precisely the continuous functions from M to G, $C(M, G)$.

NOTE 2.7. When BH is an H–space then $H^1(M, \underline{H})$ is a group and the reductions of structure group of P (should they exist) correspond to members of a left coset of $H^1(M, \underline{H})$ by exactness,

$$i^{-1}([P]) = [R].d(C(M, G/H)).$$

Once again using exactness we may identify

$$d(C(M, G/H)) = C(M, G/H)/i(C(M, G)).$$

NOTE 2.8. It would be nice to claim the equivalence between the reductions of the structure group of P and homotopy classes of maps $\hat{f}$ such that $f = \pi_{EG/H} \circ \hat{f}$ over all of SP_0 but this result currently eludes me. However, since obstruction theory will be one of the primary approaches I use in what follows and obstruction theory is best understood for maps with domain in CW_0, the results of (IV) will be sufficient for many purposes.

As I mentioned, the advantage of the situation where $\pi : G \to G/H$ defines a principal bundle is that we can realise $B_{CW_0} = B_{CW_0}H(B_{CW_0}H, G/H, G/S)$ as a G/H–fibre bundle over $B_{CW_0}G$. Whereas, in general we do not know the homotopy type of the fibre of $g : BH \to BG$ considered as a fibration in HSP_0. The following lemma, however, will allow me to gain enough information about the homotopy groups of the fibre of $[g]$ for all my purposes.

LEMMA 2.3. *Let $i : H \hookrightarrow G$ be an inclusion of topological groups. Then there is a commutative diagram of homotopy groups for all $q \geq 0$.*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_q(BH) & \xrightarrow{\delta} & \pi_{q-1}(H) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle g_{*,q}} & & \downarrow{\scriptstyle i_{*,q}} & & \downarrow \\ 1 & \longrightarrow & \pi_q(BH) & \xrightarrow{\delta} & \pi_{q-1}(G) & \longrightarrow & 1 \end{array}$$

PROOF. Setting $B := (EH \times G)/H = B(BH, G, f)$ let g' be the bundle morphism $g' : B \to EG$ covering g and let i be the obvious bundle morphism $i : EH \to B$ covering $id_{EG/H}$. Then $g' \circ i : EH \to EG$ is a bundle morphism covering g. The commutative diagram above is just the commutative diagram of the long exact sequences of the fibrations $EG(BG, G)$ and $EH(BH, H)$ with fibre map $g' \circ i$ (including the weak contractibility of EG and EH). □

NOTE 2.9. If $i : H \hookrightarrow G$ is a weak homotopy equivalence then the commutativity of the diagram in Lemma 2.3 entails that $g_{*,q}$ is a weak homotopy equivalence and hence $BH \sim BG$. It follows that $B_{CW_0}H \simeq B_{CW_0}H$ and that isomorphism classes of principal G–bundles and isomorphism classes of H–bundles coincided in CW_0.

Before, however, we can apply Theorem 2.2 usefully we need to know some extra theory with regard to Theorem 2.2 and the obstructions to lifting maps from the base space of a fibre bundle to the total space. Fortunately, this problem has been studied in depth when the domain of the map is a CW-complex (see Steenrod pg 177 – 181 or Whitehead pp 297 – 305.) Briefly, let us assume that M is a CW–complex and that we are trying to lift a map $f : M \to B$ to the total space of the fibre bundle $E(B, F, G, \pi_E)$ such that the lift, $\hat{f}$ satisfies $f = \pi_E \circ \hat{f}$. We define $\hat{f}$ over the zero skeleton of M by lifting f arbitrarily. Extending over the 1–skeleton of M is only a problem if the fibre, F, is not connected. In general, there is no difficulty in extending a map from the n–skeleton to the $(n+1)$–skeleton of M if $\pi_n(F)$ is zero. One can be very precise about the obstruction of lifting f which arises at the first non–trivial homotopy group of F.

THEOREM 2.4. *(THE PRIMARY OBSTRUCTION TO A LIFTING) Let $f : M \to B$ be a continuous map whose domain, M, is a CW complex and $E(B, F, G)$ a fibre bundle over M with connected fibre F and structure group G. Suppose further that $\pi_n(F)$ is the first non–vanishing homotopy group of F, $(n > 0)$. Then*

1) There exists $\hat{f} : M_n \to E$, a lift of f on the n–skeleton on M.

2) There exists a cohomology class, $o(f, E) \in H^{n+1}(M, \pi_n(\mathcal{F}))$, which depends only on the homotopy class of f with the property that f has a lift, $\hat{f}$, on the $n+1$–skeleton of M if and only if $o(f, E) = 0$.

3) If $g : M' \to M$ is continuous, then

$$o(f \circ g, E) = g^*(o(f, E)) \in H^{n+1}(M', \pi_n(\mathcal{F})).$$

NOTE 2.10. This is proved in Whitehead pp 302.

NOTE 2.11. Obstruction theory is a part of homotopy theory and applies to the more general notion of a fibering, $\pi : E \longrightarrow B$. All locally trivial fibre bundles are fiberings.

NOTE 2.12. The script F, $\mathcal{F}$, is used to denote cohomology taking values not simply in $\pi_n(F)$ but in a possibly twisted $\pi_n(F)$ bundle over B. However, when this bundle is trivial we recover standard cohomology and this is the case at least when B is simply connected which will cover almost all of my purposes.

NOTE 2.13. (3) of Theorem 2.4 is critical for the purposes of this exposition since it makes the primary obstruction to lifting the classifying map of a principal G–bundle $P(M, G, f)$, up to a fibre bundle, $E(B_{CW_0}G, F, G)$ over $B_{CW_0}G$ into a characteristic class of that bundle. (Typically, E will be the classifying space of either a subgroup or extension of G). The universal class is the primary obstruction to lifting the identity map on $B_{CW_0}G$ to E, $o(id_{BG}, EG)$. One has the following diagram.

$$\begin{array}{ccc} E & \xrightarrow{id} & E \\ \downarrow & & \downarrow \\ M \xrightarrow{f} B_{CW_0}G & \xrightarrow{id} & B_{CW_0}G \end{array}$$

$$o(P) := o(f, E) = f^*(o(id_{B_{CW_0}G}, E)).$$

NOTE 2.14. If the fibre of E, F, is an Eilenberg–Maclane space then the primary obstruction is the only obstruction and f lifts to some $\hat{f}$ defined on all M, if and only if $o(f, E) = 0$.

NOTE 2.15. The classification of possible lifts of f has been completely solved for the case of vertical homotopy classes of lifts. (A vertical homotopy of a map $\hat{f} : M \to E$, $\hat{f}_t$, is one for which $\pi \circ \hat{f}_t = f$ for all $t \in [0, 1]$.) In this case, if $\hat{f}$ and $\hat{f}'$ are different lifts of f, there is a difference cocyle, $d_n(\hat{f}, \hat{f}', E)$ in $H^n(M, \pi_n(\mathcal{F}))$ which measures their difference and depends only on the vertical homotopy classes of $\hat{f}$ and $\hat{f}'$. Furthermore, for any fixed lift of f, $\hat{f}_0$, the correspondence

$$\hat{f} \to d_n(\hat{f}_0, \hat{f}, E)$$

defines a one to one correspondence between the vertical homotopy classes of lifts of f and the elements of $H^n(M, \pi_n(\mathcal{F}))$. This fact along with Note 2.12 is the essence of why Eilenberg–MacLane spaces are the classifying spaces for abelian cohomology. However, these results on vertical homotopy classes of lifts are of little use for the purpose of combining this theory with Theorem 2.2 in order to classify the possible reductions of structure group of a principal G–bundle. In this instance we consider possible lifts of the classifying map of a principal G–bundle, $P(M, G, f)$, to BH. Here we are interested in the absolute homotopy classes of possible lifts since these correspond to different isomorphism classes of H–bundles which reduce the structure group of P. Taking absolute homotopy classes constitutes a further equivalence relation on $H^n(M, \pi_n(\mathcal{F}))$ (which we are considering as the space of vertical homotopy classes of lifts). Hence, I can find no more effective, general method for classifying possible reductions of structure group than that described in Theorem 2.2.

I shall now use this theory to investigate a series of well known examples from bundle theory.

2. METRICS ON VECTOR BUNDLES

The canonical starting structure group for a complex vector bundle is $G = GL(n, C)$ since this is the largest subgroup of $Homeo(C^n)$ whose action on C^n preserves its vector space structure. Consider a rank n complex vector bundle $V(M, C^n, GL(n, C))$ with transition functions $\{g_{\alpha,\beta}, U_\alpha\}$. A choice of inner product on V is a continuous, symplectic linear form

$$\sigma : V \oplus V \to C$$

where $V \oplus V$ is the rank $2n$ complex vector bundle with fibre $(V \oplus V)_x := V_x \oplus V_x$ and transition functions $\{g_{\alpha,\beta} \times g_{\alpha,\beta}, U_\alpha\}$. Given an hermitian metric on V we can restrict the structure group of V to those elements of $GL(n, C)$ which leave σ invariant on all the fibres of $V \oplus V$. This new subgroup is, of course, the unitaries, $H = U(n)$. Conversely, if the structure group reduces to $U(n)$ then V can be equipped with an hermitian metric. Now is it well known that via the polar decomposition $GL(n, C)$ (considered merely as a topological space) can be written as direct product $GL(n, C) = U(n) \times W$ where W is homeomorphic to the space $M_n^*(C)$ of self adjoint matricies. W is a Euclidean space and hence contractible. We let $P(M, GL(n, C), f)$ be the frame bundle of V. Reducing the structure group of V to $U(n)$ corresponds to lifting f to $B_{CW_0}U(n)$. But from Theorem 2.2 (IV) (since U(n) is compact) $B_{CW_0}U(n)$ is a W–fibre bundle over $B_{CW_0}GL(n, C)$. Since W is contractible there is no obstruction to lifting any map into $B_{CW_0}GL(n)$ to $B_{CW_0}U(n)$. Hence every isomorphism class of rank n complex vector bundles has a unique reduction of structure group to $U(n)$. Note that this also shows that every complex vector bundle has an inner product. The situation is more general than this since every finite dimensional Lie group, G, is homeomorphic to $H \times E$, where H is a maximal compact subgroup and E is a Euclidean space. In this case all principal G–bundles can be considered as principal H–bundles. In particular, when $G = GL(n, R)$, $H = O(n)$ and every real vector bundle can be given a Euclidean metric.

3. SEMI-DIRECT PRODUCTS WITH DISCRETE GROUPS

Let us suppose that G has the form of a semi-direct product $G = H \times_s \mathbb{Z}_n$. G has n disconnected components and H is the identity component of G. (n may be infinite but shall always be assumed to be countable, $\mathbb{Z}_\infty = \mathbb{Z}$).

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\rho} \mathbb{Z}_n \longrightarrow 1$$

In such circumstances it is natural to seek to reduce the structure group of a principal G–bundle to H. Since H is normal in G and $G/H = \mathbb{Z}_n$, $B_{CW_0}H$ is a principal $\mathbb{Z}_n$–bundle over $B_{CW_0}G$. In fact, since $\pi_1(B_{CW_0}G) = \pi_0(G) = \mathbb{Z}_n$ and similarly $\pi_1(B_{CW_0}H) = 0$, $B_{CW_0}H$ is an n–covering space for $B_{CW_0}G$. Since the general obstruction theory of Theorem 2.4 does not apply to bundles with disconnected fibre we must rely on non–abelian cohomology. We have the long exact sequence for $B_{CW_0}G$,

$$1 \longrightarrow C(B_{CW_0}G, H) \longrightarrow C(B_{CW_0}G, G) \xrightarrow{\rho} C(B_{CW_0}G, \mathbb{Z}_n)$$

$$\xrightarrow{d} H^1(B_{CW_0}G, \underline{H}) \xrightarrow{i} H^1(B_{CW_0}G, \underline{G}) \xrightarrow{\rho} H^1(B_{CW_0}G, \mathbb{Z}_n).$$

The universal class obstructing reduction of structure group to the connected component of G is $\rho[EG]$, an element of $H^1(B_{CW_0}G, \mathbb{Z}_n)$. One also observes that $\rho : C(B_{CW_0}G, G) \to C(B_{CW_0}G, \mathbb{Z}_n)$ is onto and hence by the proof of Theorem 2.3 (III) we see that reduction to the identity component of G is unique when it occurs. An important example of this is when $G = O(n) = SO(n) \times_s \mathbb{Z}_2$. Reducing the structure group of an R^n vector bundle to $SO(n)$ corresponds to giving it an orientation. Reducing the structure group of the tangent bundle, $TM(M, R^n, O(n))$, of an n–dimensional manifold to $SO(n)$ is the definition of the manifold's orientability. The obstruction to orientation is W_1, the first Steiffel–Whitney class. Its universal class is the generator of $H^1(B_{CW_0}O(n), \mathbb{Z}_2) = \mathbb{Z}_2$ and, in general, $W_1(P) \in H^1(M, \mathbb{Z}_2)$ for a principal–$O(n)$ bundle, $P(M, O(n))$.

4. ALMOST COMPLEX STRUCTURES

Given a complex, n–dimensional manifold N, one knows that N also has the structure of an oriented, real, $2n$–dimensional manifold. The question in the converse situation, whether an oriented, real, $2n$–dimensional manifold, M, also has the structure of a complex manifold, is far more difficult and splits into two sub–questions. In what follows we shall see how the first and easier of the two questions, whether M has an almost complex structure, can be answered using the principal bundle theory so far developed. Starting with M we consider its tangent bundle TM. If M is to be a complex manifold then there must exist a continuous choice of complex structure for each of the tangent planes in its tangent bundle. The space of complex structures on R^{2n} compatible with the usual inner product, $J(2n)$, is the subspace of matrices $\{j \in M_n(R^{2n}) : j^2 = 1, j^T = -j\}$. Choosing a j we can make the indentification

$$U(n) = \{u \in M_n(R^{2n}) : uj = ju\}$$

which defines a closed embedding of $U(n)$ in $SO(2n)$. $J(2n)$ may be identified with the coset space of $U(n)$ in $SO(2n)$. Then we have

$$J(2n) \cong SO(2n)/U(n) \tag{2}$$

If we take the frame bundle of TM, $FM(M, SO(2n), f)$, which is a principal $SO(2n)$–bundle, and mod out by the action of $U(n)$ on its fibres then we arrive at a $J(2n)$-bundle, $FM/U(n)$. A section of $FM/U(n)$ is precisely the continuous choice of complex structure mentioned above and, by Theorem 2.2(3), a necessary and sufficient condition for the reduction of the structure group of FM to $U(n)$. A section of $FM/U(n)$ is called an almost complex structure for M and not all oriented, real, $2n$–dimensional manifolds possess one (e.g. $S^{2n}, n \geq 4$).

Since $U(n)$ is not normal in $SO(2n)$, $J(2n)$ is not a group and we cannot avail ourselves of non–abelian cohomology to proceed further. Obstruction theory and the use of Theorem 2.2(4) requires knowledge of the homotopy groups of $J(2n)$ which can be discovered by judicious use of the homotopy exact and spectral sequences of 2 considering $SO(2n)$ as a principal $U(n)$ bundle over $J(2n)$. One obtains that $J(2n)$ is simply connected and $\pi_2(J(2n)) = \mathbb{Z}$ for $n > 1$ (when $n = 1$ J is a point and all oriented, real two dimensional manifolds have an almost complex structure). Redrawing the lifting diagram for this situation,

$$\begin{array}{c} BU(n) \\ \downarrow \\ M \xrightarrow{f} BSO(2n) \end{array}$$

we see that the universal, primary obstruction to an almost complex structure is the generator of

$$H^3(BSO(2n), \mathbb{Z}) = \mathbb{Z}_2.$$

This class is W_3, the third Steifel–Whitney class. Note that $J(2n)$ ($n > 2$) is not an Eilenberg–MacLane space and its higher homotopy groups will act as further obstructions to lifting f. Hence the vanishing of $W_3(FM)$ is a necessary but not sufficient condition for the existence of an almost complex structure on M.

5. THE CHERN CLASSES

As I mentioned in Chapter 1, the calculation of the cohomology ring of classifying spaces is an important labour of bundle theory. I shall conclude Chapter 2 with a quick calculation of $H^*(BU(n), \mathbb{Z})$. (Another method, using the Gysin sequence, can be found in Milnor and Stasheff). The action of $U(n)$ on S^{2n-1} (the latter regarded as the unit sphere in C^n) yields the useful decomposition of $U(n)$ as a principal $U(n-1)$–bundle over S^{2n-1}, $U(n) = U(n)(S^{2n-1}, U(n-1))$. Furthermore, the sphere bundle of the associated classifying spaces, $BU(n-1) = BU(n-1)(BU(n), S^{2n-1})$, provides a simple definition of the universal nth Chern class. If e is the generator of $H^{2n-1}(S^{2n-1}, \mathbb{Z})$ considered as an element of $E_{2n}^{(0,2n-1)}(BU(n-1))$ and d_{2n} is the $2n$th differential in the spectral sequence of $BU(n-1)$ then the universal nth Chern class is given by

$$c_n = d_{2n}(e).$$

PROPOSITION 2.5. *(THE COHOMOLOGY RING OF $BU(n)$) The integral cohomology ring of $BU(n)$ is the polynomial ring freely generated by the n Chern classes of the universal bundle.*

$$H^*(BU(n), \mathbb{Z}) = [c_1, ..., c_n]$$

PROOF. (by induction on n) In the case $n = 1$, $BU(1) = BS^1$ and the proposition was proven in Chapter 1. Suppose that

$$H^*(BU(n-1), \mathbb{Z}) = [c_1, ..., c_{n-1}].$$

Then consider the $E_{2n}(BU(n-1))$. Since d_r is a graded algebra homomorphism for all r, $H^p(BU(n-1), \mathbb{Z})$ is zero for p odd and $2n-1$ is odd we see that

$$H^{2n}(BU(n-1), \mathbb{Z}) = H^{2n}(BU(n), \mathbb{Z}) / < c_n > .$$

The result now follows. □

CHAPTER 3

EXTENSION OF STRUCTURE GROUP

We now turn to the question of extending the structure group of a principal G–bundle. In the previous chapter largely mathematical questions about the existence of extra structure on bundles — metrics, orientations, almost complex structures — were translated into question about the reduction of the structure group of those bundles. In this chapter the desire, largely of mathematical physicists, to put extra structure on some specific bundles will lead to the question of extending the structure group of bundles.

1. TERMINOLOGY AND GENERAL SETTING

Let

(3) $$1 \to N \to \hat{G} \to G \to 1$$

be the short exact sequence of a central extension of a Lie group G by a compact Lie group N to yield a third Lie group $\hat{G}$. We regard N as the kernel of the canonical projection

$$\rho : \hat{G} \longrightarrow \hat{G}/N = G$$

and we identify G with $\hat{G}/N$. Since N is a compact subgroup of $\hat{G}$, $\hat{G}$ has a principal N–bundle structure, $\hat{G} = \hat{G}(G, N)$. We shall be concerned with cases where $\hat{G}$ is a non–trivial bundle over G for otherwise the extension problem is trivially solved. As we saw in Chapter 2, the bundle structure of $\hat{G}$ allows us to regard any principal G-bundle, $Q(M, \hat{G})$ as a bundle of bundles and, in particular, as a principal N–bundle over the homogenous space $P := Q/N$. P is itself a principal G–bundle over M and we write

$$Q = Q(M, \hat{G}) \quad Q = Q(P, N) \quad P = P(M, G) = Q/N.$$

We shall also use the symbol "ρ" to refer to the bundle map

$$\rho : Q \longrightarrow P$$

$$p \to [p] = pN \quad \text{for all } p \in Q.$$

DEFINITION 3.1. (EXTENSION OF STRUCTURE GROUP) Given a principal G–bundle $P(M, G)$, if there exists a principal $\hat{G}$–bundle, $Q(M, \hat{G})$ such that

$$P = \rho(Q) = Q/N.$$

then we say that the structure group of P extends to $\hat{G}$. An extension of the structure group of P to $\hat{G}$ is the isomorphism class of principal $\hat{G}$–bundles, $< Q >$. The situation is pictured in the following diagram.

$$\begin{array}{ccc} N \hookrightarrow \hat{G} & \xrightarrow{\rho} & G \\ \downarrow & & \downarrow \\ N \hookrightarrow Q & \xrightarrow{\rho} & P \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ M & \xrightarrow{id} & M \end{array}$$

NOTE 3.1. Any such $\hat{G}$–bundle is also a principal N–bundle, $Q(P, N, \pi)$ over P with the property that, when restricted to any fibre of $P(M, G)$, it yields $\hat{G}$.

$$Q|P_x := \pi^{-1}(P_x) \cong \hat{G} \quad \text{for all } x \in M.$$

Therefore, an obvious way to begin the search for extensions of the structure group of $P(M, G)$ is to examine the principal N–bundles over P. As an example, such reasoning readily shows that the universal G–bundle, $EG(BG, G)$ has only trivial extensions of structure group. Since EG is contractible any N–bundle, $Q(EG, N)$ over it is trivial and so too is the restricted bundle on each fibre.

$$Q = Q(EG, N) = EG \times N$$

$$\rho^{-1}(EG_x) = G \times N \quad \text{for all } x \in BG.$$

NOTE 3.2. As with the case of the reduction of the structure group, the extension of the structure group of P need not be unique.

2. ASSUMPTIONS

Throughout this section I shall restrict myself to CW_0 in the sense that I will consider only bundles over CW–complexes. Hence I shall drop the subscript CW_0 and B should be read as B_{CW_0} everywhere.

Since I shall be examining only cases in which $N = S^1$ or $\mathbb{Z}_n$ I shall make some further assumptions on N and G. However, what follows can be used as the basis for a general theory of the problem of extending the structure group of a principal G–bundle. Let us assume the following extra conditions.

1) $N = S^1 \sim K(\mathbb{Z}, 1)$ or $N = \mathbb{Z}_n \sim K(\mathbb{Z}_n, 0)$, $(n < \infty)$.

We have already remarked that $PU(\mathcal{H})_u$ and $PU(\mathcal{H})_{so}$ are BS^1's. (Henceforth, I shall drop the $\mathcal{H}$ and simply write U_u etc.) Similarly $U_u/\mathbb{Z}_n$ and $U_{so}/\mathbb{Z}_n$ realise $B\mathbb{Z}_n$. Since I am restricting my attention to bundles over CW-complexes, the difference between U_u and U_{so} is often not important and in that case I shall write U to refer to either. However, at other times the difference shall be crucial (since U_{so} is not a Banach Lie group) and then the subscripts shall return.

2) There is an inclusion

$$i : \hat{G} \hookrightarrow U$$

which induces an inclusion

$$i_{/N} : G \hookrightarrow QU := U/N$$
$$i(\hat{G})/N = i_{/N}(G).$$

Under these circumstances it follows that

(a) $[U] \in H^1(QU, \underline{N})$ is a generator

(b) $[\hat{G}] = i^*_{/N}([U]) \in H^1(G, \underline{N})$.

3) We shall further assume that $H^1(G, \underline{N})$ is singly generated with $[\hat{G}]$ one generator.

4) G has no integral cohomology in degree one.

Given a principal G–bundle, $P(M, G)$, there are at least three independent ways to arrive at a class $D(P) \in H^2(M, \underline{N})$ which depends only on the isomorphism class of P and has the properties

(i) $D(P)$ is trivial if and only if the structure group of P extends to $\hat{G}$.

(ii) $D(f^*P) = f^*(D(P))$ for any map $f : M' \to M$.

I shall sketch each of these three briefly in turn.

3. NON–ABELIAN COHOMOLOGY

Since N is abelian the long exact sequence in cohomology induced by (3) can be extended to $H^2(M, \underline{N})$. For any space M, we have

$$1 \to C(M,N) \to C(M,\hat{G}) \to C(M,G) \tag{4}$$
$$\xrightarrow{d} H^1(M,\underline{N}) \xrightarrow{i} H^1(M,\underline{\hat{G}}) \xrightarrow{\rho} H^1(M,\underline{G}) \xrightarrow{D} H^2(M,\underline{N})$$

Just as in Chapter 2, exactness of (4) and the correspondence of isomorphism classes of principal G–bundles to elements of $H^1(M, \underline{G})$ yields that the structure group of P extends to $\hat{G}$ if and only if $D([P]) \in H^2(M, \underline{N})$ is zero. Also by exactness, the possible extensions are classified by $i(H^1(M, \underline{N}))$ which we identify with $H^1(M, \underline{N})/d(C(M, G))$.

4. CLASSIFYING SPACES

Just as with the case of reduction of the structure group, there is a bundle relation between the classifying spaces of the groups involved which allows an obstruction theoretic approach to the problem of the extension of the structure group.

PROPOSITION 3.1. *(THE RELATION OF BG, BN AND $B\hat{G}$) We may take $B\hat{G}$ to be a BN–fibre bundle over BG.*

PROOF. From Chapter 2 and the closed inclusion of $N \hookrightarrow \hat{G}$ we may take BN to be a principal G–bundle over $B\hat{G}$, $BN = BN(B\hat{G}, G)$. As the CW_0-G-universal bundle, $EG(BG, G)$ is certainly a principal G–bundle we may form the G–reduced product.

$$B := (BN \times EG)/G$$

As we saw in 1.7, B leads a double life as two different fibre bundles.

$$B = B(BG, BN, G)$$

$$B = B(B\hat{G}, EG, G)$$

The first decomposition of B is as a BN-fibre bundle over BG with structure group G. Since the fibre of the second decomposition of B is EG which is contractible it follows that B has the homotopy type of $B\hat{G}$ and hence is another realisation of the classifying space for $\hat{G}$. This proves the proposition. □

NOTE 3.3. To avoid confusion between the two universal $\hat{G}$–bundles we shall denote the first as

$$E'\hat{G}(B'\hat{G}, \hat{G}).$$

The second, which is the pullback of the first by the canonical projection

$$\pi_1 : B \longrightarrow B'\hat{G}$$

we shall denote by $E\hat{G}(B\hat{G} = B, \hat{G}) = \pi_1^*(E'\hat{G}(B'\hat{G}, \hat{G}))$.

THEOREM 3.2. *(OBSTRUCTION THEORETIC APPROACH TO THE PROBLEM OF THE EXTENSION OF THE STRUCTURE GROUP) Let $P(M, G, f, \pi_P)$ be a principal G–bundle. There is a one to one correspondence between extensions of the structure group of P to $\hat{G}$ and homotopy classes of maps, $\hat{f} : M \to B\hat{G}$, such that $f \simeq \pi_2 \circ \hat{f}$. (Where $\pi_2 : B \to BG$ is the projection of $B = B\hat{G}$ onto BG coming from B's structure as a BN-fibre bundle over BG.) We have the following diagram.*

$$\begin{array}{ccc}
 & & B = B\hat{G} \hookleftarrow BN \\
 & & \downarrow{\scriptstyle \pi_2} \\
M & \xrightarrow{f} & BG
\end{array}$$

PROOF. I begin by showing the following equality of G–bundles

$$E\hat{G}/N = {\pi_2}^* EG(BG, G)$$

which we can see from careful inspection of the diagram. (Note that maps have the fibre of each mapping shown and where relevant, its name.)

$$\begin{array}{ccccc}
E\hat{G}' & \longleftarrow & E\hat{G} & & \\
{\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & \\
BN & \xleftarrow{EG} & EG \times BN & \xrightarrow{BN} & EG \\
{\scriptstyle G}\downarrow & & {\scriptstyle G}\downarrow & & {\scriptstyle G}\downarrow \\
B\hat{G}' & \xleftarrow{EG,\pi_1} & (EG \times BN)/G & \xrightarrow{BN,\pi_2} & BG
\end{array}$$

We have

$$E\hat{G} = {\pi_1}^* E\hat{G}'$$

and hence

$$E\hat{G}/N = \pi_1{}^*(E\hat{G}'/N) = \pi_1{}^*(BN) = EG \times BN.$$

But from the other side of the diagram,

$$EG \times BN = \pi_2^* EG(BG, G)$$

and the claimed identity holds.

Now, suppose such an $\hat{f}$ exists. Set $Q := \hat{f}^* E\hat{G}$. Then

$$\begin{aligned} Q/N &= (\hat{f}^* E\hat{G})/N \\ &= \hat{f}^*(E\hat{G}/N) \\ &= \hat{f}^* \pi_2^* EG \\ &= f^* EG \\ &\cong P. \end{aligned}$$

Conversely, given a principal $\hat{G}$–bundle $Q(M, \hat{G}, f')$ such that $P = Q/N$ then

$$\begin{aligned} f^* EG &\cong P \\ &= Q/N \\ &= f'^*(E\hat{G}/N) \\ &= f'^* \pi_2^* EG \\ &= (\pi_2 \circ f')^* EG. \end{aligned}$$

Thus, since EG is a universal bundle, $f \simeq \pi_2 \circ f'$ and extensions of the structure group of P correspond bijectively with homotopy classes of such $\hat{f}$.

□

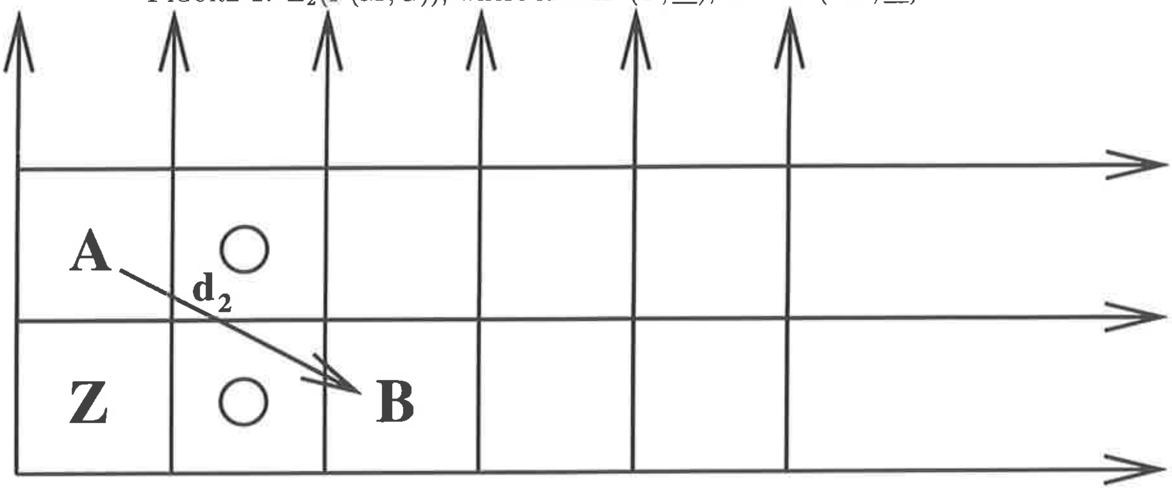
NOTE 3.4. This treatment is summarised in Coquereaux and Pilch pp 368 - 371 (and they attribute it to A. Haefliger in "Sur l'extension du groupe structural d'un espace fibre" in C.R. Acad. Sci. 234, 558-60 (1956)).

NOTE 3.5. As for the reduction case, the possible extensions of the structure group of P are given by the homotopy classes of the set of lifts $\{\hat{f} : M \to B\hat{G}, f = \pi_2 \circ \hat{f}\}$ and we can do no better with homotopy theory than the classification obtained using non–abelian cohomology.

NOTE 3.6. Since, under assumption 1, BN is an Eilenberg MacLane space $K(n+1, A)$, the primary obstruction to lifting f, $o(f, B\hat{G}) \in H^{n+2}(M, \pi_{n+1}(BN)) = H^{n+2}(M, A)$, is the only obstruction. So the structure group of P extends to $\hat{G}$ if and only if $o(f, B\hat{G}) = 0$. From our review of obstruction theory we have that for $g : N \to M$

$$o(g \circ f, B\hat{G}) = g^*(o(f, B\hat{G})).$$

FIGURE 1. $E_2(P(M,G))$; where $A \simeq H^1(G,\underline{N})$, $B \simeq H^2(BG,\underline{N})$



Hence $o(f,B\hat{G})$ satisfies (i) and (ii) with the universal obstruction class being $o(id_{BG},BG) \in H^{n+2}(BG,A)$. Now,

$$H^{n+2}(M,A) \cong H^2(M,\underline{N})$$

since in the case $N = \mathbb{Z}_n$ both are $H^2(M,\mathbb{Z}_n)$ and in the case $N = S^1$, both are $H^3(M,\mathbb{Z})$. This shows that $o(f,B\hat{G}) \in H^2(M,\underline{N})$.

5. SPECTRAL SEQUENCE APPROACH

Starting with a principal G–bundle $P(M,G,f)$ and the sequence (3) the spectral sequence approach to the extension problem follows the line of thought mentioned in the introduction of this chapter. We consider the principal N–bundles over P. Isomorphism classes of these correspond bijectively with the elements of $H^1(P,\underline{N})$. Starting with a cocycle, $\xi \in [\hat{G}] \in H^1(G,\underline{N})$, and the inclusion of the standard fibre

$$i : G \cong P_x \hookrightarrow P, \quad x \in M$$

we seek to extend ξ over all of P to $\eta \in Z^1(P,\underline{N})$ so that

$$D(\eta) = 0.$$

If we can do this then $\eta \in H^1(P,\underline{N})$ defines a principal N–bundle over P which restricts to $\hat{G}(G,N)$ on each fibre of P and hence defines an extension of the structure group of P to $\hat{G}$.

Happily the machinery of spectral sequences measures exactly the obstruction to extending ξ. It requires, however, that we use cohomology with discrete co-efficients which is possible in the cases that $N = \mathbb{Z}_n$ and $N = S^1$ since the first case is already discrete and in the second case we may exploit the well known correspondence between S^1–bundles and elements of second integral cohomology, $H^1(M,\underline{S^1}) \cong H^2(M,\mathbb{Z})$. To keep track of the two cases let $r = 2$ in the case $N = \mathbb{Z}_n$ and $r = 3$ in the case $N = S^1$ where we shall use integral cohomology and $H^*(\_,\underline{N})$ should be read as $H^{*+1}(\_,\mathbb{Z})$.

Consider $[\hat{G}] \in H^1(G, \underline{N}) = E_2^{(0,r-1)}(P)$. From the theory of spectral sequences we have that $[\hat{G}]$ extends to a global cohomology class of P if and only if $d_r^P([\hat{G}]) = 0$. In the case where P is the universal bundle $EG(BG, G)$, the contractibility of EG entails that

$$d_r^{EG} : E_2^{(0,r-1)} = H^1(G, \underline{N}) \to E_2^{(r,0)}$$

must be an isomorphism and hence $d_r^{EG}([\hat{G}])$ must be a generator of $H^2(BG, \underline{N})$ and, in particular, is non–zero. Hence we recover the result of the introduction that EG has only trivial extensions of structure group. Since the differentials of spectral sequences behave well under pullback

$$d_r^P(f^*([\hat{G}])) = f^*(d_r^{EG}([\hat{G}])$$

and we see that $d_r^{EG}([\hat{G}]) \in H^2(BG, \underline{N})$ is a universal obstruction to the extension of the structure group of a principal G–bundle to $\hat{G}$. Specifically it satisfies conditions (i) and (ii) above.

6. THREE "IDENTICAL" CHARACTERISTIC CLASSES

I would like to prove that the three characteristic classes defined in the previous sections are all the same. However, I cannot since that would require checking the consistency of the three definitions given for any given universal bundle which is beyond my technical prowess and would yield no important information for my purposes. I shall prove instead that all three classes generate $H^2(BG, \underline{N})$ and then we can speak of the Dixmier–Douady class of a principal G–bundle $P(M, G)$ as revealed by either of the three methods above and know that this class is precisely defined up to possibly an automorphism of $H^2(M, \underline{N})$.

LEMMA 3.3. *("IDENTITY" OF THE THREE UNIVERSAL CLASSES FOR THE CASE $G = QU$) In the case that $G = QU$ the three classes $D[EQU]$, $o(id_{BQU}, BU)$ and $d_r^{EQU}[U])$ all generate $H^2(BQU, \underline{N})$.*

PROOF. Assumption 3 of section 2 and the previous discussion of the spectral sequence of $EQU(BQU, QU)$ implies that $H^2(BQU, \underline{N})$ is singly generated by $d_r^{EQU}([U])$. Hence the lemma is true for $d_r^{EQU}([U])$. The following formal procedure will prove the lemma for the other two classes.

$D[EG]$: Since $G = QU = U/N$, $\hat{G} = U$ which is contractible and it follows that $H^1(M, \underline{U}) = 0$. Furthermore, this situation allows us to extend the sequence 3 one group further by the trivial group and we have for all spaces M that

$$H^1(M, QU) \cong H^2(M, \underline{N}).$$

We let $M = BG$ and consider $[EG] \in H^1(BG, \underline{U/N})$. Let

$$D[EG] = m$$

Now consider the principal QU–bundle over BG corresponding to 1 in $H^1(BG, \underline{QU})$. Lets denote it by $P(BG, G, f)$. We have

$$P = f^* EG \text{ and } [P] = f^*[EG]$$

and hence

$$1 = f^*(m)$$

Since f^* is a homomorphism of the the singly generated group $H^1(BG, \underline{QU})$ into itself, it follows that $m = \pm 1$. That is $[EG]$ generates $H^1(BG, \underline{QU})$ and hence $D[EG]$ generates $H^2(BG, \underline{N})$ as required.

$o(id_{BQU}, BU)$: Since U is contractible, U can be taken to be its own universal bundle with base space a point. We take QU as a realisation of BN and the bundle $BU(BQU, BN)$ is just $EQU(BQU, QU)$. Suppose that $o(id_{BQU}, EQU) = m$ which does not generate $H^2(BG, \underline{N})$. Let X be a space such that $H^2(X, \underline{N}) = \mathbb{Z}_m$. (Take $X = K(\mathbb{Z}_{(n/m)}, 2)$ for $N = S^1$ or $X = K(\mathbb{Z}_m, 1)$ for $N = \mathbb{Z}_n$). Then $1 \in H^2(X, \underline{N})$ defines a map

$$1 : X \to BQU$$

since BQU is the relevant Eilenberg MacLane space. We have the following lifting diagram.

$$\begin{array}{ccc} EQU & & \\ \downarrow & & \\ BQU & \xleftarrow{1} & X \end{array}$$

Now $1^*(EQU)$ is a principal QU–bundle over X and since

$$o(1, EQU) = 1^*(o(id_{BQU}, EQU)) = m = 0 \in \mathbb{Z}_m$$

$1^*(EQU)$ must have a section by Theorem 2.4(2) and hence 1^*EQU is the trivial bundle. This means that

$$D[1^*EQU] = D[X \times QU] = 0.$$

But we know that

$$D[1^*EQU] = 1^*(D[EQU]) = 1$$

from directly above. So via contradiction we have proved that $o(id_{BQU}, EQU)$ generates $H^2(BG, \underline{N})$. □

PROPOSITION 3.4. *("IDENTITY" OF THE THREE UNIVERSAL CLASSES FOR ALL GROUPS G SATISFYING 3.1) For all groups satisfying the assumptions of Definition 3.1 the three classes* $D[EG]$, $o(id_{BG}, B\hat{G})$ *and* $\hat{d}_r^E G([\hat{G}])$ *all generate* $H^2(BG, \underline{N})$.

PROOF. Consider, once again, $B\hat{G}$, and $B := (EG \times QU)/G$. $B = B(BG, QU, f)$ is the principal QU–bundle over BG obtained by regarding the transition function of EG as elements of QU. It follows that via non–abelian cohomology B and EG give rise to the same class in $H^2(BG, \underline{N})$. We have

$$D[EG] = D[B] = f^*(D[EQU]). \tag{5}$$

Now we already have that $d_r^{EG}([\hat{G}])$ is a generator of $H^2(BG, \underline{N})$ from the arguments already reviewed in spectral theory.

Since $B = (EG \times QU)/G$ we have the obvious bundle inclusion

$$i : EG \hookrightarrow B.$$

This induces (see 1.21) a pull back mapping between the spectral sequences of B and EG.

$$i^* : E_r^{(p,q)}(B) \to E_r^{(p,q)}(EG)$$

Considering $[U] \in E_2^{(0,r-1)}(B)$ and $[\hat{G}] \in E_2^{(0,r-1)}(EG)$ and noting that i^* is the identity when $q = 0$ we obtain the following.

$$\begin{aligned} d_r^{EG}([\hat{G}]) &= d_r^E G(i^*[U]) \\ &= i^*(d_r^B([U]) \\ &= d_r^B([U]) \\ &= f^*(d_r^{EQU}([U]) \end{aligned}$$

Finally, to compute $o(id_{BG}, B\hat{G})$ we have that $B\hat{G} = B$ and hence

$$o(id_{BG}, B\hat{G}) = o(id_{BG}, f^*EQU) = f^*(o(id_{BQU}, EQU)).$$

Hence each universal class for G is the pullback by f of the corresponding universal class for QU and since we know that the latter generate $H^2(BQU, \underline{N})$ and that $d_r^{EG}([\hat{G}]) = f^*(d_r^{EQU}([U]))$ generates $H^2(BG, \underline{N})$ it follows that $D[EG]$ and $o(id_{BG}, EG)$ also generate $H^2(BG, \underline{N})$. □

I now move to apply this general theory to specific examples.

7. $Spin(n)$ STRUCTURES ON $SO(n)$ BUNDLES

We consider the central $\mathbb{Z}_2$–extension of Lie groups

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1 \quad (n > 2).$$

In physics the Dirac operator is defined over a Hilbert space of spinors which are the square integrable sections of a $Spin(n)$ bundle. It is therefore often important to extend the structure group of a $SO(n)$–bundle $P(M, SO(n), f)$ — typically the frame bundle of the tangent bundle of an oriented manifold M — to $Spin(n)$ in order to define the Dirac operator on M. From above we see that the obstruction to such an extension is a class in $H^2(M, \mathbb{Z}_2)$. In fact, the universal obstruction class is a generator of

$$H^2(BSO(n), \mathbb{Z}_2) \cong < W_2(ESO(n)) > \cong \mathbb{Z}_2.$$

W_2 is the second Steifel-Whitney class of a principal $SO(n)$–bundle and $W_2(P) = f^*(W_2(ESO(n)))$.

8. CIRCLE EXTENSIONS — THE DIXMIER DOUADY CLASS

The rest of this thesis shall be concerned with obstructions to circular extensions of the structure group of a principal G–bundle, i.e. $N = S^1$. All such obstructions may be generically called Dixmier–Douady classes and I shall digress briefly to explain the history of this name.

Dixmier and Douady's work (collated in Chapter 7 of Dixmier's C^*–Algebras (1977)) dealt not with locally trivial fibre bundles but with related objects called continuous fields of algebras which they used, in particular, to investigate the structure of type I C^*–algebras. A continuous field of elementary C^*–algebras with fixed fibre $LC(\mathcal{H})$ (the compact operators on $\mathcal{H}$) and which satisfies Fell's condition is precisely the set of sections of a locally trivial bundle of elementary C^*–algebras with structure group PU. (Since we are working in CW_0 in this chapter we can consider either PU_u or PU_{so} as PU_u is a CW–approximation to PU_{so} so we apply Lemma 2.3 to $i : PU_u \hookrightarrow PU_{so}$ and conclude that $BPU_u \simeq BPU_{so}$. Thus, over CW–complexes, isomorphism classes of principal PU_u-bundles and principal PU_{so}-bundles coincide). Dixmier and Douady discovered a third integral cohomology class, D, which vanished if and only if the continuous field of elementary C^*–algebras arose from a continuous field of Hilbert spaces. From the bundle theoretic perspective of this thesis this is equivalent to extending the structure group of the related elementary C^*–algebra bundle to $U(\mathcal{H})$. Using non–abelian cohomology, Dixmier and Douady were the first to discover the isomorphism

$$D : H^1(M, \underline{PU}) \cong H^3(M, \mathbb{Z}) \tag{6}$$

which gives a geometrical correspondent, an elementary C^*–algebra bundle or a principal PU–bundle, for every element in $H^3(M, \mathbb{Z})$. They also showed that the group multiplication in $H^3(M, \mathbb{Z})$ corresponded to tensoring the associated bundles of elementary C^*–algebras.

The work of Dixmier and Douady with the projective unitaries has a pleasing explanation in homotopy theory. The key is to realise that the projective unitaries are a realisation of $K(\mathbb{Z}, 2)$ and hence that any BPU is a $K(\mathbb{Z}, 3)$. So the correspondences derived in Chapter 1 provides a proof of 6 via

$$H^1(M, \underline{PU}) = Bun_M(PU)/\sim\ \cong [M, K(\mathbb{Z}, 3)] \cong H^3(M, \mathbb{Z}).$$

Given the importance of PU and principal PU–bundles in the theory of circular extensions, it would be nice to discover a concrete example of a CW_0–universal PU–bundle. The following proposition discovers an $EPU_u(BPU_u, PU_u)$ as a homogeneous space and will allow us to do the same for $EG(BG, G)$ when $G \hookrightarrow PU_u$ and $\pi : PU_u \to PU_u/G$ define a principal G-bundle.

Proposition 3.5. *(A HOMEOMORPHISM OF PU_u INTO A CLOSED SUBGROUP OF THE FULL UNITARY GROUP) There exists a homeomorphism of $PU(\mathcal{H})_u$ onto its closed image in $U(\mathcal{T})_u$, the unitary group of the Hilbert space of Hilbert–Schmidt operators on $\mathcal{H}$.*

Proof. Given $[a] \in PU_u$, choose a representative $a \in U$. Then define

$$i : PU_u \to U(\mathcal{T})_u$$

$$[a] \mapsto Ad(a)$$

where

$$Ad(a) : \mathcal{T} \to \mathcal{T}$$

$$t \mapsto a.t.a^*$$

The conjugation kills the scalar difference between the possible choices of representatives for a and hence i is well defined. i is injective because the identity representation of the Hilbert Schmidts on $\mathcal{H}$ is irreducible. If $i([a]) = i([b])$ it follows that ab^* commutes with every $x \in T$ and hence

$$ab^* = \lambda.1 \quad i.e.\ [a] = [b].$$

Since PU_u is a metric space, I show the continuity of i by considering a convergent sequence $([a_n])_{n=1}^{\infty} \to [a]$ in PU_u. By taking n large enough we may assume that the $[a_n]$ lie in a neighbourhood of $[a]$ over which $U_u(PU_u, S^1)$ is locally trivial. Hence we may assume that there is a sequence $(a_n)_{n=N}^{\infty} \to a$ in U_u. From the fact that $\|(a_n - a)\|_{B(\mathcal{H})} \to 0$ and the equations

$$(a_n - a)^*.(a_n - a) = 2.1 - a_n^*.a - a^*.a_n$$

$$(a_n - a)^*.(a_n + a) = a_n^*.a - a^*.a_n$$

it follows that $a_n^*.a \to 1$ and $a^*.a_n \to 1$. Choose N' such that $a_n^*.a = 1 + \epsilon$ and $a^*.a_n = 1 + \epsilon^*$ for $n \geq N'$ where $\|\epsilon\|_{B(\mathcal{H})}$ is small. Then for $n \geq N'$

$$\|Ad(a_n) - Ad(a)\|_{B(\mathcal{T})}^2$$

$$= Sup(\|t\|_{\mathcal{T}} = 1)\ \|(Ad(a_n) - Ad(a))t\|_{\mathcal{T}}^2$$

$$= Sup(\|t\|_{\mathcal{T}} = 1)\ Tr((a_n.t.a_n^* - a.t.a^*)^*.(a_n.t.a_n^* - a.t.a^*))$$

$$= Sup(\|t\|_{\mathcal{T}} = 1)\ Tr(a_n.t^*.t.a_n^* + a.t^*.t.a^* - a_n.t^*.a_n^*.a.t.a^* - a.t^*a^*.a_n.t.a_n^*)$$

$$= Sup(\|t\|_{\mathcal{T}} = 1)\ Tr(a_n.t^*.t.a_n^* + a.t^*.t.a^* - a_n.t^*.t.a^* - a.t^*.t.a_n^* - a_n^*.t^*.\epsilon.t.a * -a.t^*.\epsilon^*.t.a_n^*)$$

$$\leq Sup(\|t\|_{\mathcal{T}} = 1)\ Tr((a_n - a).t^*.t.(a_n - a)^*) + 2\|\epsilon\|_{B(\mathcal{H})}$$

$$\leq Sup(\|t\|_{\mathcal{T}} = 1)\ \|(a_n - a)\|_{B(\mathcal{H})}^2.\|t\|_{\mathcal{T}}^2 + 2.\|\epsilon\|_{B(\mathcal{H})}$$

$$= \|(a_n - a)\|_{B(\mathcal{H})}^2 + 2.\|\epsilon\|_{B(\mathcal{H})}.$$

Since $\|a_n - a\|_{B(\mathcal{H})} \to 0$ and $\|\epsilon\|_{B(\mathcal{H})} \to 0$ as $n \to \infty$, $\|Ad(a_n) - Ad(a)\|_{B(T)} \to 0$ as $n \to \infty$ and thus i is continuous.

To see that the image of i is closed consider a sequence $i([a_n]) \to b$ where $b \in U(T)_u$. Define a $*-$automorphism of $\mathcal{T}$ by

$$b'(t) = lim_{n \to \infty} Ad(a_n)t.$$

One can verify that b' is a $*-$automorphism of $\mathcal{T}$, the Hilbert-Schmidt operators on $\mathcal{H}$. Since $\mathcal{T}$ is uniform norm dense in $L(\mathcal{H})$, b' defines a $*-$automorphism of $LC(\mathcal{H})$ and is thus of the form $Ad(a)$ for some $a \in U(\mathcal{H})_u$. Hence $b = Ad(a) = i([a])$ and the image of i is closed.

Finally, to see that i defines a homeomorphism we begin with the metric, ρ, which defines the topology on $PU(\mathcal{H})_u$.

$$\rho([a], [b]) = Inf(\lambda \in S^1)\|a - \lambda.b\|_{B(\mathcal{H})}$$

So,

$$\begin{aligned}\rho([a],[b])^2 &= (Inf(\lambda \in S^1)\|a-\lambda.b\|_{B(\mathcal{H})})^2\\ &= Inf(\lambda\in S^1)Sup(\|u\|_{\mathcal{H}}=1)((a-\lambda.b)u,(a-\lambda.b)u)\\ &= Inf(\lambda\in S^1)Sup(\|u\|_{\mathcal{H}}=1)(2.\|u\|_{\mathcal{H}}-2.Re((au,\lambda.bu))\end{aligned}$$

where $(\_,\_)$ is the inner product on $\mathcal{H}$. Now, as is customary, let $\Theta_{u,v}$ be the linear operator

$$\Theta_{u,v}:\mathcal{H}\to\mathcal{H}$$

$$w\mapsto(v,w).u$$

The following properties are easy to check.

(a): $\theta_{u,v}\in\mathcal{T}$.
(b): $\Theta_{\lambda^{-1}.u,\bar{\lambda}.u}=\Theta_{u,v}$ for all non-zero complex λ.
(c): $\Theta_{u_1,v_1}.\Theta_{u_2,v_2}=(v_1,u_2)\Theta_{u_1,v_2}$.
(d): $\|\Theta_{u,v}\|_{B(\mathcal{H})}=\|u\|_{\mathcal{H}}.\|v\|_{\mathcal{H}}$.
(e): $Ad(a).\Theta_{u,v}=\Theta_{au,av}$.
(f): $(\Theta_{u,v})^*=\Theta_{v,u}$.
(g): $Tr(\Theta_{u,v})=(v,u)$.

The map

$$u\otimes v\mapsto\Theta_{u,v}$$

extends to an isomorphism of $\overline{\mathcal{H}}\otimes\mathcal{H}$ with the Hilbert-Schmidt operators on $\mathcal{H}$. Here the bar denotes the complex conjugate Hilbert space $\mathcal{H}$. The operator $Ad(a)$ becomes $\bar{a}\otimes a$ where $\bar{a}$ denotes the action of a on the conjugate space. To prove our result it suffices to work in a neighbourhood of the identity in $U\mathcal{T}$. Now for $\bar{a}\otimes a$ to be close to the identity operator the spectrum of a must contain a gap (for if the spectrum is the whole circle then it is not possible for $\bar{a}\otimes a$ to be close to the identity. That being the case we can assume -1 is not in the spectrum of a by multiplying by a phase if necessary. Assume we have a sequence $a_n\in U(\mathcal{H})$ with

$$\|Ad(1)-Ad(a_n)\|_{B(\mathcal{T})}\to 0.$$

Then there is a sequence of self adjoint operators K_n on $\mathcal{T}$ with $a_n=\exp(iK_n)$ and the spectrum of K_n a subset of the interval $[\gamma_n,\delta_n]\subset[-\pi,\pi]$. In fact we may assume

$$\gamma_n=\inf_{(\|u\|_{\mathcal{H}}=1)}\{(u,K_nu)\},$$

$$\delta_n\ \sup_{(\|u\|_{\mathcal{H}}=1)}\{(u,K_nu)\}.$$

Then

$$||Ad(a_n) - Ad(1)|| = \sup\{|\exp i(\lambda - \mu) - 1| \mid \lambda, \mu \in [\gamma_n, \delta_n]\}$$
$$= |\exp i(\delta_n - \gamma_n) - 1|$$

On the other hand

$$\inf_\lambda ||a_n - \lambda 1|| = |\exp[i(\delta_n - \gamma_n)/2] - 1|$$
$$= ||Ad(a_n) - Ad(1)||.$$

Thus, if $||Ad(a) - Ad(a_n)||_{B(\mathcal{H})} \to 0$ as $n \to \infty$ then $\rho([a],[a_n]) \to 0$. Hence $i^{-1} : i(PU(\mathcal{H})_u) \to PU(\mathcal{H})_u$ is continuous and thus i is a homeomorphism.

□

NOTE 3.7. Identifying $PU(H)_u$ with $i(PU(H)_u)$ we have that $PU(H)_u$ is a closed sub-Banach Lie group of $U(\mathcal{T})_u$. Thus $U(\mathcal{T})_u(U(\mathcal{T})_u/PU(H)_u, PU(H)_u)$ is a CW_0–universal $PU(H)_u$–bundle and $U(\mathcal{T})_u/PU(\mathcal{H})_u$ is a $BPU(H)_u$ Similarly, if $G \hookrightarrow PU(H)_u$ induces a principal G–bundle structure, $PU(\mathcal{H})_u(PU(H)_u/G, G)$ on $PU(H)_u$ (if, for example, the image of i, a homeomorphism is a Banach Lie subgroup of $PU(H)$), then by composing inclusions, $U(\mathcal{T})_u(U(\mathcal{T}_u/G, G)$ is a realisation of the CW_0–universal G–bundle.

9. THE FINITE PROJECTIVE UNITARIES

Principal $PU(n)$–bundles arise naturally as the principal bundles associated to matrix bundles, that is bundles with $M_n(C)$ as fibre. The projective unitaries act on $M_n(C)$ via conjugation. The theory of projective unitary bundles has been succinctly described by Plymen and Hayden (1981) and what follows is a summary of their paper. For an arbitrary connected space M the following commutative diagram of cohomology groups comes from the corresponding diagram of group extensions and inclusions.

$$\begin{array}{ccccc} H^1(M, \underline{U(n)}) & \longrightarrow & H^1(M, \underline{PU(n)}) & \xrightarrow{D_1} & H^2(M, \underline{S^1}) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(M, \underline{SU(n)}) & \longrightarrow & H^1(M, \underline{PU(n)}) & \xrightarrow{D_2} & H^2(M, \underline{\mathbb{Z}_n}) \end{array}$$

The last vertical map comes from the identification of $\mathbb{Z}_n \subseteq S^1$ as the nth roots of unity. For a principal $PU(n)$–bundle $P(M, PU(n))$ the commutativity of the diagram yields

$$D_1[P] = D_2[P].$$

Now the latter of these clearly is clearly torsion of degree n and so all Dixmier–Douady classes of principal $PU(n)$–bundles are torsion of degree n. Interestingly, the converse is also true. (See A. Grothendieck, Le Groupe de Brauer, I. Seminaire Bourbaki, 290 (1965) pp 1-21.) The Brauer group of M, $B(M)$, is defined to be the torsion part of $H^3(M, \mathbb{Z}) = H^2(M, \underline{S^1})$. For any $x \in B(M)$ we can find a principal $PU(n)$–bundle with Dixmier–Douady class x. As in the infinite case, the correspondence of elements of

$H^1(M, \underline{PU(n)})$ with isomorphism classes of elementary C^*–algebra bundles (the fibre is now $M_n(C)$ not $LC(\mathcal{H})$), allows the construction of a geometrical model for the Brauer group. Elements of $B(M)$ are isomorphism classes of $M_n(C)$–bundles for some n, modulo the equivalence of tensor products with the trivial bundle and multiplication is again given by tensoring the bundles together. Note that the tensor product of an $M_n(C)$ bundle with an $M_m(C)$ bundle is an $M_{n.m}(C)$ bundle and hence we have not discovered a group structure on $H^1(M, \underline{PU(n)})$ but on the infinite union

$$\cup_{n=1}^{\infty}(H^1(M, \underline{PU(n)}))/\sim$$

where "$\sim$" denotes the equivalence relation already mentioned.

10. $Spin^C(n)$ STRUCTURES ($n > 2$).

As well as the double cover of $SO(n)$ by $Spin(n)$ there is also a circle bundle over $SO(n)$ corresponding to the generator of $H^2(SO(n), \mathbb{Z}) = \mathbb{Z}_2$. This bundle is the complex Lie group $Spin^C(n)$ which sits in the following long exact sequence of Lie groups.

$$1 \to S^1 \to Spin^C(n) \to SO(n) \to 1$$

A $Spin^C(n)$ structure for a principal $SO(n)$–bundle, $P(M, SO(n), f)$ is an extension of the structure group of P to $Spin^C(n)$. The universal obstruction to the existence of a $Spin^C(n)$ structure is the generator of $H^3(BSO(n), Z) =< \beta(W_2) >= \mathbb{Z}_2$, where β is the Bockstein map (see Hayden and Plymen p 18 for a definition) and W_2 is the second Steifel–Whitney class. This is precisely the primary obstruction to the existence of an almost complex structure for M in the case that n is even and $P = FM$. It follows that every oriented, real $2m$–dimensional manifold ($m > 1$) with an almost complex structure has a $Spin^C(n)$ structure.

11. STRING STRUCTURES

We now come to the first of the major examples motivated by mathematical physics. String theory is based on the idea of modelling particles not by points in the manifolds of space or space-time but by loops into these manifolds. The bundles which now become important to string theorists are the bundles, called loop bundles, created by looping a principal G–bundle where G is a compact Lie group. From a principal G–bundle $P(M, G, f)$ one forms the bundle $L_dP(L_dM, L_dG, Lf)$ where, in general, L_dX denotes the space of differentiable loops into a finite dimensional manifold X.

$$L_dX := \{\gamma : S^1 \to X, \gamma \text{ is differentiable}\}$$

It is well known (see Pressley and Segal Ch 6) that L_dG has a canonical central, circular extension, $\widehat{L_dG}$, induced from an embedding of L_dG in the restricted unitary group (of which much more in the next and final example) which in turn embeds in the projective unitaries of a second Hilbert space $\mathcal{H}_\pi$. Henceforth U and PU will refer respectively to the unitaries and projective unitaries over $\mathcal{H}_\pi$.

$$L_dG \hookrightarrow U_{res} \hookrightarrow PU_{so}$$

$$\widehat{L_dG}(L_dG, S^1) = i^*U_{so}(PU_{so}, S^1)$$

$$[\widehat{L_dG}] \text{ generates } H^2(L_dG, \mathbb{Z})$$

In string theory one typical starts with a principal $SO(n)$–bundle, $P(M, SO(n), f)$ $(n > 2)$, which is usually the frame bundle of a tangent bundle, TM, and which has a $Spin(n)$–structure, $Q(M, Spin(n), \hat{f})$. We then form

$$L_dQ(L_dM, L_dSpin(n), L_d\hat{f})$$

and P is said to have a string structure if and only if the structure group of L_dQ extends to $L_d\widehat{Spin(n)}$. A string structure is required since $LSpin(n)$ only has projective unitary representations and one needs the full unitary representation of $L_d\widehat{Spin}(n)$ to define the generalised Dirac–Ramond operator. Of course, the Dixmier Douady class of L_dQ, $D[L_dQ]$, is the obstruction to the existence of a string structure for P. Killingback, in his brief paper "World–sheet Anomalies and Loop Geometry" outlined the above theory of string structures and proposed that twice $D[L_dQ]$ was in fact the transgression of the Pontryagin class of P. Since then Carey and Murray (1991) produced a rigorous proof of Killingback's thesis in the case of based loops which are smooth except possibly at the base point as did MacLaughlin (1992) in the case of differentiable free loops. In what follows I shall summarise these two papers and use the homotopy between continuous and differentiable loops to link them.

12. QUICK REVIEW OF THE TECHNICAL ASPECTS OF LOOP SPACES, LOOP GROUPS AND LOOP BUNDLES

12.1. NOTATION. Throughout X and M will be finite differentiable manifolds unless noted and G will be a compact, simply connected Lie group.

$$\Omega_d(X, x_0)$$

shall denote the based, differentiable loops into X

$$\Omega_d(X, x_0) := \{\gamma \in L_d(X) : \gamma(0) = \gamma(1) = x_0\}$$

When the base point is unimportant I shall supress it. (When G is a Lie group x_0 is taken to be the identity.) L_cX and Ω_cX shall be used to denote the spaces of continuous loops and continuous based loops respectively while L_pX and Ω_pX shall be used to denote the spaces used by Carey and Murray consisting of piecewise differentiable loops and piecewise differentiable based loops respectively. When X is a topological (Lie) group then $L_cX(L_dX)$ has the structure of a topological (Lie) group under pointwise multiplication of loops. I shall later show that the differentiable and continuous loop spaces are homotopic and hence they share many properties. When dealing with facts and properties equally applicable to either the differentiable, piecewise differentiable or continuous loops I shall drop the subscripts and use LX and ΩX.

12.2. THE LOOP MAP. If X and Y are two manifolds and f is a continuous (differentiable) map

$$f : X \to Y$$

then there is a continuous (differentiable) map, the loop of f, defined by

$$Lf : LX \to LY$$

$$\gamma \mapsto f \circ \gamma$$

If $P(M, G, f)$ is a locally trivial principal G–bundle then $LP(LM, LG, Lf)$ is a locally trivial principal LG–bundle. Furthermore when P is the universal bundle, $EG(BG, G)$, we see that LEG is also a contractible space and hence

$$BLG = LBG.$$

All of this holds mutatis mutandis for the based loops. In the case of differentiable loops, both L_dX and L_pX have the structure of differentiable Frechet manifolds when given the Frechet topology (See Carey and Murray (1991)).

12.3. TRANSGRESSION. The slant product (see Spanier pg 287) and the evaluation map can be used to define a homomorphism, called transgression, between the cohomologies of a space and its loop space,

$$\tau^q : H^{q+1}(M, A) \to H^q(LM, A).$$

Let $ev : LM \times S^1 \to M$ be the evaluation map and if, i is the fundamental class of $H^1(S^1, A)$, define I^q by

$$I^q : H^{q+1}(LM \times S^1, A) \to H^q(LM, A)$$

$$y \mapsto y/i.$$

Then

$$\tau^q = I^q \circ ev^*.$$

12.4. THE PATH FIBRATION. Working with the loop spaces, L_cX and Ω_cX (or L_pX and Ω_pX) , has the advantage that one can use the path fibration to realise a CW_0–universal bundle for Ω_cX (resp. Ω_pX) if $X = G$ is a compact, simply connected Lie group. Let

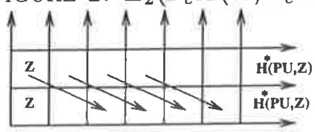
$$P_{c(d)}X := \{\delta : [0,1] \to X, \delta \text{ is continuous (differentiable and } \delta(0) = x_0\}$$

Then there is a fibration with fibre Ω_cX (resp. Ω_pX).

$$\pi : P_{c(d)}X \longrightarrow X$$

$$\delta \mapsto \delta(1)$$

FIGURE 2. $E_2(P_cX(X, \Omega_c X))$



In the case where $X = G$ $P_{c(d)}G(G, \Omega_{c(d)}G)$ is a locally trivial principal $\Omega_{c(d)}G$–bundle and since $P_{c(d)}G$ is contractible, this is a CW_0-universal $\Omega_{c(d)}G$-bundle. Under the assumption that X is p–connected $(p > 1)$ $\Omega_{c(d)}X$ is 1–connected and so $E_2^{(p,q)}(P_cX) = H^p(X, H^q(\Omega_c X, \mathbb{Z}))$. We have the graded algebra differentials, d_p and d_{p+1} which I draw on the one grid in the Figure.

The contractibility of P_cX entails

$$d_q : H^{q-1}(\Omega_c X, \mathbb{Z}) \cong H^q(X, \mathbb{Z}) \quad q = p+1 \text{ or } p+2 \tag{7}$$

13. KILLINGBACK'S RESULT

In this section I continue to confine my attention to cases where G is a compact, connected and simply connected Lie group and I consider string structures for bundles with fibre $\Omega_c G$ We can consider merely continuous based loops and maintain the full generality of the situation for the following reasons. Firstly, as I shall show, the obvious inclusions

$$\Omega_d G \hookrightarrow \Omega_p G \hookrightarrow \Omega_c G$$

are homotopy equivalences. This means that, for a Lie group G, isomorphism classes of $\Omega_d G$, $\Omega_p G$ and $\Omega_c G$ bundles are in 1–1 correspondence via the obvious bundle inclusions. It follows, in particular, that the problem of finding a string structure is identical in the case of $\Omega_d G$ and $\Omega_p G$. The following commutative diagram makes this clear:

$$\begin{array}{ccc} H^1(M, \Omega_p G) & \cong & H^1(M, \Omega_d G) \\ {\scriptstyle D}\downarrow & & {\scriptstyle D}\downarrow \\ H^2(M, \underline{S^1}) & \cong & H^2(M, \underline{S^1}) \end{array}$$

For a principal G-bundle over a manifold M, $P(M, G)$, $D[\Omega_p P] = 0$ if an only if $D[\Omega_d P] = 0$. This links the work of Carey and Murray and MacLauglin Moreover since LG is homeomorphic to $\Omega G \times G$ we need only consider based loops. So if ϕ is the obvious projection $\phi : LG \to \Omega G$ the correspondence between circle bundles and second integral cohomology entails that

$$\widehat{L_{d(p)}}(L_{d(p)}G, S^1) = \phi^* \widehat{\Omega_{d(p)}G}(\Omega_{d(p)}G, S^1).$$

Now ϕ can be extended to a fibre-map for any $L_d(p)G$ bundle and it follows that $L_d(p)P$ has a string structure if $\Omega_{d(p)}P$ has one.

PROPOSITION 3.6. *(HOMOTOPY TYPE OF $\Omega_c X$ AND $\Omega_d X$) Let X be a differentiable manifold of finite or infinite dimension, then $\Omega_c X$ $\Omega_p X$ and Ω_d have the same homotopy type.*

PROOF. I shall show that the obvious inclusions $i : \Omega_d X \hookrightarrow \Omega_p X$, $j : \Omega_p X \hookrightarrow \Omega_c X$ and $j \circ i$ are weak homotopy equivalences. Then, since $\Omega_c X$, $\Omega_p X$ and $\Omega_d X \in CW_0$, it will follow that they are of the same homotopy type. Firstly, we start with some standard notation and the case of $j \circ i$.

$$I^n := \{(y_0, \dots, y_{n-1}) \in R^n : 0 \le y_i \le 1\}$$

$$dI^n := \{(y_0, \dots, y_{n-1}) \in R^n : y_i = 0 \text{ or } 1 \text{ for some } i\}$$

$$C((X, A), (Y, B)) = \{f \in C(X, Y) : f(A) \subset B\}$$

Then $\pi_q(X) = [(I^q, dI^q), (X, x_0)]$. Recall the 1–1 correspondence between the sets of maps

$$\phi : C((I^n, dI^n), (\Omega_c X, x_0)) \to C((I^{n+1}, dI^{n+1}), (X, x_0)).$$

$$\phi(f)(y_0, y_1 \dots, y_n) = f(y_1, \dots, y_n)(y_0)$$

(Here x_0 denotes both the base point of X and the constant loop onto it.) It is well known that ϕ descends to a isomorphism on the homotopy groups

$$\phi_* : \pi_n(\Omega_c X) \cong \pi_{n+1}(X).$$

Observe also that if $g \in C((I^{q+1}, dI^{q+1}), (X, x_0))$ is differentiable then $\phi^{-1}(g) \in C((I^q, dI^q), (\Omega_d X, x_0))$. So now we can show that

$$(j \circ i)_* : \pi_q(\Omega_d X) \to \pi_q(\Omega_c X)$$

is bijective. From 17.8 and 17.8.1 of Bott and Tu, it follows that there is a differentiable map, g, in the homotopy class of $\phi(f)$ (surjectivity of $(j \circ i)_*$) and that any two differentiable maps, $\phi(f_0)$ and $\phi(f_1)$ which are continuously homotopic are homotopic via a path of differentiable maps (injectivity of $(j \circ i)_*$). This argument also shows that j is a weak homotopy equivalence and thus so too is i. □

Let us now turn to the general situation for $\Omega_p G$. Start with a principal $SO(n)$–bundle, $P(M, SO(n), f)$ $(n > 2)$, (typically P is the frame bundle of the tangent bundle of a Spin manifold M) that has a $Spin(n)$–structure $Q(M, Spin(n), \hat{f})$. Note that $Spin(n)$ is 2-connected and so satisfies the hypothesis for equation (7). The universal Dixmier–Douady class for $\Omega_c Q$–bundles, call it μ following MacLauglin, generates $H^3(B\Omega_p Spin(n), \mathbb{Z}) = \mathbb{Z}$. In order to show the commutativity of the diagram we will need for Killingback's thesis we first need to consider the classifying map of Q, $\hat{f} : M \to BSpin(n)$. Taking continuous path spaces as a functor on CW_0 (the "path of a map", $P_d f$, is defined in the obvious manner via composition) we obtain the following diagram of fibrations.

$$\begin{array}{ccc} \Omega_p M \hookrightarrow P_d M & \xrightarrow{P_d \hat{f}} & E = P_d BSpin(n) \hookleftarrow \Omega_p BSpin(n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\hat{f}} & BSpin(n) \end{array}$$

One now applies Note 1.21. Further, assume that M is 2–connected itself which entails that

$$Pf^* | E_r^{(0,3)}(E) = \Omega_p \hat{f}^* \text{ and } Pf^* | E_r^{(4,0)}(E) = \hat{f}^*.$$

Thus

$$d_4^{P_d M} \circ \Omega_p \hat{f}^* = \hat{f}^* \circ d_4^E.$$

The crucial diagram is comes from simply looping $\hat{f} : M \to BSpin(n)$ and moving to cohomology.

$$\begin{array}{ccc} H^3(B\Omega_p Spin(n), \mathbb{Z}) & \xrightarrow{d_4^E} & H^4(BSpin(n), \mathbb{Z}) \\ \Omega_c \hat{f}^* \downarrow & & \hat{f}^* \downarrow \\ H^3(\Omega_p M, \mathbb{Z}) & \xrightarrow{d_4^{P_c M}} & H^4(M, \mathbb{Z}) \end{array}$$

We have just shown that this is commutative. Since $BSpin(n)$ is 3–connected and M is assumed 2–connected, d_4^E and $d_4^{P_d M}$ are isomorphisms by equation (7). Hence $d_4^E(\mu)$ $(= x$ say) generates $H^4(BSpin(n), \mathbb{Z})$. Now,

$$\begin{aligned} \text{P has a string structure} &\iff \Omega_c \hat{f}^*(\mu) = 0 \\ &\iff d_4^{P_d M}(\Omega_p \hat{f}^*(\mu)) = 0 \\ &\iff \hat{f}^*(d_4^E(\mu)) = 0 \\ &\iff \hat{f}^*(x) = 0. \end{aligned}$$

Now, MacLauglin in his Lemma 2.2 shows by analysing the spectral sequence of the bundle

$$BSO(n)(BSpin(n), B\mathbb{Z}_2)$$

that

$$2.\hat{f}^*(x) = P_1(P),$$

where $P_1(P)$ is the first Pontryagin class of P. Thus when M is 2–connected, the vanishing of half the Pontryagin class is necessary and sufficient for the existence of a string structure for P.

In the case where M is not 2–connected we encounter technical difficulties with spectral sequences so that while functorality will ensure that $D(\Omega_p Q) = \Omega_p \hat{f}^*(\mu)$ stays in the domain of $d_4^{P_d M}$ and $d_4^{P_d M}$ will remain injective, the range of $d_4^{P_d M}$ is now a coset space of $H^4(M, \mathbb{Z})$, $H^4(M, \mathbb{Z})/B$ say. We have only that

$$2.d_4(\Omega_p \hat{f}^*(\mu)) = [P_1(P)] = P_1(P) + B.$$

Thus, the vanishing of $(1/2)P_1(P)$ is now only a sufficient but not necessary condition for the existence of a string structure on P.

One drawback of the above method is that differentials of spectral sequences are notoriously difficult to calculate. The transgression homomorphism, in this case τ^4, is much more amenable to calculation (as seen in Carey and Murray say). MacLauglin shows in general that for M p–connected, τ^q is injective for $q \leq p+2$ and in specific that

$$\tau^4 : H^4(BSpin(n), \mathbb{Z}) \cong H^3(\Omega_p BSpin(n), \mathbb{Z}).$$

Up to sign τ^4 is thus the inverse of d_4. In particular when M is 2–connected we see that the Dixmier–Douady class of $\Omega_c Q$ is given by

$$2.D(\Omega_c Q) = \tau^4(P_1(P))$$

and that the vanishing of $P_1(P)$ is sufficient for the existence of a string structure on P otherwise.

14. THE RESTRICTED UNITARIES

My final example, the restricted unitaries, is an interesting group in quantum field theory and quantum statistical mechanics as well as pure mathematics. One of its most important features is that it is the subgroup of the automorphisms of the CAR–algebra which are implementable in the Dirac representation of the latter as an algebra of particle creation and annihilation operators. To make this precise I shall spend a little time reminding the reader of the central definitions associated with abstract quantum field theory for fermions.

In what follows it is crucial to polarise standard Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by infinite dimensional subspaces $\mathcal{H}+$ and $\mathcal{H}_-$ which are the range of the self adjoint projections P_+ and P_- respectively, $Id_{\mathcal{H}} = P_+ + P_-$. The restricted unitary group relative to a polarisation is defined by

$$U_{res}(\mathcal{H}, P_+) = \{u \in U(\mathcal{H}) : P_{\pm} u P_{\mp} \text{ is Hilbert Schmidt}\}.$$

Typically the Hilbert space and polarisation are understood and omitted from the notation. U_{res} is not equipped with any of subspace topologies it might have received from $U(\mathcal{H})$. There are in fact two other topologies of interest on U_{res}. The first of these (I will write U_{res1}) is defined by the metric ρ,

$$\rho(u_1, u_2) = |P_+(u_1 - u_2)P_+| + |P + -(u_1 - u_2)P + -|$$
$$+ |P_+(u_1 - u_2)P_-|_{HS} + |P_-(u_1 - u_2)P_+|_{HS}.$$

Where $|\quad|_{HS}$ denotes the Hilbert-Schmidt norm on the Hilbert–Schmidts. It is known (see Pressley and Segal p 80) that U_{res1} is a Banach Lie group. The second topology on U_{res} (I shall write U_{res2}) is defined using the strong operator topology on the on-diagonal components and the Hilbert-Schmidt topology on the off-diagonal components. It makes U_{res2} into a topological group (see Carey (1984) section 2 for a full account). The first topology is finer than the second and thus the identity $id : U_{res1} \to U_{res2}$ is continuous. It can be shown that id is a weak homotopy equivalence. When considering merely the set U_{res} or when discussing properties which apply equally to U_{res} in either topology I shall simply write U_{res}.

If (,) denotes the inner product on $\mathcal{H}$, then the CAR (canonical anti–commutation relation) algebra over $\mathcal{H}$, $CAR(\mathcal{H})$ is defined to be the C^*–algebra generated by the set

$$\{a(f), a^*(f), f \in \mathcal{H}\}$$

whose elements satisfy the canonical anti–commutation relations

$$a(f).a(g) + a(g)a(f) = 0$$

$$a(f).a^*(g) + a(g^*).a(f) = (f, g).$$

The $a^*(f)$ correspond to creating a fermion in the state f, the $a(f)$ correspond to the annihilation of such a fermion. The anti–commutation relations capture the Pauli exclusion principal for fermions. Any unitary $u \in U(\mathcal{H})$ allows one to define an automorphism of $CAR(\mathcal{H})$ in the obvious manner by

$$\alpha_u((a(f)) = a(u.f) \quad \alpha_u((a^*(f)) = a^*(u.f)$$

Automorphisms of this form are called Bogolibuov transformations. If we take $\mathcal{H}$ to be the solution space to the Dirac equation, $\mathcal{H}^+$ the space of positive energy solutions (particles) and $\mathcal{H}^-$ the space of negative energy solutions (anti–particles) then the Dirac representation of $CAR(\mathcal{H})$ makes the vacuum state, ω, into a state containing all anti–particle states and no particle states. Mathematically this is given by a state on $CAR(\mathcal{H})$

$$\omega(a^*(f_1)...a^*f_m)a(g_n)...a(g_1) = \delta_{(m,n)}det(g_i, P^- f_j)$$

Since $CAR(\mathcal{H})$ is simple, any representation is trivial or faithful and since the GNS representation, π , defined by ω, is not trivial it maps faithfully into $B(\mathcal{H}_\pi)$. Additionally, π is an irreducible representation. The precise statement of my introductory remarks is the theorem (see Shale and Stinespring 1965) that, given a Bogoliubov transformation $\alpha(u)$, there exists a unitary $W(u) \in U(\mathcal{H}_\pi)$ such that

$$\pi(\alpha(u)(a(f)) = \pi(a(u.f)) = Ad(W(u))(\pi(a(f)) = W(u)\pi(a(f))W(u)^*$$

if and only if $u \in U_{res}(\mathcal{H})$. Since π irreducible, $W(u)$, is uniquely defined up to a scalar which is killed by the adjoint. Hence the above defines an embedding (continuous in either topology on U_{res})

$$i : U_{res} \hookrightarrow PU(\mathcal{H}_\pi)_{so}$$

of the restricted unitaries of $\mathcal{H}$ in the projective unitaries on $\mathcal{H}_\pi$. It is a corollary of a proof of Carey's (see Carey 1984 Lemma 2.10) that $i(U_{res})$ is closed in $PU(\mathcal{H}_\pi)_{so}$. In fact, $i_2 : U_{res2} \to i(U_{res2})$ is a homeomorphism of topological groups. This and the fact that $id : U_{res1} \to U_{res_2}$ is a weak homotopy equivalence means that $i_1 : U_{res1} \to i_1(U_{res1})$ is a continuous homomorphism of topological groups onto it's closed image which is a weak homotopy equivalence.

We shall see below that $H^2(U_{res}, \mathbb{Z}) = \mathbb{Z}$. The canonical central extension of U_{res}, $\hat{U}_{res}$, defined by the generator of $H^2(U_{res}, \mathbb{Z})$, is given by

$$\hat{U}_{res}(U_{res}, S^1) = i^*U(\mathcal{H}_\pi)_{so}(PU(\mathcal{H}_\pi)_{so}, S^1).$$

Hence the assumptions of Section 2 are fulfilled for either topology on U_{res}. Finally, note that U_{res} is a disconnected group with $\mathbb{Z}$ path connected components (see Carey and O'Brien 1981 for a definition of a topological index, in: $U_{res} \longrightarrow \mathbb{Z}$) and we denote the connected component by U^0_{res}. From now on I shall drop reference to the different (yet as always isomorphic) Hilbert spaces over which U_{res} and PU are defined and it shall be understood that PU refers to the projective unitaries on $\mathcal{H}_\pi$ and not $\mathcal{H}$. Furthermore it will not matter whether we are considering U_{res1} or U_{res2} (for example, isomorphism classes of principal bundles with these two groups as structure group are in bijective correspondence in CW_0) and so I shall simply write U_{res} from now on. Also PU shall denote PU_{so} for the final sections.

That ends the requisite introduction and I now turn to consider the homotopy properties of U_{res}, its role as a classifying space for $U(\infty)$ and the relation between U_{res} and PU bundles.

15. U_{res} AS A CLASSIFYING SPACE

The group of unitaries with determinant, $\mathcal{D}$ consists of those operators of the form 1+ trace class. By considering $\mathcal{D}(\mathcal{H}+)$, Pressley and Segal (see Ch 6) show that there is a principal $\mathcal{D}$–bundle over U^0_{res} with contractible total space and hence U^0_{res} is a $B\mathcal{D}$. So U^0_{res} is a CW_0–classifying space for $\mathcal{D}$ and $U(\infty)$,

$$U^0_{res} \simeq B\mathcal{D} \simeq BU(\infty).$$

It is known (see Pressley and Segal Ch 6) that $\mathcal{D}$ is of the same homotopy type as the direct limit of the finite unitaries:

$$\mathcal{D} \simeq U(\infty) = \lim_{n\to\infty} (U(n)).$$

Since the homotopy groups of $U(\infty)$ are well known by Bott periodicity we have that

$$\pi_q(U_{res}) = \begin{cases} \mathbb{Z}, & q \text{ even}, \\ 0 & q \text{ odd}. \end{cases}$$

(This result has elsewhere been proven via methods more closely tied to U_{res}'s structure as a group of operators, see Carey (1983).) $U(\infty)$ and $BU(\infty)$ are extremely important spaces in algebraic topology because they are the classifying spaces for reduced K–theory.

Proposition 3.7 (The homotopy type of U_{res} & BU_{res}).

$$BU_{res} \simeq U(\infty) \quad U_{res} \simeq \Omega_c U(\infty).$$

Proof. It is known that the embedding of $\Omega_d U(n) \hookrightarrow U_{res}$ extends to $i : \Omega_d U(\infty) \hookrightarrow U_{res}$ and one can check that this is a weak homotopy equivalence and hence a homotopy equivalence (see Pressley and Segal pp 82-5 for these details). Now, by Proposition 3.6 $\Omega_d U(\infty) \simeq \Omega_c U(\infty)$, so $U_{res} \simeq \Omega_d U(\infty)$. Remember also that via the path fibration $B\Omega_c G = G^0$ and that (restricting as we are to CW_0) $H \sim G$ entails $BH \simeq BG$. Thus

$$BU_{res} \simeq B\Omega_d U(\infty) \simeq B\Omega_c U(\infty) \simeq U(\infty).$$

□

NOTE 3.8. Now $\tilde{\mathcal{K}}_1(X)$ of a space, X, is defined to be the stable isomorphism classes of vector bundles over the reduced suspension of X, ΣX. Thus,

$$\begin{aligned}\tilde{\mathcal{K}}_1(X) &= [\Sigma X, BU(\infty)] \\ &= [X, \Omega_c BU(\infty)] \\ &= [X, B\Omega_c U(\infty)] \\ &= [X, BU_{res}] \\ &= Bun_X(U_{res}).\end{aligned}$$

Elements of $\tilde{\mathcal{K}}_1$ correspond bijectively with $U(\infty)$-bundles over ΣX whilst U_{res}-bundles correspond bijectively with $\Omega_c U(\infty)$-bundles. So our correspondence becomes a mapping between $U(\infty)$–bundles over the reduced suspension of a space and $\Omega_c U(\infty)$–bundles over that space. We now exploit this observation.

16. THE DIXMIER DOUADY CLASS AND THE SECOND CHERN CLASS

Regarding U_{res} as a subgroup of PU via the inclusion mentioned in Section 11, we may ask when can we reduce the structure group of a PU–bundle, $P(M, PU, f)$ to U_{res}? Now, PU is not a Banach Lie group in the strong operator topology so there is no guarantee that $\pi : PU \to PU/U_{res}$ defines a principal U_{res}–bundle. Hence, I use the theory of the more general setting of Theorem 2.4 4 and translate the question of the reduction of structure group from PU to U_{res} into a search for maps $\hat{f}$ such that $f = g \circ \hat{f}$. Where we take $g : BU_{res} \to BPU$ to be a fibration with fibre F.

$$\begin{array}{c} BU_{res} \simeq U(\infty) \\ g \big\downarrow \\ BPU \simeq K(\mathbb{Z}, 3) \xleftarrow{\;f\;} M. \end{array}$$

In general we know that if there were a section of g, s, then this would entail the existence of group homomorphisms

$$g^* : H^*(BPU, \mathbb{Z}) \to H^*(BU_{res}, \mathbb{Z})$$

$$s^* : H^*(BU_{res}, \mathbb{Z}) \to H^*(BPU, \mathbb{Z})$$

such that

$$s^* \circ g^* = (g \circ s)^* = id.$$

It is a group theoretic result that this implies that $H^*(BPU, \mathbb{Z})$ would be a direct summand of $H^*(BU_{res}, \mathbb{Z})$. But we know (See Bott and Tu pp 245–246) that $H^*(BPU, \mathbb{Z})$ has torsion where as $H^*(BU_{res}, \mathbb{Z})$ is a free group. Therefore the sought after section cannot exist and the structure groups of some PU–bundles do not reduce to U_{res}.

The situation in specific instances depends in part on the homotopy groups of the fibre, F, which we can compute in this case by noting that $i_{*,q} : \pi_q(U_{res}) \to \pi_q(PU)$ is an isomorphism for $q = 2$ and null otherwise. It follows by Lemma 2.3 that $g_{*,q} : \pi_q(BU_{res}) \to \pi_q(BPU)$ is an isomorphism for $q = 3$ and null otherwise. By considering the long exact homotopy sequence of the fibration

$$F \hookrightarrow BH \xrightarrow{g} BG$$

we see that

$$\pi_q(F) = \begin{cases} \mathbb{Z}, & q \text{ odd} \neq 3, \\ 0 & q \text{ even or } 3. \end{cases}$$

Now the cohomology, $H^n(K(\mathbb{Z},3))$, of $K(\mathbb{Z},3)$ is zero for $n = 1$ and torsion for $n > 3$ (see Bott and Tu pp 245–246). Hence obstructions to lifting f can lie only in $H^{2n+4}(M,\mathbb{Z})$ ($n \geq 1$). So the structure group of any PU–bundle over a space with free, even (greater than fourth) cohomology groups reduces to U_{res}.

We can cast this question of reducing the structure group of a U_{res} bundle in a slightly different light by exploiting the correspondence between U_{res}–bundles over a space M and $\tilde{\mathcal{K}}_1(M)$. There is a well known isomorphism of cohomology,

$$\Sigma^q : H^q(M,\mathbb{Z}) \cong H^{q+1}(\Sigma M, Z)$$

which one can obtain from the Mayer-Vietoris sequence for $(\Sigma M, CM, CM)$ (where "CM" denotes the cone of M) or by using the adjoint relation between Σ and Ω_c considered as functors on CW_0

$$(M \xrightarrow{f} \Omega_c K(\mathbb{Z}, q+1) = K(\mathbb{Z},q)) \longleftrightarrow (\Sigma M \xrightarrow{\Sigma f} K(\mathbb{Z}, q+1)).$$

The suspension isomorphism can be used to link the Dixmier–Douady class of a U_{res}–bundle over M to the second Chern class of the associated $U(\infty)$–bundle over ΣM as follows.

Proposition 3.8. — *THE DIXMIER DOUADY CLASS AND THE SECOND CHERN CLASS: Let $P(M, U_{res}, f)$ be a principal U_{res}–bundle over M and $\Sigma P(\Sigma M, U(\infty))$ the associated $U(\infty)$–bundle (element of $\tilde{\mathcal{K}}_1(M)$) over M. Then*

$$\Sigma^3(D(P)) = \pm c_2(\Sigma P).$$

Proof. Include $U(1) \hookrightarrow U(\infty)$ by $\lambda \mapsto \lambda . Id_{U(\infty)}$. Notice that this inclusion kills the first homotopy group of $U(\infty)$ to make $U(\infty)/U(1)$ 2–connected. The second Chern class of a $U(\infty)$–bundle can be defined in an analogous manner to that for a $U(n)$–bundle as the primary obstruction to lifting the the classifying map of ΣP, Σf, to the total space of the bundle $BU(1)(BU(\infty),\ U(\infty)/U(1))$.

$$\begin{array}{ccc} & & BU(1) \hookleftarrow U(\infty)/U(1) \\ & & \downarrow \\ \Sigma M & \xrightarrow{\Sigma f} & BU(\infty) \end{array}$$

$$c_2(\Sigma P) = o(\Sigma f, BU(1)) = \Sigma f^*(o(id_{BU(\infty)}, BU(1))) \tag{8}$$

Now, the adjoint relation between Σ and Ω_c and the construction of the primary obstruction to a lift relates (via Σ^3) $o(\Sigma f, BU(1))$ to the primary obstruction to lifting defined in the following diagram.

$$\begin{array}{ccc} \Omega_c(U(\infty)/U(1)) \hookrightarrow \Omega_c BU(1) & & \\ \downarrow & & \\ BU_{res} = \Omega_c BU(\infty) & \overset{f}{\longleftarrow} & M \end{array}$$

(9) $$\Sigma^3(o(f, \Omega_c BU(1))) = \Sigma^3(f^*(o(id_{\Omega_c BU(\infty)}, \Omega_c BU(1)))) = o(\Sigma f, BU(1)) = c_2(\Sigma P)$$

Now consider the case where $M = S^3$ (thus ΣM is homotopic to S^4). One can show by considering long exact sequences of the fibration $U(n)(S^{2n-1}, U(n-1))$ (n large) that the Hurewicz homomorphism is an isomorphism on $\pi_3(BU_{res}) = \pi_3(U(\infty))$. Using duality between homology and cohomology, let $[f] \in \pi_3(BU_{res})$ correspond to, D, the universal Dixmier Douady class. Note that f induces isomorphisms, f_* and f^*, on third homotopy and cohomology respectively. So we have that $f^*(D)$ generates $H^3(S^3, \mathbb{Z})$.

By the adjointness of Σ and Ω_c, $[\Sigma f]$ generates $\pi_4(BU(\infty))$ and defines an isomorphism, Σf_*, on fourth homotopy. One can also show that

$$\Sigma f^* : H^4(BU(\infty), \mathbb{Z}) \longrightarrow H^4(S^4, \mathbb{Z})$$

is a surjection taking c_2, the universal second Chern class, to a generator of $H^4(S^4, \mathbb{Z})$. Setting $P = f^*EU_{res}$ we see that $c_2(\Sigma P) = \Sigma f^*(c_2)$ and thus $c_2(\Sigma P)$ is a generator of $H^4(S^4, \mathbb{Z})$. Now $\Sigma^3 : H^3(S^3, \mathbb{Z}) \cong H^4(S^4, \mathbb{Z})$ so, applying equation 9, we see that $f^*(o(id_{\Omega_c BU(\infty)}, \Omega_c BU(\infty)))$ generates $H^3(S^3, \mathbb{Z})$. Thus

$$f^*(o(id_{\Omega_c BU(\infty)}, \Omega_c BU(\infty))) = \pm f^*(D).$$

Since f^* is an isomorphism, it follows that $D = \pm o(id_{\Omega_c} BU(\infty), \Omega_c BU(\infty))$. Now applying equation 9 we see that $\Sigma^3(D) = \pm c_2$ as required. □

NOTE 3.9. To sum up, the structure of group of a PU–bundle, $Q(M, PU)$ reduces to U_{res} if and only if there is a U_{res} bundle, $P(M, U_{res})$ whose Dixmier–Douady class coincides with that of Q. This, we have just seen, happens if and only if there is a $U(\infty)$–bundle, $\Sigma P(\Sigma M, U(\infty))$ over ΣM such that $c_2(\Sigma P) = \Sigma^3(D(Q))$. We know from above that one cannot, in general, construct a $U(\infty)$–bundle with an arbitrary second Chern class on any given space. This differs from the case for the first Chern class where one can always find a line bundle, and hence a $U(\infty)$–bundle, for any given element of $H^2(M, \mathbb{Z})$. Note that any U_{res}–bundle, $P(M, U_{res})$ defines a class in $H^1(M, \mathbb{Z})$ which is the obstruction to reducing the structure group of P to U^0_{res}. By similar arguments to those above on can show that this corresponds (via Σ^1) to the first chern class of the $U(\infty)$-bundle over ΣM which corresponds to P.

17. CONCLUDING REMARKS

This thesis has heavily exploited the power of the categorical approach to homotopy theory to produce a summary of the theory of principal bundles. A systematic consideration of the problems of the reduction and the extension of structure group has therefore been possible using non-abelian cohomology, obstruction theory and spectral theory. I have also briefly sketched the mathematical reasons for being interested in such problems and some of the information contained in the structure group of a bundle. For example, my final example, U_{res} is an important group in the quantum field theory of fermions since it is a natural choice of a physically meaningful structure group for a CAR–bundle. Its relation to the functors of K–theory and the mathematical machinery there developed may prove informative. For the reader's time and attention, many thanks.

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